Optimal Myopia
an Axiomatic Approach to Bounded Rationality

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Abstract

We introduce an decision-theoretic characterization of a model (originating in Diasakos [10]) of endogenous bounded rationality due to the presence of costs in foreseeing the structure of a dynamic decision problem. The decision maker is not required to know the entire structure of the problem when making choices. She can think ahead, through costly search, to reveal more of its details. However, the costs of search are not assumed exogenously; they are inferred from revealed preferences through choices. Thus, bounded rationality and its extent emerge endogenously: as problems become simpler or as the benefits of deeper search become larger relative to its costs, the choices more closely resemble those of a rational agent. For a fixed decision problem, the costs of search will vary across agents. For a given decision maker, they will vary across problems. The model explains, therefore, why the disparity, between observed choices and those prescribed under rationality, varies across agents and problems. It also suggests, under reasonable assumptions, an identifying prediction: a relation between the benefits of deeper search and the depth of the search. In decision problems with structure that allows the optimal foresight of search to be revealed from choices of plans of action, the relation can be tested on any agent-problem pair, rendering the model falsifiable. Moreover, the relation can be estimated allowing the model to make predictions with respect to how, in a given problem, changes in the terminal payoffs affect the depth of search and, consequently, choices.

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1 Introduction

Most economic models assume rational decision-making. This requires, at a minimum, that the agent understands the entire decision problem, including all possible plans of action and the terminal consequences of each, and chooses optimally according to well-defined preferences over final outcomes. For all but the simplest problems, however, being able to make choices in this way amounts to having access to extraordinary cognitive and computational abilities and resources. It is not surprising, therefore, that the predictions of a variety of models of rational decision-making are systematically violated across a spectrum of experimental and actual decision makers.[1] Theories of bounded rationality attempt to address these discrepancies by accounting for agents with limited abilities to formulate and solve complex problems or to process information, as first advocated by Simon [27] in his pioneering work.

Bounded rationality in economics has been widely incorporated but in models that take the cognitive, computational, or information-processing limitations as fixed and exogenous.[2] Yet, a striking feature of bounded rationality in practice is that its nature and extent vary across both decision makers and problems. More importantly, the variation can depend systematically on the decision problem itself. This suggests that it is the interaction between the limited abilities of the agent and the precise structure of the problem that determines the extent of bounded rationality and, hence, the resulting choices. That is, our understanding of boundedly-rational decision-making remains incomplete until we can relate the heterogeneity in decision-making abilities to the heterogeneity in decision problems. Ultimately, the nature and extent of bounded rationality should themselves be determined within the model, explaining (i) why a given agent departs from rational decision-making in some problems but not in others, and (ii) why, in a given decision problem, different agents exhibit different degrees of bounded rationality. Ideally, the variation should be connected to observable features, yielding concrete implications which could, in principle, be tested empirically.

Diasakos [10] takes a step towards these desiderata for finite, sequential decision problems with a model of endogenous bounded rationality. The decision maker is not required to know the entire structure of the problem when making choices but can think ahead, through search that may be costly, to reveal more of its details. The novelty of the approach lies in

[1] The extant literature on violations of rational decision-making is vast. For a representative (yet, by no means comprehensive) review, see Benartzi and Thaler [5], Carbone and Hey [7], Charness and Rabin [8], Forsythe *et al.* [11], Henrich *et al.* [13], Hey [14], Hey and Lee [15], Kahneman and Tversky [16]; Loewenstein and Elster [18], and Thaler [31], [32].

[2] To give but a few examples, bounded rationality models have been introduced in game theory (Abreu and Rubinstein [1], Piccione and Rubinstein [22], Osborne and Rubinstein [20], Rosenthal [23], Camerer *et al.* [6]), bargaining (Sabourian [24]), auctions (Crawford and Iriberri [9]), macroeconomics (Sargent [25]), and contracting (Anderlini and Felli [4], Lee and Sabourian [17]).

constructing a theoretical framework for the complexity costs of search to be inferred from revealed preferences through observed choices. Specifically, it considers decision-making under limited foresight, a restriction on the forward depth of search in the decision-tree that depicts the extensive form of the problem at hand. For a given depth, the horizon of search is given by the collection of the chains of information sets whose length does not exceed the depth. For a given finite sequential problem, the decision-tree defines a finite sequence of nested search horizons.

To abstract from discrepancies with rational decision-making due to sources unrelated to costly search (errors in choice, misperceptions of payoffs, behavioral or psychological biases in preferences or information-processing, complete misunderstanding of the problem, etc.), the agent is taken to perceive correctly the utility payoffs of the terminal nodes within her horizon of search and to be unrestricted in her ability to optimize across them. Beyond her search horizon, however, the structure of the continuation problem or the possible terminal payoffs from different plans of action may not be known exactly. Diasakos assumed that her expected continuation payoff, from following an available plan of action within the horizon, is given by a weighting-average of the corresponding best and worst terminal payoffs. This paper provides an axiomatization of his representation via a framework which views the agent's limited knowledge regarding the terminal contingencies of a current action as ambiguity. The agent assigns a set of lotteries over the terminal outcomes that may realize beyond the current search horizon from each plan of action available within the horizon. Her expected continuation payoff, from following an available plan of action within the horizon, is of the α-MEU form where the weight α between the worst and best lottery depends on the foresight of the agent's search.

The resulting choice behavior under limited foresight amounts to a two-stage optimization (Theorem 1). The agent chooses first the optimal plans of action for each value of foresight that can be defined on the decision-tree; then, she chooses the optimal level of foresight. In general, this choice process will lead to quite different choices than those prescribed by rational decision-making (unlimited foresight) if the cognitive, computational, or information-processing limitations that necessitate search also constrain its horizon to one of inadequate foresight. It does embed rational decision-making, however, as a special case (Theorem 2) and, thus, can be used not only to depict boundedly-rational choices but also to compare them to rational ones.

Such comparisons are based on features of the representation. They translate readily, therefore, to welfare comparisons with respect to some underlying utility representation of preferences over terminal outcomes in the decision-tree under consideration. Welfare comparisons between observed choices and those of rational decision-making correspond in turn to the implied complexity costs of bounded rationality; they are given by the welfare loss relative to rational decision-making. This allows the extent of bounded rationality to emerge endogenously. As problems become simpler or as the benefits of deeper search become larger relative to its costs, the choices more closely resemble those of a rational decision maker. In fact, for a given agent, if the decision problem is sufficiently simple or the benefits of search outweigh its costs, her choices become indistinguishable from rational ones. For a given problem, the costs will vary across agents, explaining why the disparity between observed and rational choices differs across decision makers.

As Diasakos points out, this approach establishes, for any decision maker-problem pair, a monotone relation between the perceived benefits of deeper search and the depth of search. More precisely, in decision problems with structure that allows the optimal foresight of search to be revealed from choices of plans of action, our representation determines lower (upper) bounds for the percentage change in the perceived continuation payoff when the foresight of search is extended from the current optimal, say n^*, to $n^* + 1$ (from $n^* - 1$ to n^*). These bounds imply that changes in the terminal utility payoffs, which increase (decrease) sufficiently the percentage change in the maximal perceived continuation payoff, induce the agent to lengthen (shorten) her foresight of search and, thus, possibly change her choice of plan of action (Section 3.3).

The monotonicity relation, between the perceived benefits of deeper search and the depth of search, offers a powerful test that renders the model falsifiable. Moreover, the bounds of the percentage changes in the agent's perceived continuation payoff, with respect to a one-step extension in her foresight of search, are identifiable. By varying the marginal benefit of further search along the given decision tree, the bounds can be sharpened until they become binding and, thus, turn into estimates for the percentage changes in the perceived continuation payoffs. Under our representation, the estimation of these bounds corresponds directly to calibrating the model's implied search cost function. That is, it allows the model to make predictions with respect to how, in the given problem (or in others similar in complexity), changes in the terminal payoffs affect the depth of search and, consequently, choices.

Perhaps the most appealing feature of our approach to modeling bounded rationality is perhaps its versatility. Three different applications in Diasakos [10] show that it is consistent with violations of timing-independence in temporal framing problems, with dynamic inconsistency and diversification bias in sequential versus simultaneous choice problems, and with plausible but contrasting risk attitudes across small- and large-stakes gambles. The cases under study correspond to three well-known decision-making paradoxes which are seemingly unrelated and obtain for obviously different reasons. Accordingly, a variety of explanations have been suggested in the literature based on case-specific models and arguments. Yet, the endogenously-implied complexity costs approach provides a common framework for depicting the underlying limitations that force choices to depart from those prescribed by rationality. In each case, paradoxical behavior results when the costs of search are large relative to its perceived benefits, preventing a boundedly-rational agent from considering the decision problem in its entirety.

2 Limited Foresight

Unlike the vast majority of the extant literature on decision-making under ambiguity, this paper investigates dynamic rather than static decision-problems. We employ a decision-making under ambiguity approach to analyze boundedly-rational decision-making in problems depicted by finite decision-trees. Such trees correspond to a finite set S of underlying states of the world (states of nature) and a finite set X of terminal consequences (terminal outcomes) whose typical elements will be denoted by s and x, respectively. Adopting the framework of Anscombe and Aumann [3], we assume that the decision maker (DM) has preferences over compound lotteries. A lottery is a device for deciding which prize the DM will receive on the basis of a single observation that records which one of a set of mutually-exclusive and collectively-exhaustive uncertain events took place. A "roulette lottery" (prize) is one in which each of these uncertain events is associated with a known chance (physical probability), an element of the probability simplex $\Delta(X) = \left\{p \in \mathbb{R}_+^{|X|} : \sum_{x \in X} p_x = 1\right\}$. In a "horse lottery," on the other hand, logical probabilities (probabilities proper) are assigned to at least some of the uncertain events - the corresponding chances for such events are not known or cannot even be assigned at all. A compound lottery is a lottery whose prizes are other lotteries - it may be compounded from roulette lotteries only or from roulette lotteries and horse lotteries.

Under unbounded rationality (perfect foresight), the DM is called upon to choose amongst standard Anscombe-Aumann acts. These are <u>crisp</u> mappings $S \mapsto \Delta(X)$ assigning to a state $s \in S$ a probability distribution over X as the terminal consequence. In contrast, under bounded rationality (limited foresight), the DM's choice-set of acts depends fundamentally on the foresight and structure of the search process itself (on how and to what extend the decision-tree has been truncated by search). Our search process truncates the decision-tree into <u>horizons</u>, a horizon of <u>foresight</u> n consisting of all chains of information sets with length at most n. Obviously, in a given finite decision-tree $n \in \mathcal{N}_h = \{1, \ldots, N\}$ for some $N < \infty$. As demonstrated by the example below, a search horizon of foresight n defines (i) a partition $\pi_{h(n)}$ of the underlying state space S, and (ii) a collection $\mathcal{A}_{h(n)}$ of $\pi_{h(n)}$-measurable acts, the set of choices actually available to the DM.

Under our search process, the gradual resolution of uncertainty regarding Nature's "moves" along a decision-tree amounts to revealing successive refinements of partitions of S. Let $\pi = \{E_k\}_{k=1,\ldots,K\leq|S|}$ be such a partition. In the spirit of Ghirardato [12] and Mukerjie [19], we will capture the boundedly-rational DM's current ambiguity regarding the terminal consequences of her future actions and Nature's future moves by having her envision a *set* of terminal outcomes $X_k \in 2^X \setminus \varnothing$ as possible consequences following from the event E_k. In our setting, the extent of the DM's search in a given decision problem dictates the corresponding truncation of the decision-tree and the partition π. It defines also the set of acts the DM can choose from. These are π-measurable mappings $f : S \mapsto 2^X \setminus \varnothing$ that associate to each $s \in E_k$ the corresponding collection X_k of the terminal outcomes the agent believes possible under the act in the event E_k.

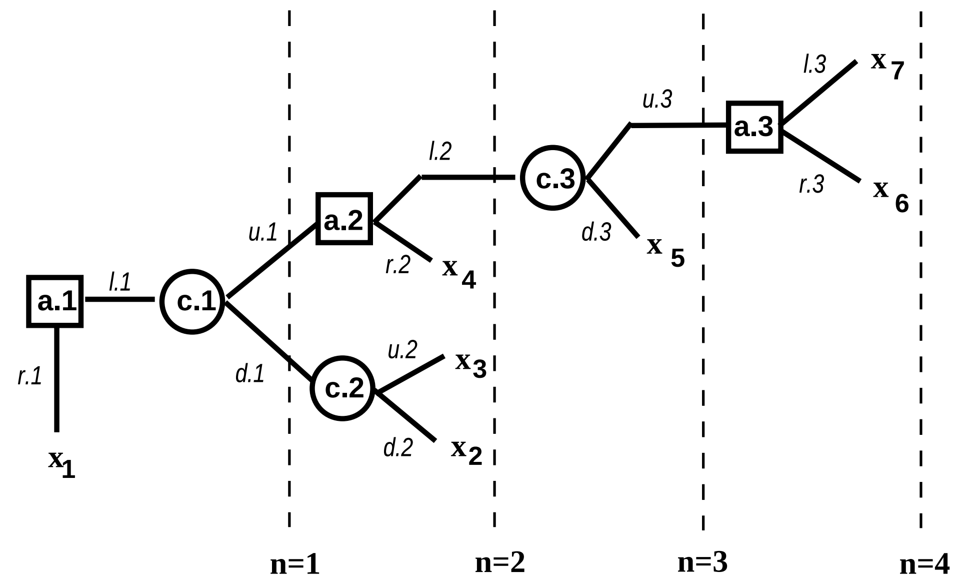

Figure 1: The foresight of search induces a sequence of sets of acts and a filtration

To illustrate, consider the problem in Figure 1 where decision and chance nodes are shown by squares and circles, respectively. The terminal nodes (the elements of X) are indexed by $x_{i=1,\ldots,7}$. The elements of S, indexed by $s_{i=1,\ldots,8}$, are given by the corresponding complete "plans of moves" by Nature. Specifically, the state space in this example is

$$S = \{s_i\}_{i=1}^{8} = \{d_1d_2d_3, d_1d_2u_3, d_1u_2d_3, d_1u_2u_3, u_1d_2d_3, u_1d_2u_3, u_1u_2d_3, u_1u_2u_3\}$$

At her initial node $a.1$, the agent must choose a plan of action for some specification of probability distributions over the possible "moves" by Nature at each chance node. Under unbounded rationality, this specification is known with certainty at every chance node along the tree while perfect foresight corresponds to the finest partition of the state space $\pi = S$. That is, the DM is called upon to choose amongst a set of S-measurable crisp acts. In fact, every act in the choice-set of a perfectly-foresighted DM maps each $s \in S$ to an element of X, a degenerate probability distribution in $\Delta(X)$. Consider, for instance, the plan of action that dictates choosing the upper branch at every decision node along the tree. This plan is depicted by the act $f_{l_1l_2l_3} \in \Delta(X)^S$ with $f_{l_1l_2l_3}(s_1) = f_{l_1l_2l_3}(s_2) = x_2$, $f_{l_1l_2l_3}(s_3) = f_{l_1l_2l_3}(s_4) = x_3$, $f_{l_1l_2l_3}(s_5) = f_{l_1l_2l_3}(s_7) = x_5$, and $f_{l_1l_2l_3}(s_6) = f_{l_1l_2l_3}(s_8) = x_7$. Let also the specification of Nature "moves" along the tree render, for instance, each of the branches stemming from each of the chance nodes $\{c_k\}_{k=1}^{3}$ equiprobable. This is depicted by the uniform probability measure $p = \left(\frac{1}{8}, s_{j=1,\ldots,8}\right) \in \Delta(S)$. In this case, the agent's plan of action described above offers the prize $\left(0, x_1; \frac{1}{4}, x_2; \frac{1}{4}, x_3; 0, x_4; \frac{1}{4}, x_5; 0, x_6; \frac{1}{4}, x_7\right)$. If the agent's preferences over $\Delta(X)$ admit a von Neumann-Morgenstern representation with Bernoulli utility index u on X, the expected utility payoff of this prize is the utility value of the corresponding act: $U(f_{l_1l_2l_3}) = \sum_{s \in S} p_s u(g(s))$.

Under limited foresight, our search process truncates the tree of Figure 1 into horizons of foresight $n \in \mathcal{N}_{a.1} = \{1, 2, 3, 4\}$ with $n = 4$ depicting perfect foresight. In the figure, the increasing sequence of nested horizons $\{a.1(n)\}_{n=1}^{4}$ is depicted by the subtrees starting at $a.1$ and extending to the corresponding vertical dashed boundaries. In particular, the horizon of foresight $n = 1$ starting at $a.1$ is given by the collection $a.1(1) = \{a.1, c.1\}$; similarly, $a.1(2) = a.1(1) \cup \{c.2, a.1\}$, $a.1(3) = a.1(2) \cup \{c.3\}$, and $a.1(4) = a.1(3) \cup \{a.3\}$. Each horizon $a.1(n)$ defines also the partition $\pi_{a.1(n)}$ of the state space S and the set $\mathcal{A}_{a.1(n)}$ of

$\pi_{a.1(n)}$-measurable acts that the DM may choose from, as follows.

$$\begin{aligned}
\mathcal{A}_{a.1(1)} &= \{f_{l_1}, f_{r_1}\} \\
\mathcal{A}_{a.1(2)} &= \{f_{l_1 l_2}, f_{l_1 r_2}, f_{r_1 l_2}, f_{r_1 r_2}\} \\
\mathcal{A}_{a.1(3)} &= \mathcal{A}_{a.1(2)} \\
\mathcal{A}_{a.1(4)} &= \{f_{l_1 l_2 l_3}, f_{l_1 l_2 r_3}, f_{l_1 r_2 l_3}, f_{l_1 r_2 r_3}, f_{r_1 l_2 l_3}, f_{r_1 l_2 r_3}, f_{r_1 r_2 l_3}, f_{r_1 r_2 r_3}\} \\
\pi_{a.1(1)} &= \{\{s_1, s_2, s_3, s_4\}, \{s_5, s_6, s_7, s_8\}\} \\
\pi_{a.1(2)} &= \{\{s_1, s_2\}, \{s_3, s_4\}, \{s_5, s_6\}, \{s_7, s_8\}\} \\
\pi_{a.1(3)} &= \pi_{a.1(4)} = S
\end{aligned}$$

When her foresight of search is n, we will assume that the agent knows with certainty her utility payoffs at but only at the terminal consequences that immediately succeed nodes within the horizon $h(n)$. Regarding continuation consequences that do not immediately succeed from $h(n)$, the DM is faced with ambiguity; she can only assign a *set* of possible continuation outcomes to a current event. When her foresight of search is $n = 1$ (Figure 2), the agent knows precisely her payoff at the terminal node x_1; she is certain about the consequence of her act f_{r_1} in any state $s \in S$. Yet, this is not the case with respect to her other act f_{l_1} whose set of possible terminal outcomes is $X_1 = \{x_2, x_3\}$ and $X_2 = \{x_4, \ldots, x_7\}$, respectively, in the event $E_1 = \{s_1, \ldots, s_4\}$ and $E_2 = \{s_5, \ldots, s_8\}$ of $\pi_{a.1(1)}$. From her $a.1(1)$ perspective, this act is a correspondence: $f_{l_1}(s) = X_1$ for $s \in E_1$ and $f_{l_1}(s) = X_2$ for $s \in E_2$. Notice also that the specification for Nature's "moves" $p \in \Delta(S)$ given previously defines now a probability measure on $\pi_{a.1(1)}$, $p_1 = \left(\frac{1}{2}, E_1; \frac{1}{2}, E_2\right)$.

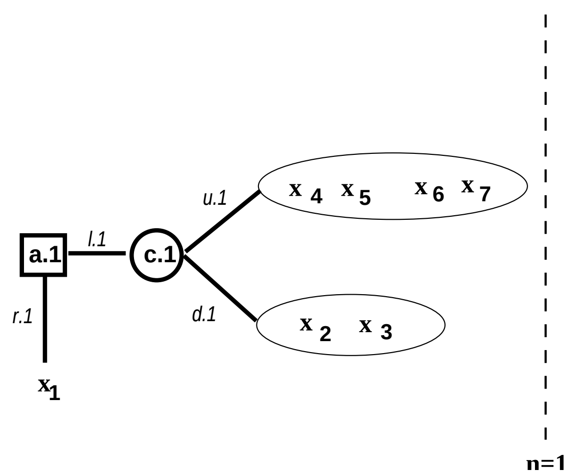

Figure 2: Partition and Acts for n=1

A horizon of longer foresight, such as $a.1(2)$, gives the agent more precise information about some of the consequences of her choices. When $n = 2$ (Figure 3), she knows the consequences of the acts $f_{r_1 l_2}$, $f_{r_1 r_2}$, and $f_{l_1 r_2}$ under any state in S. These acts map S into degenerate distributions in $\Delta(X)$. Specifically, $f_{r_1 r_2}(s) = f_{r_1 l_2}(s) = x_1$ $\forall s \in S$ while $f_{l_1 r_2}(s_1) = f_{l_1 r_2}(s_2) = x_2$, $f_{l_1 r_2}(s_3) = f_{l_1 r_2}(s_4) = x_3$, and $f_{l_1 r_2}(s_j) = x_4$ for $j \in \{5, \ldots, 8\}$. The act $f_{l_1 l_2}$, on the other hand, maps into a degenerate distribution in $\Delta(X)$ only under the events $E_{21} = \{s_1, s_2\}$ and $E_{22} = \{s_3, s_4\}$ of $\pi_{a.1(2)}$. In the events $E_{23} = \{s_5, s_6\}$ and $E_{24} = \{s_7, s_8\}$, its images are non-singleton subsets of X. We have $f_{l_1 l_2}(s_1) = f_{l_1 l_2}(s_2) = x_2$, $f_{l_1 l_2}(s_3) = f_{l_1 l_2}(s_4) = x_3$ but $f_{l_1 l_2}(s) = \{x_5, x_6, x_7\}$ $\forall s \in E_{23} \cup E_{24}$. Moreover, the (conditional on $\pi_{a.1(2)}$) specification for Nature's "moves" given by p above becomes now $p_2 = \left(\frac{1}{4}, E_{21}; \frac{1}{4}, E_{22}; \frac{1}{4}, E_{23}; \frac{1}{4}, E_{24}\right)$.

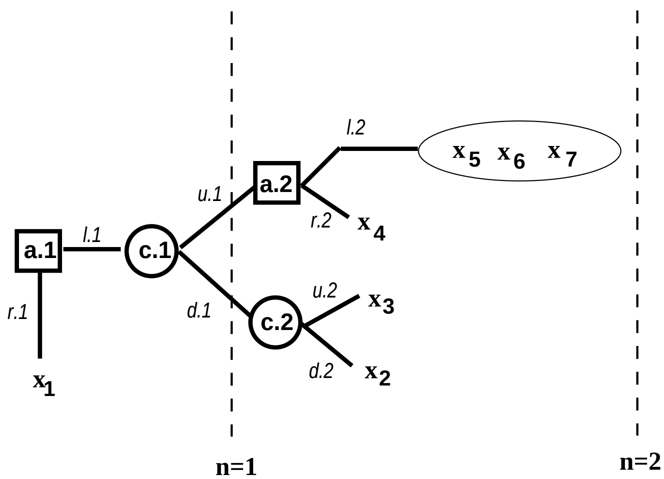

Figure 3: Partition and Acts for n=2

When $n = 3$ (Figure 4), the partition is $\pi_{a.1(3)} = S$ and the assumed specification for Nature's "moves" becomes now fully-revealed as p. The acts $f_{r_1 l_2}$, $f_{r_1 r_2}$, and $f_{l_1 r_2}$ remain unchanged between $\mathcal{A}_{a.1(2)}$ and $\mathcal{A}_{a.1(3)}$ as they were already crisp in the former set. The act $f_{l_1 l_2}$ becomes now measurable with respect to the new partition $\pi_{a.1(3)}$. It is given by $f_{l_1 l_2}(s_1) = f_{l_1 l_2}(s_2) = x_2$, $f_{l_1 l_2}(s_3) = f_{l_1 l_2}(s_4) = x_3$, $f_{l_1 l_2}(s_5) = f_{l_1 l_2}(s_7) = x_5$, and $f_{l_1 l_2}(s_6) = f_{l_1 l_2}(s_8) = \{x_6, x_7\}$. Finally, foresight $n = 4$ suffices to cover the entire decision-tree. Hence, each act $f \in \mathcal{A}_{a.1(4)}$ is a standard Anscombe-Aumann act: $f(s)$ is a degenerate element of $\Delta(X)$ $\forall s \in S$.

As illustrated by our example, the acts on a search-truncated finite decision-tree associate each $s \in S$ to a subset of the set of terminal outcomes X - i.e., to a collection of degenerate probability distributions in $\Delta(X)$. Nevertheless, purely for technical reasons in obtaining

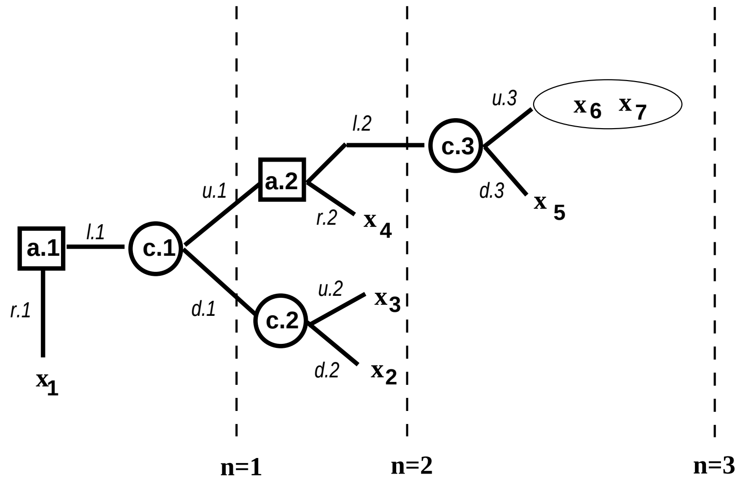

Figure 4: Partition and Acts for n=3

a representation, we will take the DM as having well-defined preferences over more general acts. These are mappings from S to subsets of $\Delta(X)$ - i.e. to collections of not necessarily degenerate probability distributions on X. More precisely, we consider a collection $\mathcal{X}$ of non-empty subsets of $\Delta(X)$, containing $\Delta(X)$ itself and all singletons in $\Delta(X)$, with typical element Z. [3] Given a partition $\pi = \{E_k\}_{k=1,\ldots,K<|S|}$, an act is a π-measurable mapping $S \mapsto \mathcal{X}$ associating the partition with a horse-lottery correspondence $[Z_1, \ldots, Z_K]$; $f(s) = Z_k$ $\forall s \in E_k$ with $Z_k \subseteq \Delta(X) \backslash \varnothing$ being a set of roulette-lotteries on the set of all possible terminal outcomes X.

