

Learning, Cascades and Transaction Costs*

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Abstract

This paper analyzes the effect of transaction costs on the social learning in an asset market with asymmetric information, sequential trading and competitive price mechanism. Both fixed and proportional transaction costs reduce the informational content of trading orders and lead to informational cascades. If transaction costs are very high, an informational cascade can occur not only when beliefs converge to a specific asset value, but also when in the market there is complete uncertainty about the asset's fundamental value. Finally, if the asset value in the bad state is sufficiently low, proportional transaction costs lead to an informational cascade only when prices are very high.

1 Introduction

In most market microstructure literature, it is accepted that when agents are asymmetrically informed and trades occur sequentially, the market gradually learns fundamentals and, eventually, prices converge to fundamental value.¹ In other words, in the long run, asset markets are viewed as being informational efficient.² However, recent theoretical literature has advanced new models of asset market behavior that are not in accordance with the efficient market

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¹See Glosten and Milgrom [6], Kyle [9].

²Informational efficiency refers to how much information is revealed by the price process. This is important in economies where information is dispersed among many individual. Prices are informational efficient if they fully and correctly reflect the relevant information. If prices do not correctly and fully reflect public information, then there would be a profitable trading opportunity for individuals. In general, this is ruled out in models with rational utility maximizing agents.

hypothesis.³ In particular, the literature on informational cascades shows that sequential actions can produce herd behavior by rational agents and lead to a complete or partial information blockage.⁴

The term "informational cascade" was coined by Bikhchandani, Hirshleifer and Welch [2] to describe a situation in which, in a sequential trade framework, every agent, based on the observation of previous agents, makes the same choice independent of his private information. The idea they illustrate by their simple model is the following. When individuals observe imperfect private information and make publicly observed actions, every individual will try to infer his predecessors' information from their choices, provided these predecessors based their choices on their private signals. If early actions show a clear pattern, later agents may optimally choose to imitate the action of previous agents regardless of their private information. Hence, once an informational cascade starts, the private information of subsequent individuals never joins the public pool of knowledge. Bikhchandani, Hirshleifer and Welch [2] show that this type of behavioral convergence is: idiosyncratic, because the behavior of many followers depends on the first few individuals; path dependent, because the order of moves and information arrival affects the outcomes; fragile, because the arrival of a little new information may breakdown easily a cascade.

The basic cascades model applies in fixed-price contexts. Thus, it cannot be directly applied to asset markets, where prices change to take into account the information revealed by trades. Avery and Zemsky [1] prove that, in the standard Glosten-Milgrom [6] setting, price adjustments prevent informational cascades. Besides, they find that with multidimensional uncertainty, individuals may optimally act in opposition to their private information. Intuitively, this happens because price is a single-dimensional instrument and it only assures that the market learns about one dimension of uncertainty at a time. Hence, in a market with multiple dimensions of uncertainty, a short run mispricing may occur because market makers and informed traders interpret differently the history of trades. However, in the Avery and Zemsky model, the flow of information to the market never stops because the market continues to learn about at least one of the dimensions of uncertainty. Therefore, herding does not

³See Kortian [8] for a survey of recent theoretical literature on asset price formation.

⁴See Hirshleifer and Teoh [7] for a survey on the theoretical and empirical literature on herd behavior and cascades in capital markets.

impede prices to converge to the fundamental value, that is, in the long run, the market is informational efficient.

Decamps and Lovo [4] however find that, with a sequential trading structure, differences in risk aversion between market makers and informed traders can generate informational cascades if the minimum and maximum size of trade per period are fixed. More specifically, they analyze the case of risk averse traders and risk neutral market makers. In their model, an informational cascade occurs because as prices become more informative, the traders' informational advantage vanishes and orders only reflect the inventory imbalance of traders. Cipriani and Guarino [3] reach similar results by considering heterogeneous traders in a multiple security setting. Moreover, they find that informational cascades can spill over from one asset to the other pushing prices far from fundamentals value, even in the long run.

Since market microstructure is mainly about the effect of trading frictions on price formation, it is natural to ask whether such frictions are more conducive to informational cascades. Indeed, Lee [10] prove that in a stock market with fixed transaction costs and sequential trading, the market may fail to aggregate information and partial or total informational cascades occur. However, in Lee [10] the market maker is not maximizing the profit. At each trading round, he is assumed to set the price equal to the expected asset value conditional on the history of past trades.

In this paper, we relax this hypothesis and analyze a Glosten - Milgrom type model with fixed transaction costs.⁵ We prove that when the bid and ask prices are set by competitive market makers who bear a fixed cost per transaction, the market is not informational efficient. In other words, the price mechanism cannot prevent the occurrence of an informational cascade. In tune with the finding of Lee [10], we show that during the informational cascade, no informed traders will trade. This implies that in a cascade, the adverse selection component of bid-ask spread disappears and the spread decreases.

We also find a positive correlation between bid-ask spreads and trading volume. This is because traders endowed with private signals with moderate informational content place high weight on the

⁵Glosten and Milgrom ([6]) introduce fixed transaction costs in their model to show that if the market maker pay a fixed processing cost then transaction prices changes exhibit negative serial correlation.

previous price history in their investment decisions. Therefore, they will prefer to refrain from trading because of transaction costs. Namely, before the cascade occurs, the trading volume will gradually decrease and it will reach the minimum as the cascade develops.

Moreover, we show that if transaction costs are high enough, an informational cascade may occur even when the market is completely uncertain about the fundamental value. This is in contrast with results in the previous literature on informational cascades in stock markets which highlights that cascades develop when there is a convergence of beliefs in the market.

Finally, we extend the analysis to include also the case of a market characterized by proportional transaction costs. We find that in this setting, if the true asset value in the bad state of nature is sufficiently low, then the probability of an informational cascade approaches zero when the prices are low. This has the interesting implication that cascades will tend to be asymmetric. They will seldom emerge in depressed markets, while they are more likely to develop in bull markets. As a consequence, they are more likely to result in market crashes than in price jumps.

2 The environment

We consider a sequential trade model similar to Glosten and Milgrom [6]. The market is for a single risky asset whose value \tilde{V} depends on the state of nature. If the true state of nature is good, the asset value is \bar{V} ; if it is bad, the value of the asset is \underline{V} , with $\bar{V} > \underline{V} \geq 0$. We assume that the initial prior probability $\pi_0 = P(\bar{V})$ of the high value is non-degenerate, that is, $\pi_0 \in (0, 1)$. The asset is exchanged among traders and market makers who are responsible for quoting prices. Trades take place in a sequential fashion, with one trader allowed to transact at any point in the time. Before a trader arrives, market makers simultaneously announce the bid and ask prices at which they are willing to buy and sell one unit of the asset. We assume that market makers are risk neutral and act competitively. In addition, we suppose that they incur a fixed order-processing cost c . The trader arriving in the market observes the prices and has the option to sell or buy one unit of the asset at the most attractive bid and ask prices, or to refrain from trading. The trader leaves the market after he had the opportunity to trade.

He may trade further, but only after returning to the pool of traders and being selected again to trade.

A fraction μ of traders are uninformed liquidity traders while $1-\mu$ are informed⁶. Liquidity traders trade for reasons exogenous to the model. To simplify the analysis, we assume that they choose to sell, buy, or refrain from trading with equal probability⁷. Informed traders are risk neutral, price taking agents⁸ that privately observe a signal θ correlated with the asset value. They trade to maximize their expected profit. We denote $\Theta = \{\theta_1, \theta_2, \dots, \theta_N\}$ the set of private signals, and assume that the private signals are conditionally independent and satisfy the monotone likelihood property:

$$0 < \lambda_1 < \lambda_2 < \dots < 1 < \dots < \lambda_{N-1} < \lambda_N < \infty$$

where $\lambda_n = \frac{P(\theta_n | \tilde{V} = \underline{V})}{P(\theta_n | \tilde{V} = \bar{V})}$.

The signal θ_n is denoted “good” if the probability that a trader observes θ_n when the true asset value is \bar{V} , exceeds the probability of θ_n conditional on \underline{V} ; that is, $\lambda_n < 1$. It is denoted “bad” in the opposite case. For simplicity, we assume that good and bad signals are symmetric, that is $\lambda_1 = \frac{1}{\lambda_N}$, $\lambda_2 = \frac{1}{\lambda_{N-1}}$, and so on.

Both market makers and traders are Bayesians who know the structure of the market. We denote by π_t the probability of \bar{V} conditional on the publicly observable history of trades up until time t . Thus, the public belief about the true asset value at t is:

$$E_t \left[\tilde{V} \right] = \pi_t \cdot \bar{V} + (1 - \pi_t) \cdot \underline{V} = \underline{V} + \pi_t \cdot (\bar{V} - \underline{V})$$

3 The equilibrium

Prior to each trading round, market makers set their prices. Bertrand competition and risk neutrality lead, in equilibrium, market makers

⁶The presence of liquidity traders is needed to guarantee that trading occurs. Indeed, without the presence of traders who trade for reasons other than speculation, the no-trade theorem of Milgrom and Stokey [11] applies and the market breaks down.

