Asking Questions*

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Abstract

We examine a model with limited communication in which the seller is selling a single good to two potential buyers. Limited communication is modeled as follows: in each of the finite number of periods the seller asks one of the two buyers a binary question. After the final answer the allocation and the transfers are executed. The model sheds light on the communication protocols that arise in the welfare maximizing mechanisms.

Among other things, we show that when the total number of questions is bounded the welfare optimal mechanism requires the seller to start with questioning one of the buyers and proceed by a single last question to the other buyer.

1 Introduction

"*[T]he literature on incentive compatibility is now quite extensive. However, with only a few exceptions, it is assumed that agents can transmit messages that are sufficiently detailed to describe fully all their private information.*" Green and Laffont (1987)

The comment applies just as well as it did more than two decades ago. Bulk of the literature assumes no restrictions on communication, enabling it to apply the revelation principle and reduce communication to a simple one shot procedure. Observed communication is rarely costless, almost never instantaneous. It tends to proceed through a sequence of exchanges which take both time and effort. Rarely is the private information revealed completely, be it because agents do not want

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to reveal it, or because they have to carefully choose what they will convey in the limited time they have to disposal. Milgrom (2009) reports several examples where reporting buyers's private information would be too complicated. For example in FCC auctions with many licences.

Our model explores effects of limited communication in a setup where a seller is selling a single indivisible good to privately informed buyers. More precisely, we seek for an (ex ante) welfare maximizing mechanism in which the seller sells an object to one of the two buyers whose valuations are independently distributed over $[0, 1]$. Unlike in the most of mechanism design literature where buyers can fully communicate private information, we assume that communication proceeds through a sequence of binary questions the seller can ask. Binary questions are interpreted as: "Does your valuation belong to a set C?", where C is a Borel measurable set in $[0, 1]$. We call an uninterrupted sequence of questioning of one buyer a round.[1]

We first consider a framework in which the seller commits to asking buyer i at most k_i question. In each period the seller asks one buyer a question, possibly depending on the questions and answers in the past. However, before the mechanism starts the seller commits to who will be questioned in which period and what questions will be asked. After the last answer the allocation rule and the transfers are executed. We start the analysis by assuming that all the buyers report truthfully. This enables us to obtain an upper bound on the welfare to be achieved under limited communication. We show that under truthful reporting, when each buyer has at least two questions, a welfare optimal mechanism entails three rounds. In the first round one of the two buyers is asked all questions assigned to him, but one. In the second round the other buyer is asked all of the questions intended for him, and finally in the last period the first buyer is asked one last question.

Next we consider the setup in which the seller only commits to a total number of questions. An interpretation of such a setup is that questions and answers take time, moreover, the same amount of time regardless of who the seller is talking to. In our main result we show that in any welfare optimal equilibrium communication proceeds through two rounds. In the first round all but one question are used on one of the buyers, in the second round the remaining buyer is asked one last question. We characterize questions asked in welfare optimal equilibria and show how one provides incentives for buyers to report truthfully. In particular, the optimal mechanism requires the seller to elicit information from one buyer for a longer period of time and to make a take it or leave it

[1] For example, if buyer 1 is asked a question in periods 1 and 2, and buyer 2 in periods 3 to 5 the protocol has two rounds.

offer depending on the previously obtained information to the other buyer in the last period. In the case the offer is refused the buyer questioned first receives the object.

We do not lay claim to have modeled any particular occurrence of communication in economic activity, nor do we claim that optimal communication will always have the properties shown to hold in our framework. Our objective was to explore the effects of sequential limited communication in a well understood model and to contrast them with the results obtained when no limits on communication are imposed.

We would like to point out that although we model the bound on communication as exogenously given, the model can be looked at from a somewhat broader perspective. One could specify a more general problem where the buyers' or the seller's utility include some cost as a function of the number of questions for each buyer. Consequently, the welfare function would include these costs. The methodology we developed enables one to compute such a welfare for every possible combination of the number of questions for the two buyers. But then after comparing the welfare over all the combinations one arrives at the optimal number of questions for each buyer; provided the solution exists.

Finally, it should be noted that our approach extends to the problem of revenue maximization. Blumrosen, Nisan and Segal (2007) have shown how a problem of revenue maximization under simultaneous limited communication can be converted into a problem equivalent to welfare maximization with simultaneous limited communication with fictional buyers whose valuations are virtual valuations of the buyers in the original problem. This result, as extended in Kos (2009), allows one to solve the problem of revenue maximization when dynamic communication is allowed for by applying the methodology developed in the present paper.

1.1 Related Literature

The importance of limited communication in trading environments was recognized by Green and Laffont (1987). They consider a model with an agent and a central resource-allocation unit. The decision of a central unit is a vector in $\mathbb{R}^n$, parameters relevant to both players' objectives are a vector in $\mathbb{R}^m$ and the message the sender can send is a vector in $\mathbb{R}^l$. Limited communication is modeled by assuming that $l < \min\{m, n\}$. The problem with modeling limited communication this way is that $\mathbb{R}^m$ can be bijectively encoded into $\mathbb{R}^l$ and the latter can be decoded into $\mathbb{R}^n$. To prevent such coding and decoding, and to get limited communication, authors had to assume that the central resource-allocation unit may only use differentiable mechanisms. We, on the other hand, propose that limited communication be modeled by having message spaces of smaller cardinality than the spaces of private information. That way limited communication arises without a need to impose extraneous assumptions on the allocation rules or transfers.

Closest papers to ours are Blumrosen, Nisan and Segal (2007) and Kos (2009). They study auction environment in which buyers can only report one of the finite number of messages although their private information is in a compact interval in $\mathbb{R}$. Most of their analysis is concerned with simultaneous communication. Blumrosen, Nisan and Segal (2007), however, provide an example showing that with the same amount of communication one can achieve a higher level of welfare by using sequential communication.