At any node of a given decision-tree, the set of choices actually available to the DM will be some subset of the space of our acts $\mathcal{X}^S$. As we have already seen, under unbounded rationality (perfect foresight), the set of choices contains only crisp acts $f \in \Delta(X)^S \subseteq \mathcal{X}^S$. Under bounded rationality (limited foresight), the choice-set is given by the structure and extent of the DM's search: a search horizon $h(n)$ defines (i) the partition $\pi_{h(n)}$ of the state space S, and (ii) the collection $\mathcal{A}_{h(n)} \subset \mathcal{X}^S$ of the $\pi_{h(n)}$-measurable acts the DM may choose from.

To conclude the presentation of how the collections of acts from $\mathcal{X}^S$ and the partitions of S evolve with the foresight of our search-process, we point out some properties that will be of use in the sequel. They are obvious in the preceding example and trivial to check in general.

Remark 1 *Let $\geq$ denote the refinement relation for partitions. For $n \in \mathcal{N}_h \setminus \{N\}$, we have*

[3] As usual, an algebra is a class of subsets which is closed with respect to complements and finite unions. For any set Z, 2^Z denotes the algebra generated by all of its subsets. For concreteness, the algebra $\mathcal{A}$ in the text can be thought of as the set $2^{\mathcal{X}}$, though nothing to follow depends on such an assumption.

(a) $\pi_{h(n+1)} \geq \pi_{h(n)}$ *where* $\pi_{h(0)} = \{S\}$ *and* $\pi_{h(n)} = S$. *That is,* $\geq$ *is complete on* $\Pi = \left\{\pi_{h(n)}\right\}_{n=0}^{N}$ *and* Π *is a filtration.*

(b) *The following are equivalent.*

- $\pi_{h(n+1)} > \pi_{h(n)}$
- $h(n+1) \backslash h(n)$ *contains at least one chance node. That is, some of the uncertainty regarding Nature's "moves" is resolved as the foresight extends from* n *to* $n+1$.

(c) *The following are equivalent.*

- $\mathcal{A}_{h(n+1)} \neq \mathcal{A}_{h(n)}$
- $h(n+1) \setminus h(n)$ *contains at least one decision-node. That is, the DM is called upon to act at some node as the foresight extends from* n *to* $n+1$.

In obtaining a representation for the DM's preferences over her available acts when her horizon of search is $h(n)_{n \in \mathcal{N}_h}$, two collections of acts will be of such use that they deserve particular notation. The first is that of constant acts. Consider the set

$$\mathcal{X}_{h(n)} = \Delta(X) \cup \left\{Z \subseteq \Delta(X) : \exists E \in \pi_{h(n)} \text{ and } f \in \mathcal{A}_{h(n)} \text{ s.t. } f(E) = Z\right\}$$

consists of $\Delta(X)$ and those of its subsets that are images of some act in $\mathcal{A}_{h(n)}$. When the foresight of search is n, the set of constant acts consists of all the constant mappings $S \mapsto \mathcal{X}_{h(n)}$. Formally,

$$\mathcal{F}^{c}_{h(n)} = \left\{f \in \mathcal{X}^{S}_{h(n)} : f(s) = f(s') \ \ \forall s, s' \in S\right\}$$

The other set of acts that is of interest selects from the set of crisp acts $\Delta(X)^S$ those that are $\pi_{h(n)}$-measurable. The collection of $\pi_{h(n)}$-measurable crisp acts will be denoted by $\mathcal{F}^{cr}_{h(n)}$.

Remark 2 *For all* $n, n' \in \mathcal{N}_h$,[4]

(a) $\mathcal{F}^{cr}_{h(n)} \subseteq \bigcap_{n' \geq n} \mathcal{F}^{cr}_{h(n')}$. *In words, a crisp act that is measurable with respect to a given partition is also measurable with respect to any refinement. An immediate corollary is that* $\mathcal{F}^{cr}_{h(1)} \subseteq \ldots \mathcal{F}^{cr}_{h(n)} \subseteq \cdots \subseteq \mathcal{F}^{cr}_{h(n)}$.

[4]Henceforth, we will abuse notation slightly, using the terminal outcome $x \in X$ to denote the degenerate lottery 1_x, an element of $\Delta(X)$, but also the constant crisp act $S \mapsto \{x\}$, a member of $\Delta(X)^S$. To abuse notation slightly more, we will let $z \in \Delta(X)$ and $Z \in \mathcal{X}_{h(n)}$ denote (apart from their standard interpretation), respectively, the constant acts $S \mapsto \{z\}$ and $S \mapsto \{Z\}$. The former abbreviated notation is used, for instance, in the second part of Remark 2(b). The distinction between lowercase z (necessarily a singleton) and uppercase Z (possibly containing more than one element) should be kept in mind.

(b) $\mathcal{F}^{c}_{h(n)} \cap \mathcal{F}^{cr}_{h(n)} = \left\{ f \in \Delta(X)^{S} : f(s) = f(s') \ \ \forall s, s' \in S \right\} \equiv \Delta(X)$

(c) The global set of constant acts $\cup_{n \in \mathcal{N}_h} \mathcal{F}^{c}_{h(n)}$ *is measurable with respect to* $\pi_{h(n)}$.

3 Decision-making under Limited Foresight

As Anscombe and Aumann did (and is standard practise in obtaining representation theorems), we will assume that the DM has a rational preference relation over pairs of acts, some of which might be counterfactual to her current decision-making situation (or not in her real choice set). To be precise, we will require that, given the search horizon $h(n)$, she can in principle rank any two acts in the collection

$$\mathcal{F}_{h(n)} = \mathcal{F}^{cr}_{h(n)} \cup \mathcal{F}^{c}_{h(n)} \cup \mathcal{A}_{h(n)}$$

Apart from the acts in $\mathcal{A}_{h(n)}$, which are defined by her current decision-making situation and form her available choice-set, $\mathcal{F}_{h(n)}$ includes also all the $\pi_{h(n)}$-measurable crisp acts as well as all the constant acts $S \mapsto \mathcal{X}_{h(n)}$. The DM has, thus, a preference relation $\succsim_{h(n)}$ on $\mathcal{F}_{h(n)}$ (with $\sim_{h(n)}$ and $\succ_{h(n)}$ denoting, as usual, its symmetric and asymmetric component).

We will endow $\succsim_{h(n)}$ with some properties that are by now standard in the decision-theory literature, albeit appropriately adopted to our setting.

A. 1 *Weak Order: The relation* $\succsim_{h(n)}$ *satisfies*

(i) Completeness: $\forall f, g \in \mathcal{F}_{h(n)} \quad f \succsim_{h(n)} g$ *or* $g \succsim_{h(n)} f$

(ii) Transitivity: $\forall f, g, h \in \mathcal{F}_{h(n)} \quad f \succsim_{h(n)} g$ *and* $g \succsim_{h(n)} h \Rightarrow f \succsim_{h(n)} h$

A. 2 *Crisp Independence:* $\forall f, h, r \in \mathcal{F}^{cr}_{h(n)}$ *and* $\forall \alpha \in (0,1)$
$f \succ_{h(n)} h \ \Leftrightarrow \ \alpha f + (1-\alpha) r \succ_{h(n)} \alpha h + (1-\alpha) r$

A. 3 *Extended-Archimedean Property:* $\forall f, h \in \mathcal{F}^{cr}_{h(n)}$ *and* $\forall g \in \mathcal{F}^{c}_{h(n)}$
$f \succ_{h(n)} g \succ_{h(n)} h \Rightarrow \exists \alpha, \beta \in (0,1)$ *s.t.* $\alpha f + (1-\alpha) h \succ_{h(n)} g \succ_{h(n)} \beta f + (1-\beta) h$

A. 4 *Crisp Monotonicity:* $\forall f, g \in \mathcal{F}^{cr}_{h(n)} \quad f(s) \succsim_{h(n)} g(s) \ \forall s \in S \Rightarrow f \succsim_{h(n)} g$

A. 5 *Non-triviality:* $\exists x', x'' \in X$ *s.t.* $x' \succ_{h(n)} x''$

By a trivial application of the argument in Schmeidler [29], the axioms A.1-A.5 are necessary and sufficient for the existence of an affine function $u_{h(n)} : \Delta(X) \mapsto \mathbb{R}$, unique up to a positive linear transformation, and a unique finitely-additive probability measure $p_{h(n)}$ on the sigma-algebra induced by the partition $\pi_{h(n)}$ such that the function $U^{cr}_{h(n)} : \mathcal{F}^{cr}_{h(n)} \mapsto \mathbb{R}$ defined by

$$U^{cr}_{h(n)}(f) = \sum_{E \in \pi_{h(n)}} p_{h(n)}(E)\, u_{h(n)}(f(E))$$

represents $\succsim_{h(n)}$ on the set $\mathcal{F}^{cr}_{h(n)}$.[5]

To extend the representation to the set $\mathcal{F}_{h(n)} \setminus \mathcal{F}^{cr}_{h(n)}$ of non-crisp $\pi_{h(n)}$-measurable acts, we will proceed in two steps. First, we will extend it to the set $\mathcal{F}^{c}_{h(n)}$ of constant acts. To this end, we assume that the relation $\succsim_{h(n)}$ satisfies also

A. 6 *Boundedness:* $\forall Z \in \mathcal{X}_{h(n)}\ \ \exists z', z'' \in \Delta(X)$: $z' \succsim_{h(n)} Z \succsim_{h(n)} z''$

Under boundeness, there exists a unique function $\alpha_{h(n)} : \mathcal{X}_{h(n)} \mapsto [0,1]$ (see Lemma 4 in the Appendix) such that the function $U^{c}_{h(n)} : \mathcal{F}^{cr}_{h(n)} \cup \mathcal{F}^{c}_{h(n)} \mapsto \mathbb{R}$ defined by

$$U^{c}_{h(n)}(f) = \sum_{k=1}^{K} p_{h(n)}(E_k) \left(\alpha_{h(n)}(f(E_k)) \inf_{z \in f(E_k)} u_{h(n)}(z) + \left[1 - \alpha_{h(n)}(f(E_k))\right] \sup_{z \in f(E_k)} u_{h(n)}(z) \right)$$

represents the preference relation $\succsim_{h(n)}$ on the set $\mathcal{F}^{cr}_{h(n)} \cup \mathcal{F}^{c}_{h(n)}$.

Consider now a general (not necessarily constant) non-crisp act $f \in \mathcal{F}_{h(n)}$ and let the partition $\pi_{h(n)}$ be given by $\{E_k\}_{k=1,\ldots,K \leq |S|}$. Since f is $\pi_{h(n)}$-measurable, it must be of the form $f(s) = Z_k\ \forall s \in E_k$, for some sets of lotteries $Z_k \in \mathcal{X}_{h(n)}$ and for all k. By the monotonicity assumption (A.4), it is without loss of generality to consider instead the act $f^* = \sum_{k=1}^{K} 1_{E_k} Z_k$ where 1_{E_k} is the indicator function of the event E_k and Z_k is the constant act that maps every state in S to the set of lotteries Z_k. By repeated applications of (A.4),

[5]Since the set of terminal outcomes X is finite here, the set of probability distributions over X with finite supports is the entire simplex $\Delta(X)$. Since, moreover, our state space S is finite, taking $\Sigma_{h(n)}$ to be the sigma-algebra induced by the partition $\pi_{h(n)}$, Schmeidler's set L_0 of all $\Sigma_{h(n)}$-measurable finite-valued functions $S \mapsto \Delta(X)$ corresponds to our set $\mathcal{F}^{cr}_{h(n)}$ of all $\pi_{h(n)}$-measurable crisp acts. That is, in the text, we are referring to Schmeidler's statement of the Ancsombe-Aumann theorem (pp. 578). The independence and continuity axioms in Schmeidler's exposition correspond here to crisp independence and extended-Archimedean, respectively. We have monotonicity instead of strict monotonicity but, for crisp acts on a finite state space, a monotone weak order is equivalent to a strictly monotone weak order (see Schmeidler's observation, pp.576).

moreover, it is straightforward to establish inductively the following indifference.

$$
\begin{aligned}
f & \sim_{h(n)} \\
f^* & \sim_{h(n)} \begin{cases} \alpha_{h(n)}(Z_1)\inf_{y\in Z_1} u_{h(n)}(z) + \left[1-\alpha_{h(n)}(Z_1)\right]\sup_{z\in Z_1} u_{h(n)}(z) & \text{if } s\in E_1 \\ \sum_{k=2}^K 1_{E_k} Z_k & \text{if } s\in S\setminus E_1 \end{cases} \\
& \sim_{h(n)} \begin{cases} \alpha_{h(n)}(Z_1)\inf_{z\in Z_1} u_{h(n)}(z) + \left[1-\alpha_{h(n)}(Z_1)\right]\sup_{z\in Z_1} u_{h(n)}(z) & \text{if } s\in E_1 \\ \alpha_{h(n)}(Z_2)\inf_{z\in Z_2} u_{h(n)}(z) + \left[1-\alpha_{h(n)}(Z_2)\right]\sup_{z\in Z_2} u_{h(n)}(z) & \text{if } s\in E_2 \\ \sum_{k=3}^K 1_{E_k} Z_k & \text{if } s\in S\setminus (E_1\cup E_2) \end{cases} \\
& \dots \\
& \sim_{h(n)} \begin{cases} \alpha_{h(n)}(Z_1)\inf_{z\in Z_1} u_{h(n)}(z) + \left[1-\alpha_{h(n)}(Z_1)\right]\sup_{z\in Z_1} u_{h(n)}(z) & \text{if } s\in E_1 \\ \dots & \\ \alpha_{h(n)}(Z_K)\inf_{z\in Z_K} u_{h(n)}(z) + \left[1-\alpha_{h(n)}(Z_K)\right]\sup_{z\in Z_K} u_{h(n)}(z) & \text{if } s\in E_K \end{cases}
\end{aligned}
$$

Since the last act on the right-hand side is crisp, the representation $U^{cr}_{h(n)}$ applies valuing this act (and, thus, also f) according to $\sum_{k=1}^K p_{h(n)}(E_k)\, U^c_{h(n)}(Z_k)$. To complete the argument, it should be noted that the axioms A.1-A.6 are not only sufficient but also necessary. The necessity of A.1-A.5 for the $U^{cr}_{h(n)}$ representation is a standard result while boundedness is implied immediately by the $U^c_{h(n)}$ representation.

Theorem 1 *A binary relation $\succsim_{h(n)}$ defined on the set of acts $\mathcal{F}_{h(n)}$ satisfies axioms A.1-A.6 if and only if there exists an affine function $u_{h(n)} : \Delta(X) \mapsto \mathbb{R}$, unique up to a positive linear transformation, a unique finitely-additive probability measure $p_{h(n)}$ on the sigma-algebra induced by the partition $\pi_{h(n)}$, and a unique function $\alpha_{h(n)} : \mathcal{X}_{h(n)} \mapsto [0,1]$ such that*

$$\forall f, g \in \mathcal{F}_{h(n)} : \ f \succsim_{h(n)} g \ \Leftrightarrow \ U_{h(n)}(f) \geq U_{h(n)}(g)$$

where $U_{h(n)} : \mathcal{F}_{h(n)} \mapsto \mathbb{R}$ is defined by[6]

$$U_{h(n)}(f) = \sum_{E\in\pi_{h(n)}} p_{h(n)}(E)\left(\alpha_{h(n)}(f(E)) \inf_{z\in f(E)} u_{h(n)}(z) + \left[1-\alpha_{h(n)}(f(E))\right] \sup_{z\in f(E)} u_{h(n)}(z)\right)$$

3.1 Endogenous Limited Foresight

In dynamic decision problems, any theoretical framework in which the extent of the DM's search plays a role for the determination of her optimal choice gives rise to a rather subtle

[6] In contrast with the representation in Olszewski [21], $\alpha_{h(n)}$ need not be a constant function since we assume Crisp, not Constant Independence. The latter stronger axiom is essentially Olszewski's Set S-independence, appropriately adapted to our setting (see Appendix B for a detailed discussion).

conceptual conundrum. Namely, how to compare the decision-making prescriptions of a search-based model, which depend fundamentally on the structure and extent of search, with those of the rational paradigm which does not involve search. To address this issue in an inherently consistent way, we ought to be able to compare the DM's preferences across the different levels $n \in \mathcal{N}_h$ of her foresight of search.

To this end, a most obvious interpretation of the relations $\left\{\succsim_{h(n)}\right\}_{n\in\mathcal{N}_h}$ is to regard them as depicting the preferences of different "selves" of a given DM who vary in their information about the decision-making situation at hand as search proceeds further along a given problem from the same starting point $h \in H$. To compare these "selves," we need to impose some structure. We will consider, thus, a preference relation $\succsim_h$ on the global set

$$\mathcal{F}_h = \left\{(n, f) : n \in \mathcal{N}_h, f \in \mathcal{F}_{h(n)}\right\}$$

of search-horizon and corresponding-act pairs. For this relation, we will assume the following properties.

B. 1 *Weak Order: The relation $\succsim_h$ satisfies*

(i) Completeness: $\forall n_1, n_2 \in \mathcal{N}_h$ *and* $\forall (f, g) \in \mathcal{F}_{h(n_1)} \times \mathcal{F}_{h(n_2)}$
$(n_1, f) \succsim_h (n_2, g)$ *or* $(n_2, g) \succsim_h (n_1, f)$

(ii) Transitivity: $\forall n_1, n_2, n_3 \in \mathcal{N}_h$ *and* $\forall (f, g, h) \in \mathcal{F}_{h(n_1)} \times \mathcal{F}_{h(n_2)} \times \mathcal{F}_{h(n_3)}$
$(n_1, f) \succsim_h (n_2, g)$ *and* $(n_2, g) \succsim_h (n_3, h) \Rightarrow (n_1, f) \succsim_h (n_3, h)$

B. 2 *Solvability:* $\forall n_1, n_2 \in \mathcal{N}_h$ *and* $\forall z_1, z_2 \in \Delta(X)$ *the following two conditions*

(i) $\exists z' \in \Delta(X) : (n_1, z_1) \succ_h (n_1, z')$ *or* $(n_2, z') \succ_h (n_2, z_2)$

(ii) $(n_1, z_1) \succ_h (n_2, z_2)$

together imply
$\exists z'' \in \Delta(X) : (n_1, z_1) \succ_h (n_1, z'') \succ_h (n_2, z_2)$ *or* $(n_1, z_1) \succ_h (n_2, z'') \succ_h (n_2, z_2)$

B. 3 *Vertical Consistency:* $\forall n \in \mathcal{N}_h$ *and* $\forall f, g \in \mathcal{F}_{h(n)}$ $\quad (n, f) \succsim_h (n, g) \;\Leftrightarrow\; f \succsim_{h(n)} g$

B. 4 *Horizontal Consistency:* $\forall n \in \mathcal{N}_h \setminus \{N\}$, *the restrictions of* $\succsim_{h(n)}$ *and* $\succsim_{h(n+1)}$ *on the collection of crisp or constant acts that are common across the corresponding search horizons* $h(n)$ *and* $h(n+1)$ *coincide. Formally,*

$$\succsim_{h(n)} |_{\mathcal{F}^{cr}_{h(n)} \cup \left(\mathcal{F}^{c}_{h(n)} \cap \mathcal{F}^{c}_{h(n+1)}\right)} \equiv \succsim_{h(n+1)} |_{\mathcal{F}^{cr}_{h(n)} \cup \left(\mathcal{F}^{c}_{h(n)} \cap \mathcal{F}^{c}_{h(n+1)}\right)}$$

For the preference relation $\succsim_h$ to be representable, its rationality (B.1) is indispensable. This is true also for its solvability (B.2), an equally important albeit purely technical property. As for the two consistency conditions, the very aim of our analysis renders their inclusion into our set of axioms *condicio sine qua non* since a preference relation on the set $\mathcal{F}$ amounts actually to a two-stage decision-making exercise. First, one chooses the optimal act for each level of search foresight $n \in \mathcal{N}_h$. Then, by comparing the respective optimal acts, the optimal n is determined. Vertical consistency is necessary to guarantee that the first-stage optimization depicts the preferences $\succsim_{h(n)}$ on the DM's multiple "selves". Horizontal consistency then is required in the second stage to tie together the first-stage "selves" and, hence, ensure that they all emanate from the same original DM.

As demonstrated by the example in Section 2, extending the DM's foresight of search from n to $n+1$ matters because it can improve her information about the decision-making situation she is facing along two dimensions. On the one hand, further search may reveal subsequent moves by Nature, a refinement of the current informational partition $\pi_{h(n)}$. On the other, it could enable the DM to better specify the continuation paths of at least some acts in $\mathcal{A}_{h(n)}$, a more precise specification of the consequences of the currently available acts. There are acts in $\mathcal{F}_{h(n)}$, however, for which further search provides no additional information in either dimension. These are the acts that are common across subsequent search horizons and not affected by partition-refinements; i.e., the crisp acts that are $\pi_{h(n)}$- as well as $\pi_{h(n+1)}$-measurable, $f \in \mathcal{F}^{cr}_{h(n)} \cap \mathcal{F}^{cr}_{h(n+1)} = \mathcal{F}^{cr}_{h(n)}$ (recall Remark 2(a)), and the common constant acts, $f \in \mathcal{F}^{c}_{h(n)} \cap \mathcal{F}^{c}_{h(n+1)}$.[7] Horizontal consistency requires that the relations $\succsim_{h(n)}$ and $\succsim_{h(n+1)}$ agree on these acts. Since extending the foresight of search from n to $n+1$ has no contribution in improving their evaluation, the ranking of such acts should be the same for the nth and $(n+1)$th "self," dictated solely by the underlying DM.

The axioms B.1-B.4 are necessary and sufficient for a representation that allows for the extent of search to obtain endogenously in a finite dynamic decision problem. The two consistency axioms, in particular, have important implications for the properties of the representation. By vertical consistency, for each $n \in \mathcal{N}_h$, the ranking of the constant crisp acts $(n, \cdot)$ on $\Delta(X)$ should be given by the preference relation $\succsim_{h(n)}$. In addition, due to horizontal consistency, the relations $\left\{\succsim_{h(n)}\right\}_{n\in\mathcal{N}_h}$ must all agree on the probability simplex (recall Remark 2(b)). Since all satisfy the expected utility axioms on $\Delta(X)$, they all stem from a common underlying expected utility preference relation $\succsim$ on $\Delta(X)$. This is represented some affine function $u_h : \Delta(X) \mapsto \mathbb{R}$, unique up to a positive linear transformation, and $\forall n \in \mathcal{N}_h$, $\forall z', z'' \in \Delta(X)$, $(n, z') \succsim_h (n, z'')$ iff $z' \succsim_{h(n)} z''$ iff $z' \succsim z''$. Horizontal consis-

[7]In Figure 1, this is the case for the prizes in $\Delta(X)$ and the act f_{r_1} for all n, for the act $f_{l_1 r_2}$ for $n \geq 2$.

tency requires also that the functions $\left\{\alpha_{h(n)}\right\}_{n\in\mathcal{N}_h}$ all agree on the constant acts that are common across search horizons. It imposes further consistency in updating the probability process $\left\{p_{h(n)}\right\}_{n\in\mathcal{N}_h}$. In Figure 1, for instance, we ought to have the same $p_3(s) = p_4(s)$ for all $s \in S$. In addition, it must be $p_2(E_{2j}) = p_3(s_j) + p_3(s_{j+1})$ for $j \in \{1,3,5,7\}$, $p_1(E_1) = p_2(E_{21}) + p_2(E_{22})$, and $p_1(E_2) = p_2(E_{23}) + p_2(E_{24})$.

Theorem 2 *For each $n \in \mathcal{N}_h$, let the preference relations $\succsim_{h(n)}$ on the sets $\mathcal{F}_{h(n)}$ satisfy axioms A.1-A.6. Suppose, moreover, that $\succsim_h$ on the set $\mathcal{F}_h$ satisfies axioms B.1-B.4. Then, there exist a representation $U_h : \mathcal{F}_h \mapsto \mathbb{R}$ for $\succsim_h$ on $\mathcal{F}_h$*

$$\forall n_1, n_2 \in \mathcal{N}_h \ \forall (f,g) \in \mathcal{F}_{h(n_1)} \times \mathcal{F}_{h(n_2)} : \ (n_1, f) \succsim_h (n_2, g) \ \Leftrightarrow \ U_h(n_1, f) \geq U_h(n_2, g)$$

given by

$$\begin{aligned} U_h(n,f) &= \lambda_{h(n)} U_{h(n)}(f) + k_{h(n)} \qquad (1) \\ U_{h(n)}(f) &= \sum_{E\in\pi_{h(n)}} p_{h(n)}(E) \left(\alpha_{h(n)}(f(E)) \inf_{z\in f(E)} u_h(z) + \left[1 - \alpha_{h(n)}(f(E))\right] \sup_{z\in f(E)} u_h(z) \right) \end{aligned}$$

for some collection of pairs of reals $\left\{\lambda_{h(n)}, k_{h(n)}\right\}_{n\in\mathcal{N}_h}$, with $\lambda_{h(n)} > 0$ and $k_{h(n)} \in \mathbb{R}$, and an affine function $u_h : \Delta(X) \mapsto \mathbb{R}$, unique up to a positive linear transformation. Moreover, for any $n_1, n_2 \in \mathcal{N}_h$,

(i) the functions $\alpha_{h(n)} : \mathcal{X}_{h(n)} \mapsto [0,1]$ are such that

$$\alpha_{h(n_1)}(Z) = \alpha h(n_2)(Z) \quad \forall Z \in \mathcal{X}_{n_1} \cap \mathcal{X}_{n_2} \qquad (2)$$

(ii) if $n_2 > n_1$ and $E = \{E_k\}_{k=1,\ldots,K<\infty}$ is the refinement of the event $E \in \pi_{h(n_1)}$ in the subsequent partition $\pi_{h(n_2)}$, the probability distributions $p_{h(n)}$ are such that

$$p_{h(n_1)}(E) = p_{h(n_2)}\left(\sum_{k=1}^{K} E_k\right) \qquad (3)$$

Furthermore, if there exists a functional $U_h : \mathcal{F}_h \mapsto \mathbb{R}$ as in (1) and such that (2) and (3) hold, then the induced preference relation on the set of horizon-act pairs $\mathcal{F}_h$ satisfies axioms B.1-B.4 while the induced relations on the sets $\mathcal{F}_{h(n)}$ satisfy axioms A.1-A.6.