⁷In Glosten and Milgrom [6], liquidity traders trade as long as the reservation value dictated by their liquidity needs exceed the ask (bid) price for the buy (sell) order. As a consequence, if there are too many informed traders, then the market maker may have to set a spread so large as to preclude any trading at all. In order to avoid market collapse, we assume that the probability that liquidity traders buy and sell in each period is sufficiently high and is stationary.

⁸This assumption simplify the analysis by ruling out strategic behavior of informed traders. It is reasonable given that the trader's chance to trade again is zero since there are infinitely many informed traders in the pool.

to earn zero expected profit for every possible trade. After prices are set, a trader is randomly selected to trade. If he is informed, he chooses his optimal strategy given the prices. By assuming that market makers act competitively and that informed traders are price takers, we rule out all strategic behavior. Therefore, any trading round can be viewed as a four-stage game.

At stage 1, nature chooses the true asset value. Agents do not observe the realization of \tilde{V} : they know only the probability π of $\tilde{V} = \bar{V}$, which is common knowledge.

At stage 2, nature selects a trader to transact. With probability μ the trader will be uninformed and with probability $1 - \mu$ he will be informed. Informed traders differ from one another in the private signal $\theta \in \Theta$ they observe. There are N types of signals and then N types of informed traders. The signal distribution depends on the true asset value. The conditional distributions $P(\theta | \tilde{V})$ are common knowledge. Agents do not observe the type of the selected trader.

At stage 3, market makers announce the price B_t at which they are willing to buy the asset and the price A_t at which they are willing to sell the asset. Perfect competition restricts the market makers to set these prices so as they earn zero expected profit.

At stage 4, the selected trader observes the prices and plays his strategy. He can choose to place a sell order (*SO*) at the highest bid price, to place a buy order (*BO*) at the lowest ask price, to refrain from trading (*RT*). We denote by $\mathcal{A} \equiv \{SO, BO, NT\}$ the traders' action space. Uninformed traders trade for exogenous reasons and submit sell and buy orders in the ex-ante specified probabilistic way. Informed traders choose the strategy that maximizes their expected profit given the price schedule. We denote $\sigma \equiv \{\sigma_\theta\}_{\theta \in \Theta}$ the informed traders' strategies, where σ_θ is the mixed strategy of the informed if he observes θ . Clearly:

$$\sigma_\theta \equiv (\sigma_{\theta,SO}, \sigma_{\theta,BO}, \sigma_{\theta,RT})$$

where $\sigma_{\theta,i}$ is the probability of i , with $i \in \mathcal{A}$, if the informed observes θ , $\sum_{i \in \mathcal{A}} \sigma_{\theta,i} = 1$, and $\sigma_{\theta,i} \geq 0 \quad \forall i \in \mathcal{A}$.

The expected profit of a market maker is equal to $E_t[\tilde{V} | SO \text{ at } B] - B - c$, if he buys at B , and it is equal to $A - E_t[\tilde{V} | BO \text{ at } A] - c$, if he sells at A . The expected profit of a trader endowed with signal θ , when the price schedule is $P = \{B, A\}$ and he plays the strategy $\sigma \in \Delta(\mathcal{A})$, is $E_t[\Pi_\theta(\sigma | P)] = \sigma_{\theta,SO}(B - E_t[\tilde{V} | \theta]) + \sigma_{\theta,BO}(E_t[\tilde{V} | \theta] - A)$.

Next, we define the equilibrium in the asset market as a sequentially rational Nash equilibrium.

Definition 1 *At each trading round t , the equilibrium in the asset market consists of a trading strategy correspondence $\sigma_\theta^*(P|t) : R_+^2 \rightarrow \Delta(\mathcal{A})$ for each informed trader $\theta \in \Theta$, and a price schedule $P_t^* = \{B_t^*, A_t^*\}$ such that:*

1. $\sigma_\theta^*(P|t) = \arg \max_{\sigma \in \Delta(\mathcal{A})} E_t[\Pi_\theta(\sigma|P)], \quad \forall P \in R_+^2, \quad \forall \theta \in \Theta$
2. $B_t^* \in E_t[\tilde{V}|SO \text{ at } B_t^*, \sigma^*(P_t^*|t)] - c$
3. $A_t^* \in E_t[\tilde{V}|BO \text{ at } A_t^*, \sigma^*(P_t^*|t)] + c$
4. $B_t^* \leq A_t^*$

where $\sigma^*(P_t^*|t) = \{\sigma_\theta^*(P|t)\}_{\theta \in \Theta}$.

The first equilibrium condition means that informed traders maximize their expected profit given the price schedule, trade by trade. The second and third conditions imply that market makers determine the price schedule such that they anticipate zero expected profit from each trade. It is clear that, if there are no fixed transaction costs, the equilibrium bid and ask prices are equal to the conditional expectation of \tilde{V} given a sell or a buy order respectively. The last condition says that at the equilibrium the price at which market makers buy the asset must be lower than (or at most equal to) the price at which they sell the asset.

We can now provide an useful characterization of the equilibrium in term of informed traders' optimal strategy, and then prove the existence and the uniqueness of zero profit equilibrium prices. We will see that equilibrium bid and ask prices straddle the public belief about the true asset value.

The last condition of definition 1 states that the equilibrium bid price does not exceed the equilibrium ask price. Proposition 1 describes the optimal strategies of informed traders when they face a price schedule that satisfies this condition.

Proposition 1 For any $P = \{B, A\}$ such that $B \leq A$, the optimal strategy of the traders endowed with the private signal $\theta \in \Theta$ is $\sigma_\theta^*(P|t)$ equal to:

$$\begin{cases} \sigma_{\theta, SO}^*(P|t) = 1 & \text{if } E_t[\tilde{V}|\theta] < B \\ \sigma_{\theta, SO}^*(P|t) = \alpha & \sigma_{\theta, RT}^*(P|t) = 1 - \alpha & \text{if } E_t[\tilde{V}|\theta] = B \\ \sigma_{\theta, RT}^*(P|t) = 1 & \text{if } B < E_t[\tilde{V}|\theta] < A \\ \sigma_{\theta, BO}^*(P|t) = \beta & \sigma_{\theta, RT}^*(P|t) = 1 - \beta & \text{if } E_t[\tilde{V}|\theta] = A \\ \sigma_{\theta, BO}^*(P|t) = 1 & \text{if } E_t[\tilde{V}|\theta] > A \end{cases}$$

where α and β are any real number belonging to the interval $[0, 1]$.

Proof: See Appendix.

Informed traders are profit maximizing agents. Hence, for any price schedule $P = \{B, A\}$ such that $B \leq A$, traders observing the private signal $\theta \in \Theta$ optimally prefer to sell the asset if their conditional expectation of its value is below the bid price; to place a buy order if their conditional expectation is above the ask price; to refrain from trading if the bid and ask prices straddle their valuation. Finally, if their valuation is equal to the bid or to the ask prices, they are indifferent between all mixed strategies defined on the simplex $\Delta(SO, RT)$ in the first case or on the simplex $\Delta(BO, RT)$ in the second case.

The next lemma establishes a link between the optimal trading strategies of traders endowed with different signals.

Lemma 1 If $\sigma_{\theta_n, SO}^*(P|t) \neq 0$, then $\sigma_{\theta_{n+i}, SO}^*(P|t) = 1$ for any $i \in \{1, 2, \dots, N - n\}$. If $\sigma_{\theta_n, BO}^*(P|t) \neq 0$, then $\sigma_{\theta_{n-j}, BO}^*(P|t) = 1$ for any $j \in \{1, 2, \dots, n - 1\}$.

Proof: See Appendix.

Informed traders maximize the expected profit given their information set. The information set includes both the whole history of trades and the trader's private signal. For the maximum likelihood

ratio property of signals, the expected asset value of traders observing the signal θ_i exceeds the expectation of traders observing θ_j for any $j < i$. Hence, if it is profitable for traders endowed with signal θ_n to sell the asset when the price schedule is P , then all traders observing signals θ_{n+i} , with $i \in \{1, 2, \dots, N - n\}$, optimally prefer to sell the asset as well. Similarly, if the traders endowed with the signal θ_n prefer to buy the asset, then all traders endowed with signals θ_{n-j} , with $j \in \{1, 2, \dots, n - 1\}$, prefer to do the same.

Let define the informational content of a trading order as the likelihood ratio of that order. More precisely, the informational content λ_t^{SO} of a sell order arriving at t is :

$$\lambda_t^{SO} \equiv \frac{P_t(SO \text{ at } B | \underline{V})}{P_t(SO \text{ at } B | \bar{V})}$$

and the informational content λ_t^{BO} of a buy order arriving at t is :

$$\lambda_t^{BO} \equiv \frac{P_t(BO \text{ at } A | \underline{V})}{P_t(BO \text{ at } A | \bar{V})}.$$

It is straightforward to notice that an order indicates \bar{V} if the probability of its occurrence is greater in the good state of nature, that is, the likelihood ratio is lower than 1. It indicates the low asset value if the likelihood ratio is greater than 1. Finally, a trading order is uninformative about the true asset value if the probability of observing it does not depend on the true asset value, that is, likelihood ratio is exactly equal to 1. Clearly, the more the likelihood ratio differs from 1, the more informative the order is about the true asset value.