Lately limited (costly) communication started attracting more attention in contract theory, for example Battigalli and Maggi (2002) and Mookherjee and Tsumagari (2007). For more references see the latter paper.

Last, we point out the link between our paper and the revelation principle; see Myerson (1979) and for a more general version Myerson (1986). In the light of our paper the revelation principle can be seen as stating that when buyers are able to fully communicate their private information in several periods one looses nothing by letting them simultaneously communicate in one period. The amount of information one is able to convey does not change. When communication resources are limited, however, it is of great benefit to make communication sequential. Sequentiality enables one to convey the information most relevant given what has been previously disclosed.

2 Framework

A seller is selling a single good to two buyers, $I = \{1, 2\}$, who have independently distributed valuations over $[0, 1]$. The corresponding distribution functions F_i are assumed to have positive density on all of $[0, 1]$. Each buyer maximizes quasilinear utility function of the form

$$q_i v_i - m_i,$$

where v_i is the privately known valuation, q_i the probability of obtaining a good and m_i the expected transfer the buyer pays. While the preceding assumptions are well established and populate most of the output in mechanism design, and particularly auctions, the following is a stark departure.

Communication proceeds through a sequence of binary questions the seller can ask. The seller commits to asking each buyer at most k_i questions, let $K = (k_1, k_2)$, specifies the sequence in which the buyers are questioned, the actual questions, the allocations and the transfers.

Formally, the horizon is finite with time indexed by $t \in T = \{1, ..., k_1 + k_2\}$. The mechanism specifies who is asked a question in each period[2]:

$$\iota : T \to \{1, 2\},$$

with $\left|\iota^{-1}(i)\right| = k_i$, and the question the buyer is asked given the history:

$$\eta_t : H^{t-1} \to B,$$

where B is the Borel sigma algebra on $[0, 1]$. After each history the specified buyer is asked a question of a type: "Does your valuation belong to a set C?", where C is a Borel set in $[0, 1]$. H^{t-1} is the set of histories at the beginning of period t, with a generic element $h^{t-1} = (h_1, h_2, ..., h_{t-1})$ and the convention $H^0 = \emptyset$. History here is a sequence of answers, thus with our notation $h_t \in \{0, 1\}$. When the final history $h^{k_1+k_2}$ is realized the allocation rule Q and the transfer rule M are executed:

$$\begin{aligned} Q &: H^{k_1+k_2} \to [0, 1]^3, \\ M &: H^{k_1+k_2} \to \mathbb{R}_+^2, \end{aligned}$$

where Q_0 is interpreted as the probability that the seller keeps the object. A mechanism in our setup is a tuple (K, ι, η, Q, M). Often we will be interested in a set of mechanism with a fixed set of parameters; $\{(K), (K, \iota), (K, \iota, \eta)\}$ covers those of greater importance. For example, we denote the set of all mechanisms for which communication is fully determined and given by K, ι and η by $G_{K,\iota,\eta}$. These mechanisms differ only in allocation and transfer rules. We use letter g to denote a particular mechanism.

[2] We assume that a single person is asked a question in each period. It is easy to see that our characterizations would remain unaltered if we allowed for simultaneous questioning.

Buyer i's pure strategy[3] $\left(\mu_t^i\right)_{t\in\iota^{-1}(T)}$:

$$\mu_t^i : H^{t-1} \times [0,1] \to \{0,1\},$$

is to answer 1 or 0, yes or no respectively, when he is asked a question, i.e. for t such that $\iota(t) = i$, given his valuation and the history. History h^t describes all the answers up to including the period $t-1$. We are assuming that all the communication is observed by all the participants in the mechanism. This is purely for expositional purposes.

Throughout most of the paper the objective will be maximization of ex ante welfare

$$E\left[\sum_i Q_i v_i\right];$$

i.e. the mechanism should award the object to a buyer with the highest value.

The equilibrium concept we apply is perfect Bayesian equilibrium. The analysis will proceed somewhat unconventionally, though. At first we will neglect any kind of incentives on the side of buyers; i.e. buyers will be assumed to report truthfully whenever called upon:

$$\mu_t^i\left(h^{t-1}, v_i\right) = \mathbf{1}_{\left[v_i \in \eta_t\left(h^{t-1}\right)\right]},$$

for t such that $\iota(t) = i$. We call a mechanism that achieves the highest welfare under truthful reporting **informationally optimal**.[4] In the second stage of analysis we will show how such a mechanism can be incentivized. The welfare achieved in mechanism g, when bidders always report truthfully, is denoted

$$w^{io}(g).$$

Furthermore, given the set of parameters $P \in \{(K), (K,\iota), (K,\iota,\eta), (K,\iota,Q)\}$ we denote the highest welfare achieved in a mechanism with these parameters under truthful reporting by

$$w_P^{io*} = \sup_{g\in G_P} w^{io}(g).$$

[3] In the Discussion section we argue that considering only pure strategies is without loss of generality for our results.

[4] The term was coined by Blumrosen and Feldman (2007).

2.1 Alternative Interpretation

We make an explicit assumption of questions being asked as a part of a mechanism. Instead of talking about abstract message spaces spanning through several periods we have the seller facilitating communication through directed questioning. This is, in a sense, without loss of generality since any equilibrium of the model with questions can be embedded as a Bayesian equilibrium in the model without questions (and vice versa) where bidders choose one of the two messages when they are called upon to report.

Results from *the mechanisms with limited simultaneous communication* will be of great benefit to us here. In such mechanisms buyers report simultaneously, each buyer choosing one of the finite number of messages in his message space, although the valuation space is the interval $[0, 1]$. Upon reports the allocation and the transfers are executed. For a broader exposition see Kos (2009). We denote optimal welfare achieved in such a mechanism by

$$w^{s*}_{k_1,k_2},$$

where k_1 and k_2 stand for cardinalities of buyer 1 and 2's message space, respectively. As a reminder we state a result from Kos (2009) showing that in a simultaneous reporting mechanism with limited communication and 2 buyers welfare cannot be increased by increasing the cardinality of the buyer with the higher cardinality. Reader should consult Kos (2009) for the proof and intuition.