3.1.1 Functional Properties

A variety of limited-horizon and costly search decision-making models that have been widely used in the literature can be nested by our representation in (1). Its linearity makes it analytically convenient and enables facilitation of empirical applications. It is important, therefore, to have some understanding of the behavioral properties of the linear coefficients $\left\{\lambda_{h(n)}, k_{h(n)}\right\}_{n\in\mathcal{N}_h}$. Their operational bite is to be found in the way they relate with one another the utility images of the probability simplex $U_{h(n)}\left(\Delta\left(X\right)\right)$ across the different levels of search foresight. Starting with a common underlying representation $u_h : \Delta\left(X\right) \mapsto \mathbb{R}$, the pair $\left(\lambda_{h(n)}, k_{h(n)}\right)$ transforms the utility interval $u_h\left(\Delta\left(X\right)\right)$ with respect to the vertical axis. Changing one's search foresight from n_1 to n_2, this interval is enlarged (shrunk) if $\lambda_{h(n_1)} < \lambda_{h(n_2)}$ ($\lambda_{h(n_1)} > \lambda_{h(n_2)}$) and displaced up (down) whenever $k_{h(n_1)} < k_{h(n_2)}$ ($k_{h(n_1)} > k_{h(n_2)}$).

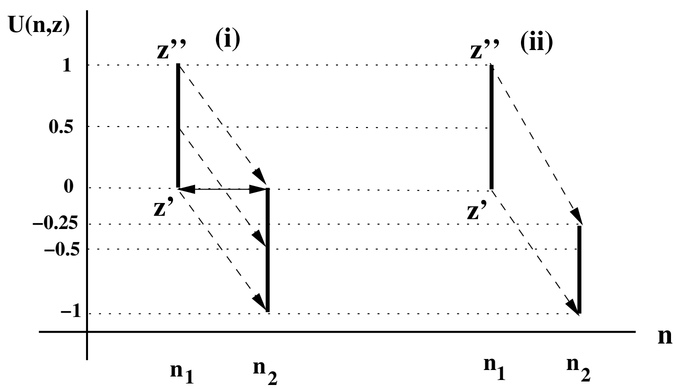

Figure 5: Utility Intervals without Comparability

Such transformations are shown in Figures 5 and ??. For each $n \in \{n_1, n_2\}$, the utility interval $U_h\left(n, \Delta\left(X\right)\right)$ is depicted by a vertical segment.The relative displacement of each prize $z \in \Delta\left(X\right)$ in the $U_h\left(n, z\right)$-space as the foresight changes from $n1$ to n_2 is shown by a dotted arrow. The double-sided solid arrows indicate situations of indifference across the two horizons; in Figure 5(i), for instance, we have $\left(n_1, z'\right) \sim_h \left(n_2, z''\right)$: the agent is indifferent between receiving the worst prize z' and limiting her search horizon to $h\left(n_1\right)$ or receiving the best prize z'' and having, though, to extend her search to $h\left(n_2\right)$. Part (i) of Figure 5 depicts an example where $\lambda_{h(n_2)} = \lambda_{h(n_1)}$ and $k_{h(n_2)} = k_{h(n_1)} - 1$ whereas $\lambda_{h(n_2)} = 0.75\lambda_{h(n_1)}$ and $k_{h(n_2)} = k_{h(n_1)} - 1$ in Figure 5(ii). In Figure 6(i), we have $\lambda_{h(n_2)} = 2\lambda_{h(n_1)}$ and $k_{h(n_2)} = k_{h(n_1)}$ while $\lambda_{h(n_2)} = \lambda_{h(n_1)}$ and $k_{h(n_2)} = k_{h(n_1)} - 0.5$ in Figure 6(ii).

The examples of Figure 6 will be of particular interest in the subsequent analysis for they exhibit an overlap in the utility intervals $U\left(n, \Delta\left(X\right)\right)$ across the two search horizons. In Fig.6(i), we have $U_h\left(n_1, z'\right) = U_h\left(n_2, 7/8z' + 1/8z''\right)$ and $U_h\left(n_1, z''\right) = U_h\left(n_2, 3/8z' + 5/8z''\right)$.

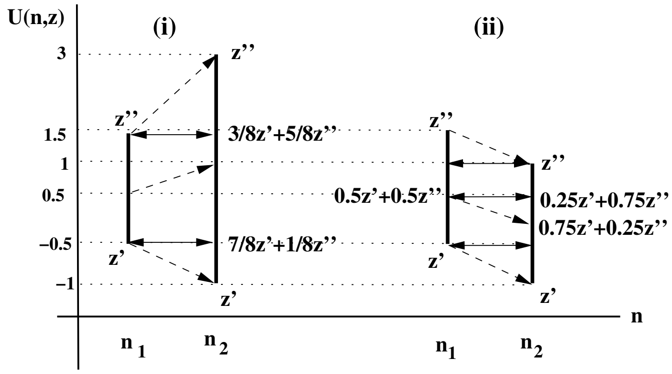

Figure 6: Utility Intervals with Comparability

The DM is indifferent between receiving the prize z', when her search foresight is n_1, or extending her foresight to n_2 in order to receive the lottery $7/8z'+1/8z''$. In fact, every prize in $\Delta(X)$ has its corresponding convex combination of the prizes $7/8z'+1/8z''$ and $3/8z'+5/8z''$ such that the DM is indifferent between receiving the prize itself, having searched up to the horizon $h(n_1)$, or the combination once se has extended her search to $h(n_2)$. In Fig.6(ii), we have $(n_1, z') \sim_h (n_2, 3/4z' + 1/4z'')$ and $(n_1, 1/4z' + 3/4z'') \sim_h (n_2, z'')$.

In general, a utility overlap across two search horizons takes places if there are two corresponding pairs of distinct lotteries such that the global DM is indifferent between the two search horizons if she were to receive the corresponding prize. Formally,

B. 5 *Comparability.* $\forall n_1, n_2 \in \mathcal{N}_h$, $\exists z_1^*, \widetilde{z}_1, z_2^*, \widetilde{z}_2 \in \Delta(\mathcal{X})$ *s.t.*

$$z_1^* \succ \widetilde{z}_1, \quad (n_1, z_1^*) \sim_h (n_2, z_2^*), \quad \textit{and} \quad (n_1, \widetilde{z}_1) \sim_h (n_2, \widetilde{z}_2)$$

With respect to our statement of the comparability property, notice that we could have instead $z_2^* \succ \widetilde{z}_2$ - the horizontal and vertical consistency axioms ensure that the latter is an equivalent requirement.[8] Observe also that, for any $\alpha \in [0,1]$, we have $(n_1, \alpha z_1^* + (1-\alpha)\widetilde{z}_1) \sim_h (n_2, \alpha z_2^* + (1-\alpha)\widetilde{z}_2)$.[9] In what follows, we treat comparability as an axiomatic assumption for it is of crucial importance. It ensures that utility comparisons across different levels of search foresight are consistent in the sense that utility differences have meaning; in other words, our representation is cardinal.

[8]To see the "only if", suppose otherwise (the argument for the "if" statement proceeds in a trivially similar way). Let, thus, $z_1^* \succ \widetilde{z}_1$ but $\widetilde{z}_2 \succsim z_2^*$. By vertical consistency, we must have $(n_1, z_1^*) \succ_h (n_1, \widetilde{z}_1)$ and $(n_2, \widetilde{z}_2) \succsim_h (n_2, z_2^*)$ By the remaining of the comparability axiom then, $(n_1, z_1^*) \succ_h (n_1, \widetilde{z}_1) \sim_h (n_2, \widetilde{z}_2) \succsim_h (n_2, z_2^*)$, a contradiction. Given completeness of the relation $\succsim$, it must be $z_2^* \succ \widetilde{z}_2$.

[9]This is immediate from the linearity of the representation in (1): $\forall n \in \mathcal{N}_h$ and $\forall z, z' \in \Delta(X)$, we have $U_h(n, \alpha z + (1-\alpha)\widetilde{z}) = \alpha U_h(n, z) + (1-\alpha) U_h(n, \widetilde{z})$.

Proposition 1 ***(Cardinal Representation)*** *Let the preference relations $\left\{\succsim_{h(n)}\right\}_{n\in\mathcal{N}_h}$ on the sets of acts $\left\{\mathcal{F}_{h(n)}\right\}_{n\in\mathcal{N}_h}$ satisfy axioms A.1-A.6. Suppose also that the relation $\succsim_h$ satisfies axioms B.1-B.5 on the set $\mathcal{F}_h$. Then, the representation in (1) is cardinal. That is, for any two functionals $U_h(n,\cdot) = \lambda_{h(n)}u_h(\cdot) + k_{h(n)}$ and $U_h^0(n,\cdot) = \lambda^0_{h(n)}u^0_h(\cdot) + k^0_{h(n)}$ representing $\succsim_h$ on $\mathcal{N}_h \times \Delta(X)$, $\forall z_1, z_1', z_2, z_2' \in \Delta(X)$ and $\forall n_1, n_2 \in \mathcal{N}_h$, the following are equivalent*

(i) $U_h(n_1, z_1) - U_h(n_1, z_1') = U_h(n_2, z_2) - U_h(n_2, z_2')$

(ii) $U^0_h(n_1, z_1) - U^0_h(n_1, z_1') = U^0_h(n_2, z_2) - U^0_h(n_2, z_2')$

Furthermore, the relative coefficients $\frac{\lambda_{h(n_1)}}{\lambda_{h(n_2)}}$ in (1) are uniquely determined

$$\frac{\lambda_{h(n_1)}}{\lambda_{h(n_2)}} = \frac{\lambda^0_{h(n_1)}}{\lambda^0_{h(n_2)}} \tag{4}$$

while

$$k^0_{h(n_2)} - k^0_{h(n_1)} = \frac{\lambda^0_{h(n_1)}}{\lambda_{h(n_1)}}\left(\beta\left[k_{h(n_2)} - k_{h(n_1)}\right] + \gamma\left[\lambda_{h(n_1)} - \lambda_{h(n_2)}\right]\right) \tag{5}$$

for positive affine transformations $u_h(\cdot) = \beta u^0_h(\cdot) + \gamma$ with $\beta > 0$ and $\gamma \in \mathbb{R}$.

The proposition establishes only sufficiency between comparability and cardinality. For a complete treatment of their relation, it remains to investigate necessity.

Claim 1 *Let the preference relations $\left\{\succsim_{h(n)}\right\}_{n\in\mathcal{N}_h}$ on the sets of acts $\left\{\mathcal{F}_{h(n)}\right\}_{n\in\mathcal{N}_h}$ satisfy axioms A.1-A.6. Suppose also that the relation $\succsim_h$ satisfies axioms B.1-B.4 on the set $\mathcal{F}_h$ and its representation in (1) is cardinal. Then $\succsim_h$ fails to obey axiom B.5 on $\mathcal{F}_h$ only if one of the following are true*

(i) If $(n_1, \underline{z}) \succsim_h (n_2, \overline{z})$, the coefficients in (1) are s.t. $k_{h(n_1)} \geq k_{h(n_2)}$, with $u_h(\cdot) \neq 0$ when $k_{h(n_1)} = k_{h(n_2)}$, $\lambda_{h(n_1)}u_h(\underline{z}) \geq \lambda_{h(n_2)}u_h(\overline{z})$, and $u_h(\cdot)$ is unique up to positive linear transformations, $u^0_h(\cdot) = \beta u_h(\cdot)$ for $\beta > 0$.

(ii) The coefficients $\left\{\lambda_{h(n)}\right\}_{n\in\mathcal{N}_h}$ are constant, $\lambda_{h(n)} = \lambda > 0$ $\forall n \in \mathcal{N}_h$. That is, the representation is of the form

$$U_h(n, f) = U_{h(n)}(f) + k_{h(n)} \tag{6}$$

with $k_{h(n_1)} > k_{h(n_2)}$ if $(n_1, \underline{z}) \succsim_h (n_2, \overline{z})$.

As defined here, cardinality appears to be concerned only with utility differences, not necessarily rankings, across search horizons. Yet, as often the case, appearances are deceptive for, under comparability, our representation is guaranteed to be consistent also in the sense of ranking comparisons across different levels of foresight. For ordinal consistency, we require $U_h(n_1, z) > U_h(n_2, z')$ to be equivalent to $U_h^0(n_1, z) > U_h^0(n_2, z')$ for any $z, z' \in \Delta(X)$. That is, we want

$$\lambda_{h(n_1)} u_h(z_1) - \lambda_{h(n_2)} u_h(z_2) > k_{h(n_2)} - k_{h(n_1)} \tag{7}$$

to be equivalent to $\lambda^0_{h(n_1)} u^0_h(z_1) - \lambda^0_{h(n_2)} u^0_h(z_2) > k^0_{h(n_2)} - k^0_{h(n_1)}$. Horizontal consistency, however, restricts $u^0_h(\cdot)$ to be a positive affine transformation of $u_h(\cdot)$: $u^0_h(\cdot) = \beta u_h(\cdot) + \gamma$ for some $\beta > 0$ and $\gamma \in \mathbb{R}$. Therefore, the last inequality reads

$$\beta\left[\lambda^0_{h(n_1)} u_h(z_1) - \lambda^0_{h(n_2)} u_h(z_2)\right] + \gamma\left[\lambda^0_{h(n_1)} - \lambda^0_{h(n_2)}\right] > k^0_{h(n_2)} - k^0_{h(n_1)} \tag{8}$$

which, under (5), reduces to (7).

We are now in position to examine the functional properties of our representation. As it turns out, they depend fundamentally on the existence (and multiplicity) of prizes in $\Delta(X)$ for which the preference relation $\succsim_h$ is insensitive to the extent of the search undertaken in the decision problem. We will be referring to a prize $z^* \in \Delta(X)$ such that

$$(n_1, z^*) \sim_h (n_2, z^*) \quad \forall n_1, n_2 \in \mathcal{N}_h$$

as a reference prize for the preference relation $\succsim_h$ on $\mathcal{N}_h \times \Delta(X)$. The term is due to the fact that such lotteries can be used as reference points for calibrating the transformations in the utility functions $u_{h(n)}(\cdot)$ of Theorem 1 which become embedded in the functional $U_h(n, \cdot)$ of Theorem 2 as the level of search foresight $n \in \mathcal{N}_h$ changes.[10]

Of course, as attested by the examples in Figures 5 and 6, a reference prize may not exist. In this case, our representation could take one of two general functional forms described formally by parts (i) and (ii) of the following claim. In terms of the examples themselves, the ones in Figures 5(ii) and 6(i) are representable by the form given in Claim 2(i). Those in Figures 5(i) and 6(ii) fall under Claim 2(ii).

Claim 2 ***(Additive Comparability)*** *Let the preference relations* $\left\{\succsim_{h(n)}\right\}_{n\in\mathcal{N}_h}$ *on the sets of acts* $\left\{\mathcal{F}_{h(n)}\right\}_{n\in\mathcal{N}_h}$ *satisfy axioms A.1-A.6. Suppose also that the preference relation* $\succsim_h$

[10] In proving Lemma 5 and Theorem 2 (see Appendix A), we established that the linear coefficients in (1) are given by $\lambda_{h(n)} = \mu_{h(n)}\beta_{h(n)}$ and $k_{h(n)} = \mu_{h(n)}\gamma_{h(n)} + \nu_{h(n)}$, for some constants $\mu_{h(n)} > 0, \nu_{h(n)} \in \mathbb{R}$, under the transformation $u_{h(n)}(\cdot) = \beta_{h(n)} u_h(\cdot) + \gamma_{h(n)}$ for some constants $\beta_{h(n)} > 0, \gamma_{h(n)} \in \mathbb{R}$.

satisfies axioms B.1-B.4 on the set $\mathcal{F}_h$. The following are equivalent.

(a) *No Reference Prize.* $\nexists z \in \Delta(\mathcal{X})$ *s.t.* $(n_1, z) \sim_h (n_2, z)$ $\forall n_1, n_2 \in \mathcal{N}_h$.

(b) The representation in (1) takes one of two forms

(i) *The coefficients are s.t.* $k_{h(n_1)} \geq k_{h(n_2)}$, *with* $u_h(\cdot) \neq 0$ *when* $k_{h(n_1)} \neq k_{h(n_2)}$, *and* $\left(\lambda_{h(n_1)} - \lambda_{h(n_2)}\right) u_h(\cdot) \geq 0$ *with* $u_h(\cdot)$ *unique up to positive linear transformations. This gives* $(n_1, z) \succ_h (n_2, z)$ $\forall z \in \Delta(X)$.

(ii) *As in (6), giving* $(n_1, z) \succ_h (n_2, z)$ $\forall z \in \Delta(X)$ *if* $k_{h(n_1)} > k_{h(n_2)}$.

When reference prizes do exist, we need to distinguish between two possibilities. Suppose first that there are actually multiple (preferentially distinct) reference prizes for $\succsim_h$ on $\mathcal{N}_h \times \Delta(X)$. It is easy to see that this can be only if every prize is actually a reference prize.[11] In terms of the representation in (1), this translates into constant coefficients.

Claim 3 ***(Identical Comparability)*** *Let the preference relations* $\left\{\succsim_{h(n)}\right\}_{n\in\mathcal{N}_h}$ *on the sets of acts* $\left\{\mathcal{F}_{h(n)}\right\}_{n\in\mathcal{N}_h}$ *satisfy axioms A.1-A.6. Suppose also that the preference relation* $\succsim_h$ *satisfies axioms B.1-B.4 on the set* $\mathcal{F}_h$. *The following are equivalent.*

(a) *Multiple Reference Prizes.* $\exists z^*, z^{**} \in \Delta(X)$ *s.t.* $z^* \succ z^{**}$, $(n_1, z^*) \sim_h (n_2, z^*)$, *and* $(n_1, z^{**}) \sim_h (n_2, z^{**})$ $\forall n_1, n_2 \in \mathcal{N}_h$.

(b) *The coefficients* $\left\{\lambda_{h(n)}, k_{h(n)}\right\}_{n\in\mathcal{N}_h}$ *in (1) are constant:* $\lambda_{h(n)} = \lambda > 0$ *and* $k_{h(n)} = k \in \mathbb{R}$ $\forall n \in \mathcal{N}_h$. *That is, the representation in (1) reads*

$$U(n, f) = U_{h(n)}(f) \tag{9}$$

When the reference prize is unique, the linear coefficients are not necessarily constant. Yet, the representation in (1) admits an elegant and analytically convenient form. As this will be used throughout the remaining of our analysis, we present the uniqueness property as an axiom.

B.5* Unique Reference Prize. $\exists! z^* \in \Delta(X)$ s.t. $(n_1, z^*) \sim_h (n_2, z^*)$ $\forall n_1, n_2 \in \mathcal{N}_h$.

Figure 7 shows different cases where the uniqueness axiom is met (in Figure 7(i)-(iii), z^*

[11]Let $z^*, z^{**} \in \Delta(X)$ be such that $z^* \succ z^{**}$. Given any other $z \in \Delta(X)$, there exists a unique number $\alpha \in [0,1]$ such that $z \sim \alpha z^* + (1-\alpha) z^{**}$ (see Lemma 2 in the Appendix). By vertical consistency, therefore, we ought to have $(n, z) \sim_h (n, \alpha z^* + (1-\alpha) z^{**})$ $\forall n \in \mathcal{N}_h$. Recall, now, that the representation for $\succsim_h$ on $\mathcal{F}_h$ in (1) is linear. Thus, $\forall n \in \mathcal{N}_h$, $(n, \alpha z^* + (1-\alpha) z^{**}) \sim_h \alpha (n, z^*) + (1-\alpha)(n, z^{**})$. But $(n_1, z) \sim_h \alpha (n_1, z^*) + (1-\alpha)(n_1, z^{**})$ implies, under monotonicity and z^*, z^{**} being reference prizes, $(n_1, z) \sim_h (n_2, z)$.

is, respectively, $\overline{z}$, $\underline{z}$, and $1/2\underline{z} + 1/2\overline{z}$). It is easy to see, of course, that B.5* is a stronger condition than cardinality.[12]

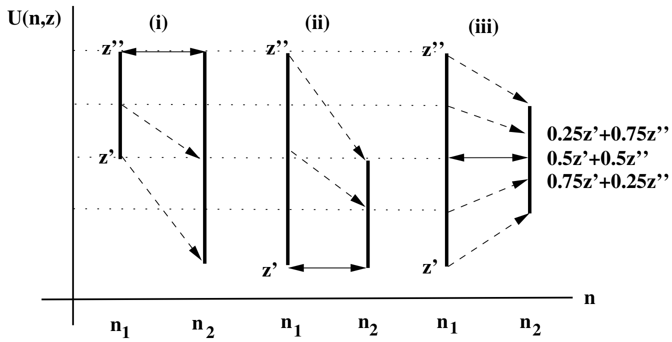

Figure 7: Utility Intervals with Unique Reference Prize

Claim 4 ***(Linear Comparability)*** *Let the preference relations $\{\succsim_{h(n)}\}_{n\in\mathcal{N}_h}$ on the sets of acts $\{\mathcal{F}_{h(n)}\}_{n\in\mathcal{N}_h}$ satisfy axioms A.1-A.6. Suppose also that $\succsim_h$ satisfies axioms B.1-B.4 on $\mathcal{F}_h$. The following are equivalent.*

(i) $\succsim_h$ satisfies axiom B.5 on $\mathcal{N}_h \times \Delta(X)$.*

(ii) The coefficients $\{k_{h(n)}\}_{n\in\mathcal{N}_h}$ in (1) are constant, $k_{h(n)} = k \in \mathbb{R}\ \forall n \in \mathcal{N}_h$. That is, there exists a normalized utility function $u : \Delta(X) \mapsto \mathbb{R}$, namely assigning $u(z^*) = 0$, so that the functional takes the form

$$U(n, f) = \lambda_{h(n)} U_{h(n)}(f) \tag{10}$$

with uniquely determined relative coefficients $\left\{\frac{\lambda_{h(n)}}{\lambda_{h(n+1)}}\right\}_{n\in\mathcal{N}_h\setminus\{N\}}$.

More generally, whether or not the functional form in (10) is adopted, the uniqueness axiom (B.5*) leads to two important characterizations for the representation in (1). The first establishes a connection between the monotonicity of the coefficients $\{\lambda_{h(n)}\}_{n\in\mathcal{N}_h}$, the unique reference prize being an extreme prize on $\Delta(X)$, and the preference relation $\succsim_h$ being itself monotone across the level of search foresight $n \in \mathcal{N}_h$. The concept of increasing (decreasing) horizon monotonicity refers to an unambiguous preference for receiving any prize $z \in \Delta(X)$ under a larger (smaller) horizon of search. Formally, we will say that $\succsim_h$ exhibits Increasing

[12] Recall also of our previous observation that multiplicity of reference prizes requires every prize to be a reference one. Clearly, the very existence of a reference prize is a stronger assumption than comparability.

[Decreasing] Horizon Monotonicity (IHM) [(DHM)] on $\mathcal{N}_h \times \Delta(X)$ if, for any $n \in \mathcal{N}_h \setminus \{N\}$, we have

$$(n+1, z) \succsim_h (n, z) \;\; [(n, z) \succsim_h (n+1, z)] \quad \forall z \in \Delta(X)$$

with strict preference for at least one prize.

Claim 5 *Let the preference relations $\{\succsim_{h(n)}\}_{n \in \mathcal{N}_h}$ on the sets of acts $\{\mathcal{F}_{h(n)}\}_{n \in \mathcal{N}_h}$ satisfy axioms A.1-A.6. Suppose also that $\succsim_h$ satisfies axioms B.1-B.5* on the set $\mathcal{F}_h$. For the following statements, (i) is equivalent to (ii) whereas (iii) is equivalent to (iv).*

(i) $\succsim_h$ *exhibits IHM on* $\mathcal{N}_h \times \Delta(X)$.

(ii) $\lambda_{h(n+1)} > \lambda_{h(n)}$ $\left(\lambda_{h(n+1)} < \lambda_{h(n)}\right)$ $\forall n \in \mathcal{N}_h \setminus \{N\}$ *while* z^* *is minimal (maximal) on* $\Delta(X)$.

(iii) $\succsim_h$ *exhibits DHM on* $\mathcal{N}_h \times \Delta(X)$.

(iv) $\lambda_{h(n+1)} > \lambda_{h(n)}$ $\left(\lambda_{h(n+1)} < \lambda_{h(n)}\right)$ $\forall n \in \mathcal{N}_h \setminus \{N\}$ *while* z^* *is maximal (minimal) on* $\Delta(X)$.

The second characterization is complementary to the one just given. It provides an ineluctable link between horizon monotonicity and an extreme position for the unique reference prize in the utility ranking on $\Delta(X)$.