An implication of Lemma 1 is that a sell order generally indicates \underline{V} , while a buy order generally indicates \bar{V} . To see that, let define the “marginal selling trader” at t , given the price schedule $P = \{B, A\}$ with $B \leq A$, as the trader endowed with signal $\theta_{n_t^b(B)}$ such that:

$$E_t[\tilde{V} | \theta_{n_t^b(B)}] \leq B \quad \text{and} \quad E_t[\tilde{V} | \theta_{(n_t^b(B)-1)}] > B$$

and the “marginal buying trader” at t as the trader endowed with signal $\theta_{n_t^a(A)}$ such that:

$$E_t[\tilde{V} | \theta_{n_t^a(A)}] \geq A \quad \text{and} \quad (E_t[\tilde{V} | \theta_{n_t^a(A)+1}]) < A.$$

Given the traders' strategy described above, the likelihood ratio of a sell order arriving at t , when the price schedule is $P = \{B, A\}$, is equal to:

$$\lambda_t^{SO}(B) = \frac{\frac{\mu}{3} + (1 - \mu) \sum_{i=n_t^b(B)}^N P(\theta_i|\underline{V})}{\frac{\mu}{3} + (1 - \mu) \sum_{i=n_t^b(B)}^N P(\theta_i|\bar{V})}$$

and the likelihood ratio of a buy order is equal to:

$$\lambda_t^{BO}(A) = \frac{\frac{\mu}{3} + (1 - \mu) \sum_{i=1}^{n_t^a(A)} P(\theta_i|\underline{V})}{\frac{\mu}{3} + (1 - \mu) \sum_{i=1}^{n_t^a(A)} P(\theta_i|\bar{V})}$$

where $\frac{\mu}{3}$ is the probability that the trading order comes from a liquidity trader, and $\theta_{n_t^b(B)}$ and $\theta_{n_t^a(A)}$ are respectively the signals of marginal selling and buying traders⁹.

Since $\sum_{i=n}^N P_t(\theta_i|\underline{V}) \geq \sum_{i=n}^N P_t(\theta_i|\bar{V})$ for all $n = 1, 2, \dots, N$, the probability of a sell order is greater in the bad state of nature. Similarly, since $\sum_{i=1}^n P_t(\theta_i|\underline{V}) \leq \sum_{i=1}^n P_t(\theta_i|\bar{V})$ for all $n = 1, 2, \dots, N$, the probability of a buy order is greater in the good state of nature. This shows that a sell order generally indicates \underline{V} , and a buy order generally indicates \bar{V} .

We now analyze the equilibrium behavior of market makers. From Lemma 1, it follows that if equilibrium bid and ask prices exist, they straddle the unconditional expected asset value, which is the price that would prevail in the absence of adverse selection and fixed transaction costs.

Proposition 2 *If a price schedule $P_t^* = \{B_t^*, A_t^*\}$ satisfying conditions 2 and 3 of Definition 1 exists, the equilibrium bid and ask prices are such that:*

$$B_t^* \leq E_t[\tilde{V}] \leq A_t^*.$$

Proof: See Appendix.

Market makers know that orders may come from either a liquidity trader or an informed trader, but they cannot tell them apart. When

⁹For simplicity, here we suppose that the marginal traders choose to place an order also if they are indifferent between trade and to refrain from trading.

they are trading with informed traders, they lose on average. Hence, they have to balance the losses to the informed traders with the gains from the liquidity traders. Moreover, market makers pay a fixed cost c to process each trading order. To recover the losses due to both the adverse selection and the order processing cost, market makers set a spread between the price at which they are willing to sell the asset and the price at which they are willing to buy the asset. Namely, if the probability that an informed trader places a trading order is greater than 0 and/or the fixed transaction cost is strictly positive, then $B_t^* \leq E_t[\tilde{V}] \leq A_t^*$.

The next proposition states the existence and the uniqueness of equilibrium bid and ask prices.

Proposition 3 *In each period t there exists a unique price schedule $P_t^* = \{B_t^*, A_t^*\}$ that satisfies conditions 2 and 3 of Definition 1.*

Proof: See Appendix.

4 Informational cascades with fixed transaction costs

In this section we study the occurrence of informational cascades in the asset market just described.

If all informed traders make the same choice regardless of their private signal, no new information reaches the market. Hence, during an informational cascade, the market equilibrium is such that: $\sigma_\theta^*(P_t^*|t) = \sigma^*(P_t^*|t)$ for all $\theta \in \Theta$, and $B_t^* = E_t[\tilde{V}] - c$ and $A_t^* = E_t[\tilde{V}] + c$ since $\lambda_t^{SO}(B_t^*) = \lambda_t^{BO}(A_t^*) = 1$.

It is straightforward to notice that in equilibrium traders endowed with a bad signal never place a buy order, and traders endowed with a good signal never place a sell order. Indeed, the asset valuation of traders with a bad signal is always lower than the unconditional expected asset value, while that of traders observing a good signal always exceeds it. Proposition 2 dictates that the unconditional expected asset value is the upper bound of the equilibrium bid price and the lower bound of the equilibrium ask price. Therefore, traders

endowed with a bad signal will never find worthwhile to buy the asset, and traders endowed with a good signal will never find worthwhile to sell the asset. This implies that informational cascades characterized by all informed traders placing orders in the same direction will never occur in the market equilibrium. Nevertheless, if fixed transaction costs are positive, the equilibrium price schedule may be such that all informed traders choose to refrain from trading, since transaction costs induce higher bid ask spreads. On the other hand, the expectation of market makers and that of informed traders converge as the number of transactions increases.¹⁰ Hence, there will be a moment when the informational advantage of the informed is so small relative to the fixed transaction cost that it is no longer profitable for any informed trader to place an order.

A no-trade informational cascade starts if $E_t [\tilde{V}|\theta] > B_t^*$ and $E_t [\tilde{V}|\theta] < A_t^*$ for any $\theta \in \Theta$. Clearly, if an informational cascade occurs, the trading orders have no informational content and hence $B_t^* = E_t [\tilde{V}] - c$ and $A_t^* = E_t [\tilde{V}] + c$.

Proposition 4 $B_t^* = E_t [\tilde{V}] - c$ if, and only if:

$$E_t [\tilde{V}] - E_t [\tilde{V}|\theta_N] \leq c$$

and $A_t^* = E_t [\tilde{V}] + c$ if, and only if:

$$E_t [\tilde{V}|\theta_1] - E_t [\tilde{V}] \leq c.$$

Proof: See Appendix.

Proposition 4 establishes that when (and only when) the informational advantage of traders with the most informative bad signal θ_N is lower than the fixed transaction cost, the equilibrium bid price is equal to the public belief about the true asset value minus the transaction cost c . Symmetrically, when (and only when) the informational advantage of traders endowed with the most informative

¹⁰See Glosten and Milgrom [6].

good signal θ_1 is lower than the fixed transaction cost, the equilibrium ask price is equal to the public belief about the true asset value plus c .

The valuation of traders endowed with bad signals less informative than θ_N , exceeds that of traders observing θ_N . This means that:

$$E_t [\tilde{V}] - E_t [\tilde{V}|\theta_N] \leq c \implies E_t [\tilde{V}] - E_t [\tilde{V}|\theta] \leq c$$

for all bad signals $\theta \in \Theta$. Likewise, the valuation of traders endowed with good signals less informative than θ_1 , is lower than that of traders who observe it. This means that:

$$E_t [\tilde{V}|\theta_1] - E_t [\tilde{V}] \leq c \implies E_t [\tilde{V}|\theta] - E_t [\tilde{V}] \leq c$$

for all good signals $\theta \in \Theta$. Hence, if the informational advantage of all informed traders is lower than the transaction cost, no trader observing a private signal will place a trading order. Since all informed traders act alike, no new information reaches the market and an informational cascade starts.

It is straightforward that until the uncertainty is resolved, the informational advantage of informed traders is strictly positive. As a consequence, in the absence of fixed transaction costs, an informational cascade would never occur. On the other hand, if for any possible history of trades the fixed transaction cost is greater than the advantage of informed traders, then trading orders will never be information-based.