Lemma 1 *Let $k_1 \geq k_2$ and $k = \min\{k_1, k_2 + 1\}$, then*

$$w^{s*}_{k_1,k_2} = w^{s*}_{k,k_2}.$$

The above Lemma shows that the lower of the two cardinalities is crucial for welfare when it comes to limited communication with simultaneous reporting. For example, suppose $K = (7, 3)$, so that buyer 1 can choose among 7 messages and buyer 2 among 3. Lemma states $w^{s*}_{7,3} = w^{s*}_{4,3}$. That is, the highest welfare that can be achieved in a mechanism with simultaneous communication with bounds on communication given by K is equal to the highest welfare that can be achieved in a simultaneous communication mechanism with bounds given by $(4, 3)$.

On the other hand, if the cardinality of the message space of the buyer with the smaller cardinality is increased or if cardinalities of both bidders are raised, strictly higher welfare can be achieved. Again, the reader is advised to see Kos (2009) for the proof.

Lemma 2 *Let* $K = (k_1, k_2)$, $K' = (k'_1, k'_2)$, $k = \min\{k_1, k_2\}$ *and* $k' = \min\{k'_1, k'_2\}$. *If* $k > k'$ *then*

$$w^{s*}_{k_1,k_2} > w^{s*}_{k'_1,k'_2}.$$

The mechanism with the higher lowest cardinality achieves the higher welfare. For example, if $K = (6, 3)$ and $K' = (2, 7)$ then $w^{s*}_{6,3} > w^{s*}_{2,7}$. The first step towards this observation are the equalities $w^{s*}_{6,3} = w^{s*}_{4,3}$ and $w^{s*}_{2,7} = w^{s*}_{2,3}$. Somewhat stronger result than the one stated in Lemma 2 holds. Namely the following example is not covered by the Lemma. Suppose $K = (6, 3)$ and $K' = (3, 3)$. One can still show that $w^{s*}_{6,3} = w^{s*}_{4,3} > w^{s*}_{3,3}$. Yet differently, as soon as cardinality of the messages space of a buyer with the lowest cardinality is increased a higher welfare can be achieved.

3 Welfare Maximization

First we show that a mechanism in $G_{K,\iota,\eta}$ that uses an allocation rule awarding the object to a buyer with the highest expected value given the questions and the answers is informationally optimal in $G_{K,\iota,\eta}$.

Lemma 3 *Let* $g^* \in G_{K,\iota,\eta}$ *be a mechanism that allocates the object to a buyer with the highest expected value given the questions and the answers. Then* g^* *achieves the welfare* $w^{io*}_{K,\iota,\eta}$.

We provide a simple example to demonstrate working of the above Lemma.

Example 1 *Suppose buyers* 1 *and* 2 *have valuations distributed according to the uniform distribution on the interval* $[0, 1]$. *Furthermore, suppose there are only two periods. In the first period buyer* 1 *is asked whether his valuation is at least* 0.5. *If he answeres with yes, buyer* 2 *is asked whether his valuation is at least* 0.75, *otherwise whether his valuation is at least* 0.25. *With our notation this means* $K = (1, 1)$, $\iota = (1, 2)$ *and* $\eta = (0.5, 0.75, 0.25)$. *Clearly among all the mechanisms with these parameters any welfare maximizing mechanims allocates the object to buyer* 2 *if both of them answer positively, to buyer* 1 *if both of them answered negatively and so forth.*

We omit a formal proof. By the definition of $G_{K,\iota,\eta}$ all the mechanisms in it have the same sequence of questioning and the same questions. Remember, in each period one of the buyers is asked whether his valuation belongs to a certain set. At the end of questioning the seller knows each buyer's valuation is in the intersection of the sets the buyer claimed his valuation is in.[5] On the technical side, the intersection will be nonempty since the buyers are assumed to report truthfully. To maximize welfare one is merely left to compute the expected value corresponding to the deduced set for each of the buyers and awarding the object to a buyer with the highest expected value. Since buyers are assumed to report truthfully the design of mechanism has no effect on incentives, but solely on welfare. Ties can therefore be broken arbitrarily. Finally, one can prove a somewhat stronger lemma, stating that a mechanism achieves the welfare $w^{io*}_{K,\iota,\eta}$ if and only if it allocates the object to a bidder with the highest valuation with ex ante probability 1. Potentially, an ex ante welfare maximizing mechanism could award the object to the buyer with the lower valuation, as long as that event has probability zero.

We define *threshold questions* to be questions of the type $A_i = [a_i, 1]$ for $a_i \in [0, 1]$; or $A_i = [0, a_i)$. Threshold strategies, natural analog to threshold questions, are the the crux of the analysis in Blumrosen, Nisan and Segal (2007) and Kos (2009). The origins of the following lemma can be traced to Blumrosen, Nisan and Segal (2007); see the Theorem 6.1 on the page 260 and the discussion preceding it.

Lemma 4 *There exists a mechanism with threshold questions in $G_{K,\iota,Q}$ that achieves $w^{io*}_{K,\iota,Q}$, and a mechanism with threshold questions in $G_{K,\iota}$ that achieves $w^{io*}_{K,\iota}$.*

Proof. *Proof of this and the subsequent results can be found in the Appendix.* ■

In the first part of Lemma 4 we fix the number of questions, who is questioned in each period, and the allocation rule, and show that considering only threshold questions is without loss of generality. This should be rather intuitive since the seller's objective is to award the object to a bidder with the highest valuation. Second part shows the same for a more general class of mechanisms.