Claim 6 *Let the preference relations $\{\succsim_{h(n)}\}_{n \in \mathcal{N}_h}$ on the sets of acts $\{\mathcal{F}_{h(n)}\}_{n \in \mathcal{N}_h}$ satisfy axioms A.1-A.6. Suppose also that $\succsim_h$ satisfies axioms B.1-B.5* on $\mathcal{F}_h$. The following are equivalent.*

(i) *Betweeness*. $\exists z', z'' \in \Delta(X)$ *s.t.* $z' \succ z^* \succ z''$

(ii) *Horizon Non-monotonicity*. $\exists z', z'' \in \Delta(X)$ *s.t.* $\forall n_1, n_2 \in \mathcal{N}_h$ *with* $n_1 \neq n_2$

$$(n_1, z') \succ_h (n_2, z') \;\Leftrightarrow\; (n_2, z'') \succ_h (n_1, z'')$$

3.2 Rationality under Endogenous Foresight

Our approach is based fundamentally upon comparisons of the (observed) choices under search-based decision-making with those prescribed by rationality (perfect foresight). Of course, such comparisons can be made in a coherent and consistent way only if rational decision-making itself can be depicted as (a special case of) decision-making under limited foresight. In this section, we specify the necessary and sufficient conditions for this to be the case.

Recall that $\mathcal{A}_{h(n)}$ is the collection of acts defined on the given decision tree under search-foresight of level n. This is the set of acts *actually* available to the nth "self" of our underlying DM, a "self" who chooses according to the preference relation $\succsim_{h(n)}$. For $n_1, n_2 \in \mathcal{N}_h$ let f^* and g^* be maximal acts in $\mathcal{A}_{h(n_1)}$ and $\mathcal{A}_{h(n_2)}$ under the relations $\succsim_{h(n_1)}$ and $\succsim_{h(n_2)}$, respectively. By vertical consistency, the restriction of the underlying "global" preference relation $\succsim_h$ coincides with the relation $\succsim_{h(n)}$ on $\mathcal{A}_{h(n)}$. Which is to say that the horizon-act pairs (n_1, f^*) and (n_2, g^*) are optimal, according to $\succsim_h$, over all pairs (n_1, f) and (n_2, g), respectively, for any $r \in \mathcal{A}_{h(n_1)}$ and any $\widetilde{r} \in \mathcal{A}_{h(n_2)}$. Horizontal consistency, on the other hand, restricts the relative ranking of (n_1, f^*) and (n_2, g^*) - and, consequently, of f^* and g^* themselves - but only if f^* and g^* are both common across the sets $\mathcal{A}_{h(n_1)}$ and $\mathcal{A}_{h(n_2)}$.

Yet, in order to construct a platform for comparisons between optimal choices prescribed by limited and unlimited foresight, we must endow our theoretical framework with some structure for the relative ranking of the acts f^* and g^* even when they are not common across $\mathcal{A}_{h(n_1)}$ and $\mathcal{A}_{h(n_2)}$. The axiom that follows achieves this by imposing some (minimal in our view) consistency across acts that are defined under different search horizons.

B. 6 *Continuation Consistency. Let $n \in \mathcal{N}_h \setminus \{N\}$, $f \in \mathcal{A}_{h(n)}$, and $g \in \mathcal{A}_{h(n+1)}$ be arbitrary.*

(i) $\exists f^+ \in \mathcal{A}_{h(n+1)} : \; f \succsim_{h(n)} z \Rightarrow f^+ \succsim_{h(n+1)} z \quad \forall z \in \Delta(X)$

(ii) $\exists g^- \in \mathcal{A}_{h(n)} : \; g \succsim_{h(n+1)} z \Rightarrow g^- \succsim_{h(n)} z \quad \forall z \in \Delta(X)$

The intuition for requiring continuation consistency can be understood as deriving from the filtration property of the sequence of partitions formed by our search process as it proceeds along the given decision tree - recall Remark 1(a). Suppose that $n_2 > n_1$ and, for the succeeding partitions $\pi_{h(n_1)}$ and $\pi_{h(n_2)}$, let

$$\mathcal{E}_{h(n_2)}(E) = \left\{E' \in \pi_{h(n_2)} : E' \subseteq E\right\}$$

denote the collection of the events in $\pi_{h(n_2)}$ that are sub-events of the event $E \in \pi_{h(n_1)}$. For an act $f \in \mathcal{A}_{h(n_1)}$, we will refer to the act $g \in \mathcal{A}_{h(n_2)}$ as a continuation act if g assigns to the sub-events in $\pi_{h(n_2)}$ subsets of the sets of lotteries that f assigns to each event in $\pi_{h(n_1)}$. More precisely, if

$$g(E') \subseteq f(E) \qquad \forall E' \in \mathcal{E}_{h(n_2)}(E) \;\; \forall E \in \pi_{h(n_1)}$$

If g is a continuation act for f, we will refer also to the act f as a predecessor act of g. Obviously, due to the filtration property that our structured search process imposes on consecutive partitions, for any $\forall n_1 \in \mathcal{N}_h \setminus \{N\}$, every act in $\mathcal{A}_{h(n_1)}$ has at least one continuation act in $\mathcal{A}_{h(n_2)}$ for each $n_2 = n_1 + 1, \ldots, N$. Conversely, for any $n_2 \in \mathcal{N}_h \setminus \{1\}$, every act in $\mathcal{A}_{h(n_2)}$ has some predecessor act in $\mathcal{A}_{h(n_1)}$ for each $n_1 = 1, \ldots, n_2 - 1$.

Using the concepts of continuation and predecessor acts, the axiom (B.6) can be re-stated as follows. Given a current level of foresight n_1 and an act $f \in \mathcal{A}_{h(n_1)}$, the collection $\mathcal{A}_{h(n_2)}$ of those acts that would be available to the DM were her search foresight to extend to n_2 should include a continuation act g such that the corresponding non-ambiguous equivalents, $g \sim_{h(n_2)} z_{g'}$ and $z_f \sim_{h(n_1)} f$, are (globally) related as $z_g \succsim z_f$.[13] In addition, given a current level of foresight n_2 and an act $g' \in \mathcal{A}_{h(n_2)}$, the collection $\mathcal{A}_{h(n_1)}$ of acts that were available to the DM when her search foresight was limited at n_1 must have included a predecessor act f' such that $z_{f'} \succsim z_{g'}$.

As read on the given decision tree, under the event $E \in \pi_{h(n_1)}$, an act f corresponds to some set, say $X_{f_E} \subseteq X$, of outcomes the DM's "self" with search foresight n_1 regards as possible terminal consequences if she chooses the act f and a state $s \in E$ realizes. For technical reasons, nevertheless, our analysis has allowed f to be a more general mapping, one that assigns a set of lotteries $f(E) \subseteq \Delta(X)$ in the event E. The existence of a continuation act g such that $z_g \succsim z_f$ guarantees that the DM's current assessment of the future value of the act f is consistent with the future possibilities of the underlying tree in the sense that it impedes the DM from being too optimistic. For we require the DM to believe that, were her current "self" able to extend the search foresight to n_2 from n_1, it would have been revealed to her n_2-level "self" that she has available an act g which, from the perspective of the n_2-level "self," is at least as-good-as f in the event E. Similarly, the requirement of a predecessor act f' such that $z_{f'} \succsim z_{g'}$ guarantees that the DM's past assessment (when her search foresight was limited to n_1) of the future value of the act f' was consistent with the future possibilities of the underlying tree in the sense that the DM could not have been too pessimistic. For the DM ought to have believed back then that, were her then current "self" able to extend the search foresight to n_2 from n_1, it would have been revealed to her n_2-level "self" that there is an act g which, from the perspective of the n_2-level "self," is at

[13]For the notion and construction of the non-ambiguous equivalent prize, $z_f \in \Delta(X) : z_f \sim_{h(n)} f$, to an act $f \in \mathcal{F}_{h(n)}$ under the relation $\succsim_{h(n)}$, see Lemma 4 in the Appendix and the paragraph immediately-preceding Theorem 1 in the main text. Notice that, for reasons of expositional clarity, we could have taken the contrapositive statement of that in the axiom. As both relations $\succsim_{h(n_1)}$ and $\succsim_{h(n_2)}$ are complete on $\Delta(X)$, this is given by "$\forall z \in \Delta(X) : z \succ_{h(n_2)} g \Rightarrow z \succ_{h(n_1)} f$," implying, for the relation between the non-ambiguous equivalents, that "$\nexists z \in \Delta(X) : z_f \succsim_{h(n_1)} z \succ_{h(n_2)} z_g$." Since, though, the non-ambiguous equivalents are prizes (constant crisp acts), the subscript can be dropped; by Remarks 2(b)-(c) and vertical and horizontal consistency, the relation is respected by all preferences $\left\{\succsim_{h(n)}\right\}_{n \in \mathcal{N}_h}$ as well as by $\succsim$. Of course, the statements "$\nexists z \in \Delta(X) : z_f \succsim z \succ z_g$" and "$z_g \succsim z_f$" are equivalent (see Lemma 6 in the Appendix).

least as-bad-as f in the event E.

To illustrate, let us revisit the example of Figure 1. Under search foresight $n = 1$, the act f_{l_1} is viewed as assigning the sets of terminal consequences $X_1 = \{x_2, x_3\}$ and $X_2 = \{x_4, x_5, x_6, x_7\}$ in the events $E_1 = \{s_1, \ldots, s_4\}$ and $E_2 = \{s_5, \ldots, s_8\}$, respectively (Figure 2). Let us denote the corresponding sets of terminal lotteries (probability distributions on X_1 and X_2), respectively, by $f_{l_1}(E_1)$ and $f_{l_1}(E_2)$. Extending the agent's foresight to $n = 2$ reveals two candidate continuation acts, $f_{l_1 l_2}$ and $f_{l_1 r_2}$ (Figure 3). Both assign the degenerate lotteries 1_{x_2} and 1_{x_3}, corresponding to the terminal consequence singletons $\{x_2\}$ and $\{x_3\}$, in the events $E_{21} = \{s_1, s_2\}$ and $E_{22} = \{s_3, s_4\}$, respectively. Under either of the events $E_{23} = \{s_5, s_6\}$ and $E_{24} = \{s_7, s_8\}$, however, $f_{l_1 r_2}$ corresponds to the lottery 1_{x_4} whereas $f_{l_1 l_2}$ results in the set of possible terminal consequences $X_2 = \{x_5, x_6, x_7\}$. If the corresponding set of terminal lotteries, $f_{l_1 l_2}(E_{23}) = f_{l_1 l_2}(E_{24})$ and $f_{l_1 r_2}(E_{23}) = f_{l_1 r_2}(E_{24}) = 1_{x_4}$, are both contained in $f_{l_1}(E_2)$, then both acts are indeed continuations of f_{l_1}. In this case, to rule out over-optimism and over-pessimism at foresight $n = 1$ relative to $n = 2$, it is necessary that the non-ambiguous equivalents, $z_{f_{l_1}} \sim_{a.1(1)} f_{l_1}$, $\left(x_2, x_3, x_4; \frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right) \sim_{a.1(2)} f_{l_1 r_2}$, and $z_{f_{l_1} l_2(E_{23})} \sim_{a.1(2)} f_{l_1 l_2}(E_{23})$, are related as follows[14]

$$\begin{array}{c} 0.25\left[u(x_2) + u(x_3)\right] \\ +0.5 \min\left\{u(x_4), u\left(z_{f_{l_1} l_2(E_{23})}\right)\right\} \end{array} \leq u\left(z_{f_{l_1}}\right) \leq \begin{array}{c} 0.25\left[u(x_2) + u(x_3)\right] \\ +0.5 \max\left\{u(x_4), u\left(z_{f_{l_1} l_2(E_{23})}\right)\right\} \end{array} \quad (11)$$

Under foresight $n = 3$ (Figure 4), a continuation of $f_{l_1 l_2}$, say the act $g_{l_1 l_2} \in \mathcal{A}_3$, would assign the degenerate lottery 1_{x_5} in the states s_5 and s_6 and a set of lotteries $g_{l_1 l_2}(s_7) = g_{l_1 l_2}(s_8) \subseteq f_{l_1 l_2}(E)$, corresponding to the outcome set $\{x_6, x_7\}$, in the states s_6 and s_8. Continuation consistency requires now the existence of continuation acts $g'_{l_1 l_2}, g''_{l_1 l_2} \in \mathcal{A}_3$ for which the corresponding non-ambiguous equivalents, $z_{g'_{l_1 l_2}(s_8)} \sim_{a.1(3)} g'_{l_1 l_2}(s_8)$ and $z_{g''_{l_1 l_2}(s_7)} \sim_{a.1(3)} g''_{l_1 l_2}(s_7)$,

[14] $z_{f_{l_1 l_2}(E_{23})}$ is the non-ambiguous equivalent of the corresponding set of terminal lotteries $f_{l_1 l_2}(E_{23})$ when this is viewed as a constant act. Recall also that $f_{l_1 l_2}(E_{23}) = f_{l_1 l_2}(E_{24})$. We restrict our attention to the events E_{23} and E_{24} for there is no need to analyze the continuation acts of $f_{l_1 l_2}$ under any other event in the partition $\pi_{a.1(2)}$. For in any other event, $f_{l_1 l_2}$ as well as any of its continuation acts assign degenerate lotteries, corresponding to singleton outcomes. That is, in any event $E \in \pi_{a.1(2)} \setminus \{E_{23}, E_{24}\}$, $f_{l_1 l_2}$ is a crisp act, coinciding with any of its continuation acts. By vertical and horizontal consistency, therefore, $f_{l_1 l_2}(E)$ and its continuation acts must be evaluated in exactly the same way by any relation $\succsim_{a.1(n):n \geq 2}$ as well as by $\succsim_{a.1}$.

are such that[15]

$$0.5u\left(x_5\right)+0.5u\left(z_{g''_{l_1l_2}(s_8)}\right)\leq u\left(z_{f_{l_1l_2}(E_{23})}\right)\leq 0.5u\left(x_5\right)+0.5u\left(z_{g'_{l_1l_2}(s_8)}\right) \tag{12}$$

If the foresight of search were to be extended to $n=4$ (Figure 1), there would be only two continuations of $f_{l_1l_2}$ in $\mathcal{A}_4$, $g_{l_1l_2l_3}$ and $g_{l_1l_2r_3}$. Both assign the same degenerate lottery as $f_{l_1l_2}$ does in any state $s\notin\{s_6,s_8\}$. For $s\in\{s_6,s_8\}$, they result, respectively, in the degenerate lotteries 1_{x_6} and 1_{x_7}. If $x_7\succ x_6$, therefore, we must have

$$0.5u\left(x_5\right)+0.5u\left(x_6\right)\leq u\left(z_{g'_{l_1l_2}(s_8)}\right),u\left(z_{g''_{l_1l_2}(s_8)}\right)\leq 0.5u\left(x_5\right)+0.5u\left(x_7\right) \tag{13}$$

Conditions (11)-(13) are necessary for continuation consistency. As we argue below, however, they are not sufficient for the full force of the axiom. A sufficient (but not necessary) condition for continuation consistency obtains through the interaction between horizon monotonicity and global optimality. Specifically, under DHM, an act $g\in\mathcal{A}_{h(n_2)}$ that is optimal under $\succsim_h$ (on the global set of horizon-pair acts $\mathcal{A}_h$) is automatically the at-least-as-good-as act in axiom (B.6) of any act $f\in\mathcal{A}_{h(n_1)}$ and for any $n_1<n_2$. Under IHM, on the other hand, if the act f is optimal under $\succsim_h$, then any act $g\in\mathcal{A}_{h(n_2)}$ for any $n_2>n_1$ is automatically the at-least-as-bad-as act in the axiom.[16]

Claim 7 *$\forall n_1,n_2\in\mathcal{N}_h$ with $n_2>n_1$ the following are true.*

(i) Let $g\in\mathcal{A}_{h(n_2)}$ be such that (n_2,g) is optimal for $\succsim_h$ on $\mathcal{A}_h$. Under DHM,

$$\forall z\in\Delta\left(X\right)\ \forall f\in\mathcal{F}_{h(n_1)}:\quad f\succsim_{h(n_1)}z\Rightarrow g\succsim_{h(n_2)}z$$

(ii) Let $f\in\mathcal{A}_{h(n_1)}$ be such that (n_1,f) is optimal for $\succsim_h$ on $\mathcal{A}_h$. Under IHM,

$$\forall z\in\Delta\left(X\right)\ \forall g\in\mathcal{F}_{h(n_2)}:\quad z\succ_{h(n_1)}f\Rightarrow z\succ_{h(n_2)}g$$

With continuation consistency, our theoretical framework is able to compare the optimal choices prescribed by limited and unlimited foresight in a coherent manner. To see this, recall that the relation $\succsim_{h(n)}$, deployed by the nth "self" of the DM to choose from the set

[15] $z_{g'_{l_1l_2}(s_8)}$ and $z_{g''_{l_1l_2}(s_8)}$ are, respectively, the non-ambiguous equivalents of the corresponding sets of terminal lotteries $g'_{l_1l_2}\left(s_8\right)$ and $g''_{l_1l_2}\left(s_8\right)$ when each set is viewed as a constant act. Recall that $g'_{l_1l_2}\left(s_8\right)=g'_{l_1l_2}\left(s_6\right)$ and $g''_{l_1l_2}\left(s_8\right)=g''_{l_1l_2}\left(s_6\right)$.

[16] In this case, we apply the continuation consistency criterion on the act $g\in\mathcal{A}_{h(n_2)}$. That is, we require that there is an act $f\in\mathcal{A}_{h(n_1)}$ s.t. $\forall z\in\Delta\left(X\right):g\succsim_{h(n_2)}z\Rightarrow f\succsim_{h(n_1)}z$. The latter statement is simply the contrapositive of that in part (ii) of the claim.

$\mathcal{A}_{h(n)}$ of acts that are available on the given decision tree, is rational. It induces, thus, a well-defined (in the sense that it satisfies the weak axiom of revealed preference) choice rule, the correspondence $\mathcal{C}\left(\cdot, \succsim_{h(n)}\right) : \mathcal{A}_{h(n)} \mapsto 2^{\mathcal{A}_{h(n)}}$ which assigns the set of optimal acts under $\succsim_{h(n)}$,

$$f \in \mathcal{C}\left(\mathcal{A}_{h(n)}, \succsim_{h(n)}\right) \subseteq \mathcal{A}_{h(n)} \quad \text{iff} \quad f \succsim_{h(n)} h \;\; \forall h \in \mathcal{A}_{h(n)}$$

To consistently compare choices made under different rules, one has to deploy a benchmark that is common, in a consistent way, across the corresponding preference relations $\left\{\succsim_{h(n)}\right\}_{n \in \mathcal{N}_h}$. Clearly, this can be nothing else than the "global" preference relation $\succsim_h$ of the underlying common DM. With respect to choice rules induced by different levels of search, $n_1, n_2 \in \mathcal{N}_h$, we will say that $\mathcal{C}\left(\mathcal{A}_{h(n_1)}, \succsim_{h(n_1)}\right)$ and $\mathcal{C}\left(\mathcal{A}_{h(n_2)}, \succsim_{h(n_2)}\right)$ are $\succsim_h$-equivalent if

$$f \sim_h g \qquad \forall f \in \mathcal{C}\left(\mathcal{A}_{h(n_1)}, \succsim_{h(n_1)}\right), \forall g \in \mathcal{C}\left(\mathcal{A}_{h(n_2)}, \succsim_{h(n_2)}\right)$$

In the light of our analysis thus far, the notion of $\succsim_h$-equivalence should be quite intuitive. For suppose that the underlying DM is indifferent between two acts, f and g, which are chosen by two "selves" with different levels of search foresight, n_1 and n_2 with, say, $n_2 > n_1$. Then, the change in information, about the structure of the given decision problem, obtained by extending the level of our search process from n_1 to n_2 does not affect the welfare of the DM. But in decision theory (and, more generally, in classical economics), we can only go as far as distinguishing between choices, or between the sources that dictate them, if they differ in implied welfare (revealed preference). In our setting, this means distinguishing up to $\succsim_h$-equivalence because this is exactly what identifies the equivalence classes of search, indeed the source of structural information that determines choices in our setting.

It is in this sense that our endogenous limited foresight framework can be used to compare the choices of search-based decision-making with those prescribed by the rational choice rule (unlimited foresight). Under endogenous limited foresight, a rational DM can be thought of as one for whom the level of search is not binding for her welfare. That is, one for whom any of her possible "selves" with search foresight $n < N$ would dictate choices that are indistinguishable ($\succsim$-equivalent) to those made by the "self" with search level N. As it turns out, rationality can be depicted as a special case of the representation in (1).

Theorem 3 *Suppose that the preference relations $\left\{\succsim_{h(n)}\right\}_{n \in \mathcal{N}_h}$ and $\succsim_h$ satisfy axioms A.1-A.6 and B.1-B.4&B.6, respectively. The statement*

(i) $\forall n \in \mathcal{N}_h \setminus \{N\}$, $\forall \mathcal{A}_{h(n)}$, and $\forall \mathcal{A}_{h(N)}$: $\mathcal{C}\left(\mathcal{A}_{h(n)}, \succsim_{h(n)}\right)$ and $\mathcal{C}\left(\mathcal{A}_{h(N)}, \succsim_{h(N)}\right)$ are $\succsim_h$-equivalent.

holds if so does the statement

(ii) For the representation of $\succsim_h$ in (1), $\lambda_{h(n)} = \lambda_{h(N)}$ and $k_{h(n)} = k_{h(N)}$ $\forall n \in \mathcal{N}_h \setminus \{N\}$.

The two statements become equivalent if $\succsim_h$ satisfies, in addition, horizon monotonicity.

To demonstrate the necessity of our continuation consistency criterion for Theorem 3, consider again the example of Figure 1. As before, Nature chooses each of its available actions at each chance node with equal probability. Let us, though, now specify some terminal utility payoffs: $u(x_i) = i$ for $i = 2, \ldots, 7$ and $u(x_1) = x$. At the opening decision node $a.1$, a rational agent (with unlimited foresight) assigns the expected utility value 4.25 to the act $f_{l_1 l_2 l_3}$ (the optimal continuation act following $l.1$). A boundedly-rational agent with foresight $n = 1$, however, views the act f_{l_1} as resulting in the sets of terminal consequences $X_1 = \{x_2, x_3\}$ and $X_2 = \{x_4, x_5, x_6, x_7\}$ in the events $E_1 = \{s_1, \ldots, s_4\}$ and $E_2 = \{s_5, \ldots, s_8\}$, respectively (Figure 2). From her perspective, if she assigns full-support on each of X_1 and X_2 for the corresponding sets of possible terminal probability distributions (lotteries), the non-ambiguous equivalent $z \sim_{a.1(1)} f_{l_1}$ will be such that $u(z) = 0.5 [\alpha_{11} u(x_2) + (1 - \alpha_{11}) u(x_3) + \alpha_{12} u(x_4) + (1 - \alpha_{12}) u(x_7)] = 0.5 [10 - \alpha_{11} - 3\alpha_{12}]$ for some $\alpha_{11}, \alpha_{12} \in [0, 1]$.

Similarly, an agent with foresight $n = 2$ views the act $f_{l_1 l_2}$ as resulting in the degenerate lottery 1_{x_2} in the event $E_{21} = \{s_1, s_2\}$, in the lottery 1_{x_3} in the event $E_{22} = \{s_4, s_5\}$ and the set of terminal consequences $X_2 = \{x_5, x_6, x_7\}$ in either of the events $E_{23} = \{s_5, s_6\}$ and $E_{24} = \{s_7, s_8\}$ (Figure 3). From the perspective of this agent, if she assigns full-support on X_2 for the set of possible terminal lotteries, the non-ambiguous equivalent $z' \sim_{a.1(2)} f_{l_1 l_2}$ will be such that $u(z') = 0.25 [u(x_2) + u(x_3)] + 0.5 [\alpha' u(x_5) + (1 - \alpha') u(x_7)] = 0.5 (9.5 - 2\alpha')$ for some $\alpha' \in [0, 1]$. Under foresight $n = 3$, on the other hand, $f_{l_1 l_2}$ corresponds to the lottery 1_{x_5} and the set $X_3 = \{x_6, x_7\}$ in the states $s \in \{s_5, s_7\}$ and $s \in \{s_6, s_8\}$, respectively (Figure 4). From this perspective, if the agent assigns full-support on X_3, the non-ambiguous equivalent $z'' \sim_{a.1(3)} f_{l_1 l_2}$ will be given by $u(z'') = 0.25 [u(x_2) + u(x_3) + u(x_5) + \alpha'' u(x_6) + (1 - \alpha'') u(x_7)]$ $= 0.25 (17 - \alpha'')$ for some $\alpha'' \in [0, 1]$.

Suppose now that condition (ii) of Theorem 3 is met so that the constants λ and k can be disregarded in the utility representations under different levels of search foresight. Rationality prescribes choosing $r.1$ at $a.1$ as long as $x > 4.25$. Nevertheless, the boundedly-rational agent with foresight $n = 1$ will choose $l.1$ at $a.1$ as long as $\alpha_{11} + 3\alpha_{12} < 10 - 2x$. If her foresight is extended to $n = 2$, the agent views the act $f_{l_1 l_2}$ (rather than $f_{l_1 r_2}$) as the optimal continuation of choosing $l.1$ at $a.1$ since $u\left(f_{l_1 l_2 (E_{23})}\right) = 7 - 2\alpha' > 4 = u\left(f_{l_1 r_2 (E_{23})}\right)$ $\forall \alpha' \in [0, 1]$. The continuation value of choosing $l.1$ at $a.1$ is $4.75 - \alpha'$ and this is preferred to choosing $r.1$ as long as $\alpha' < 4.75 - x$. The act $f_{l_1 l_2}$ remains the optimal continuation of choosing $l.1$ at $a.1$ also under foresight $n = 3$ as $6 - 0.5\alpha'' > 4$ $\forall \alpha'' \in [0, 1]$. Choosing now

$l.1$ at $a.1$ dominates $r.1$ as long as $\alpha'' < 17 - 4x$.