To determine the minimum level of c such that no informed trader prefers to trade, define the functions $h_\theta(\pi)$ as:

$$h_\theta(\pi) \equiv \left| E_t [\tilde{V}|\theta] - E_t [\tilde{V}] \right| = \frac{\pi - \pi^2}{\pi + (1 - \pi) \lambda_\theta} \cdot |1 - \lambda_\theta| \cdot (\bar{V} - \underline{V}).$$

$h_\theta(\pi)$ gives the informational advantage of traders observing θ for any $\pi \in [0, 1]$. Since signals are assumed to be symmetric, it is easy to see that:

$$\max h_{\theta_N}(\pi) = \max h_{\theta_1}(\pi) = \frac{|1 - \sqrt{\lambda_1}|}{1 + \sqrt{\lambda_1}} \cdot (\bar{V} - \underline{V}) \equiv \bar{c}.$$

We know that the informational advantage of traders endowed with the signals θ_N and θ_1 always exceeds that of traders observing a

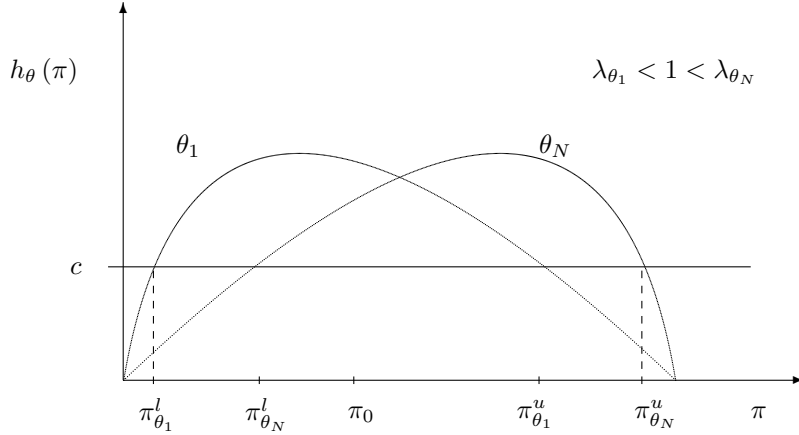


Figure 1: *Informational cascades in a market with positive fixed transaction costs.*

different signal. Hence, if $c > \bar{c}$, all informed traders will prefer to refrain from trading whatever the history of trades is.

The next proposition states that if transaction costs are below \bar{c} then an informational cascade occurs as the public belief approaches the extreme values of the distribution of the true asset value.

Proposition 5 *If $c \in (0, \bar{c})$ there exist unique $\pi_{\theta_1}^l$ and $\pi_{\theta_N}^u$, with $\pi_{\theta_1}^l < \pi_{\theta_N}^u$, such that when $\pi \in [0, \pi_{\theta_1}^l) \cup (\pi_{\theta_N}^u, 1]$ all informed traders refrain from trading in equilibrium.*

Proof: See Appendix.

Proposition 5 establishes that, if the transaction cost is small enough, there exist a lower bound $\pi_{\theta_1}^l$ and an upper bound $\pi_{\theta_N}^u$ of public beliefs such that if the unconditional expected asset value is lower than $E[\tilde{V}|\pi_{\theta_1}^l]$ or greater than $E[\tilde{V}|\pi_{\theta_N}^u]$, then no informed trader prefers to trade.

The last result is a consequence of the beliefs convergence. Suppose that at $t = 0$ the unconditional expected asset value $E_0 [\tilde{V}]$ is such that at least traders endowed with signals θ_1 and θ_N prefer to trade (see figure 1). Suppose also that a sequence of buy orders arrive. Because of the new information that reaches the market, the public belief about the true asset value moves toward \bar{V} . Hence the bid and ask prices increase as well as the asset assessment of informed traders. If the public belief rises too much, the informational advantage of traders endowed with signal θ_1 becomes so small that it is not worthwhile for them to buy the asset. Moreover, if $E_0 [\tilde{V}]$ is low enough, at the beginning of the arrival process the expected profit from selling of traders endowed with signal θ_N increases because the bid price rises. Nevertheless, if the buy orders are very numerous, the expected profit from selling begins to fall off until it becomes negative. This occurs because in the traders' asset assessment the relative weight of the information inferred from the history of trades grows with respect to the traders' private information. If enough buy orders are placed, so that the unconditional expected asset value grows bigger than $E [\tilde{V} | \pi_{\theta_N}^u]$, no informed trader places a sell order although the bid price is very high. At that point, an informational cascade occurs. A similar argument can be used to show the occurrence of an informational cascade when the public belief tends to \underline{V} .

An informational cascade develops because the asset assessment of each informed trader depends not only on his private signal but also on the history of actions taken by previous traders. If in the market there are perfectly informed traders, an informational cascade never occurs. Indeed, if signals θ_1 and θ_N are perfectly informative (that is, $P(\theta_1 | \underline{V}) = P(\theta_N | \bar{V}) = 0$) then when the public belief tends to \underline{V} , traders endowed with signal θ_1 will always buy the asset, and when the public belief tends to \bar{V} , traders endowed with signal θ_N will always sell the asset.

The next proposition states that if transaction costs are high enough, an informational cascade may start even when the market is completely uncertain about the true asset value, that is π close to $\frac{1}{2}$. To see this, notice that if the most informative good and bad signals are symmetric then $h_{\theta_N}(\frac{1}{2}) = h_{\theta_1}(\frac{1}{2}) = \frac{|1-\lambda_1|}{2(1+\lambda_1)} \cdot (\bar{V} - \underline{V}) = \underline{c}$.

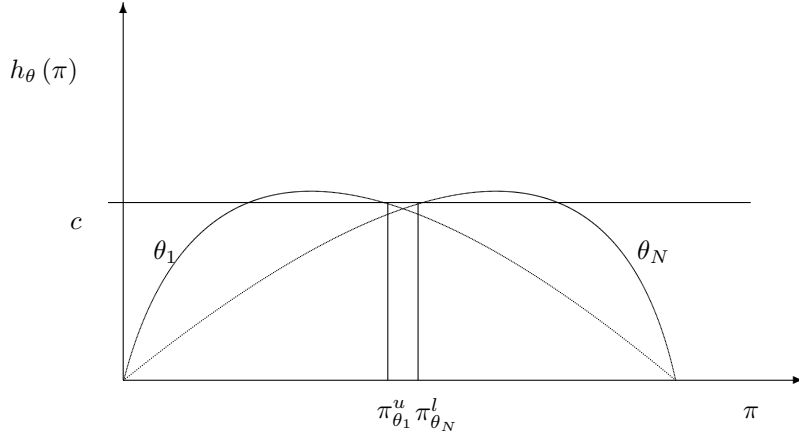


Figure 2: *Informational cascades in a market with high fixed transaction costs.*

That is, if the market is completely uncertain about the true asset value, then the informational advantage of traders endowed with signal θ_1 is equal to that of traders observing signal θ_N . Proposition 6 establishes that if the fixed transaction cost exceeds this bound, there exists a neighborhood of $\frac{\bar{V}+V}{2}$ such that an informational cascade occurs also if the unconditional expected asset value, $E_t[\tilde{V}]$, is within this neighborhood.

Proposition 6 *If $\underline{c} < c < \bar{c}$ then there exist $\pi_{\theta_1}^u \in (\pi_{\theta_1}^l, \frac{1}{2})$ and $\pi_{\theta_N}^l \in (\frac{1}{2}, \pi_{\theta_N}^u)$ such that an informational cascade occurs when $\pi \in (\pi_{\theta_1}^u, \pi_{\theta_N}^l)$.*

Proof: See Appendix.

To understand intuitively why cascades can occur under the conditions described in Proposition 6, consider the following argument. The informational advantage of traders endowed with a good signal exceeds that of traders observing an equally informative bad signal

when $\pi < \frac{1}{2}$, and it is lower in the opposite case. This implies that if the transaction cost is high, no trader with a bad signal chooses to place a sell order when $E_t[\tilde{V}]$ is below $\frac{\bar{V}-V}{2}$, and no trader observing a good signal chooses to buy the asset when $E_t[\tilde{V}]$ exceeds $\frac{\bar{V}-V}{2}$ (see figure 2). As a consequence, when the bid price is low, sell orders are uninformative about the true asset value and when the ask price is high, buy orders are uninformative. Moreover, if $E_t[\tilde{V}]$ is into the interval $(E[\tilde{V}|\pi_{\theta_1}^u], E[\tilde{V}|\pi_{\theta_N}^l])$ no informed trader will place an order and an informational cascade occurs. Therefore, if transaction costs are large enough, an informational cascade may develop even when the public belief is not concentrated on \bar{V} or \underline{V} . This result contrasts with the typical finding of the literature that informational cascades tend to occur when there is convergence of beliefs.

5 Informational content of trading orders and fixed transaction costs

In this section we analyze the effect of fixed transaction costs on the informational content of trading orders in equilibrium. To this aim, we first prove that in the absence of order processing costs, the competitive price mechanism leads to equilibrium bid and ask prices that maximize the informational content of trading orders. Then we show that positive transaction costs reduce the information that the market can infer from both buy and sell orders.

The informational content of a trading order depends on the set of informed traders who prefer to place the order. From Lemma 1 it follows that, given the price schedule $P = \{B, A\}$, the set of informed traders who sell the asset is $\Theta_t^b(B) = \{\theta_n \in \Theta : n \geq n_t^b(B)\}$ and the set of informed traders who buy the asset is $\Theta_t^a(A) = \{\theta_n \in \Theta : n \leq n_t^a(A)\}$, with $n_t^b(B)$ and $n_t^a(A)$ being the signals respectively of the marginal selling and buying traders.