We are still left to see what is the optimal way to sequence questions. That is, we are left to determine the optimal ι given the fixed vector K. Analysis from the simultaneous communication mechanisms will be of great value here. Before we proceed to the formal analysis we present a simple example.

[5] Suppose buyer 1 is asked in the first period whether his valuation belongs to the interval $[0.5, 0.6)$ and in the third whether it belongs to the interval $[0.57, 0.62)$. Furthermore, suppose he answered affirmatively in both periods. Then it is clear that his valuation is in the interval $[0.57, 0.6)$.

Example 2 *Let, $k_1 = k_2 = 2$ and $\iota = (1, 2, 2, 1)$. In the first period buyer 1 is asked a question which can be described by a threshold. For each of the two answers of buyer 1 there is a threshold (question) for buyer 2 in period two. By the end of the second period there are four possible histories and for each of those there is another threshold (question) for buyer 2 in period three. Therefore there are altogether 6 thresholds (corresponding to threshold questions) for buyer 2 (two in the second period and four in the third). Finally, at the beginning of the third period there are 8 possible histories, thus eight thresholds, which together with the first period threshold yields 9 thresholds for buyer 1. The above analysis implies that whatever expected welfare can be achieved in our sequential mechanism can also be achieved in a mechanism with simultaneous communication in which buyer 1 uses 9 thresholds and buyer 2 uses 6. That is, in a simultaneous communication mechanism in which cardinalities of the message spaces of buyer 1 and 2 are 10 and 7, respectively. Thus we have $w^{io*}_{(2,2),\iota} \leq w^{s*}_{(10,7)}$. By Lemma 1 $w^{s*}_{(10,7)} = w^{s*}_{(8,7)}$, hence $w^{io*}_{(2,2),\iota} \leq w^{s*}_{(8,7)}$. There exists a profile of thresholds $c^1 = \left(c^1_1, c^1_2, ..., c^1_7\right)$, $c^2 = \left(c^2_1, c^2_2, ..., c^2_6\right)$ for buyers 1 and 2 respectively that together with an appropriate allocation rule and the transfers achieves the welfare $w^{s*}_{(8,7)}$.* [6, 7] *These thresholds can be naturally embedded into the dynamic setup. Let*

$$\begin{aligned}
\eta_1(\emptyset) &= \left[c^1_4, 1\right], \\
\eta_2(1) &= \left[c^2_2, 1\right],\ \eta_2(0) = \left[c^2_6, 1\right], \\
\eta_3(1,1) &= \left[c^2_1, 1\right],\ \eta_3(1,0) = \left[c^2_3, 1\right],\ \eta_3(0,1) = \left[c^2_5, 1\right],\ \eta_3(0,0) = \left[c^2_7, 1\right], \\
\eta_4(1,1,1) &= \left[c^1_1, 1\right], \eta_4(1,1,0) = \left[c^1_2, 1\right], \eta_4(1,0,1) = \left[c^1_3, 1\right], \eta_4(1,0,0) = \left[c^1_4, 1\right], \\
\eta_4(0,1,1) &= \left[c^1_5, 1\right], \eta_4(0,1,0) = \left[c^1_6, 1\right], \eta_4(0,0,1) = \left[c^1_7, 1\right], \eta_4(0,0,0) = \left[c^1_8, 1\right].
\end{aligned}$$

Now notice that $(\eta_t)_t$ achieves the expected welfare $w^{s*}_{(8,7)}$; therefore $w^{io*}_{(2,2),\iota} = w^{s*}_{(8,7)}$.

We call an uninterrupted sequence of questioning of one buyer a *round*. For example, if g is some mechanism with $\iota = (1, 1, 2, 2, 1, 1)$, then the mechanism has three rounds, in each of which a buyer is asked two questions. The following lemma shows it is inefficient to use more than one question in the final round (if there are at least three rounds).

Lemma 5 *Let g be a mechanism in G_K with at least three rounds, and more than one question in*

[6] For details see Kos (2009).

[7] We addopt the convention $c^i_j \geq c^i_{j+1}$ for each i. c^i_1 is the highest threshold of buyer i, c^i_2 the second highest, etc.

the last round. Then there exists an alternative mechanism $g' \in G_K$, with one question in the last round, such that $w^{io}(g) \leq w^{io}(g')$.

After the penultimate round the seller has all information he will get about the buyer who was questioned in that round (penultimate buyer). At that point the seller could compute the set the penultimate buyer's valuation belongs to, and the expected value corresponding to it. Since the objective is welfare maximization, all that seller cares about from there on is whether the other buyer's valuation is above or below that expected value. A matter that can be settled by a single question. By redistributing the remaining questions from the last round to earlier rounds one can achieve higher welfare.[8]

Next we show that it is optimal to have three rounds; every additional round decreases welfare. The optimal sequence of questioning ι^* is either of the type $(1, 1, ..., 1, 2, 2, ..., 2, 1)$ or with the roles of buyers 1 and 2 reversed. Furthermore, it is always optimal to start questioning the buyer who is assigned the smaller number of questions.

Theorem 1 *An informationally optimal mechanism in G_K exists and entails at most three rounds. If $k_i > k_{-i} \geq 2$, for some $i \in \{1, 2\}$, then it is informationally optimal to first ask buyer $-i$, $k_{-i} - 1$ questions, then ask buyer i k_i questions and finally ask buyer $-i$ one last question. If* $\min\{k_1, k_2\} = 1$, *then informationally optimal mechanism has two rounds, buyer with* $\min\{k_1, k_2\}$ *being questioned in the second round.*

The later the question is asked the more preceding histories it has. Since for each history a question creates a threshold, more preceding histories translates into more thresholds. More thresholds, in turn, into a better idea of the buyer's valuation, thus possibility of achieving a higher welfare. Consequently one would like to ask questions assigned to the penultimate buyer as late as possible, meaning in the penultimate round. The penultimate buyer will therefore optimally only be questioned in the penultimate round. Since Lemma 1 implies the other buyer is asked only one question in the last round his remaining questions have to be asked in the first round. Delivering three rounds altogether. Turning to the case where each buyer is asked at least two questions and $k_1 \neq k_2$, we show that the buyer with the larger number of questions should be questioned in the second round. This way the number of thresholds for the penultimate buyer is maximized. Finally,

[8] This is formally shown in the following Theorem.

reducing the number of rounds to two would be inefficient. As we showed in Lemma 5 all the relevant information for welfare maximization in last round can be obtained by one question. The other questions, would therefore be better utilized if asked at the beginning.