If $x > 4.25$, under condition (ii) of Theorem 3, the rational and boundedly-rational choice rules coincide under search foresight $n \geq 3$. For $n < 3$, however, a discrepancy will arise if the α-coefficients are sufficiently small. That is, the statements (i) and (ii) of the theorem will fail to be equivalent, in this example, if the boundedly-rational agent is too optimistic, in her assessment of the continuation values of her available acts, relative to the rational agent (who knows the true probability distribution that Nature's play imposes on the state space and who assigns probability one on her own optimal act at each of the future decision nodes on the tree). In this case, however, the set of conditions (11)-(13), which are necessary for axiom (B.6), also fail demonstrating that such overoptimism is ruled out by continuation consistency.[17] If $x = 4.25 - \epsilon$ for some $\epsilon \geq 0$, the rational and boundedly-rational choice rules coincide under search foresight $n = 1, 2, 3$ as long as $\alpha_{11} + \alpha_{12} \leq 1.5 + \epsilon$, $\alpha' \leq 0.5 + \epsilon$, and $\alpha'' \leq \epsilon$, respectively. In this case, the necessary conditions (11)-(13) do not have bite. They are not sufficient for the full force of axiom (B.6) as they account only for the relation between a current act and its continuations. They do not restrict the converse relation, between a current act and its predecessors. With respect to the latter relation, continuation consistency imposes upper bounds on the α's, ruling out overpessimism by the boundedly-rational agent relative to the rational one.[18]

3.3 Bounded Rationality

Our analysis has presented a framework of decision-making under endogenous limited foresight that can be used to compare bounded rationality with rational decision-making under some minimal structure imposed on the underlying choice rules.

Definition 1 *The choice rule $\mathcal{C}(\cdot, \succsim^0)$ is* <u>*admissible*</u> *if $\exists n \in \mathcal{N}_h$ such that $\mathcal{C}(\overline{\mathcal{A}}, \succsim^0) = \mathcal{C}(\mathcal{A}_{h(n)}, \succsim_{h(n)})$. An admissible choice rule $\mathcal{C}(\cdot, \succsim^0)$ is* <u>*boundedly-rational*</u> *if $\mathcal{C}(\overline{\mathcal{A}}, \succsim^0) = \mathcal{C}(\mathcal{A}_{h(n)}, \succsim_{h(n)})$ and $\max_{f \in \mathcal{A}_{h(n)}} U_{h(n)}(f) < \max_{f \in \mathcal{A}_{h(n)}} U_{h(n)}(f)$, for some $n \in \mathcal{N}_h \setminus \{N\}$.*

This allows for an empirically testable and robust relation between the terminal utility payoffs of the decision problem at hand and the agent's optimal horizon. To see this, denote the

[17] Given that $f_{l_1 l_2}(s) = \Delta(X_3)$ for $s \in \{s_6, s_8\}$ when $n = 3$, under the assumed terminal utility payoffs, (12) and (13) together require $5.5 \leq 7 - 2\alpha' \leq 6$. Equivalently, $0.5 \leq \alpha' \leq 0.75$, the left-hand side of which rules out overoptimism at foresight $n = 2$. Since $u(f_{l_1 l_2 (E_{23})}) = 7 - 2\alpha' \geq 5.5 > 4$, (11) reads $6.5 \leq 10 - \alpha_{11} - 3\alpha_{12} \leq 9.5 - 2\alpha'$ or $0.5 + 2\alpha' \leq \alpha_{11} + 3\alpha_{12} \leq 3.5$. Given that $\alpha' \geq 0.5$, the left-hand side inequality implies $\alpha_{11} + 3\alpha_{12} \geq 1.5$, ruling out overoptimism when $n = 1$.

[18] If $x = 4.25 - \epsilon$, the rational agent does not choose $r.1$. Since there is some act with value higher than x at foresight $n = 4$, applying the axiom on this act requires the existence of some at-least-as-good-as predecessor act in $\mathcal{A}_{h(n)}$ for each $n = 1, 2, 3$. Thus, the corresponding non-ambiguous equivalents of these acts, z, z', and z'', should give $u(\widetilde{z}) \geq x$ for $\widetilde{z} \in \{z, z', z''\}$, resulting in the upper bounds for the α's in the text.

value functional $u\left(z^*_{h(n)}\right) = \max_{f\in\mathcal{A}_{h(n)}} U_{h(n)}(f)$ and observe that

$$\begin{aligned}\Delta_{h(n)}\left[\lambda_{h(n)} u\left(z^*_{h(n)}\right)\right] &= \lambda_{h(n+1)} u\left(z^*_{h(n+1)}\right) \lambda_{h(n)} u\left(z^*_{h(n)}\right) \\ &= \lambda_{h(n+1)}\left[u\left(z^*_{h(n+1)}\right) - u\left(z^*_{h(n)}\right)\right] + \left[\lambda_{h(n+1)}\lambda_{h(n)}\right] u\left(z^*_{h(n)}\right) \quad (14)\end{aligned}$$

At the optimal foresight $\overline{n}$, moreover, $\Delta_{h(n)}\left[\lambda_{\overline{n}} V(\overline{n})\right] \leq 0$. Since $\Delta_{h(n)} \max_{f\in\mathcal{A}_{h(n)}} U_{h(n)}(f) \geq 0$, it is immediate that

Corollary 1 *For the representation in Th.2, let* $\overline{n}$ *be optimal on* $\mathcal{N}_h$*. Then,* $\Delta_{h(n)}\lambda_{\overline{n}} \leq 0$*.*

Moreover, after normalizing $u(\cdot)$ on $\Delta(\mathcal{X})$ such that $u(y) > 0$ $\forall y \in \Delta(\mathcal{X})$, we must have

$$\frac{\Delta_{h(n)} V(\overline{n})}{V(\overline{n})} \leq -\frac{\Delta_{h(n)}\lambda_{\overline{n}}}{\lambda_{\overline{n}+1}}$$

Since the numbers $\left\{\lambda_{h(n)}\right\}_{n\in\mathcal{N}_h}$ do not depend upon the terminal utility payoffs, the validity of the above inequality is completely determined by the left-hand side quantity. If this increases sufficiently, other things remaining unchanged, the inequality will no longer be valid.[19] Similarly, by the optimality of $\overline{n}$, we have $\Delta_{h(n)}\left[\lambda_{\overline{n}-1} V(\overline{n}-1)\right] \geq 0$ and, thus,

$$\frac{\Delta_{h(n)} V(\overline{n}-1)}{V(\overline{n}-1)} \geq -\frac{\Delta_{h(n)}\lambda_{\overline{n}-1}}{\lambda_{\overline{n}}}$$

If the quantity $\frac{\Delta_{h(n)} V(\overline{n}-1)}{V(\overline{n}-1)}$ decreases sufficiently, ceteris paribus, the inequality will cease to hold.[20]

4 Related Literature

Our decision-theoretic set-up employs an Anscombe-Aumann framework in which acts assign sets of lotteries rather than singletons. In the sense, therefore, that preferences over acts induce preferences over sets of lotteries, our approach is in the same spirit as those in Ahn [2] and Olszewski [21]. However, both papers do explicitly away with a state space which is crucial for search within decision trees. As our introductory example illustrates, changing the extent of one search on a tree changes not only the partition of the underlying state-space but also the very set of acts the decision maker must choose from. It should be pointed

[19] *Other things remaining unchanged* refers here to the quantities $\frac{\Delta_{h(n)} \max_{f\in\mathcal{A}_{h(n)}} U_{h(n)}(f)}{U_{h(n)}(f)}$, for $n \in \mathcal{N}_h \setminus \{\overline{n}\}$.

[20] *Ceteris paribus* refers to the quantities $\frac{\Delta_{h(n)} \max_{f\in\mathcal{A}_{h(n)}} U_{h(n)}(f)}{U_{h(n)}(f)}$, for $n \in \mathcal{N}_h \setminus \{\overline{n}-1\}$.

out, moreover, that Ahn's analysis focuses explicitly on regular sets of lotteries, a restriction that excludes finite sets and renders the approach inapplicable on finite trees. Olszewski's representation is actually the same as that of Theorem 1 with $\alpha_{h(n)}(\cdot)$ being a constant function (see Appendix B).

A Proofs

The main argument for Theorem 1 derives from Lemma 4 whose proof is build upon the following preliminary results.

Lemma 1 *Suppose that the preference relation $\succsim_{h(n)}$ on the set $\mathcal{F}_{h(n)}$ satisfies the weak order (A.1) and crisp independence (A.2) axioms. Then,*
$\forall f, h \in \mathcal{F}^{cr}_{h(n)}$ *and* $\forall \alpha, \beta \in [0,1]$ $f \succ_{h(n)} h$ *implies* $\beta f + (1-\beta) h \succ_{h(n)} \alpha f + (1-\alpha) h$ *iff* $\alpha < \beta$.

Proof. For the "if" part of the statement, suppose that $\alpha < \beta$. Let first $\beta = 1$. Since $f \succ_{h(n)} h$, there is nothing to show if $\alpha = 0$. If $\alpha > 0$, then $\alpha \in (0,1)$ and, by crisp independence, $f \succ_{h(n)} h$ implies $f = \alpha f + (1-\alpha) f \succ_{h(n)} \alpha f + (1-\alpha) h$. Let now $\beta < 1$. Since $0 \leq \alpha < \beta < 1$, we have $\beta, \frac{\beta-\alpha}{1-\alpha} \in (0,1)$. For $\alpha = 0$, by crisp independence, $f \succ_{h(n)} h$ implies $\beta f + (1-\beta) h \succ_{h(n)} \beta h + (1-\beta) h = h$ as required. If $\alpha > 0$, we have shown above that $f \succ_{h(n)} \alpha f + (1-\alpha) h$. Hence, by crisp independence, $\left(\frac{\beta-\alpha}{1-\alpha}\right) f + \left(1 - \frac{\beta-\alpha}{1-\alpha}\right) [\alpha f + (1-\alpha) h] \succ_{h(n)} \left(\frac{\beta-\alpha}{1-\alpha}\right) [\alpha f + (1-\alpha) h] + \left(1 - \frac{\beta-\alpha}{1-\alpha}\right) [\alpha f + (1-\alpha) h] = \alpha f + (1-\alpha) h$. But $\left(\frac{\beta-\alpha}{1-\alpha}\right) f + \left(1 - \frac{\beta-\alpha}{1-\alpha}\right) [\alpha f + (1-\alpha) h] = \beta f + (1-\beta) h$ and we are done.

For the "only if," consider the contrapositive statement and let $\alpha \geq \beta$. By completeness, we need to show that $\alpha f + (1-\alpha) h \succsim_{h(n)} \beta f + (1-\beta) h$. If $\alpha = \beta$, this holds trivially as indifference. For $\alpha > \beta$, reverse the roles of α and β in the "if" statement above to get $\alpha f + (1-\alpha) h \succ_{h(n)} \beta f + (1-\beta) h$. ■

Lemma 2 *Let $\succsim_{h(n)}$ on the set $\mathcal{F}_{h(n)}$ satisfy the weak order (A.1), crisp independence (A.2), and extended-Archimedean (A.3) axioms. Then* $\forall f, h \in \mathcal{F}^{cr}_{h(n)}$ *and* $\forall g \in \mathcal{F}^{c}_{h(n)}$
$f \succ_{h(n)} g \succ_{h(n)} h$ *implies* $\exists! \alpha_g \in [0,1]$ *s.t.* $g \sim_{h(n)} \alpha_g f + [1-\alpha_g] h$.

Proof. Let $\alpha_g = \inf \left\{ \alpha \in [0,1] : \alpha f + (1-\alpha) h \succsim_{h(n)} g \right\}$. Since $f \succ_{h(n)} g$, the quantity α_g is the infimum of a non-empty set (it contains $\alpha = 1$); it is well-defined and unique by construction.

By the extended-Archimedean axiom, since $f \succ_{h(n)} g \succ_{h(n)} h$, $\exists \beta_1 \in (0,1)$ such that $g \succ_{h(n)} \beta_1 f + (1-\beta_1) h$. By Lemma 1, $\beta_1 f + (1-\beta_1) h \succ_{h(n)} \gamma f + (1-\gamma) h$ $\forall \gamma \in [0, \beta_1)$. By transitivity, $g \succ_{h(n)} \gamma f + (1-\gamma) h \ \forall \gamma \in [0, \beta_1]$. Clearly, it must be $\alpha_g \geq \beta_1 > 0$. A trivially similar argument establishes $\alpha_g \leq \beta_2$ for some $\beta_2 \in (0,1)$.

We will now show $g \succsim_{h(n)} \alpha_g f + (1-\alpha_g) h$, by contradiction. Let it be false. By completeness, it must be $\alpha_g f + (1-\alpha_g) h \succ_{h(n)} g$. Since also $g \succ_{h(n)} h$, by the extended-Archimedean axiom, $\exists \beta_3 \in (0,1)$ such that $\beta_3 (\alpha_g f + (1-\alpha_g) h) + (1-\beta_3) h \succ_{h(n)} g$. Rearranging, $\beta_3 \alpha_g f + (1-\beta_3 \alpha_g) h \succ_{h(n)} g$. But $\beta_3 \alpha_g < \alpha_g$ which is absurd given the definition of α_g.

For $\alpha_g f + (1-\alpha_g) h \succsim_{h(n)} g$, we argue again by contradiction. Assume, to the contrary, $g \succ_{h(n)} \alpha_g f + (1-\alpha_g) h$. Since also $f \succ_{h(n)} g$, $\exists \beta_4 \in (0,1)$ such that $g \succ_{h(n)} \beta_4 f + (1-\beta_4)(\alpha_g f + (1-\alpha_g) h)$. Rearranging, $g \succ_{h(n)} (\alpha_g + \beta_4 (1-\alpha_g)) f + (1-(\alpha_g + \beta_4 (1-\alpha_g))) h$. Since $\beta_4 (1-\alpha_g) > 0$, by Lemma 1 and transitivity, $g \succ_{h(n)} (\alpha_g + \gamma_3) f + (1-(\alpha_g+\gamma_3)) h$ $\forall \gamma_3 \in [0, \beta_4 (1-\alpha_g)]$ which is absurd given the definition of α_g.

To complete the argument, it should be noted that, as indicated by the notation we have used above, the unique quantity α_g depends upon the act g in question. ■

Lemma 3 *Suppose that the preference relation $\succsim_{h(n)}$ on the set $\mathcal{F}_{h(n)}$ satisfies the weak order (A.1), monotonicity (A.4), and non-triviality (A.5) axioms. Then,*
$\forall Z \in \mathcal{X}_{h(n)}$, $\exists x', x'' \in X$: $x' \succsim_{h(n)} z \succsim_{h(n)} x'' \ \forall z \in Z$.

Proof. Since the set of terminal consequences X is finite, the weak order and non-triviality axioms guarantee the existence of $\succsim_{h(n)}$-best and $\succsim_{h(n)}$-worst elements in X. That is, $\exists \overline{x}, \underline{x} \in X : \overline{x} \succsim_{h(n)} x \succsim_{h(n)} \underline{x} \ \forall x \in X$. By monotonicity, $\overline{x}$ and $\underline{x}$ are, respectively, $\succsim_{h(n)}$-best and $\succsim_{h(n)}$-worst lotteries in $\Delta(X)$ (the simplex viewed, of course, as the set of constant crisp acts). Since $Z \subseteq \Delta(X)$, $\overline{x}$ and $\underline{x}$ can be taken, respectively, as the required consequences x' and x''. ■

Lemma 4 *For $Z \in \mathcal{X}_{h(n)}$, let $\overline{z} = \sup_{z \in Z} u_{h(n)}(z)$ and $\underline{z} = \inf_{z \in Z} u_{h(n)}(z)$. There exists a unique $\alpha_Z \in [0,1]$ such that $Z \sim_{h(n)} \alpha_Z \overline{z} + [1-\alpha_Z] \underline{z}$.*

Proof. By the preceding lemma, we can find $x', x'' \in X$ s.t. $x' \succsim_{h(n)} z \succsim_{h(n)} x'' \ \forall z \in Z$. Since $u_{h(n)}(\cdot)$ represents $\succsim_{h(n)}$ on $\Delta(X)$, we must have $u_{h(n)}(x'') \leq u_{h(n)}(z) \leq u_{h(n)}(x')$ $\forall z \in Z$. Clearly, the quantities $\overline{z}$ and $\underline{z}$ are well-defined. They are unique by construction.

By boundedness, we can find $z', z'' \in Z$ s.t. $z' \succsim_{h(n)} Z \succsim_{h(n)} z''$. By the definition of the lotteries $\overline{z}$ and $\underline{z}$, it must be $u_{h(n)}(\underline{z}) \leq u_{h(n)}(z'')$ and $u_{h(n)}(\overline{z}) \geq u_{h(n)}(z')$. Since $u_{h(n)}(\cdot)$ represents $\succsim_{h(n)}$ on $\Delta(X)$, this implies $z'' \succsim_{h(n)} \underline{z}$ and $\overline{z} \succsim_{h(n)} z'$. By transitivity, therefore, it must be $\overline{z} \succsim_{h(n)} Z \succsim_{h(n)} \underline{z}$. If either relation is indifference (which includes the case of Z

being singleton), the claim is true for $\alpha_Z = 1$ or $\alpha_Z = 0$. If both relations are strict, the claim follows from Lemma 2.

To complete the argument, notice that, as indicated by our notation, the unique quantity α_Z depends upon the constant act Z in question. In the main text, we make use of the induced functional relation $\alpha_{h(n)} : \mathcal{X}_{h(n)} \mapsto [0, 1]$. ■

We now turn to Theorem 2 for which the following result will be of substantial use.

Lemma 5 *Suppose that the preference relations $\succsim_{n\in\mathcal{N}_h}$ and $\succsim_h$ satisfy axioms A.1-A.6 and B.3-B.4, respectively. Then,*

(i) there exists an affine function $u_h : \Delta(X) \mapsto \mathbb{R}$, unique up to a positive affine transformation, and a collection $\{\beta_{h(n)}, \gamma_{h(n)}\}_{n\in\mathcal{N}_h}$ of pairs of reals, with $\beta_{h(n)} > 0$ and $\gamma_{h(n)} \in \mathbb{R}$ $\forall n \in \mathcal{N}_h$, such that the functions $u_{h(n)}(\cdot)$ of Theorem 1 are given by

$$u_{h(n)}(\cdot) = \beta_{h(n)} u_h(\cdot) + \gamma_{h(n)}$$

(ii) for any $n_1, n_2 \in \mathcal{N}_h$, the functions $\alpha_{h(n)}(\cdot)$ of Theorem 1 are such that

$$\alpha_{h(n_1)}(Z) = \alpha_{h(n_2)}(Z) \quad \forall Z \in \mathcal{X}_{h(n_1)} \cap \mathcal{X}_{h(n_2)}$$

(iii) for any $n_1, n_2 \in \mathcal{N}_h$ with $n_2 > n_1$, let $E = \{E_k\}_{k=1,\ldots,K<\infty}$ be the refinement of the event $E \in \pi_{h(n_1)}$ in the subsequent partition $\pi_{h(n_2)}$. The probability distributions $p_{h(n)}(\cdot)$ of Theorem 1 are such that

$$p_{h(n_1)}(E) = p_{h(n_2)}\left(\sum_{k=1}^{K} E_k\right)$$

Proof. Part (i) follows by the finiteness of $\mathcal{N}_h$ and the horizontal consistency (B.4) axiom. Recall that any prize $z \in \Delta(X)$ can be viewed as a constant crisp act, a member of the set $\left\{g \in \Delta(X)^S : g(s) = g(s') \ \forall s, s' \in S\right\}$. Since $z \in \cap_{n\in\mathcal{N}_h} \mathcal{F}^c_{h(n)}$ (Remark 2(b) in the text), by horizontal consistency, there must exist pairs of reals $\left\{\beta'_{h(n)}, \gamma'_{h(n)}\right\}_{n\in\mathcal{N}\setminus\{N\}}$ with $\beta'_{h(n)} > 0$ and $\gamma'_{h(n)} \in \mathbb{R}$ $\forall n \in \mathcal{N}_h \setminus \{N\}$ such that

$$u_{h(n)} = \beta'_{h(n)} u_{h(N)} + \gamma'_{h(n)}$$

To complete the argument, take a positive affine transformation of $u_{h(N)}$, $u_h = \beta_h u_{h(N)} + \gamma_h$ for some $\beta_h > 0$ and $\gamma_h \in \mathbb{R}$, and set $\beta_{h(N)} = \frac{1}{\beta_h}$, $\gamma_{h(N)} = -\frac{\gamma_h}{\beta_h}$, $\beta_{h(n)} = \frac{\beta'_{h(n)}}{\beta_h}$, and $\gamma_{h(n)} =$

$\gamma'_{h(n)} - \frac{\beta'_{h(n)}\gamma_h}{\beta_h}$.

For (ii), take $Z \in \mathcal{F}^c_{h(n_1)} \cap \mathcal{F}^c_{h(n_2)}$ and let $z^* \in \Delta(X)$ be a non-ambiguous equivalent of Z under the relation $\succsim_{h(n_1)}$: $z^* \sim_{h(n_1)} Z$. As $\Delta(X) \subseteq \mathcal{F}^c_{h(n_1)} \cap \mathcal{F}^c_{h(n_2)}$, by horizontal consistency, we must also have $z^* \sim_{h(n_2)} Z$. Thus, for $n \in \{n_1, n_2\}$,

$$u_{h(n)}(z^*) = U_{h(n)}(Z) = \alpha_{h(n)}(Z) \inf_{z \in Z} u_{h(n)}(z) + \left[1 - \alpha_{h(n)}(Z)\right] \sup_{z \in Z} u_{h(n)}(z)$$

Equivalently, by part (i),

$$\begin{aligned} \beta_{h(n)} u_h(z^*) + \gamma_{h(n)} &= \alpha_{h(n)}(Z) \inf_{z \in Z} \left[\beta_{h(n)} u_h(z) + \gamma_{h(n)}\right] + \left[1 - \alpha_{h(n)}(Z)\right] \sup_{z \in Z} \left[\beta_{h(n)} u_h(z) + \gamma_{h(n)}\right] \\ &= \beta_{h(n)} \left(\alpha_{h(n)}(Z) \inf_{z \in z} u_h(z) + \left[1 - \alpha_{h(n)}(Z)\right] \sup_{z \in z} u_h(z)\right) + \gamma_{h(n)} \end{aligned}$$

i.e.

$$u_h(z^*) = \alpha_{h(n)}(Z) \inf_{z \in Z} u_h(z) + \left[1 - \alpha_{h(n)}(Z)\right] \sup_{z \in Z} u_h(z)$$

which can hold for both $n = n_1$ and $n = n_2$ only if $\alpha_{h(n_1)}(Z) = \alpha_{h(n_2)}(Z)$.

For part (iii), take $n_1, n_2 \in \mathcal{N}_h$ with $n_2 > n_1$ and let $\{E_k\}_{k=1,\dots,K<\infty}$ be the refinement of the event $E \in \pi_{h(n_1)}$ in the partition $\pi_{h(n_2)}$. Consider a non-constant crisp act $f \in \mathcal{F}^{cr}_{h(n_1)}$ given by $f = 1_E z' + 1_{S \setminus E} z''$ for some $z', z'' \in \Delta(X)$ with $z' \succ z''$. By the vertical consistency (B.3) axiom, we must have $z' \succ_{h(n)} z''$ for $n \in \{n_1, n_2\}$. Let now $\widetilde{z} \in \Delta(X)$ be a non-ambiguous equivalent of f under the relation $\succsim_{h(n_1)}$: $\widetilde{z} \sim_{h(n_1)} f$. Since $f \in \mathcal{F}^{cr}_{h(n_2)}$ (recall Remark 2(a) in the text), by horizontal consistency, it must also be $\widetilde{z} \sim_{h(n_2)} f$. Therefore, for $n \in \{n_1, n_2\}$,

$$u_{h(n)}(\widetilde{z}) = U^c_{h(n)}(f) = p_{h(n)}(E)\, u_{h(n)}(z') + \left[1 - p_{h(n)}(E)\right] u_{h(n)}(z'')$$

Equivalently, by part (i),

$$\begin{aligned} \beta_{h(n)} u_h(\widetilde{z}) + \gamma_{h(n)} &= p_{h(n)}(E) \left[\beta_{h(n)} u_h(z') + \gamma_{h(n)}\right] + \left[1 - p_{h(n)}(E)\right] \left[\beta_{h(n)} u_h(z'') + \gamma_{h(n)}\right] \\ &= \beta_{h(n)} \left[p_{h(n)}(E)\, u_h(z') + \left[1 - p_{h(n)}(E)\right] u_h(z'')\right] + \gamma_{h(n)} \end{aligned}$$

i.e.

$$u_h(\widetilde{z}) = p_{h(n)}(E)\, u_h(z') + \left[1 - p_{h(n)}(E)\right] u_h(z'')$$

As $u_h(z') > u_h(z'')$, this holds for $n = n_1$ and $n = n_2$ only if $p_{h(n_1)}(E) = p_{h(n_2)}(E)$. ■

Proof of Theorem 2

Sufficiency. By Theorem 1, we have already a utility representation on prizes $u_{h(n)} : \Delta(X) \mapsto \mathbb{R}$ for each $n \in \mathcal{N}_h$. The Countable Order Dense Theorem (CODT) ensures the existence of a countable $\succ_{h(n)}$-order dense set, $Q_{h(n)} \subseteq \Delta(X)$, such that $\forall z', z'' \in \Delta(X)$ $z' \succ_{h(n)} z''$ implies $\exists z \in Q_{h(n)} : z' \succsim_{h(n)} z \succsim_{h(n)} z''$ with at least one relation being strict.[21] By vertical consistency, this can be stated also as follows. For each $n \in \mathcal{N}_h$, there exists a countable set $Q_{h(n)} \subseteq \Delta(X)$ s.t.

$$\forall z', z'' \in \Delta(X), \ (n, z') \succ_h (n, z'') \text{ implies } \exists z \in Q_{h(n)} : (n, z') \succsim_h (n, z) \succsim_h (n, z'')$$

with at least one preference being strict.