Lemma 2 *There exist a bad signal θ_{n^b} and a good signal θ_{n^a} such that, for all trading histories:*

- the informational content of a sell order is maximum if $n_t^b(B) = n^b$
- the informational content of a buy order is maximum if $n_t^a(A) = n^a$.

Proof: See Appendix.

Lemma 2 establishes that there exist a set of selling traders, $\Theta_{n^b} = \{\theta_n \in \Theta : n \geq n^b\}$, and a set of buying traders, $\Theta_{n^a} = \{\theta_n \in \Theta : n \leq n^a\}$, which maximize the informational content respectively of sell and buy orders, for any given history of trades. Then the price schedule $P = \{B, A\}$ maximize the information of orders at t if $\Theta_t^b(B) = \Theta_{n^b}$ and $\Theta_t^a(A) = \Theta_{n^a}$.

The next proposition states that in the absence of fixed transaction costs, perfect competition among market makers leads to equilibrium bid and ask prices such that: $\Theta_t^b(B_t^*) = \Theta_{n^b}$ and $\Theta_t^a(A_t^*) = \Theta_{n^a}$ for all trading histories.

Proposition 7 *If $c = 0$, the equilibrium bid and ask prices, B_t^* and A_t^* , are such that $\Theta_t^b(B_t^*) = \Theta_{n^b}$ and $\Theta_t^a(A_t^*) = \Theta_{n^a} \forall t$.*

Proof: See Appendix.

To gain intuition about this result, denote $E_t[\tilde{V}|\Theta_{n^a}]$ the expected asset value conditional on a buy order, when the set of informed buying traders is Θ_{n^a} . Clearly: $E_t[\tilde{V}|BO \text{ at } A] \leq E_t[\tilde{V}|\Theta_{n^a}] \leq E_t[\tilde{V}|\theta_{n^a}]$ for all ask price A . In the absence of order processing costs, the expected profit of market makers is equal to the difference between the ask price and the conditional expectation. It is easy to see that if traders observing θ_{n^a} refrain from trading, that is the ask price exceeds $E_t[\tilde{V}|\theta_{n^a}]$, the expected profit from selling of market makers is positive. On the other hand, if traders observing good signals less informative than θ_{n^a} buy the asset, that is A is below $E_t[\tilde{V}|\theta_{(n^a+i)}]$, the information that market makers can infer from a buy order is more precise than the signal $\theta_{(n^a+i)}$. This implies that $E_t[\tilde{V}|BO \text{ at } A]$ exceeds the valuation of marginal traders and

hence the expected profit from selling of market makers is negative. As a consequence, the equilibrium ask price belongs to the interval $(E_t [\tilde{V}|\theta_{(n^a+1)}], E_t [\tilde{V}|\theta_{n^a}])$, and then $n_t^a(A_t^*) = n^a$.

The results of Proposition 7 suggest that in an asset market with competitive price mechanism, if the adverse selection is the only source of the bid-ask spread, the informational content of both sell and buy orders is always maximum in equilibrium.

When market makers pay a fixed cost to process orders, the equilibrium bid price is lower than the conditional expectation of the asset value, given a sell order, and the equilibrium ask price is greater than the conditional expectation of the asset value, given a buy order. As a consequence, fixed transaction costs reduce the informational content of both sell and buy orders, for any given history of trades.

An implication of Proposition 7 is that, in the absence of transaction costs, the set of informed traders whose asset assessment is lower than the equilibrium bid price, and the set of informed traders whose asset assessment is greater than the equilibrium ask price do not depend on the trading history and are constant over time. As a result, the occurrence probability of a trading order, conditional on the true asset value, is the same at each t . If we consider the expected trading volume as the occurrence probability of a trading order, Proposition 7 suggests that in equilibrium, the expected trading volume is constant over time and does not depend on the public belief about the true asset value.

This result does not apply to a market characterized by fixed order processing costs. We know that when an informational cascade occurs, no trading order comes from an informed trader. This means that, during an informational cascade, the expected trading volume is equal to the probability that a liquidity trader is selected to trade and places an order. Moreover, before an informational cascade starts, the probability of an order gradually decreases because traders endowed with less informative signals prefer to refrain from trading. To show this, suppose that at $t = 0$ traders endowed with signal θ_n prefer to buy the asset, that is:

$$E_0 [\tilde{V}|\theta_n] - A_0^* > 0.$$

Suppose then, that a sequence of buy orders arises. The informa-

tional advantage of traders endowed with signal θ_n decreases as the public belief about the true asset value approaches \bar{V} . In particular, it becomes negative before the public belief grows larger than $E[\tilde{V}|\pi_{\theta_1}^u]$ because the valuation of traders observing θ_n is lower than the valuation of traders endowed with θ_1 . Because the ask price exceeds the public belief, traders endowed with signal θ_n prefer to refrain from trading before an informational cascade starts. Therefore, the probability of a trading order is not constant over time. We conclude that, in an asset market with fixed transaction costs, the trading volume decreases as the public belief approaches \bar{V} or \underline{V} , and it is positively correlated with the bid-ask spread.

6 The case of proportional transaction costs

In this section, we discuss the occurrence of informational cascades when the market makers pay a proportional transaction cost to execute each trading order. The only difference relative to the setting of the previous sections is in the expected profit of market makers. Because of proportional transaction costs, the expected profit of a market maker is equal to $E_t[\tilde{V}|SO \text{ at } B] - (1+c)B$ if he buys at B , and it is equal to $(1-c)A - E_t[\tilde{V}|BO \text{ at } A]$ if he sells at A .

Perfect competition leads market makers to zero expected profit on any side of the market. Hence, the equilibrium price schedule, $P_t^* = \{B_t^*, A_t^*\}$, is such that:

$$B_t^* \in \frac{E_t[\tilde{V}|SO \text{ at } B_t^*, \sigma^*(P_t^*|t)]}{1+c} \quad (1)$$

$$A_t^* \in \frac{E_t[\tilde{V}|BO \text{ at } A_t^*, \sigma^*(P_t^*|t)]}{1-c}. \quad (2)$$

Since equilibrium bid and ask prices straddle the unconditional expectation of the asset value, in equilibrium traders with a bad signal will never buy the asset, and traders with a good signal will never sell the asset. Then, as before, informational cascades characterized by all informed traders placing orders in the same direction will never occur in equilibrium. But, the equilibrium price schedule may be such that all informed traders refrain from trading. If this occurs, the trading orders have no informational content and hence $B_t^* = \frac{E_t[\tilde{V}]}{1+c}$ and $A_t^* = \frac{E_t[\tilde{V}]}{1-c}$.

Proportional transaction costs do not affect the optimal correspondence strategies of informed traders. Hence, no informed trader wishes to sell the asset when the bid price is lower than the valuation of traders with the most informative bad signal, that is $E_t[\tilde{V}|\theta_N] > B_t^*$. Symmetrically, no informed trader wishes to buy the asset when the ask price exceeds the valuation of traders with the most informative good signal, that is $E_t[\tilde{V}|\theta_1] < A_t^*$.

Proposition 8 $B_t^* = \frac{E_t[\tilde{V}]}{1+c}$ if, and only if:

$$\frac{E_t[\tilde{V}] - E_t[\tilde{V}|\theta_N]}{E_t[\tilde{V}|\theta_N]} \leq c$$

and $A_t^* = \frac{E_t[\tilde{V}]}{1-c}$ if, and only if:

$$\frac{E_t[\tilde{V}|\theta_1] - E_t[\tilde{V}]}{E_t[\tilde{V}|\theta_1]} \leq c.$$

Proof: See Appendix.

Proposition 8 states that if (and only if) the relative informational advantage of traders observing the most informative bad signal θ_N is lower than the c , the equilibrium bid price is equal to the public belief about the true asset value times $\frac{1}{1+c}$. Symmetrically, if (and only if) the relative informational advantage of traders observing the most informative good signal θ_1 is lower than the transaction cost, the equilibrium ask price is equal to the public belief about the true asset value times $\frac{1}{1-c}$. Clearly, an informational cascade develops when the proportional transaction cost c exceeds the relative informational advantages of both traders with θ_N and traders with θ_1 .

In the previous model, where transaction costs are fixed, the condition for the occurrence of informational cascades concerns the absolute informational advantage of traders observing private signals. Here, the amount of the costs payed by the market maker depends on the price at which he buys or sells the asset. Higher bid

and ask prices produce higher transaction costs. So, their impact on equilibrium prices is larger when the probability that the market attaches to \bar{V} is high. As a consequence, when the transaction costs are proportional to prices, the condition for informational cascades depends on the relative informational advantage of informed traders rather than their absolute one.

If in the bad state of nature the true asset value is zero, the impact of the transaction costs on the equilibrium bid and ask prices vanishes as π converges to 0. Hence, it is worthwhile to consider separately the two cases of $\underline{V} > 0$ and $\underline{V} = 0$.

First we consider the case of $\underline{V} > 0$. If for any possible history of trades the proportional transaction cost is greater than the relative informational advantages of both traders endowed with θ_N and traders endowed with θ_1 , then trading orders will never be information-based.