We present two examples to clarify the rather abstract analysis above. First one presents the welfare maximizing equilibrium of a mechanism with limited communication and simultaneous reporting. The second shows how strategies from a mechanism with simultaneous communication can be embedded into the framework with questions.

Example 3 *Two buyers have valuations independently uniformly distributed over the interval* $[0,1]$*. Buyer 1's cardinality of the message space* k_1 *is 8 and buyer 2's,* k_2*, is 7. After buyers observe their private valuations they simultaneously report one of the messages in their message space after which the allocation and the transfers are executed. For the details of the mechanism and the analysis see Kos (2009). Optimal reporting strategies are threshold strategies, indeed they are mutually centered. Threshold strategy of the buyer with cardinality of the message space* k *can be described by* $k-1$ *thresholds. Buyer 1's threshold strategy is denoted* $c^1 = \left(c_1^1, .., c_{k_1-1}^1\right)$ *with* $c_1^1 \geq c_2^1 \geq ... \geq c_{k_1-1}^1$*, and buyer 2's* $c^2 = \left(c_1^2, c_2^2, ..., c_{k_2-1}^2\right)$ *with* $c_1^2 \geq ... \geq c_{k_2-1}^2$*. In the welfare optimal equilibrium the inequalities between thresholds are strict. To be more precise, thresholds are mutually centered:*

$$\begin{aligned} c_j^1 &= E\left[V \mid c_{j-1}^2 \geq V \geq c_j^2\right], \ j \in \{1,2,...,k_1-1\}, \\ c_j^2 &= E\left[V \mid c_j^1 \geq V \geq c_{j+1}^1\right], \ j \in \{1,2,...,k_2-1\}, \end{aligned}$$

with the convention $c_j^i = 1$ *for* $j \leq 0$ *and* $c_j^i = 0$ *for* $j \geq k_i$ *and* $i = 1,2$*. For our case of uniform distribution and* $k_1 = k_2 + 1 = 8$ *the above system of equations yields a unique solution:*

$$\begin{aligned} c_k^1 &= 1 - \frac{2k-1}{14}, \ k = 1,...,7, \\ c_k^2 &= 1 - \frac{2k}{14}, \ k = 1,...,6. \end{aligned}$$

With simultaneous communication the above threshold strategies coupled with the allocation rule that awards the object to a buyer with the highest expected value given the strategies and the reports yields the informationally optimal mechanism. Even more, if one adds the Vickrey type transfers in which the winning buyer pays the smallest valuation he could have, report according to his threshold strategy, and still win, one obtains a welfare maximizing incentive compatible equilibrium.

Next we show how to deal with the sequential binary questions.

Example 4 *Revisiting Example 1, assume two buyers have valuations distributed independently and uniformly over* $[0,1]$ *and* $k_1 = k_2 = 2$. *An informationally optimal mechanism entails* $\iota = (1,2,2,1)$ *and achieves the welfare of* $w^{s*}_{8,7}$. *This level of welfare, while assuming that buyers report truthfully, can be achieved by embedding the thresholds that achieve the highest welfare in the simultaneous communication mechanism with* $k_1 = k_2 + 1 = 8$*:*

$$
\begin{aligned}
\eta_1 &= [0.5,1] \\
\eta_2(1) &= \left[\frac{10}{14},1\right], \eta_2(0) = \left[\frac{4}{14},1\right] \\
\eta_3(1,1) &= \left[\frac{12}{14},1\right], \eta_3(1,0) = \left[\frac{8}{14},1\right], \eta_3(0,1) = \left[\frac{6}{14},1\right], \eta_3(0,0) = \left[\frac{2}{14},1\right] \\
\eta_4(1,1,1) &= \left[\frac{13}{14},1\right], \eta_4(1,1,0) = \left[\frac{11}{14},1\right], \eta_4(1,0,1) = \left[\frac{9}{14},1\right], \eta(1,0,0) = [0.5,1] \\
\eta_4(0,1,1) &= [0.5,1], \eta_4(0,1,0) = \left[\frac{5}{14},1\right], \eta_4(0,0,1) = \left[\frac{3}{14},1\right], \eta_4(0,0,0) = \left[\frac{1}{14},1\right].
\end{aligned}
$$

3.1 Distributing questions among buyers

So far we were concerned with the question of how to organize communication in order to maximize welfare when the seller commits to a certain number of questions for each buyer under the assumption that buyers report truthfully. In some instances the real constraint for the seller, however, will be the total number of questions asked. Say if the seller is time constrained and questioning each buyer is equally time costly. Therefore, the natural next step is to ask what can be done when the seller is only restricted by the total number of questions. In particular, what are the welfare optimal mechanisms when the restriction on k_1 and k_2 is $k_1 + k_2 \leq k^*$, for $k^* \geq 2$, and $k_1, k_2 \geq 1$.[9]

Let G_{k^*} be the set of all mechanisms with $k_1 + k_2 \leq k^*$, and let $w^{io*}_{k^*}$ be the lowest upper bound on welfare achieved by mechanisms in G_{k^*} under truthful reporting, i.e. $w^{io*}_{k^*} = \sup_{g \in G_{k^*}} w^{io}(g)$. We call a mechanism g^* informationally optimal in G_{k^*} if it achieves welfare $w^{io*}_{k^*}$ under truthful reporting. That is, if $w^{io}(g^*) = w^{io*}_{k^*}$.