Let $(n_1, f) \succ_h (n_2, g)$ for some $n_1, n_2 \in \mathcal{N}_h$, $f \in \mathcal{F}_{h(n_1)}$, and $g \in \mathcal{F}_{h(n_2)}$. Consider the corresponding non-ambiguous equivalents: $z_1, z_2 \in \Delta(X)$ s.t. $z_1 \sim_{h(n_1)} f$ and $z_2 \sim_{h(n_2)} g$. By vertical consistency, it must be $(n_1, z_1) \sim_h (n_1, f)$ and $(n_2, g) \sim_h (n_2, z_2)$. Hence, by the transitivity (B.1.ii) axiom, $(n_1, z_1) \succ_h (n_2, z_2)$.
Suppose first that z_1 and z_2 are not, respectively, minimal under $\succsim_{h(n_1)}$ and maximal under $\succsim_{h(n_2)}$ prizes in $\Delta(X)$. There must be, therefore, $z' \in \Delta(X)$ such that $z_1 \succ_{h(n_1)} z'$ or $z' \succ_{h(n_2)} z_2$. Equivalently, by vertical consistency, $(n_1, z_1) \succ_h (n_1, z')$ or $(n_2, z') \succ_h (n_2, z_2)$. The Solvability (B.2) axiom guarantees then the existence of some $z'' \in \Delta(X)$ such that at least one of the following cases holds

(a) $(n_1, z_1) \succ_h (n_1, z'') \succ_h (n_2, z_2)$

(b) $(n_1, z_1) \succ_h (n_2, z'') \succ_h (n_2, z_2)$

In case (a), due to the CODT, the first strict relation implies $\exists z \in Q_{h(n_1)}$ s.t. $(n_1, z_1) \succsim_h (n_1, z) \succsim_h (n_1, z'')$ with at least one preference being strict. We have, thus, $(n_1, f) \sim_h (n_1, z_1) \succsim_h (n_1, z) \succsim_h (n_1, z'') \succ_h (n_2, z_2) \sim_h (n_2, g)$ and, by transitivity, $(n_1, f) \succsim_h (n_1, z) \succ_h (n_2, g)$. In case (b), a trivially analogous argument establishes $\exists z \in Q_{h(n_2)}$ s.t. $(n_1, f) \succ_h (n_2, z) \succsim_h (n_2, g)$. In either case, therefore, $\exists (n, z) \in \{n_1, n_2\} \times \left(Q_{h(n_1)} \cup Q_{h(n_2)}\right)$ s.t. $(n_1, f) \succsim_h (n, z) \succsim_h (n_2, g)$ with at least one preference being strict.
Suppose now that z_1 and z_2 are, respectively, minimal under $\succsim_{h(n_1)}$ and maximal under $\succsim_{h(n_2)}$ prizes in $\Delta(X)$. In this case, they are actually, respectively, minimal and maximal prizes in $\Delta(X)$ under any relation $\left\{\succsim_{h(n)}\right\}_{n \in \mathcal{N}_h}$ as well as under $\succsim$.[22] Taking $z = z_1$, we

[21] See Kreps [?], Theorem 3.5, pp.25.

[22] Since $\Delta(X) = \mathcal{F}^{cr}_{h(n)} \cap \mathcal{F}^{c}_{h(n)}$ $\forall n \in \mathcal{N}_h$ (recall Remark 2(b) in the text), by horizontal consistency, the relations $\left\{\succsim_{h(n)}\right\}_{n \in \mathcal{N}_h}$ ought to give the same ranking over the prizes in $\Delta(X)$. Due to vertical consistency, given $n \in \mathcal{N}_h$, this is also the ranking for $(n, \cdot)$ on in $\Delta(X)$. In other words, we can think of an underlying expected utility preference relation $\succsim$ as the common restriction of all the above relations on $\Delta(X)$. That

have $(n_1, f) \succsim_h (n_1, z) \succsim_h (n_2, g)$ with the former relation an indifference and the latter a strict preference.

Let now $\underline{z}$ denote a minimal prize in $\Delta(X)$ and consider the countable set $\mathcal{Q}_h = \mathcal{N}_h \times \bigcup_{n \in \mathcal{N}_h} \left(Q_{h(n)} \cup \{\underline{z}\}\right)$.[23] We have just shown that, for any $n_1, n_2 \in \mathcal{N}_h$, any $f \in \mathcal{F}_{h(n_1)}$, and any $g \in \mathcal{F}_{h(n_2)}$, whenever $(n_1, f) \succ_h (n_2, g)$ then $(n_2, f) \succsim_h (n, z) \succsim_h (n_2, g)$ for some $(n, z) \in \mathcal{Q}_h$ and with at least one preference being strict. By the CODT, this suffices for the existence of a utility representation $U_h : \mathcal{F}_h \mapsto \mathbb{R}$ for the rational preference relation $\succsim_h$ on the set $\mathcal{F}_h$. Formally,

$$\forall n_1, n_2 \in \mathcal{N}_h \; \forall (f, g) \in \mathcal{F}_{h(n_1)} \times \mathcal{F}_{h(n_2)} : \; (n_1, f) \succsim_h (n_2, g) \;\Leftrightarrow\; U_h(n_1, f) \geq U_h(n_2, g)$$

The functional form in (1) follows as the representations $U_{h(n)} : \mathcal{F}_{h(n)} \mapsto \mathbb{R}$ of the relations $\left\{\succsim_{h(n)}\right\}_{n \in \mathcal{N}_h}$ are all affine in the probability measures $p_{h(n)}$ on the partitions $\pi_{h(n)}$. By the vertical consistency requirement, $U_h(n, \cdot)$ must be a positive linear transformation of $U_{h(n)}(\cdot)$ $\forall n \in \mathcal{N}_h$. Formally,

$$U_h(n, f) = \mu_{h(n)} U_{h(n)}(f) + \nu_{h(n)}$$

for some collection of pairs of reals $\left\{\mu_{h(n)}, \nu_{h(n)}\right\}_{n \in \mathcal{N}}$ with $\mu_{h(n)} > 0$ and $\nu_{h(n)} \in \mathbb{R}$ $\forall n \in \mathcal{N}$. This specification becomes (1) for $\lambda_{h(n)} = \mu_{h(n)} \beta_{h(n)}$ and $k_{h(n)} = \mu_{h(n)} \gamma_{h(n)} + \nu_{h(n)}$, the collection $\left\{\beta_{h(n)}, \gamma_{h(n)}\right\}_{n \in \mathcal{N}_h}$ taken from the preceding lemma.

Necessity. Consider a utility representation $U_h : \mathcal{F}_h \mapsto \mathbb{R}$ for the preference relation $\succsim_h$ on $\mathcal{F}_h$ and suppose that it is of the form in (1). That the very existence of a utility representation requires $\succsim_h$ to be rational is a standard result. In addition, the given specification renders vertical consistency immediate. For any $n \in \mathcal{N}_h$, $U_h(n, \cdot)$ is a positive affine transformation of $U_{h(n)} : \mathcal{F}_{h(n)} \mapsto \mathbb{R}$ in Theorem 1.

Take now any $n_1, n_2 \in \mathcal{N}_h$. It is without loss of generality to let $n_2 > n_1$. Conditions (1)-(3) of the theorem dictate, respectively, the same positive affine transformation relation between $U_{h(n_1)}(\cdot)$ and $U_{h(n_2)}(\cdot)$ for the constant crisp, constant non-crisp, and non-constant crisp acts in $\mathcal{F}_{h(n_1)} \cap \mathcal{F}_{h(n_2)}$. In other words, the relations $\succsim_{h(n_1)}$ and $\succsim_{h(n_2)}$ give the same

is, $\forall \widetilde{n} \in \mathcal{N}_h$ and $\forall z', z'' \in \Delta(X)$, we have $(\widetilde{n}, z') \succsim_h (\widetilde{n}, z'')$ iff $(n, z') \succsim_h (n, z'')$ iff $z' \succsim_{h(n)} z''$ iff $z' \succsim z''$ $\forall n \in \mathcal{N}_h$. It follows that the minimal and maximal prizes in $\Delta(X)$, $\underline{z}$ and $\overline{z}$, are well-defined since for any $n \in \mathcal{N}_h$ and any $z \in \Delta(X)$, we have $\overline{z} \succsim_{h(n)} z \succsim_{h(n)} \underline{z}$ iff $\overline{z} \succsim z \succsim \underline{z}$.

[23] In the argument of the preceding paragraph, we could also have taken $z = z_2$, to establish that $(n_1, f) \succsim_h (n_2, z) \succsim_h (n_2, g)$; the latter relation now being an indifference and the former a strict preference. It follows that we could consider instead the countable set $\mathcal{Q}_h = \mathcal{N}_h \times \bigcup_{n \in \mathcal{N}_h} \left(Q_{h(n)} \cup \{\overline{z}\}\right)$ where $\overline{z}$ is a maximal prize in $\Delta(X)$.

ranking on $\Delta(X) \cup \left(\mathcal{F}^{c}_{h(n_1)} \cap \mathcal{F}^{c}_{h(n_2)}\right) \cup \left(\mathcal{F}^{cr}_{h(n_1)} \cap \mathcal{F}^{cr}_{h(n_2)}\right)$ as required by horizontal consistency.

It remains to establish solvability which we do by contradiction. Suppose that (B.2) is violated. There must exist, therefore, $z_1, z_2 \in \Delta(X)$ and $n_1, n_2 \in \mathcal{N}_h$ such that the three statements below are satisfied simultaneously.

1. $\exists z' \in \Delta(X)$ s.t. $(n_1, z_1) \succ_h (n_1, z')$ or $(n_2, z') \succ_h (n_2, z_2)$

2. $(n_1, z_1) \succ_h (n_2, z_2)$

3. $\nexists z'' \in \Delta(\mathcal{X})$ s.t. $(n_1, z_1) \succ_h (n_1, z'') \succ_h (n_2, z_2)$ or $(n_1, z_1) \succ_h (n_2, z'') \succ_h (n_2, z_2)$.

Let $\delta = \lambda_{h(n_1)} u_{h(n_1)}(z_1) - \lambda_{h(n_2)} u_h(z_2) + k_{h(n_1)} - k_{h(n_2)}$. Notice that $\delta > 0$ as $U_h(n_1, z_1) > U_h(n_2, z_2)$ by the second statement above. Statements (2)-(3) together imply that one of the following must be true[24]

(a) $(n_1, z_2) \succsim_h (n_1, z_1) \succ_h (n_2, z_2) \succsim_h (n_2, z_1)$

(b) $(n_2, z_1) \succsim_h (n_1, z_1) \succ_h (n_2, z_2) \succsim_h (n_1, z_2)$

(c) $(n_1, z_2), (n_2, z_1) \succsim_h (n_1, z_1) \succ_h (n_2, z_2)$

(d) $(n_1, z_1) \succ_h (n_2, z_2) \succsim_h (n_1, z_2), (n_2, z_1)$

In case (a), by vertical consistency, $z_2 \succsim_{h(n_1)} z_1$ and, by horizontal consistency, $z_2 \succsim_{h(n)} z_1$ $\forall n \in \mathcal{N}_h$; equivalently, $u_h(z_2) \geq u_h(z_1)$ (recall footnote 22). Take now $z \in \Delta(X)$ s.t. $u_h(z) = u_h(z_2) + \varepsilon$ for some $\varepsilon \in \left(0, \frac{\delta}{\lambda_{h(n_2)}}\right)$. As long as z_2 is not maximal in $\Delta(X)$, the prize z exists by the continuity of u on $\Delta(X)$.[25] Using the given specification, $U_h(n, z) = \lambda_{h(n)} u_h(z) + k_{h(n)}$ $\forall n \in \mathcal{N}_h$ and $\forall z \in \Delta(X)$, it is trivial to verify that $U_h(n_1, z_1) > U_h(n_2, z) > U_h(n_2, z_2)$. Equivalently, $(n_1, z_1) \succ_h (n_2, z) \succ_h (n_2, z_2)$ which contradicts statement 3. If z_2 is maximal on $\Delta(X)$, then Statement 1 does not permit z_1 to be minimal. Hence, $\exists z \in \Delta(X)$ s.t. $u_h(z) = u_h(z_1) - \varepsilon$ for some $\varepsilon \in \left(0, \frac{\delta}{\lambda_{h(n_1)}}\right)$.[26] But this gives $(n_1, z_1) \succ_h (n_1, z) \succ_h (n_2, z_2)$, contradicting again statement 3.
For case (b), by vertical and horizontal consistency, $u_h(z_1) > u_h(z_2)$ which ensures that z_1 is obviously not minimal on $\Delta(X)$. There is, therefore, $z \in \Delta(X)$ s.t. $u_h(z) = u_h(z_1) - \varepsilon$

[24]Notice that each of the statements (c) and (d) is meant to include both cases $(n_1, z_2) \succsim_h (n_2, z_1)$ and $(n_2, z_1) \succsim_h (n_1, z_2)$.

[25]Recall footnote 22. If z_2 is not maximal in $\Delta(X)$, the latter part of Statement 1 applies. That is, $\exists z' \in \Delta(X)$ s.t. $z' \succ_{h(n)} z_2$ for any $n \in \mathcal{N}_h$. For the required prize z, it suffices to take a convex combination $\alpha z_2 + (1 - \alpha) z'$, with $\alpha \in (0, 1]$ appropriately close to zero.

[26]Let $z_2 \succsim_{h(n)} z$ $\forall n \in \mathcal{N}_h$ and $\forall z \in \Delta(X)$. Statement 1 requires the existence of some $z' \in \Delta(X)$ s.t. $z_1 \succ_{h(n)} z'$ for any $n \in \mathcal{N}$. For the required prize z, it suffices to take a convex combination $\alpha z_1 + (1 - \alpha) z'$, with $\alpha \in (0, 1]$ appropriately close to zero.

for some $\varepsilon \in \left(0, \frac{\delta}{\lambda_{h(n_1)}}\right)$. This gives $(n_1, z_1) \succ_h (n_1, z) \succ_h (n_2, z_2)$ contradicting statement 3. In case (c), by vertical consistency, $z_2 \succsim_{h(n_1)} z_1$ and $z_1 \succ_{h(n_2)} z_2$ which is absurd (recall footnote 22). A similar contradiction obtains in (d) as we get $z_1 \succ_{h(n_1)} z_2 \succsim_{h(n_2)} z_1$. ■

Proof of Proposition 1 Let $U_h(n, \cdot) = \lambda_{h(n)} u_h(\cdot) + k_{h(n)}$ and $U_h^0(n, \cdot) = \lambda_{h(n)}^0 u_h^0(\cdot) + k_{h(n)}^0$ be two functionals representing $\succsim_h$ on $\mathcal{N}_h \times \Delta(X)$. Cardinality means that, $\forall z_1, z_1', z_2, z_2' \in \Delta(X)$ and $\forall n_1, n_2 \in \mathcal{N}$, we ought to have $\lambda_{h(n_1)}[u_h(z_1) - u_h(z_1')] = \lambda_{h(n_2)}[u_h(z_2) - u_h(z_2')]$ iff $\lambda_{h(n_1)}^0[u_h^0(z_1) - u_h^0(z_1')] = \lambda_{h(n_2)}^0[u_h^0(z_2) - u_h^0(z_2')]$. Recall, however, that horizontal consistency restricts $u_h^0(\cdot)$ to be a positive affine transformation of $u_h(\cdot)$: $u_h^0(\cdot) = \beta u_h(\cdot) + \gamma$ for some $\beta > 0$ and $\gamma \in \mathbb{R}$.[27] Therefore, cardinality means that $\lambda_{h(n_1)}[u_h(z_1) - u_h(z_1')] = \lambda_{h(n_2)}[u_h(z_2) - u_h(z_2')]$ iff $\lambda_{h(n_1)}^0[u_h(z_1) - u_h(z_1')] = \lambda_{h(n_2)}^0[u_h(z_2) - u_h(z_2')]$ $\forall z_1, z_1', z_2, z_2' \in \Delta(X)$ and $\forall n_1, n_2 \in \mathcal{N}$.

Only if. Take arbitrary $n_1, n_2 \in \mathcal{N}$ and consider the prizes $z_1^*, z_2^*, \widetilde{z}_1, \widetilde{z}_2 \in \Delta(X)$ of the comparability axiom (B.5). Since $(n_1, z_1^*) \sim_h (n_2, z_2^*)$ and $(n_1, \widetilde{z}_1) \sim_h (n_2, \widetilde{z}_2)$, under the representation in (1), we ought to have

$$\lambda_{h(n_1)} u_h(z_1^*) - \lambda_{h(n_2)} u_h(z_2^*) = k_{h(n_2)} - k_{h(n_1)} = \lambda_{h(n_1)} u_h(\widetilde{z}_1) - \lambda_{h(n_2)} u_h(\widetilde{z}_2) \tag{15}$$

and, thus,

$$\lambda_{h(n_1)}[u_h(z_1^*) - u_h(\widetilde{z}_1)] = \lambda_{h(n_2)}[u_h(z_2^*) - u_h(\widetilde{z}_2)] \tag{16}$$

Consider also the functional $U_h^0(n, \cdot)$. Applying comparability, the corresponding version of (16) simplifies as follows

$$\lambda_{h(n_1)}^0[u_h(z_1^*) - u_h(\widetilde{z}_1)] = \lambda_{h(n_2)}^0[u_h(z_2^*) - u_h(\widetilde{z}_2)] \tag{17}$$

Recall, however, that $z_1^* \not\sim_{h(n_1)} \widetilde{z}_1$. Horizontal consistency requires then that $z_1^* \not\sim_{h(n)} \widetilde{z}_1$ $\forall n \in \mathcal{N}$. In particular, $z_1^* \not\sim_{h(N)} \widetilde{z}_1$ and, thus, $u_h(z_1^*) \neq u_h(\widetilde{z}_1)$. Since all sides of (16) and (17) are non-zero, the simultaneous validity of the two conditions means that

$$\frac{\lambda_{h(n_1)}}{\lambda_{h(n_2)}} = \frac{\lambda_{h(n_1)}^0}{\lambda_{h(n_2)}^0} \tag{18}$$

[27] In the proof of Lemma 5(i), we established that both $u_h(\cdot)$ and $u_h^0(\cdot)$ are representations of the same preference relation, $\succsim_{h(N)}$, on $\Delta(X)$.

which suffices for $\lambda_{h(n_1)}[u_h(z_1) - u_h(z_1')] = \lambda_{h(n_2)}[u_h(z_2) - u_h(z_2')]$ to be equivalent (as required for cardinality) to $\lambda^0_{h(n_1)}[u_h(z_1) - u_h(z_1')] = \lambda^0_{h(n_2)}[u_h(z_2) - u_h(z_2')]$ $\forall z_1, z_1', z_2, z_2' \in \Delta(X)$.

Consider now the corresponding version of (15) for the functional $U^0_h(n, \cdot)$. For any $z, z' \in \Delta(X)$, we ought to have

$$\begin{aligned}
k^0_{h(n_2)} - k^0_{h(n_1)} &= \lambda^0_{h(n_1)} u^0_h(z_1^*) - \lambda^0_{h(n_2)} u^0_h(z_2^*) \\
&= \lambda^0_{h(n_1)} \left(\beta \left[u_h(z_1^*) - \frac{\lambda^0_{h(n_2)}}{\lambda^0_{h(n_1)}} u_h(z_2^*) \right] + \gamma \left[1 - \frac{\lambda^0_{h(n_2)}}{\lambda^0_{h(n_1)}} \right] \right) \\
&= \lambda^0_{h(n_1)} \left(\beta \left[u_h(z_1^*) - \frac{\lambda_{h(n_2)}}{\lambda_{h(n_1)}} u_h(z_2^*) \right] + \gamma \left[1 - \frac{\lambda_{h(n_2)}}{\lambda_{h(n_1)}} \right] \right) \\
&= \frac{\lambda^0_{h(n_1)}}{\lambda_{h(n_1)}} \left(\beta \left[\lambda_{h(n_1)} u_h(z_1^*) - \lambda_{h(n_2)} u_h(z_2^*) \right] + \gamma \left[\lambda_{h(n_1)} - \lambda_{h(n_2)} \right] \right) \\
&= \frac{\lambda^0_{h(n_1)}}{\lambda_{h(n_1)}} \left(\beta \left[k_{h(n_2)} - k_{h(n_1)} \right] + \gamma \left[\lambda_{h(n_1)} - \lambda_{h(n_2)} \right] \right)
\end{aligned}$$

where the third and last equalities above use, respectively, (18) and (15). ■

Proof of Claim 1 Let the representation in (1) be cardinal but the comparability axiom fail to hold for some $n_1, n_2 \in \mathcal{N}_h$. It is straightforward to verify that this can occur only if either $(n_1, \underline{z}) \succsim_h (n_2, \overline{z})$ or $(n_2, \underline{z}) \succsim_h (n_1, \overline{z})$ where $\underline{z}$ and $\overline{z}$ are, respectively, the minimal and maximal prizes in $\Delta(X)$ under $\succsim$. We will present the argument for the former case (that for the latter being trivially similar). We have already seen above that cardinality requires $\lambda_{h(n_1)}[u_h(z_1) - u_h(z_1')] = \lambda_{h(n_2)}[u_h(z_2) - u_h(z_2')]$ to be equivalent to $\lambda^0_{h(n_1)}[u_h(z_1) - u_h(z_1')] = \lambda^0_{h(n_2)}[u_h(z_2) - u_h(z_2')]$ $\forall z_1, z_1', z_2, z_2' \in \Delta(X)$. By non-triviality, we can find prizes z_1, z_1', z_2, z_2' such that neither side of either equality is zero (in fact, we can take $z_1 = z_2 = \overline{z}$ and $z_1' = z_2' = \underline{z}$). Clearly, the equivalence between the two equalities can obtain only if (18) also holds. Now $(n_1, \underline{z}) \succsim_h (n_2, \overline{z})$ means that $U^0_h(n_1, \underline{z}) \geq U^0_h(n_2, \overline{z})$ which reads

$$\begin{aligned}
k^0_{h(n_2)} - k^0_{h(n_1)} &\leq \lambda^0_{h(n_1)} u^0_h(\underline{z}) - \lambda^0_{h(n_2)} u^0_h(\overline{z}) \\
&= \frac{\lambda^0_{h(n_1)}}{\lambda_{h(n_1)}} \left(\beta \left[\lambda_{h(n_1)} u_h(\underline{z}) - \lambda_{h(n_2)} u_h(\overline{z}) \right] + \gamma \left[\lambda_{h(n_1)} - \lambda_{h(n_2)} \right] \right)
\end{aligned}$$

We will show that this inequality cannot be remain valid for all $\beta > 0, \gamma \in \mathbb{R}$; thus, some restrictions must be imposed on the coefficients. Consider first the case $\lambda_{h(n_1)} \neq \lambda_{h(n_2)}$. Letting $\gamma \to -\infty$ whenever $\lambda_{h(n_1)} > \lambda_{h(n_2)}$ ($\gamma \to \infty$ if $\lambda_{h(n_1)} < \lambda_{h(n_2)}$) forces the right-hand side to decrease without bounds while the left-hand side remains fixed. Hence, in this case, $u_h(\cdot)$ can be unique only up to positive linear transformations, $u_h^0(\cdot) = \beta u_h(\cdot)$. Taking, however, $\gamma = 0$ above, we observe the following. First, if $\lambda_{h(n_1)} u_h(\underline{z}) < \lambda_{h(n_2)} u_h(\overline{z})$, the inequality fails for $\beta \to \infty$ as the right-hand decreases without bounds while to left-hand one remains fixed and finite. Second, the right-hand side of the inequality is negative if $u_h(\underline{z}) \leq 0$ and $u_h(\overline{z}) \geq 0$ (with at least one inequality strict of course). In this case, the inequality is valid only if $k^0_{h(n_1)} > k^0_{h(n_2)}$. For $\lambda_{h(n_1)} < \lambda_{h(n_2)}$, the right-hand side is again negative if $u_h(\underline{z}) \geq 0$ or $u_h(\overline{z}) \leq 0$ (with at least one inequality strict). In either case, therefore, we require $k^0_{h(n_1)} > k^0_{h(n_2)}$. When $\lambda_{h(n_1)} > \lambda_{h(n_2)}$, if $k^0_{h(n_1)} < k^0_{h(n_2)}$, we can take $\beta \to 0$ forcing the right-hand side to tend to zero while the left-hand side one remains fixed and positive. Again we can only allow $k^0_{h(n_1)} \geq k^0_{h(n_2)}$. Notice also that, if $k^0_{h(n_1)} = k^0_{h(n_2)}$, we cannot have simultaneously $u_h(\underline{z}) \leq 0$ and $u_h(\overline{z}) \geq 0$; thus, $u_h(z) \neq 0 \; \forall z \in \Delta(X)$. For the case $\lambda_{h(n_1)} = \lambda_{h(n_2)} = \lambda$, the right-hand side reduces to $\lambda^0 \beta (u_h(\underline{z}) - u_h(\overline{z})) < 0$. Again the inequality cannot hold unless $k^0_{h(n_2)} < k^0_{h(n_1)}$.

In conclusion, the representation in (1) is such that either (i) $k_{h(n_1)} \geq k_{h(n_2)}$ (with $u_h(z) \neq 0$ when $k_{h(n_1)} = k_{h(n_2)}$), if $(n_1, \underline{z}) \succsim_h (n_2, \overline{z})$, and $u_h(\cdot)$ unique up to positive transformations, or (ii) $\lambda_{h(n_1)} = \lambda_{h(n_2)}$ and $k_{h(n_1)} > k_{h(n_2)}$, if $(n_1, \underline{z}) \succsim_h (n_2, \overline{z})$, and $u_h(\cdot)$ unique up to affine transformations. With respect to the latter form, of course, one can normalize by setting $\beta = 1/\lambda$. ■

Proof of Claim 2 The *if* part is immediate. In (ii), notice that $\forall n_1, n_2 \in \mathcal{N}$ with $k_{h(n_1)} \neq k_{h(n_2)}$ and $\forall z \in \Delta(X)$, we have $U_h(n_1, z) - U_h(n_2, z) = k_{h(n_1)} - k_{h(n_2)} \neq 0$. In (i), $\forall n_1, n_2 \in \mathcal{N}$ with $\lambda_{h(n_1)} \neq \lambda_{h(n_2)}$ and $\forall z \in \Delta(X)$, we get $k_{h(n_2)} - k_{h(n_1)} \leq 0 \leq \left(\lambda_{h(n_1)} - \lambda_{h(n_2)}\right) u_h(z)$. By $\left(k_{h(n_1)} - k_{h(n_2)}\right) u_h(z) \neq 0$, moreover, at least one inequality is strict. That is, $U_h(n_1, z) > U_h(n_2, z)$.