To determine the minimum level of c such that no informed trader prefers to trade, define the functions $g_\theta(\pi)$ as:

$$g_\theta(\pi) \equiv \frac{|E_t[\tilde{V}|\theta] - E_t[\tilde{V}]|}{E_t[\tilde{V}|\theta]} = \frac{\pi - \pi^2}{\pi\bar{V} + (1-\pi)\lambda_\theta\underline{V}} \cdot |1 - \lambda_\theta| \cdot (\bar{V} - \underline{V}).$$

$g_\theta(\pi)$ gives the relative informational advantage of traders observing θ for any $\pi \in [0, 1]$. It is easy to see that:

$$\max g_{\theta_1}(\pi) < \max g_{\theta_N}(\pi) = \frac{\bar{V} - \underline{V}}{(\sqrt{\bar{V}} + \sqrt{\lambda_N \underline{V}})^2} \cdot (\lambda_N - 1) \equiv \bar{c}_p.$$

The relative informational advantage of traders endowed with the signals θ_N and θ_1 always exceeds that of traders observing a different signal. Hence, if $c > \bar{c}_p$, all informed traders will prefer to refrain from trading whatever the history of trades is.

The next proposition states that, if \underline{V} is strictly positive and the proportional transaction cost is below \bar{c}_p , an informational cascade develops both when the public belief about the true asset value tends to \bar{V} , and when it approaches \underline{V} . Besides, if the transaction costs are large enough and if the difference between \bar{V} and \underline{V} is not too large, an informational cascade may occur also when the probability that the market attaches to \underline{V} and \bar{V} are close.

Proposition 9 *If $c \in (0, \bar{c}_p)$ and if \underline{V} is strictly positive, there exist unique $\underline{\pi}$ and $\bar{\pi}$, with $\underline{\pi} < \bar{\pi}$, such that when $\pi \in [0, \underline{\pi}) \cup (\bar{\pi}, 1]$ all informed traders refrain from trading in equilibrium. Besides, if $c \in (\underline{c}_p, \bar{c}_p)$, with $\underline{c}_p = \frac{\bar{V}-\underline{V}}{\bar{V}+\underline{V}} \cdot \frac{1-\lambda_1}{1+\lambda_1}$, and \underline{V} is greater than $\lambda_1 \bar{V}$, there exist $\underline{\underline{\pi}} \in (\underline{\pi}, \frac{\underline{V}}{\bar{V}+\underline{V}})$ and $\bar{\bar{\pi}} \in (\frac{\underline{V}}{\bar{V}+\underline{V}}, \bar{\pi})$ such that an informational cascade occurs also when $\pi \in (\underline{\underline{\pi}}, \bar{\bar{\pi}})$.*

Proof: See Appendix.

Too large transaction costs inhibit informed traders from trading whatever is the public belief about the asset value. If transaction costs are not too large, informed traders find worthwhile to trade until their relative informational advantage exceeds c . When the low asset value is greater than zero, the relative informational advantage of traders observing a private signal decreases both when π approaches 0 and when π tends to 1, and it is zero when π is exactly equal to 0 or 1. As a result, an informational cascade occurs both when the public belief about the asset value is close to \underline{V} , and when it is near to \bar{V} .

The relative informational advantage of traders endowed with a bad signal exceeds that of traders observing an equally informative good signal for all $\pi > \frac{\underline{V}}{\bar{V}+\underline{V}}$, and it is lower than it in the opposite case.

If \underline{V} is not too small respect to \bar{V} , that is $\underline{V} > \lambda_1 \bar{V}$, the behavior of the relative informational advantages of traders is similar to the absolute ones (see figure 1). Hence, if c exceeds the relative informational advantages of traders with θ_1 and θ_N when $\pi = \frac{\underline{V}}{\bar{V}+\underline{V}}$, an informational cascade can occur even when the probability that the market attaches to \bar{V} is close to $\frac{\underline{V}}{\bar{V}+\underline{V}}$. Clearly, if the difference between \bar{V} and \underline{V} is small, $\frac{\underline{V}}{\bar{V}+\underline{V}}$ is near to $\frac{1}{2}$. We can conclude that if \bar{V} is not too small respect to \underline{V} , the results about the occurrence of an informational cascade due to proportional transaction costs, are similar to those obtained for a market characterized by fixed transaction costs.

If \underline{V} is low, the relative informational advantage of traders observing θ_1 exceeds that of traders observing θ_n only if the probability

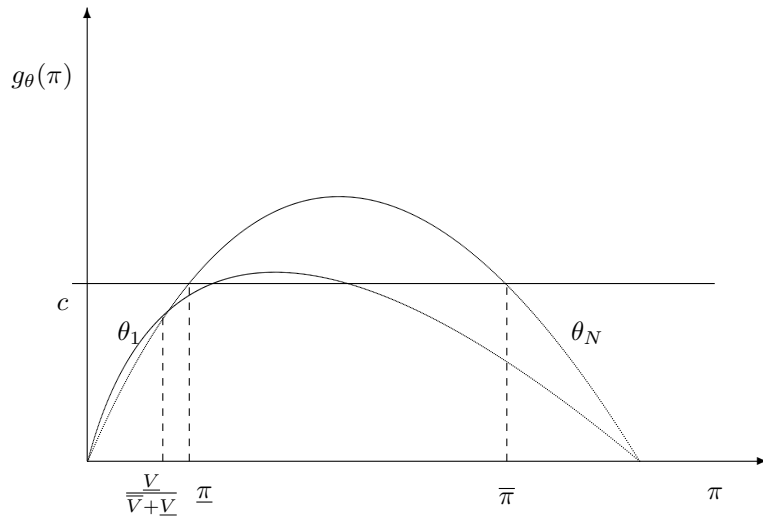


Figure 3: *Informational cascades in a market with proportional transaction costs and $\underline{V} < \lambda_1 \bar{V}$.*

that the market attaches to \underline{V} is very high (see figure 3). As a consequence, an informational cascade can occur only when π approaches 0 or 1. Moreover, if \underline{V} is close to zero, the amount of the executing cost payed by the market maker is very low when π is near to zero. Then, the probability of an informational cascade with low prices, given that true asset value is \bar{V} , approaches zero. In particular, it is exactly equal to zero in the extreme case of $\underline{V} = 0$. This result is stated in the next proposition.

Proposition 10 *If $c \in (0, \bar{c}_p)$ and if $\bar{V} = 0$, there exists unique $\bar{\pi}$ such that if, and only if, $\pi > \bar{\pi}$, all informed traders refrain from trading in equilibrium.*

Proof: See Appendix.

Intuitively, transaction costs payed by market makers reduce as π tends to zero, and they reach the minimum value, $c\underline{V}$, when $\pi = 0$. Clearly, if \underline{V} is equal to zero, proportional transaction costs vanish when $\pi = 0$. As a consequence, an informational cascade will never occur when the equilibrium prices are low.

This result has the interesting implication that cascades will tend to be asymmetric. They will almost never emerge in depressed markets, while they are more likely to be present in bull markets. As a consequence, they are more likely to result in market crashes than in price jumps.

7 Conclusions

We have presented a model of the effect of transaction costs on the informational efficiency of prices in an asset market characterized by asymmetric information and sequential trading mechanism. More precisely, we have examined the impact of both fixed and proportional trading costs on price discovery.

Standard microstructure models predict that prices ultimately converge to the true asset value. However, most of the models that analyze the price mechanism in asset markets with asymmetrically informed agents, do not take into account transaction costs. Lee [10] who does consider fixed transaction costs in a setting with asymmetric information and sequential trading, does not allow for optimizing behavior by market makers. In contrast, in this paper market maker are assumed to behave optimally and competitively in setting prices. The results show that the competitive price mechanism does not prevent the occurrence of informational cascades. Moreover, large fixed transaction costs may lead to an informational cascade not only when the market attaches a large probability to the high or to the low asset value and then prices are very high or very low, as in Lee [10], but even when there is complete uncertainty about the asset's fundamental value.

In the case of proportional transaction costs, we find that the probability of an informational cascade is greater in a context of high prices. In particular, if the true asset value in the bad state is close to zero, an informational cascade cannot develop when the asset prices are low.

As a result, cascades will tend to be asymmetric. They will almost never emerge in depressed markets, while they are more likely to be present in bull markets. Therefore, they are more likely to result in market crashes than in price jumps.