[9] The last assumption is for convenience. It is easy to see that it cannot be welfare maximizing to question only one buyer. Indeed, under our assumptions, it would be welfare optimal to award the object to this buyer without any questions. But then the welfare could be raised by adding an additional buyer and asking him whether his valuation is larger than the first buyer's expected valuation.

Theorem 2 *Let $k^* \geq 2$, $k^* \in \mathbb{N}$. An informationally optimal mechanism in G_{k^*} exists. Moreover, for any such informationally optimal mechanism g^* in G_{k^*} there exists an $i \in \{1, 2\}$ such that $k_i = k^* - 1$, $k_{-i} = 1$ and $\iota = (i, ..., i, -i)$.*

The above theorem provides existence of an informationally optimal mechanism in G_{k^*} and, what is more interesting, a characterization of such mechanisms. In particular, in any informationally optimal mechanism there are only two rounds. In the first round one of the buyers is asked all the questions but one and in the last period the other buyer is asked one last question. Using more than two rounds is wasteful, because the more thresholds the penultimate buyer has the higher welfare can be achieved. The highest number of thresholds he could possibly have is achieved by allotting him all the questions but the last one.

One of our underlying assumptions is that buyers observe all the previous questions and answers. This assumption is not necessary for the mechanism constructed in the proof of Theorem 2. Namely, in each period and at each history, all the relevant information for the buyer can be deduced from the question. Clearly, when we assume truthful reporting, observing what the other buyer reports is redundant. The same, however, applies under the equilibrium analysis. The questions reflect all the relevant information revealed in the past.

Thus far the analysis was conducted under the assumption that both buyers report truthfully in all periods. This yields an upper bound on what could possibly be achieved under limited communication. While the characterization of informationally optimal mechanisms is interesting in itself, one might worry how an optimal mechanism would look like if buyers' incentives came into play. We show that by choosing transfers appropriately one gives buyers an incentive to indeed report truthfully in an informationally optimal mechanism. Since informationally optimal mechanism with truthful reporting constitutes an upper bound on the welfare to be achieved with any combination of strategies and a mechanism, we obtain a welfare maximizing equilibrium of a mechanism with limited communication. Before proceeding to the formal result a simple example is presented.

Example 5 *Suppose two buyers have valuations distributed with the uniform distribution over the interval $[0, 1]$. Let $k_1 = 2$, $k_2 = 1$ and $\iota = (1, 1, 2)$; or in words, buyer 1 is asked a question in the first and second period and buyer 2 in the third. In what follows we will present a mechanism that achieves the informationally optimal welfare in G_K, $K = (2, 1)$, and has truthful reporting as* *a Perfect Bayesian Equilibrium. As in the previous examples one can show that* $w^{io*}_{K,\iota} \leq w^{s*}_{4,5}$*. Next we present a mechanism that achieves the welfare* $w^{s*}_{4,5}$*. First we construct the sequential questions by using the questions from the simultaneous reporting mechanism achieving* $w^{s*}_{4,5}$*:*

$$\begin{aligned} \eta_1 &= [0.5, 1], \\ \eta_2(1) &= [0.75, 1], \eta_2(0) = [0.25, 1], \\ \eta_3(1,1) &= [0.875, 1], \eta_3(1,0) = [0.625, 1], \eta_3(0,1) = [0.375, 1], \eta_3(0,0) = [0.125, 1]. \end{aligned}$$

We define the allocation rule by $Q^*_2(h_1, h_2, h_3) = 1$ *if* $h_3 = 1$*,* 0 *otherwise, and* $Q^*_1 = 1 - Q^*_2$*. Notice that such an allocation rule awards the object to the buyer with the higher expected valuation computed from the answers. Suppose the history is* $(1, 0, 1)$*; buyer* 1 *answered yes to the first question and no to the second one. Assuming that he was not laying his valuation is in the interval* $[0.5, 0.75)$*, and the expected value of his valuation given the available data is* 0.625*. Since buyer* 2 *claims that his valuation is at least* 0.625 *the object should optimally be allocated to him.*

We are left to specify the transfers which will make truthful reporting incentive compatible. Buyer 2*, provided that he wins the auction, pays the amount equal to the smallest valuation he could have had, reported truthfully, and won the auction given that buyer* 1 *reported truthfully. With other words, buyer* 2 *pays the value equal to the lower bound of his interval question in the case he won the auction. In the case of history* $(1, 0, 1)$ *he would pay* 0.625*.*

The case of buyer 1 *is somewhat trickier. Buyer* 1 *is to pay the expected value of the smallest valuation he could have had, report truthfully, and win the auction, given truthful reporting of buyer* 2*. That is, of course, when he wins the object. In the case he loses he does not pay anything. Instead of specifying the transfer rule for buyer* 1 *formally we will demonstrate the transfer for a particular history. Suppose the history was* $(1, 0, 0)$*, then* $M_1(1, 0, 0) = 0.4 * 0.5 + 0.4 * 0.25 + 0.2 * 0 = 0.3$*. From the questions and the answers we can deduce that buyer* 1 *wins the object, his valuation is in the interval* $[0.5, 0.75)$ *and buyer* 2*'s valuation is in* $[0, 0.625)$*. Given the information we have, buyer* 2*'s valuation is in* $[0.375, 0.625)$ *with probability* 0.4*. For these values of buyer* 2*, buyer* 1 *needs to have valuation of at least* 0.5 *to answer truthfully and win. This is the first part of the above sum. With probability* 0.4 *buyer* 2*'s valuation is in* $[0.125, 0.375)$ *in which case buyer* 1 *needs to be of at least type* 0.25 *to report truthfully and win. Finally, in the case buyer* 2*'s valuation is in* $[0, 0.125)$ *buyer* 1 *can be even type* 0 *and win.*

Theorem 3 *Let* $k^* \in \mathbb{N}$ *and* $k^* \geq 2$. *There exists a mechanism* $g^* \in G_{k^*}$, *with,* $k_i = k^* - 1, k_{-i} = 1$, $\iota = (i, ..., i, -i)$, *for some* $i \in \{1, 2\}$, *such that truthful reporting is a Perfect Bayesian Equilibrium of this mechanism that achieves informationally optimal welfare in* G_{k^*}.