For the *only if*, let $z \in \Delta(X)$ and $n_1, n_2 \in \mathcal{N}$ be such that $(n_1, z) \succ_h (n_2, z)$. We have $U_h(n_1, z) > U_h(n_2, z)$, equivalently, $\left(\lambda_{h(n_1)} - \lambda_{h(n_2)}\right) u_h(z) > k_{h(n_2)} - k_{h(n_1)}$. In the case $\lambda_{h(n_1)} = \lambda_{h(n_2)}$, the claim is immediate. Let $\lambda_{h(n_1)} > \lambda_{h(n_2)}$ and consider the transformation $\widetilde{U}_{h(n)}(\cdot) = \lambda_{h(n)} (\beta u_h(\cdot) + \gamma) + k_{h(n)}$ for $\beta > 0, \gamma \in \mathbb{R}$. The inequality $\left(\lambda_{h(n_1)} - \lambda_{h(n_2)}\right) \beta u_h(z) > k_{h(n_2)} - k_{h(n_1)} + \gamma \left(\lambda_{h(n_2)} - \lambda_{h(n_1)}\right)$ fails to hold for $\gamma \to -\infty$ as the right-hand side increases without bounds whereas the left-hand side remains unchanged (and finite). Clearly, $u_h(\cdot)$ can be unique only up to positive linear transformations. For $u_h(z) < 0$, however, the inequality $\left(\lambda_{h(n_1)} - \lambda_{h(n_2)}\right) \beta u_h(z) > k_{h(n_2)} - k_{h(n_1)}$ fails to hold for $\beta \to \infty$ as the left-hand side decreases without bounds while the right-hand one remains fixed (and finite). That is, we are restricted to having $u_h(z) \geq 0 \; \forall z \in \Delta(X)$. Finally, if $k_{h(n_2)} > k_{h(n_1)}$, the inequality fails for $\beta \to 0$ as the left-hand side tends to zero wile to right-hand one remains fixed and positive. It can only be, thus, $k_{h(n_2)} \leq k_{h(n_1)}$ which, in conjunction with $u_h(z) \geq 0$, ensures that $(n_1, z) \succ_h (n_2, z)$ $\forall z \in \Delta(X)$. The case $k_{h(n_1)} = k_{h(n_2)}$ is compatible only with the condition $u_h(z) \neq 0$ $\forall z \in \Delta(X)$. The argument for the case $\lambda_{h(n_1)} < \lambda_{h(n_2)}$ is trivially similar, giving $k_{h(n_2)} \leq k_{h(n_1)}$, $u_h(z) \leq 0$, $\left(k_{h(n_1)} - k_{h(n_2)}\right) u_h(\cdot) \neq 0$, and again $(n_1, z) \succ_h (n_2, z)$ $\forall z \in \Delta(X)$. ■

Proof of Claim 3 For the *only if* part, observe first that, since $U_h(n, z) = \lambda_{h(n)} u_h(z) + k_{h(n)}$ $\forall z \in \Delta(X)$ and $\forall n \in \mathcal{N}$, if $(n_1, z) \sim_h (n_2, z)$ for some $n_1, n_2 \in \mathcal{N}$, it must be $\left[\lambda_{h(n_1)} - \lambda_{h(n_2)}\right] u_h(z) = k_{h(n_2)} - k_{h(n_1)}$. Suppose now that $z^*, z^{**} \in \Delta(X)$ are such that $(n_1, z^*) \sim_h (n_2, z^*)$ and $(n_1, z^{**}) \sim_h (n_2, z^{**})$ with $z^* \succ z^{**}$. We must have $\left[\lambda_{h(n_1)} - \lambda_{h(n_2)}\right]$ $[u_h(z^*) - u_h(z^{**})] = 0$ while $u_h(z^*) > u_h(z^{**})$. Clearly, $\lambda_{h(n_1)} = \lambda_{h(n_2)} = \lambda$. Since at least one of $u_h(z^*)$ and $u_h(z^{**})$ is non-zero, it must also be $k_{h(n_1)} = k_{h(n_2)} = k$. The representation in (9) obtains as it is without loss of generality to normalize through the transformation $\widetilde{u}_h = \frac{u_h}{\lambda} - \frac{k}{\lambda}$. The *if* direction is trivial; it leads immediately to the conclusion that $(n_1, z) \sim_h (n_2, z)$ $\forall n_1, n_2 \in \mathcal{N}$ and $\forall z \in \Delta(X)$. ■

Proof of Claim 4 For the *only if*, consider the v.NM utility transformation $\widetilde{u}_h : \Delta(X) \mapsto \mathbb{R}$ defined by $\widetilde{u}_h(\cdot) = u_h(\cdot) - u_h(z^*)$. As a positive linear transformation of $u_h(\cdot)$, it continues to represent the preference $\succsim$ on $\Delta(X)$. Hence, the functional $\widetilde{U}_h(n, z) = \lambda_{h(n)} \widetilde{u}_h(z) + k_{h(n)}$ represents $\succsim_h$ on $\mathcal{N} \times \Delta(X)$. But $\widetilde{u}_h(z^*) = 0$ and, thus, $\widetilde{U}_h(n, z^*) = k_{h(n)}$ $\forall n \in \mathcal{N}_h$. Since $(n_1, z^*) \sim_h (n_2, z^*)$, it must be $\widetilde{U}_h(n_1, z^*) = \widetilde{U}_h(n_2, z^*)$; thus, $k_{h(n_1)} = k_{h(n_2)} = k \in \mathbb{R}$ $\forall n_1, n_2 \in \mathcal{N}_h$. The representation in (10) obtains as it is without loss of generality to normalize by setting $k = 0$ - in other words, by requiring that $\widetilde{U}_h(n, z^*) = 0$ $\forall n \in \mathcal{N}_h$.

It remains to show that the relative coefficients in (10) must be uniquely determined. To this end, consider two different functionals (10), $U_h(n, z) = \lambda_{h(n)} u_h(z)$ and $U_h^0(n, z) = \lambda_{h(n)}^0 u_h^0(z)$. As both represent $\succsim$ on $\Delta(X)$, by horizontal and vertical consistency, there must exist $\mu > 0$ such that $\widetilde{u}_h(\cdot) = \mu u_h(\cdot)$. By the non-triviality axiom (A.5), moreover, it cannot be $u_h(z) = 0$ $\forall z \in \Delta(X)$. For any $n_1, n_2 \in \mathcal{N}_h$, therefore, we must have $\frac{\lambda_{h(n_1)}}{\lambda_{h(n_2)}} = \frac{\lambda^0_{h(n_1)}}{\lambda^0_{h(n_2)}}$ as required.

If. Given that there exist some prize $z^* \in \Delta(X)$ that takes zero utility under $u_h(\cdot)$, it is immediate from the representation in (10) that z^* is a reference prize. For any $n_1, n_2 \in \mathcal{N}_h$ such that $\lambda_{h(n_1)} \neq \lambda_{h(n_2)}$, by Claim 3, there can be no other reference prize across the horizons $h(n_1)$ and $h(n_2)$. ■

Proof of Claim 5. In what follows, we present the argument for the IHM case (that for the DHM one is trivially similar). Our argument does not require the particular normalization

that leads to the representation in (10). For any $z \in \Delta(X)$ and any $n \in \mathcal{N}_h$, therefore, let $U_h(n,z) = \lambda_{h(n)} u_h(z) + k_{h(n)}$. Under axiom (B.5*), we have $\left[\lambda_{h(n+1)} - \lambda_{h(n)}\right] u_h(z^*) = k_{h(n+1)} - k_{h(n)}$. Hence, $U_h(n+1,z) > U_h(n,z)$ iff $\left[\lambda_{h(n+1)} - \lambda_{h(n)}\right] [u_h(z) - u_h(z^*)] > 0$.
Only if. Let $z' \in \Delta(X)$ and $n \in \mathcal{N}_h \backslash \{N\}$ be s.t. $(n+1, z') \succ_h (n, z')$. Since $U_h(n+1,z') > U_h(n,z')$, it must be $\left[\lambda_{h(n+1)} - \lambda_{h(n)}\right] [u_h(z') - u_h(z^*)] > 0$. Hence, $z' \succ z^*$ iff $u_h(z') > u_h(z^*)$ iff $\lambda_{h(n+1)} > \lambda_{h(n)}$. Similarly, $z^* \succ z'$ iff $u_h(z') < u_h(z^*)$ iff $\lambda_{h(n+1)} < \lambda_{h(n)}$. Notice also that, as long as z' exists, $\lambda_{h(n)} \neq \lambda_{h(n+1)}$. Consequently, for any $z \in \Delta(X)$, $z \sim z^*$ iff $(n+1,z) \sim_h (n,z)$.
If. This obtains trivially by signing the quantity $\left[\lambda_{h(n+1)} - \lambda_{h(n)}\right] [u_h(z) - u_h(z^*)]$ accordingly for any $z \in \Delta(X)$. Notice that, by the non-degeneracy axiom (A.5), $\exists z' \in \Delta(X)$ for which this quantity is non-zero. At least for this prize, the preference relation in the horizon-monotonicity statement is strict. ■

Proof of Claim 6. For the *only if* direction, take $\alpha \in (0,1)$ such that $\alpha z' + (1-\alpha) z'' \sim z^*$ (recall Lemma 2). We have, $u_h(z^*) = \alpha u_h(z') + (1-\alpha) u_h(z'')$. By $U_h(n_1, z^*) = U_h(n_2, z^*)$, moreover, $\lambda_{h(n_1)} u_h(z^*) + k_{h(n_1)} = \lambda_{h(n_2)} u_h(z^*) + k_{h(n_2)}$. Hence,

$$\lambda_{h(n_1)} [\alpha u_h(z') + (1-\alpha) u_h(z'')] + k_{h(n_1)} = \lambda_{h(n_2)} [\alpha u_h(z') + (1-\alpha) u_h(z'')] + k_{h(n_2)}$$

Equivalently,

$$\alpha [U_h(n_1, z') - U_h(n_2, z')] = (1-\alpha) [U_h(n_2, z'') - U_h(n_1, z'')]$$

a re-statement of what is required.
If. We have $[U_h(n_1,z') - U_h(n_2,z')] [U_h(n_1,z'') - U_h(n_2,z'')] < 0$. Moreover, $U_h(n,z) = \lambda_{h(n)} u_h(z) + k_{h(n)}$ $\forall n \in \mathcal{N}_h$ and $\forall z \in \Delta(X)$ while $U_h(n_1,z^*) = U_h(n_2,z^*)$. It is trivial to verify that, $\forall z \in \Delta(X)$, $U_h(n_1,z) > U_h(n_2,z)$ iff $\left[\lambda_{h(n_1)} - \lambda_{h(n_2)}\right] [u_h(z) - u_h(z^*)] > 0$.

Suppose that $z' \succ z^*$, equivalently, $u_h(z') > u_h(z^*)$. If $U_h(n_1,z') > U_h(n_2,z')$, we get $\lambda_{h(n_1)} > \lambda_{h(n_2)}$. But then $U_h(n_1,z'') < U_h(n_2,z'')$ requires $\left(\lambda_{h(n_1)} - \lambda_{h(n_2)}\right) [u_h(z'') - u_h(z^*)] < 0$. Hence, $u_h(z'') < u_h(z^*)$ or $z^* \succ z''$. If $U_h(n_1,z') < U_h(n_2,z')$, then $\lambda_{h(n_1)} < \lambda_{h(n_2)}$. In this case, however, $U_h(n_1,z'') > U_h(n_2,z'')$ and again $u_h(z'') < u_h(z^*)$. By a trivially similar argument, $z'' \succ z^*$ whenever $z^* \succ z'$. ■

Proof of Claim 7. Both parts can be shown by contradiction. For (i), let (n_2, g) be optimal for $\succsim_h$ on $\mathcal{A}_h$ and $z \in \Delta(X)$ be such that $f \succsim_{h(n_1)} z$ but $z \succ_{h(n_2)} g$. We have

$$(n_2, z) \succ_h (n_2, g) \succsim_h (n_1, f) \succsim_h (n_1, z)$$

where the strict and second weak preference relations follow by vertical consistency whereas the first weak preference is due to the optimality assumption. Since $n_2 > n_1$, however, $(n_2, z) \succ_h (n_1, z)$ violates DHM. For part (ii), let (n_1, f) be optimal for $\succsim_h$ on $\mathcal{A}_h$ and $z \in \Delta(X)$ be such that $z \succ_{h(n_1)} f$ but $g \succsim_{h(n_2)} z$. We have

$$(n_1, z) \succ_h (n_1, f) \succsim_h (n_2, g) \succsim_h (n_2, z)$$

But, for $n_2 > n_1$, $(n_1, z) \succ_h (n_2, z)$ violates IHM. ■

Next, we present a result that will be used in establishing Theorem 3.

Lemma 6 *Suppose that the preference relations $\succsim_{n\in\mathcal{N}}$ on the sets of acts $\mathcal{F}_{n\in\mathcal{N}}$ satisfy axioms A.1-A.6 and B.3-B.4. Consider also $n, n' \in \mathcal{N}$, $f \in \mathcal{F}_{h(n)}$, and $g \in \mathcal{F}_{h(n')}$ with $z_f, z_g \in \Delta(X)$ being, respectively, the non-ambiguous equivalents of f and g under $\succsim_{h(n)}$ and $\succsim_{h(n')}$. The following statements are equivalent.*

(i) $\forall z \in \Delta(X),\ f \succsim_{h(n)} z \Rightarrow g \succsim_{h(n')} z$

(ii) $u(z_g) \geq u(z_f)$

Proof. We will argue by contradiction. For the *only if* part, suppose that statement (i) holds and let $u(z_g) < u(z_f)$. It must be $z_f \succ_{h(n')} z_g$ by vertical and horizontal consistency. But then, $z_f \succ_{h(n')} z_g \sim_{h(n')} g$ implies, by transitivity, $z_f \succ_{h(n')} g$. Given that $f \sim_{h(n)} z_f$, this is a contradiction of (i). For the *if* part, let $u(z_g) \geq u(z_f)$ and suppose that statement (i) does not hold. There must be, therefore, some prize $z \in \Delta(X)$ such that $f \succsim_{h(n)} z \succ_{h(n')} g$. Equivalently, $z_f \sim_{h(n)} f \succsim_{h(n)} z \succ_{h(n')} g \sim_{h(n')} z_g$ which, under vertical and horizontal consistency, requires that $u(z_f) \geq u(z) > u(z_g)$. ■

Lemma 7 *Suppose that the preference relations $\succsim_{n\in\mathcal{N}}$ on the sets of acts $\mathcal{F}_{n\in\mathcal{N}}$ satisfy axioms A.1-A.6 and B.3-B.4. Consider also $n, n' \in \mathcal{N}$, $f \in \mathcal{F}_{h(n)}$, and $g \in \mathcal{F}_{h(n')}$ with $z_f, z_g \in \Delta(X)$ being, respectively, the non-ambiguous equivalents of f and g under $\succsim_{h(n)}$ and $\succsim_{h(n')}$. The following statements are equivalent.*

(i) $\forall z \in \Delta(X),\ z \succ_{h(n)} f \Rightarrow z \succ_{h(n')} g$

(ii) $u(z_g) \leq u(z_f)$

Proof. We argue again by contradiction. For the *only if* part, suppose that statement (i) holds and let $u(z_g) > u(z_f)$; thus, $g \succ_{h(n')} z_f$. As $z_f \sim_{h(n)} f$, the prize z_f does not contradict (i) now by itself. Yet, any $z \in \Delta(X)$ such that $u(z) \in (u(z_f), u(z_g))$ does (due to vertical and horizontal consistency). The existence of this prize z is ensured by the continuity of u. For the *if* part, let $u(z_g) \leq u(z_f)$ and suppose that statement (i) does not hold. There must be, therefore, some prize $z \in \Delta(X)$ such that $f \succ_{h(n)} z \succsim_{h(n')} g$ which requires, though, $u(z_f) > u(z) \geq u(z_g)$. ■

Proof of Theorem 3.

If. Suppose first that (n, f) is optimal under $\succsim_h$ on the set $\mathcal{A}_h$ of all available horizon-act pairs on the entire tree. Of course, this is so iff $U_h(n, f) \geq U_h(n', g')\ \forall g' \in C\left(\mathcal{A}_{h(n')}, \succsim_{h(n')}\right)$ and $\forall n' \in \mathcal{N}$. By the continuation consistency axiom (B.6), there must exist a continuation act $\widetilde{g} \in \mathcal{A}_{h(N)}$ such that

$$\forall z \in \Delta(X),\ f \succsim_{h(n)} z \Rightarrow \widetilde{g} \succsim_{h(N)} z$$

By Lemma (6), therefore, we ought to have $u(z_{\widetilde{g}}) \geq u(z_f)$ for the non-ambiguous equivalents of f and $\widetilde{g}$ under $\succsim_{h(n)}$ and $\succsim_{h(N)}$, respectively. But this requires that (N, g) is also optimal under $\succsim_h$ on $\mathcal{A}_h$ because

$$U_h(n, \widetilde{g}) = \lambda U_{h(n)}(\widetilde{g}) + k = \lambda u(z_{\widetilde{g}}) + k \geq \lambda u(z_f) + k = \lambda U_{h(n)}(f) + k = U_h(n, f)$$

Let now $g \in \mathcal{A}_{h(N)}$ be optimal on $\mathcal{A}_h$ under $\succsim_h$. By continuation consistency, there must exist a predecessor act $\widetilde{f} \in \mathcal{A}_{h(n)}$ such that[28]

$$\forall z \in \Delta(X),\ g \succ_{h(N)} z \Rightarrow \widetilde{f} \succ_{h(n)} z$$

Lemma 7 requires that $u\left(z_{\widetilde{f}}\right) \geq u(z_g)$ for the non-ambiguous equivalent of $\widetilde{f}$ under $\succsim_{h(n)}$. Which implies, though, that (n, f) is also optimal under $\succsim_h$ on $\mathcal{A}_h$ since

$$U_h(n, g) = \lambda u(z_g) + k \leq \lambda u\left(z_{\widetilde{f}}\right) + k = U_h\left(n, \widetilde{f}\right)$$

Only if. Let $f \in \mathcal{A}_{h(n)}$ and $g \in \mathcal{A}_{h(N)}$ be optimal under $\succsim_{h(n)}$ and $\succsim_{h(N)}$, respectively.

Assume first the existence of reference prizes. We only need to consider the case of a unique reference prize for there is nothing to show under multiplicity (Claim 3). But under horizon monotonicity, the unique reference prize z^* must be either maximal or minimal on

[28] Since the relation $\succsim_{h(n)}$ is complete, this is the contrapositive statement of (ii) in axiom (B.6). To see this, take an arbitrary $z \in \Delta(X)$ and let $g \succ_{h(N)} z$. It cannot be $z \succsim_{h(n)} f$ for all the predecessor acts in $\mathcal{A}_{h(n)}$ for this contradicts part (ii) of continuation consistency. There must exist, therefore, a predecessor act $\widetilde{f}$ such that $\widetilde{f} \succ_{h(n)} z$. Moreover, for this act and any other $z' \in \Delta(X)$, it cannot be $z' \succsim_{h(n)} \widetilde{f}$ if $g \succ_{h(N)} z'$ for this again violates part (ii) of the axiom.

$\Delta(X)$ (Claim 6). In what follows, we present the argument for decreasing horizon monotonicity (DHM) - the one for (IHM) is trivially similar.

Let z^* be maximal on $\Delta(X)$. We have two cases. If $u(z_g) = u(z_f) = u(z^*)$, consider the general representation in (1). In conjunction with the cardinality of the representation, the hypothesis that $\mathcal{C}\left(\mathcal{A}_{h(n)}, \succsim_{h(n)}\right)$ and $\mathcal{C}\left(\mathcal{A}_{h(N)}, \succsim_{h(N)}\right)$ are $\succsim_h$-equivalent requires $\left[\lambda_{h(n)} - \lambda_{h(N)}\right][\mu u(z^*) + \nu] = k_{h(N)} - k_{h(n)}$ to hold $\forall \mu > 0, \nu \in \mathbb{R}$. But this can be only if $\lambda_{h(n)} - \lambda_{h(N)} = k_{h(N)} - k_{h(n)} = 0$ as required.
If $u(z_g) < u(z_f) \leq u(z^*)$, normalize such that $u(z^*) = 0$. That is, $u(z) \leq 0 \ \forall z \in \Delta(X)$ while the representation is now given by $U_h(n', r) = \lambda_{h(n')} U_{h(n)}(r) + k \ \forall r \in \mathcal{A}_{h(n')} \ \forall n' \in \mathcal{N}$ (Claim 4). Under DHM, it must be $\lambda_{h(n)} \leq \lambda_{h(n+1)} \leq \cdots \leq \lambda_{h(N)}$ for arbitrary $n \in \mathcal{N} \setminus \{N\}$ (Claim 5). It suffices, therefore, to show that $\lambda_{h(n)} = \lambda_{h(N)}$ and we will do so by contradiction. Take, thus, $\lambda_{h(n)} < \lambda_{h(N)}$ and recall that, for the predecessor act $\widetilde{f} \in \mathcal{A}_{h(n)}$, we have $u(z_g) \leq u\left(z_{\widetilde{f}}\right)$. Since f is optimal for $\succsim_{h(n)}$, it ought to be $u(z_g) \leq u\left(z_{\widetilde{f}}\right) \leq u(z_f) \leq 0$ with at least one of the first two inequalities strict. Given that $\lambda_{h(N)} > \lambda_{h(n)} > 0$, this gives

$$U_h(n, g) = \lambda_{h(N)} U_{h(N)}(g) + k = \lambda_{h(N)} u(z_g) + k \leq \lambda_{h(n)} u\left(z_{\widetilde{f}}\right) + k \leq \lambda_{h(n)} u(z_f) + k = U_h(n, f)$$

with at least one of the inequalities strict. That is, $U_h(n, g) < U_h(n, f)$ contradicting the hypothesis that $\mathcal{C}\left(\mathcal{A}_{h(n)}, \succsim_{h(n)}\right)$ and $\mathcal{C}\left(\mathcal{A}_{h(N)}, \succsim_{h(N)}\right)$ are $\succsim_h$-equivalent.

Suppose now that z^* is minimal on $\Delta(X)$. We have again two cases. If $u(z_g) = u(z_f) = u(z^*)$, the same argument as before applies. For $u(z_g) > u(z_f) \geq u(z^*)$, normalize such that $u(z^*) = 0$. Now, $u(z) \geq 0 \ \forall z \in \Delta(X)$ while the representation is given by $U_h(n', r) = \lambda_{h(n')} U_{h(n)}(r) + k \ \forall r \in \mathcal{A}_{h(n')} \ \forall n' \in \mathcal{N}$ (Claim 4). Under DHM, $\lambda_{h(n)} \geq \lambda_{h(n+1)} \geq \cdots \geq \lambda_{h(N)}$ (Claim 5) and it suffices to show that $\lambda_{h(n)} \leq \lambda_{h(N)}$ for arbitrary $n \in \mathcal{N} \setminus \{N\}$. Given that the choice rules $\mathcal{C}\left(\mathcal{A}_{h(n)}, \succsim_{h(n)}\right)$ and $\mathcal{C}\left(\mathcal{A}_{h(N)}, \succsim_{h(N)}\right)$ are $\succsim_h$-equivalent, it must be

$$\lambda_{h(N)} u(z_g) = U_h(n, g) - k = U_h(n, f) - k = \lambda_{h(n)} u(z_f)$$

Since f is optimal under $\succsim_{h(n)}$ while $\lambda_{h(n)} > 0$, taking the continuation act $\widetilde{f}$, we have

$$\lambda_{h(n)} u(z_f) = U_h(n, f) - k \geq U_h\left(n, \widetilde{f}\right) - k = \lambda_{h(n)} u\left(z_{\widetilde{f}}\right) \geq \lambda_{h(n)} u(z_g)$$

Therefore, $\lambda_{h(N)} u(z_g) \geq \lambda_{h(n)} u(z_g)$ which suffices for the required result as $u(z_g) > 0$.