8 Appendix

Proof of Proposition 1

The proposition is an immediate consequence of definition 1. \square

Proof of Lemma 1

We prove the result for the ask side of the market. The proof for the bid side is similar. Suppose that $\sigma_{\theta_n, BO}^*(P|t) \neq 0$. This means that, when the price schedule is P and the probability of \bar{V} is π_t , the expected profit from buying of traders observing θ_n is not negative and greater or at most equal to the expected profit from selling. For the maximum likelihood ratio property of private signals, $E_t[\tilde{V}|\theta_n] < E_t[\tilde{V}|\theta_{n-j}]$. Then, the expected profit from buying of traders observing θ_{n-j} is always greater than the one of traders endowed with θ_n . This proves the thesis. \square

Proof of Proposition 2

Bertrand competition among market makers leads to equilibrium bid and ask prices respectively equal to the expected asset value conditional to a sell order minus the transaction cost and the expected asset value conditional to a buy order plus the transaction cost. When the price schedule is $P = \{B, A\}$, the expected asset value given a sell order is equal to:

$$E_t[\tilde{V}|SO \text{ at } B, \sigma^*(P|t)] = \underline{V} + (\bar{V} - \underline{V}) \frac{\pi_t}{\pi_t + (1 - \pi_t)\lambda_t^{SO}(B)}$$

and the expected asset value given a buy order is equal to:

$$E_t[\tilde{V}|BO \text{ at } A, \sigma^*(P|t)] = \underline{V} + (\bar{V} - \underline{V}) \frac{\pi_t}{\pi_t + (1 - \pi_t)\lambda_t^{BO}(A)}.$$

Since $\lambda_t^{SO}(B) \geq 1$ and $\lambda_t^{BO}(A) \leq 1$ for all π and P , the expected asset value conditional to a sell order is always lower or equal to the unconditional expectation, while the expected asset value conditional to a buy order is always greater or equal to it. \square

Lemma 3 *The marginal selling trader affects positively the informational content of a sell order if, and only if:*

- $\lambda_{\theta_{n_t^b(B)}} > \lambda_t^{SO}(B)$.

The marginal buying trader affects positively the informational content of a buy order if, and only if:

- $\lambda_{\theta_{n_t^a(A)}} < \lambda_t^{BO}(A)$.

Proof

We prove the result for the ask side. The result for the bid side follows from symmetry.

Let $f^a(n) = \frac{\frac{\mu}{3} + (1-\mu) \sum_{i=1}^n P(\theta_i|V)}{\frac{\mu}{3} + (1-\mu) \sum_{i=1}^n P(\theta_i|\bar{V})}$. In order to prove the lemma for the ask side, we have to show that:

$$f^a(n) \geq f^a(n-1) \iff \lambda_n \geq f^a(n).$$

By using algebraic calculus, it is easy to show that:

1. $f^a(n) \geq f^a(n-1) \iff \lambda_n \geq f^a(n-1)$
2. $\lambda_n \geq f^a(n-1) \iff \lambda_n \geq f^a(n)$.

By combining these results, we obtain:

$$f^a(n) \geq f^a(n-1) \iff \lambda_n \geq f^a(n).$$

Since $f^a(n_t^a(A)) = \lambda_t^{BO}(A)$, the lemma for the ask side is proved. \square

Proof of Proposition 3

We prove the proposition for the ask price. The proof for the bid price can be obtained with a symmetric argument.

For the monotone likelihood property of signals, the expected asset value conditional on a buy order is always greater or equal to the unconditional expected asset value. This implies that, if the ask price is lower than $E[y|H] + c$, the market maker's expected profit is negative. Moreover, $E[y|H, BO \text{ at } A]$ is upper bounded by \bar{V} . Hence, for the zero expected profit condition, the equilibrium ask

price cannot be greater than $\bar{V} + c$. Let define the correspondence $F_t^a : [E_t[\tilde{V}] + c, \bar{V} + c] \rightsquigarrow [E_t[\tilde{V}] + c, \bar{V} + c]$ as:

$$F_t^a(A) \equiv E_t \left[\tilde{V} | BO \text{ at } A \right] + c$$

$F_t^a(A)$ is an upper semicontinuous convex valued correspondence that maps the set $[E_t[\tilde{V}] + c, \bar{V} + c]$ in itself. For the Kakutani's fixed point theorem, $F_t^a(A)$ has a fixed point A_t^* , that is, an equilibrium ask price always exists.

The uniqueness is proved by using the results of Lemma 3, that states that the marginal buying (selling) trader affects positively the informational content of a buy (sell) order, that is the likelihood ratio of the buy (sell) order increases (decreases) when the marginal buying (selling) trader refrains from trading, if the signal that he observes is more accurate than the information that the market maker infers from a buy (sell) order.

Suppose, by way of obtaining a contradiction, that there exist two equilibrium ask prices: A_1 and A_2 , with $A_1 < A_2$. Denote $n_t^a(A_1)$ and $n_t^a(A_2)$ respectively the marginal buying trader given A_1 and the marginal buying trader given A_2 . Clearly, since we suppose $A_1 < A_2$, it has to be $\lambda_{n_t^a(A_1)} > \lambda_{n_t^a(A_2)}$. First, we notice that at the equilibrium, the asset assessment of the marginal buying trader is greater or at most equal to the market maker's conditional expected asset value. Hence $\lambda_{n_t^a(A_1)} \leq \lambda_t^{BO}(A_1)$. This implies that $\lambda_t^{BO}(A_1) < \lambda_t^{BO}(A_2)$ because, when the ask price grows to A_2 , the traders endowed with the signal $\theta_{n_t^a(A_1)}$ prefer to refrain from trading, and then the likelihood ratio of a buy order increases by lemma 3. As a consequence, if the ask price is equal to A_2 , the market maker's expected profit is strictly positive. Hence, for the zero expected profit condition, A_2 cannot be an equilibrium ask price. \square

Proof of Proposition 4

We prove the proposition for the ask price; the proof for the bid price is symmetric. First we assume that $A_t^* = E_t[\tilde{V}] + c$ and we prove that this implies $E_t[\tilde{V}] - E_t[\tilde{V}|\theta_N] \leq c$. Suppose by way of obtaining a contradiction that $E_t[\tilde{V}|\theta_1] - E_t[\tilde{V}] > c$. Since

at least the traders observing θ_1 prefer to buy the asset when the ask price is $E_t \left[\tilde{V} \right] + c$, the expected asset value conditional to a buy order at $E_t \left[\tilde{V} \right] + c$ is greater than the unconditional expected asset value. This implies that the market maker's expected profit is negative, which contradict the fact that $E_t \left[\tilde{V} \right] + c$ is the equilibrium ask price.

On the other hand, if $E_t \left[\tilde{V} | \theta_1 \right] - E_t \left[\tilde{V} \right] \leq c$ then no informed trader prefers to buy the asset. Hence $\inf \{ E_t [\tilde{V} | BO \text{ at } E_t \left[\tilde{V} \right] + c, \sigma^*(P_t^* | t)] + c \} = E_t \left[\tilde{V} \right] + c = A_t^*$. \square

Proof of Proposition 5

Consider the function h_θ defined at page 13. Notice that:

- $h_\theta(0) = h_\theta(1) = 0$
- $\max h_\theta(\pi) = \frac{|1-\sqrt{\lambda}|}{1+\sqrt{\lambda}} \cdot (\bar{V} - \underline{V})$
- $h_\theta''(\pi) = -\frac{2\lambda}{(\pi+(1-\pi)\lambda)^3} \cdot |1-\lambda_i| \cdot (\bar{V} - \underline{V}) < 0 \quad \forall \pi \in [0, 1]$.

Since by assumption $0 < c \leq \frac{|1-\sqrt{\lambda}|}{1+\sqrt{\lambda}} \cdot (\bar{V} - \underline{V})$, from the strictly concavity of $h_\theta(\pi)$ it follows that there exist unique π_θ^l and π_θ^u , with $\pi_\theta^l < \pi_\theta^u$, such that $h_\theta(\pi_\theta^l) = h_\theta(\pi_\theta^u) = c$ and $h_\theta(\pi) > c$ if, and only if, $\pi \in (\pi_\theta^l, \pi_\theta^u)$. Thus, when $\pi \in (0, \pi_\theta^l) \cup (\pi_\theta^u, 1)$, traders endowed with signal θ prefer to refrain from trading. Moreover

$$\arg \max h_{\theta_1}(\pi) = \frac{\sqrt{\lambda_1}}{1 + \sqrt{\lambda_1}} < \frac{1}{2} < \arg \max h_{\theta_N}(\pi) = \frac{\sqrt{\lambda_N}}{1 + \sqrt{\lambda_N}}$$

and

$$h_{\theta_1}(\pi) \geq h_{\theta_N}(\pi) \quad \forall \pi \leq \frac{1}{2}$$

As $\pi_{\theta_1}^l < \arg \max h_{\theta_1}(\pi)$ and $\pi_{\theta_N}^u > \arg \max h_{\theta_N}(\pi)$, then $\pi_{\theta_1}^l < \pi_{\theta_N}^l$ and $\pi_{\theta_N}^u > \pi_{\theta_1}^u$. Hence, when either $\pi < \pi_{\theta_1}^l$ or $\pi > \pi_{\theta_N}^u$, all informed traders prefer to refrain from trading. \square

Proof of Proposition 6

From the proof of Proposition 5, we know that there exist $\pi_{\theta_1}^u > \pi_{\theta_1}^l$ and $\pi_{\theta_N}^l < \pi_{\theta_N}^u$ such that $h_{\theta_1}(\pi_{\theta_1}^u) = h_{\theta_N}(\pi_{\theta_N}^l) = c$. Since $h'_{\theta_1}(\frac{1}{2}) < 0$ and $h'_{\theta_N}(\frac{1}{2}) > 0$, and since $\frac{|1-\lambda_1|}{2(1+\lambda_1)} \cdot (\bar{V} - \underline{V}) < c < \bar{c}$, then $\pi_{\theta_1}^u < \pi_{\theta_N}^l$ and both $c > h_{\theta_1}(\pi)$ and $c > h_{\theta_N}(\pi)$ for any $\pi \in [\pi_{\theta_1}^u, \pi_{\theta_N}^l]$. \square

Proof of Lemma 2

We prove the lemma for the ask side. The proof for the bid side is analogous.