The above theorem states that maximal welfare can be achieved either by first asking buyer 1 all but one question and then buyer 2 one last question or with the roles of the two buyers interchanged. Who should be questioned first will depend on the specifics of buyers' distributions of valuations. Indeed, the proof would allow for a somewhat stronger statement. Any equilibrium of any mechanism in G_{k^*} that does not have the above form will achieve strictly lower welfare. Furthermore, in the proof we construct the transfers for the mechanism so that truthful answering is incentivized. The buyer who is asked a question in the last period, in the case he wins pays the smallest valuation he could have had, answered truthfully, and still won the object (i.e., he pays the value equal to the threshold). The buyer who is questioned in the first round, and wins, pays the expected value of the smallest amount he could have had, answered truthfully, and won the auction. For details see the Example 5 and the proof of Theorem 3. Finally, we would like to point out that the equilibrium constructed in the proof of Theorem 3 is ex post individually rational.

The message of Theorem 3 is clear. When communication is costly it is optimal for the seller to spend time to learn about one buyer's private information as much as he can. Only at the end he should ask the other buyer how his private information compares to the private information of the buyer questioned first. The transfers we construct show that this can be done by seller making a take it or leave it offer to the latter buyer.

4 Discussion

We introduce a model of limited communication. Several assumptions are made that warrant an explanation. Assumption of questions being asked was commented on in the text above. One could dispense with questions altogether and assume that buyers report one of two messages in each period.

We assume that questions are binary. Binary questions for two purposes. First, since we are restricting communication to start with the results should be most striking in the most restrictive case, the one in which buyers can only answer with yes or no. Second, binary questions can be reinterpreted as bits which are well established units of information transmission in information theory; see for example Cover and Thomas (1991).

Several assumptions we made are there for ease of exposition. They would not change the main results if we dropped or suitably relaxed them. We assume that a buyer is asked a single question in each period. We could allow for both buyers to be asked a question in one period, or even both buyers to be asked several questions simultaneously within a period. It is easy to see that this would not enable one to achieve higher welfare. Such simultaneous communication would prevent one from conditioning on the information being reported in the same period. In addition, we assume that buyers use pure strategies. Again, the more general case would not help to achieve a higher welfare. For a buyer to mix between two messages for some positive measure of states he would have to be indifferent in all those states, meaning that for those two messages he would win with the same expected probability and have to pay the same transfer. This would in turn be a waste of a question.

More importantly, through the definition of ι, we assume that in each period the same buyer is questioned, regardless of the history. This is an assumption we cannot dispense with. Relaxing it causes complications we are unable to resolve.

Finally, we take into account the number of periods, or questions, to which the seller commits. In effect, this means that the cost of communication originates in the seller committing up front to a certain amount of questions and therefore time. For example, if the seller commits to five questions, he incurs the cost corresponding to five questions even if for some history he knows which of the two buyers has the highest valuation after two questions. Future research could explore how the optimal mechanism changes when one takes into account only the actual number of questions used to achieve the objective along each history.

5 Conclusion

We propose a method of modelling limited communication in a setup where a seller is selling a single indivisible object to one of the two buyers. We show how results from models with simultaneous limited communication can be used to solve the dynamic problem. In our main result we show that when the seller commits to the total number of questions the welfare maximizing protocol requires only two rounds: in the first round one of the buyers is sequentially asked all the questions but one, in the second round the other buyer is asked one last question. The welfare maximizing allocation rule awards the object to the buyer with the highest expected value computed from the revealed information.

Our analysis proceeds by providing a bound on the welfare achieved by a mechanism with questions by a mechanism with simultaneous limited communication. We show that for any mechanism with questions, there exists a mechanism with simultaneous limited communication, and an equilibrium of that mechanism, that achieves at least as high a welfare. Furthermore, the bound we provide is tight. That is, for a welfare maximizing equilibrium of a mechanism with questions there exists a mechanism with simultaneous limited communication, and an equilibrium of it, achieving the same welfare. This can be seen as an analogue of the revelation principle. In the mechanism design without limits on communication the revelation principle implies it is without loss of generality to use simultaneous reporting. Simultaneous reporting when communication is limited, however, requires a larger message space to achieve the same welfare, corresponding to the idea that if one introduced rich enough language every story, question, and answer could be described by a single word. Consequently simultaneous reporting of messages would be without loss of generality. How people would manage such a language is a different question all together.

Finally we would like to point out that the analysis in our paper carries over to revenue maximization. Blumrosen, Nisan and Segal (2007) have shown that the problem of revenue maximization with simultaneous limited communication can be transformed into the problem akin of welfare maximization with simultaneous limited communication. After such a transformation our approach of using bounds applies.

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Appendix

Proof of Lemma 4. The first part of the claim follows from Blumrosen, Nisan and Segal (2007); see the discussion preceding Theorem 6.1. For the second part observe that to achieve informationally optimal welfare in $G_{K,\iota}$ it is enough to consider deterministic allocation rules; by Lemma 3. Therefore, given the fixed K and ι we only need to consider finite number of allocation rules for each of which the informationally optimal welfare in $G_{K,\iota}$ can be achieved by threshold questions. ■

Proof of Lemma 5. The main point of the proof is rather simple. All that is relevant for welfare in the last round is whether the valuation of the questioned buyer is higher or lower than the expected valuation of the buyer questioned in the penultimate round. This can be settled by a single question, while the remaining questions can be dispensed with or moved to earlier rounds to extract more information. More precisely, suppose that bidder i is questioned in the last round of g. We can construct g' by replacing the questions in the last round of g by a single question asking bidder i whether his valuation is above (or below) the expected value of bidder $-i$ computed from the information gained from his questions and answers. To have g and g' of the same length, same number of periods, questions $[0, 1]$ can be added at the beginning of g'. The allocation rule in g' should award the object to the bidder with the highest expected value given the questions and the answers. Clearly $w(g) \leq w(g')$.