It remains to consider the case when no reference prize exists. For any $(n, f) \in \mathcal{N} \times \mathcal{A}_{h(n)}$, the representation is now given by $U_h(n, f) = U_{h(n)}(f) + k_{h(n)}$ (Claim 2). The $\succsim_h$-equivalence of $\mathcal{C}\left(\mathcal{A}_{h(n)}, \succsim_{h(n)}\right)$ and $\mathcal{C}\left(\mathcal{A}_{h(N)}, \succsim_{h(N)}\right)$ requires, thus, $u(z_g) + k_{h(N)} = u(z_f) + k_{h(n)}$. We will show that $u(z_g) \neq u(z_f)$ leads to a contradiction under horizon monotonicity. Let, for instance, $u(z_g) < u(z_f)$ (the argument for the case $u(z_g) > u(z_f)$ is trivially similar). Under DHM, we have $u(z_g) + k_{h(N)} < u(z_g) + k_{h(n)} < u(z_f) + k_{h(n)}$ which is absurd. Consider again the continuation act $\widetilde{g}$ for which $u(z_{\widetilde{g}}) \geq u(z_f)$. Since $g \in C\left(\mathcal{A}_{h(N)}, \succsim_{h(N)}\right)$, under IHM, we get $u(z_g) + k_{h(N)} \geq u(z_{\widetilde{g}}) + k_{h(N)} \geq u(z_f) + k_{h(N)} > u(z_f) + k_{h(n)}$, a contradiction. Clearly, it can only be $u(z_g) = u(z_f)$ and, thus, $k_{h(N)} = k_{h(n)}$ as desired. ■

A.1 Axiomatic Variations

Consider the following stronger version of the axiomatic setting that gives Theorem 1 in the text. Let the Extended Archimedean axiom (A.3) apply on general acts $g \in \mathcal{F}_h$ and the Boundedness axiom (A.6) read as follows

A.6* Let $f \in \mathcal{F}_{h(n)}$, $E \in \pi_{h(n)}$, *and* $f(E) = Z \in \mathcal{X}_{h(n)}$. *There exist* $z', z'' \in Z$ *s.t.*

$$\begin{pmatrix} z' & E \\ f & E^c \end{pmatrix} \succsim_{h(n)} f \succsim_{h(n)} \begin{pmatrix} z'' & E \\ f & E^c \end{pmatrix}$$

Denoting the new version of the Extended Archimedean axiom as (A.3*), it is trivial to verify that Lemma 2 applies now to general acts $g \in \mathcal{F}_{h(n)}$ as long as axiom (A.3) is replaced by (A.3*). Lemmas 1 and 3 remain unaffected by these axiomatic changes whereas Lemma 4 now becomes

Lemma 4* *Let the preference relation* $\succsim_{h(n)}$ *on the set* $\mathcal{F}_{h(n)}$ *satisfy axioms (A.1)-(A.3*) and (A.4)-(A.6*). For* $f \in \mathcal{F}_{h(n)}$ *and* $\pi_{h(n)} = \left\{E_1, \ldots, E_{K<|S|}\right\}$, *let* $f(E_k) = Z_k$ *for* $Z_k \in \mathcal{X}_{h(n)}$ *and* $k = 1, \ldots, K$. *Let also* $\overline{z}_k = \sup_{z \in Z_k} u_{h(n)}(z)$ *and* $\underline{z}_k = \inf_{z \in Z_k} u_{h(n)}(z)$. *There exists a unique number* $\alpha_{h(n)} \in [0,1]$ *such that*

$$f \sim_{h(n)} \alpha_{h(n)} \begin{pmatrix} \underline{z}_1 & E_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \underline{z}_K & E_K \end{pmatrix} + \left(1 - \alpha_{h(n)}\right) \begin{pmatrix} \overline{z}_1 & E_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \overline{z}_K & E_K \end{pmatrix}$$

Proof. Take some $k = 1, \ldots, K$. By Lemma 3, we can find $x', x'' \in X$ s.t. $x' \succsim_{h(n)} z \succsim_{h(n)} x''$ $\forall z \in Z_k$. Since $u_{h(n)}(\cdot)$ represents $\succsim_{h(n)}$ on $\Delta(X)$, we must have $u_{h(n)}(x'') \leq u_{h(n)}(z) \leq u_{h(n)}(x')$ $\forall z \in Z_k$. Clearly, the quantities $\overline{z}_k$ and $\underline{z}_k$ are well-defined for all $k = 1, \ldots, K$. They are unique by construction. By the new Boundedness axiom (A.6*), we can find

$z', z'' \in Z_k$ s.t.

$$\begin{pmatrix} z' & E_k \\ f & E_k^c \end{pmatrix} \succsim_{h(n)} f \succsim_{h(n)} \begin{pmatrix} z'' & E_k \\ f & E_k^c \end{pmatrix}$$

By the definition of the lotteries $\overline{z}_k$ and $\underline{z}_k$, it must be $u_{h(n)}(\underline{z}_k) \leq u_{h(n)}(z'')$ and $u_{h(n)}(\overline{z}_k) \geq u_{h(n)}(z')$. That is, $z'' \succsim_{h(n)} \underline{z}_k$ and $\overline{z} \succsim_{h(n)} z'_k$. By monotonicity, therefore, it must be

$$\begin{pmatrix} \overline{z}_k & E_k \\ f & E_k^c \end{pmatrix} \succsim_{h(n)} f \succsim_{h(n)} \begin{pmatrix} \underline{z}_k & E_k \\ f & E_k^c \end{pmatrix}$$

Starting with $k = 1$, the preceding argument gives

$$\begin{pmatrix} \overline{z}_1 & E_1 \\ f & E_1^c \end{pmatrix} \succsim_{h(n)} f \succsim_{h(n)} \begin{pmatrix} \underline{z}_1 & E_1 \\ f & E_1^c \end{pmatrix}$$

By monotonicity and transitivity, moreover, we also have

$$\begin{pmatrix} \overline{z}_1 & E_1 \\ \overline{z}_2 & E_2 \\ f & \overline{E_1 \cup E_2} \end{pmatrix} \succsim_{h(n)} \begin{pmatrix} \overline{z}_1 & E_1 \\ f & E_1^c \end{pmatrix} \succsim_{h(n)} f \succsim_{h(n)} \begin{pmatrix} \underline{z}_1 & E_1 \\ f & E_1^c \end{pmatrix} \succsim_{h(n)} \begin{pmatrix} \underline{z}_1 & E_1 \\ \underline{z}_2 & E_2 \\ f & \overline{E_1 \cup E_2} \end{pmatrix}$$

Since $\pi_{h(n)}$ is a finite partition, we can continue repeatedly to establish

$$\begin{pmatrix} \overline{z}_1 & E_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \overline{z}_K & E_K \end{pmatrix} \succsim_{h(n)} f \succsim_{h(n)} \begin{pmatrix} \underline{z}_1 & E_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \underline{z}_K & E_K \end{pmatrix}$$

If either relation is indifference, the claim is true for $\alpha_{h(n)} = 1$ (or $\alpha_{h(n)} = 0$). If both relations are strict, the claim follows from (the new version of) Lemma 2. To complete the argument, notice that the unique quantity $\alpha_{h(n)}$ depends upon the act f itself, giving rise to an induced functional relation $\alpha_{h(n)}(\cdot) : \mathcal{F}_{h(n)} \mapsto [0, 1]$. ■

Our main representation theorem now reads **Theorem 1*** *A binary relation $\succsim_{h(n)}$ defined on the set of acts $\mathcal{F}_{h(n)}$ satisfies axioms (A.1)-(A.3*) and (A.4)-(A.6*) if and only if there exists an affine function $u_{h(n)} : \Delta(X) \mapsto \mathbb{R}$, unique up to a positive affine transformation, a unique finitely-additive probability measure $p_{h(n)}$ on the sigma-algebra induced by the partition $\pi_{h(n)}$,* *and a unique function* $\alpha_{h(n)} : \mathcal{F}_{h(n)} \mapsto [0,1]$ *such that*

$$\forall f, g \in \mathcal{F}_{h(n)} : \; f \succsim_{h(n)} g \; \Leftrightarrow \; U_{h(n)}(f) \geq U_{h(n)}(g)$$

where $U_{h(n)} : \mathcal{F}_{h(n)} \mapsto \mathbb{R}$ *is defined by*

$$U_{h(n)}(f) = \sum_{E \in \pi_{h(n)}} p_{h(n)}(E) \left(\alpha_{h(n)}(f) \inf_{z \in f(E)} u_{h(n)}(z) + \left[1 - \alpha_{h(n)}(f)\right] \sup_{z \in f(E)} u_{h(n)}(z) \right)$$

Of course, axioms (A.3) and (A.6) are implied, respectively, by (A.3*) and (A.6*). As to be expected, the representation above is a special case of that in Theorem 1. Here, the weighting functional $\alpha_{h(n)}$ depends on the act f itself, not on the sets of lotteries $f(E)$ the act corresponds to under each $E \in \pi_{h(n)}$.

B OTHER THINGS I HAVE TRIED

B. 7 <u>*Continuation Consistency*</u>. *For arbitrary* $n, n' \in \mathcal{N}_h$ *and* $f \in \mathcal{A}_{h(n)}$, *there exist acts* $g, g' \in \mathcal{A}_{h(n')}$ *s.t.*

$$\forall z, z' \in \Delta(X), \; z \succ_{h(n)} f \succ_{h(n)} z' \; \Rightarrow \; z \succ_{h(n')} g \quad \textit{and} \quad g' \succ_{h(n')} z'$$

This is basically another way to state the axiom (B.6) in the text for we can allow $n = N$.

B. 8 <u>*Continuation Consistency*</u>. *For arbitrary* $n \in \mathcal{N}_h \setminus \{N\}$ *and* $f \in \mathcal{A}_{h(n)}$, *there exist acts* $g, g' \in \mathcal{A}_{h(N)}$ *s.t.*

$$\forall z, z' \in \Delta(X), \; z \succsim_{h(n)} f \succsim_{h(n)} z' \; \Rightarrow \; z \succ_{h(N)} g \quad \textit{and} \quad g' \succ_{h(N)} z'$$

Here I would like to make the following argument. Suppose that $g \in \mathcal{A}_{h(N)}$ is optimal on $\mathcal{A}_h$ under $\succsim_h$. By this version of the axiom , for any $z \in \Delta(X)$ and any $f \in \mathcal{A}_{h(n)}$ with $n \in \mathcal{N} \setminus \{N\}$, there must exist $g' \in \mathcal{A}_{h(N)}$ such that

$$z \succsim_{h(n)} f \Rightarrow z \succsim_{h(N)} g'$$

Take now the contrapositive statement. Since both relations $\succsim_{h(n)}$ and $\succsim_{h(N)}$ are complete, it states that, for any $g' \in \mathcal{A}_{h(N)}$, there must exist some $f \in \mathcal{A}_{h(n)}$, such that

$$\forall z \in \Delta(X), \; g' \succ_{h(N)} z \Rightarrow f \succ_{h(n)} z$$

For our given act g, therefore, there must exist an act $f \in \mathcal{A}_{h(N)}$ such that

$$\forall z \in \Delta(X), \; g \succ_{h(N)} z \Rightarrow f \succ_{h(n)} z$$

Let now $n' = N$ in Lemma 6 and reverse the roles of (n, f) and (n', g). It must be $u(z_f) \geq u(z_g)$ for the non-ambiguous equivalents of f and g under $\succsim_{h(n)}$ and $\succsim_{h(N)}$, respectively.[29] This requires, however, that (n, f) is also optimal under $\succsim_h$ on $\mathcal{A}_h$ because

$$U(N, g) = \lambda u(z_g) + k \leq \lambda u(z_f) + k = U(n, f)$$

This version of the axiom looks much nicer in terms of requirements on the tree. Yet, I am not sure that the contrapositive I am using is correct..!!

B. 9 Continuation Consistency. *$\forall n_1, n_2 \in \mathcal{N}_h$ with $n_2 > n_1$, $\forall f \in \mathcal{A}_{h(n_1)}$, and $\forall z', z'' \in \Delta(X)$,*

$$z' \succsim_{h(n_1)} f \succsim_{h(n_1)} z''$$

only if

$$\exists \; g', g'' \in \mathcal{A}_{h(n_2)} : \; z' \succsim_{h(n_2)} g' \quad \text{and} \quad g'' \succsim_{h(n_2)} z''$$

Recall that, in our setting, an act $f \in \mathcal{A}_{h(n)}$ assigns a set of lotteries $Y(X_k) \subseteq \Delta(\mathcal{X})$ to each event $E_k \in \pi_{h(n)}$ where $X_k \subseteq \mathcal{X}$ is the set of terminal outcomes the agent believes to be possible, conditional on E_k, if f is chosen. The evolution of the continuation relation between acts on a decision tree becomes both obvious and natural under a rather minimal consistency condition on the correspondence $2^{\mathcal{X}} \mapsto 2^{\Delta(\mathcal{X})}$ which assigns the sets of lotteries Y_X to the sets of possible terminal outcomes X. Specifically, subsets of sets of possible terminal outcomes ought to correspond to subsets of lotteries.

A. 7 Possibilities Consistency: *$\forall X', X'' \subseteq \mathcal{X}$, $X' \subseteq X''$ implies $Y_{X'} \subseteq Y_{X''}$.*

To illustrate, under condition A.10, the acts $u_1u_2, u_1d_2 \in \mathcal{A}_2$ in the tree of Figure 1 are continuations of $u_1 \in \mathcal{A}_1$ while $u_1u_2u_3, u_1u_2d_3 \in \mathcal{A}_4$ are continuations of $u_1u_2 \in \mathcal{A}_3$. Observe

[29] To be precise, the argument in the proof of Lemma would establish here that $z_g \sim_{h(N)} g$ whereas $z_g \succ_{h(n)} f$. Even though the prize z_g does not give the required contradiction by itself, any $z \in \Delta(X)$ such that $u(z) \in (U_n(f), u(z_g))$ does (due to vertical and horizontal consistency). The existence of this prize z is ensured by the continuity of u.

also that, by the very structure of a decision tree, every act in $\mathcal{A}_{h(n+1)}$ is in fact a continuation of some act in $\mathcal{A}_{n'}$ for $n' \in \mathcal{N}_h : n' \leq n$. Moreover, any act which is common across two immediately-succeeding partitions is a continuation of itself. In Figure 1, for example, this is the case for the act d_1 when $n \geq 2$.

An obvious logical implication of the continuation concept for acts on a decision tree is that the tendency to search deeper into the problem at hand must be connected to how an act compares with (at least some of) its continuations. The following assumption is one that builds into decision-making an innate (weak) preference for further search. It does so by restricting the comparison of ambiguity aversion between the optimal acts of immediately-succeeding partitions in the sense of Ghirardato and Marinacci [?].[30]

B.1 Discussion

$$C_{h(n)}(Y) = \left\{ y \in \Delta(\mathcal{X}) : u(y) \geq U_{h(n)}(Y) \right\}$$

for any $n \in \mathcal{N}_h$ and any $Y \in \mathcal{Y}_{h(n)}$. Of course, this set is non-empty while its very construction allows us to write $U_{h(n)}(Y) = \inf_{y \in C_{h(n)}(Y)} u(y)$.[31] By (??), therefore, the valuation of a general act $f \in \mathcal{F}_{h(n)}$ under the partition $\pi_{h(n)}$ can be written as follows

$$U_{h(n)}(f) = \sum_{E \in \pi_{h(n)}} p_{h(n)}(E) \inf_{y \in C_{h(n)}(f(E))} u(y)$$

The following lemma exploits the structure a finite decision-tree provides in refining events into a finite collection of sub-events as the extent of foresight increases. Specifically, let the foresight of search extend from n to $n+1$ and suppose that the corresponding partitions are given by $\Pi_{h(n)} = \left\{ E_1^n, \ldots, E_{K(n)}^n \right\}$ and $\Pi_{h(n+1)} = \left\{ E_1^{n+1}, \ldots, E_{K(n+1)}^{n+1} \right\}$ with $K(n) \leq K(n+1) \leq |S|$. Observe that each event $E^n \in \Pi_{h(n)}$ gets partitioned under $\Pi_{h(n+1)}$ into a finite collection of sub-events; $\forall j = 1, \ldots, K(n)$ $E_j^n = \cup_{i=1}^{k_j} E_{ji}^{n+1}$ with $E_{ji}^{n+1} \cap E_{ji'}^{n+1} = \varnothing$ for $i \neq i'$ and $\sum_{j=1}^{K(n)} k_j = K(n+1)$. Therefore, using Bayes' rule, for any pair of probability measures $p_{h(n)}$ and $p_{h(n+1)}$ on the partitions $\Pi_{h(n)}$ and $\Pi_{h(n+1)}$, respectively, we can define the marginal probability measure $\left[\lambda \left(E_{ji}^{n+1} | E_j^n \right) \right]_{i=1,\ldots,k_j \; j=1,\ldots,K(n)}$ by

[30] The representation in (??) is of the α−MEU form; hence, the preferences $\succsim_{h(n)}$ on $\mathcal{F}_{h(n)}$ are c-linearly biseparable for all $n \in \mathcal{N}_h$. Given horizon-consistency, moreover, $\succsim_{h(n)}$ and $\succsim_{n'}$ are cardinally symmetric for any $n, n' \in \mathcal{N}_h$. On its own, axiom A.11 implies that the choice of the optimal act under $\succsim_{h(n)}$ is more *uncertainty averse* (see Definition 4 of Ghirardato and Marinacci) than that under $\succsim_{h(n+1)}$. Under also axiom A.9, it implies that the former choice is more *ambiguity averse* than the latter (see their Definition 7).

[31] By the very construction of the index $U_{h(n)}(\cdot)$, we have $\sup_{y \in Y} u(y) \in C_{h(n)}(Y)$.

$\lambda_{h(n+1)}\left(E_{ji}^{n+1}|E_j^n\right)=\frac{p_{h(n+1)}\left(E_{ji}^{n+1}\right)}{p_{h(n)}\left(E_j^n\right)}$ with $\sum_{i=1}^{k_j}\lambda_{h(n+1)}\left(E_{ji}^{n+1}|E_j^n\right)=1$. That is, $\lambda\left(E_{ji}^{n+1}|E_j^n\right)$ is the "move" probability assigned to the branch of the decision-tree that connects the event E_j^n to its sub-event E_{ji}^{n+1}.

Lemma 8 *Let $f^{+1}\in\mathcal{F}_{h(n+1)}$ be a continuation act of $f\in\mathcal{F}_{h(n)}$. The following are equivalent*

(i) $U_{h(n+1)}\left(f^{+1}\right)\geq U_{h(n)}\left(f\right)$

(ii) $\left[\lambda_{h(n+1)}\left(E_k^{+1}|E\right)\right]_{k=1,\dots,K}\times C_{h(n+1)}\left(f^{+1}\left(E_k^{+1}\right)\right)\subseteq C_{h(n)}\left(f\left(E\right)\right)\quad\forall E\in\pi_{h(n)}$

The end step T must include only the true probability measure on the set $\mathcal{X}$. Let this be $\lambda\in\Delta\left(\mathcal{X}\right)$. Thus, $P_T=\{\lambda\}$. Now, $P_{T-1}=\{\lambda_{E_T}\Delta\left(E_T\right)\}$, we are going all this way backwards. Taking the minimum possible expansions, I guess the convex hulls all the way back. This should give you $\{P_t\}_{t=1}^{T}$ which is a nested sequence of probability mixture sets.

C When $\alpha_{h(n)}$ is a constant function

As in this paper, Olszewski [21] considers a finite set of outcomes X and a preference relation $\succsim$ over a collection $\mathcal{X}\subseteq 2^{\Delta(X)}$ containing all singletons (i.e., $\Delta\left(X\right)$ itself) which, restricted to single lotteries $z\in\Delta\left(X\right)$, admits an expected utility representation $u:\Delta\left(X\right)\mapsto\mathbb{R}$. Unlike us, though, he imposes axioms for the relation $\succsim$ to observe on compound sets of lotteries where, for any $Z_1,Z_2\subseteq\Delta\left(X\right)$ and any $\Lambda\subseteq[0,1]$, the compound set $\Lambda Z_1+\left(1-\Lambda\right)Z_2$ is the set of lotteries $\{\lambda z_1+\left(1-\lambda\right)z_2:z_1\in Z_1,z_2\in Z_2,\lambda\in\Lambda\}$. The resulting representation is of the form

$$U\left(Z\right)=\alpha\min_{z\in Z}u\left(z\right)+\left(1-\alpha\right)\max_{z\in Z}u\left(z\right)$$

where $\alpha\in(0,1)$ is unique for all Z.

The necessary and sufficient condition for α to be constant across the compound sets of lotteries is the Set S-independence axiom (Olszewski [21], Axiom 2):[32] $\forall Z_1,Z_2\in\mathcal{X}$ *and* $\forall z\in\Delta\left(X\right)$, $Z_2\succsim Z_1\Rightarrow\lambda Z_1+\left(1-\lambda\right)\{z\}\succsim\lambda Z_2+\left(1-\lambda\right)\{z\}$ $\forall\lambda\in[0,1]$.

To see this, notice first that, under the Set S-Solvability, Set Continuity, and Weak Disjoint Set Betweeness axioms (Axioms 1, 3, and 4(a), respectively, in Olszewski [21]), for Set S-Independence to obtain, it is necessary and sufficient for it to hold across all two-element sets of lotteries. This is due to the fact that the three axioms guarantee that, for any set of lotteries Z, the preference $\succsim$ is characterized completely by the preference for the

[32] Notice that, under completeness for the preference relation $\succsim$, the statement in the text and $Z_2\succ Z_1$ iff $\lambda Z_1+\left(1-\lambda\right)\{z\}\succ\lambda Z_2+\left(1-\lambda\right)\{z\}$ are equivalent.

two-element set consisting of the $\succsim$-worst and the $\succsim$-best lottery in Z. Take, thus, any two sets $Z_1 = \{z_1, z_1'\}$ and $Z_2 = \{z_2, z_2'\}$ where $z_1, z_2, z_1', z_2' \in \Delta(X)$ with $z_1' \succ z_1$ and $z_2' \succ z_2$. For some $\lambda \in [0,1]$, consider also the set

$$Z = \{\lambda z_1 + (1-\lambda) z_2, \lambda z_1 + (1-\lambda) z_2', \lambda z_1' + (1-\lambda) z_2, \lambda z_1' + (1-\lambda) z_2'\}$$

which is actually the compound set $\lambda Z_1 + (1-\lambda) Z_2$. Viewing the family $\mathcal{X}$ as our $\mathcal{X}_{h(n)}$, the relation $\succsim$ as our $\succsim_{h(n)}$, and the sets of lotteries Z_1, Z_2, Z as constant acts, our representation gives

$$\begin{aligned} U_{h(n)}(Z_i) &= \alpha_{h(n)}(Z_i)\, u_{h(n)}(z_i) + \left(1-\alpha_{h(n)}(Z_i)\right) u_{h(n)}(z_i') \quad \text{for } i = 1,2 \\ U_{h(n)}(Z) &= \alpha_{h(n)}(Z)\left[\lambda u_{h(n)}(z_1) + (1-\lambda) u_{h(n)}(z_2)\right] \\ &\quad + \left(1-\alpha_{h(n)}(Z)\right)\left[\lambda u_{h(n)}(z_1') + (1-\lambda) u_{h(n)}(z_2')\right] \end{aligned}$$

It is straightforward to check that Set S-Independence holds iff[33]

$$U_{h(n)}(Z) = \lambda U_{h(n)}(Z_1) + (1-\lambda) U_{h(n)}(Z_2)$$

That is, iff

$$\lambda\left[\alpha_{h(n)}(A) - \alpha_{h(n)}(A_1)\right]\left[u_{h(n)}(z_1) - u_{h(n)}(z_1')\right] = (1-\lambda)\left[\alpha_{h(n)}(A) - \alpha_{h(n)}(A_2)\right]\left[u(z_2') - u(z_2)\right]$$

But this equality holds for any $\lambda \in (0,1)$ and any $u(z_1) - u(z_1'), u(z_2') - u(z_2) \in \mathbb{R}^*$ iff $\alpha_{h(n)}(Z) = \alpha_{h(n)}(Z_i)$ for $i = 1,2$. In other words, iff $\alpha_{h(n)} : \mathcal{X}_{h(n)} \mapsto [0,1]$ is a constant function on the two-element members of $\mathcal{X}_{h(n)}$. To conclude that this is, in turn, equivalent to $\alpha_{h(n)}$ being constant on all of $\mathcal{X}_{h(n)}$, recall that, by Axioms 1, 3, and 4(a), $Z \sim_{h(n)} \{\text{argmin}_{z\in Z} u(z), \text{argmax}_{z\in Z} u(z)\}$ for any $Z \in \mathcal{X}_{h(n)}$.

The set-up in Olszewski [21] admits only constant acts (the state space S is a singleton) and elements of $\mathcal{X}_{h(n)}$ that are finite. Nevertheless, even in our more general set-up, the argument made above applies subject to the modification that we ought to consider any two sets of lotteries $Z_1, Z_2 \in \mathcal{X}_{h(n)}$ (not necessarily two-element ones) and define $z_i = \text{arginf}_{z\in Z_i} u_{h(n)}(z)$

[33] Let $z_1, z_2 \in \Delta(X)$ be such that $Z_1 \sim_{h(n)} \{z_1\}$ and $Z_2 \sim_{h(n)} \{z_2\}$. For any $\mu, \lambda \in [0,1]$ and any $z \in \Delta(X)$, we have $\mu(\lambda Z_1 + (1-\lambda) Z_2) + (1-\mu) z = \lambda(\mu Z_1 + (1-\mu) z) + (1-\lambda)(\mu Z_2 + (1-\mu) z)$. By Set S-Independence, however, $Z_1 \sim_{h(n)} \{z_1\}$ and $Z_2 \sim_{h(n)} \{z_2\}$ iff, respectively, $\mu Z_1 + (1-\mu) z \sim_{h(n)} \mu\{z_1\} + (1-\mu) z$ and $\mu Z_2 + (1-\mu) z \sim_{h(n)} \mu\{z_2\} + (1-\mu) z$. Hence, $\mu(\lambda Z_1 + (1-\lambda) Z_2) + (1-\mu) z \sim_{h(n)} \lambda(\mu\{z_1\} + (1-\mu) z) + (1-\lambda)(\mu\{z_2\} + (1-\mu) z) = \mu(\lambda\{z_1\} + (1-\lambda)\{z_2\}) + (1-\mu) z$. Clearly, $\mu(\lambda Z_1 + (1-\lambda) Z_2) + (1-\mu) z \sim_{h(n)} \mu(\lambda\{z_1\} + (1-\lambda)\{z_2\}) + (1-\mu) z$ holds for any μ and any z iff $\lambda Z_1 + (1-\lambda) Z_2 \sim_{h(n)} \lambda\{z_1\} + (1-\lambda)\{z_2\}$.

and $z_i' = \text{argsup}_{z \in Z_i} u_{h(n)}(z)$ for $i = 1, 2$.[34] Of course, with Z_1 and Z_2 being general sets of lotteries, Set S-Independence amounts to strengthening our Crisp Independence (A.2) axiom to[35]

A.2* Constant Independence: *$\forall f, h \in \mathcal{F}^{cr}_{h(n)} \cup \mathcal{F}^{c}_{h(n)}$, $\forall r \in \mathcal{F}^{cr}_{h(n)}$, and $\forall \alpha \in (0,1)$ $f \succ_{h(n)} h$ iff $\alpha f + (1-\alpha) r \succ_{h(n)} \alpha h + (1-\alpha) r$*

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[34] Olszewski's collection $\mathcal{X}$ includes only closed sets of lotteries, an assumption which ensures that z_i exists (and is a member of Z_i; hence, the use of $\min - \max$ instead of $\inf - \sup$ in the representation). As we do not impose closeness on our sets of lotteries (in fact, we do not impose any topology on the collection $\mathcal{X}_{h(n)}$), we require Boundedness in order to guarantee existence.

[35] Recall that $\Delta(X) = \mathcal{F}^{cr}_{h(n)} \cap \mathcal{F}^{c}_{h(n)} \subset \mathcal{F}^{cr}_{h(n)}$.

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