A buy order indicates the good state of nature. Hence, the informational content of a buy order is maximum when $f^a(n)$ (defined in the proof of lemma 3) reaches its minimum value. If all informed traders prefer to buy the asset, then $f^a(N) = 1 < \lambda_N$. By lemma 3, it follows that the marginal buying trader affects negatively the informational content of the buy order; that is: $f^a(N-1) < f^a(N)$. If no informed traders prefer to buy the asset, then $f^a(0) = 1 > \lambda_1$, and the marginal buying trader affects positively the informational content of the buy order; that is: $f^a(1) > f^a(0)$. For the maximum likelihood property of signals, it follows that there exists a unique signal θ_{n^a} such that:

$$f^a(n^a + 1) > f^a(n^a) \geq \lambda_{n^a} \text{ and } f^a(n^a) \leq f^a(n^a - 1).$$

If the equilibrium ask price A_t^* is such that $n_t^a(A_t^*) = n^a$, then the equilibrium informational content of a buy order is maximum. \square

Proof of Proposition 7

We prove the proposition for the ask side of the market. The proof for the bid side can be obtained by using symmetric arguments.

Since $c = 0$, the competitive ask price has to be equal to the expected asset value conditional to a buy order, that is:

$$A_t^* \in E_t[\tilde{V}|BO \text{ at } A_t^*, \sigma^*(P_t^*|t)].$$

Lemma 2 states that there exists a good signal θ_{n^a} such that $f^a(n^a) \leq f^a(n)$ for every $n \in \{1, 2, \dots, N\}$, and $\lambda_{n^a} \leq f^a(n^a)$. Moreover, by combining lemmas 3 and 2 it is easy to see that $\lambda_{(n^a+1)} > f^a(n^a + 1)$. As a consequence, for any history of trades:

$$E_t \left[\tilde{V} | \theta_{n^a+1} \right] < \underline{V} + \frac{\pi_t}{\pi_t + (1 - \pi_t) f^a(n^a + 1)} \cdot (\bar{V} - \underline{V}) \quad (3)$$

and

$$E_t \left[\tilde{V} | \theta_{n^a} \right] \geq \underline{V} + \frac{\pi_t}{\pi_t + (1 - \pi_t) f^a(n^a)} \cdot (\bar{V} - \underline{V}). \quad (4)$$

3 and 4 imply that there exists an ask price A_t^* that belongs to $(E_t \left[\tilde{V} | \theta_{n^{a+1}} \right], E_t \left[\tilde{V} | \theta_{n^a} \right])$, that satisfies the zero market maker's expected profit condition. By Proposition 3, we know that there exists a unique ask price that satisfies the zero market makers' expected profit condition. Thus, in equilibrium, $n_t^a(A_t^*) = n^a$. \square

Proof of Proposition 8

We prove the proposition for the ask price; the proof for the bid price is symmetric. First we assume that $A_t^* = \frac{E_t[\tilde{V}]}{1-c}$ and we prove that this implies $\frac{E_t[\tilde{V}|\theta_1] - E_t[\tilde{V}]}{E_t[\tilde{V}|\theta_1]} \leq c$. Suppose by way of obtaining a contradiction that $\frac{E_t[\tilde{V}|\theta_1] - E_t[\tilde{V}]}{E_t[\tilde{V}|\theta_1]} > c$. This means that $E_t \left[\tilde{V} | \theta_1 \right] > \frac{E_t[\tilde{V}]}{1-c}$. Hence, at the least the traders observing θ_1 prefer to buy the asset when the ask price is $\frac{E_t[\tilde{V}]}{1-c}$. As a consequence, the expected asset value conditional to a buy order at $\frac{E_t[\tilde{V}]}{1-c}$ is greater than the unconditional expected asset value. This implies that the market maker's expected profit is negative, which contradicts the fact that $\frac{E_t[\tilde{V}]}{1-c}$ is the equilibrium ask price.

On the other hand, if $\frac{E_t[\tilde{V}|\theta_1] - E_t[\tilde{V}]}{E_t[\tilde{V}|\theta_1]} \leq c$ then no informed trader prefers to buy the asset.¹¹ Hence $\inf \left\{ \frac{E_t[\tilde{V} | BO \text{ at } \frac{E_t[\tilde{V}]}{1-c}, \sigma^*(P_t^*|t)]}{1-c} \right\} = \frac{E_t[\tilde{V}]}{1-c} = A_t^*$. \square

Proof of Proposition 9

First we prove that there exist unique $\underline{\pi}$ and $\bar{\pi}$ such that when $\pi \in [0, \underline{\pi}) \cup (\bar{\pi}, 1]$, all informed traders prefer to refrain from trading.

Consider the function $g_\theta(\pi)$ defined at page 22. Notice that:

- $g_\theta(0) = g_\theta(1) = 0$

¹¹The relative informational advantage of informed traders increases with the signal precision.

- $\max g_\theta(\pi) = \frac{\bar{V}-\underline{V}}{(\sqrt{\bar{V}}+\sqrt{\lambda_\theta\underline{V}})^2} \cdot |\lambda_\theta - 1|$
- $g_\theta''(\pi) = -\frac{2\lambda_\theta V\bar{V}}{(\pi\bar{V}+(1-\pi)\lambda_\theta\underline{V})^3} \cdot |1 - \lambda_\theta| \cdot (\bar{V} - \underline{V}) < 0 \quad \forall \pi \in [0, 1]$.

Since by assumption $0 < c \leq \bar{c}^p$, from the strictly concavity of $g_\theta(\pi)$ it follows that there exist unique $\underline{\pi}_\theta$ and $\bar{\pi}_\theta$, with $\underline{\pi}_\theta < \bar{\pi}_\theta$, such that $g_\theta(\underline{\pi}_\theta) = g_\theta(\bar{\pi}_\theta) = c$ and $g_\theta(\pi) > c$ if, and only if, $\pi \in (\underline{\pi}_\theta, \bar{\pi}_\theta)$. Thus, $\underline{\pi} = \inf\{\underline{\pi}_\theta\}_{\theta \in \Theta}$ and $\bar{\pi} = \sup\{\bar{\pi}_\theta\}_{\theta \in \Theta}$.

In order to prove the second part of the theorem, we first notice that $g_{\theta_1}\left(\frac{V}{\bar{V}+\underline{V}}\right) = g_{\theta_N}\left(\frac{V}{\bar{V}+\underline{V}}\right) = \underline{c}_p$, and $g_{\theta_1}(\pi) < g_{\theta_N}(\pi)$ for all $\pi > \frac{V}{\bar{V}+\underline{V}}$. Moreover, if $\bar{V} > \underline{V}\lambda_1$ then $\frac{V}{\bar{V}+\underline{V}} < \arg \max g_{\theta_1}(\pi) < \bar{\pi}_{\theta_1}$. As a consequence, if $\bar{V} > \underline{V}\lambda_1$ then the relative informational advantage of traders endowed with θ_N is strictly positive for all $\pi \in [\bar{\pi}_{\theta_1}, \bar{\pi}_{\theta_N})$ whatever $c \in (0, \bar{c}_p)$. In contrast, if $\bar{V} < \underline{V}\lambda_1$ and c is greater than \underline{c}_p , then $\bar{\pi}_{\theta_1}$ is lower than $\underline{\pi}_{\theta_N}$ and both $g_{\theta_1}(\pi)$ and $g_{\theta_N}(\pi)$ are strictly negative for all $\pi \in (\bar{\pi}_{\theta_1}, \underline{\pi}_{\theta_N})$. Hence an informational cascade occurs also when $\pi \in (\bar{\pi}_{\theta_1}, \underline{\pi}_{\theta_N})$. \square

Proof of Proposition 10

If $\underline{V} = 0$, the relative informational advantage of traders endowed with the signal θ as a function of π is given by $g_\theta(\pi) = (1-\pi) \cdot |\lambda_\theta - 1|$. Since by assumption $0 < c \leq \bar{c}^p$, it follows that for any $\theta \in \Theta$ there exists a unique $\bar{\pi}_\theta$ such that $g_\theta(\bar{\pi}_\theta) = c$, and $g_\theta(\pi) > c$ if, and only if, $\pi < \bar{\pi}_\theta$. Thus, $\bar{\pi} = \sup\{\bar{\pi}_\theta\}_{\theta \in \Theta}$. \square

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