While the above proves our Lemma, clearly the $[0, 1]$ questions at the beginning can be used more productively. ■

Proof of Theorem 1. Existence of an informationally optimal mechanism in G_K is easy to establish. By Lemma 4 there exists an informationally optimal mechanism in $G_{K,\iota}$. For any fixed K there are only finitely many ι, therefore there also exists an informationally optimal mechanism in G_K. Moreover, by the same lemma, it is enough to consider only threshold questions.

If there are more than three rounds in a mechanism one can increase welfare by redistributing the questions from the earlier rounds to the later. Indeed, let g be a mechanism with at least four rounds. By Lemma 5 it is enough to consider a mechanism with one question in the last round. Let the number of rounds be $R = 2n$ for some $n \in \mathbb{N}$, $n \geq 2$ (the case of odd R is handled analogously) and let buyer 1 report in the first round. The number of questions for buyer 2 in the round $2r$ is denoted by m_r, while the number of questions for buyer 1 in the round $2r-1$ is denoted by l_r. The restrictions are $\sum l_r = k_1$ and $\sum m_r = k_2$.

Starting with the first round, in the first period buyer 1 is asked a question, creating a threshold. For each of the two answers in the first period buyer 1 is asked another question in the second period, creating two possibly distinct thresholds. At the beginning of l_1th period there are 2^{l_1-1} histories, for each of which a questions is asked. This yields at most $1+2+...+2^{l_1-1} = 2^{l_1}-1$ distinct thresholds for buyer 1 in the first round. At the beginning of the third round there are $2^{l_1+m_1}$ histories, hence at most $2^{l_1+m_1}\left(2^{l_2}-1\right)$ distinct thresholds in round three, etc. Altogether there are at most

$$x = 2^{l_1} - 1 + 2^{l_1+m_1}\left(2^{l_2}-1\right)... + 2^{l_1+m_1+...+m_{n-1}}\left(2^{l_n}-1\right)$$

thresholds for buyer 1 in the mechanism g. In the last round there are at least x thresholds created for buyer 2, and in the second at least 2. Lemma 1 then implies that we can bound the welfare achieved by g by $w^{s*}_{x+1,x+2}$.

An alternative mechanism, g', constructed by moving the last question from the round $R-3$ to the beginning of the round $R-1$ yields at most

$$\begin{aligned} x' \quad = \quad & x - 2^{l_1+...+m_{n-2}}\left(2^{l_{n-1}}-1\right) + 2^{l_1+...+l_{n-2}+m_{n-2}}\left(2^{l_{n-1}-1}-1\right) - 2^{l_1+m_1+...+m_{n-1}}\left(2^{l_n}-1\right) \\ & + 2^{l_1+m_1+...+l_{n-1}-1+m_{n-1}}\left(2^{l_n+1}-1\right) \end{aligned}$$

distinct thresholds for buyer 1. It is easy to verify that $x' > x$. Upper bound on the welfare to be achieved in g' is then $w^{s*}_{x'+1,x'+2}$. Also, by Lemma 2 $w^{s*}_{x'+1,x'+2} > w^{s*}_{x+1,x+2}$. By iterating the process one arrives at a three rounds mechanism with a higher upper bound on welfare than any mechanism with more than three rounds. Next we show that the last upper bound can be achieved.

Since we started with a mechanism with one question in the last round we have a mechanism with k_1-1 questions for buyer 1 in round 1, k_2 questions for buyer 2 in round 2 and finally one question for buyer 1 in round three. This results in at most $y = 2^{k_1-1}\left(2^{k_2}-1\right)$ distinct thresholds for buyer 2 and an upper bound on the welfare of $w^{s*}_{y+2,y+1}$. Let $c^1 = \left(c^1_1, ..., c^1_{y+1}\right)$ and $c^2 = \left(c^2_1, ..., c^2_y\right)$ be profiles of thresholds that achieve such welfare in a simultaneous communication mechanism. From Kos (2009) we know that these thresholds are mutually centered: $c^1_1 = E\left[X_2 \mid 1 \geq X_2 \geq c^2_1\right]$, $c^2_1 = E\left[X_1 \mid c^1_1 \geq X_1 \geq c^1_2\right]$, etc. In particular $c^1_1 > c^2_1 > c^1_2 > ... > c^1_{y+1}$. The construction of questions proceeds by a bisection on the level of thresholds. One starts with the middle threshold of buyer 1

$$\eta_1 = \left[c^1_{2^{k_1-1-1}\left(2^{k_2}-1\right)+1}, 1\right].$$

Suppose buyer 1 is asked a question in the second period again; otherwise proceed to the next step. If he answered positively in the first period, 1, then the threshold in period two is the middle threshold of the thresholds above and including $c^1_{2^{k_1-1-1}\left(2^{k_2}-1\right)+1}$, i.e.

$$\eta_2(1) = \left[c^1_{2^{k_1-1-2}\left(2^{k_2}-1\right)+1}, 1\right].$$

Similarly

$$\eta_2(0) = \left[c^1_{2^{k_1-1-1}\left(2^{k_2}-1\right)+2^{k_1-1-2}\left(2^{k_2}-1\right)+1}, 1\right].$$

The rest of the thresholds in the first $k_1 - 1$ periods is obtained by the same method. Formally, in the first $k_1 - 1$ periods

$$\eta_t\left(h^{t-1}\right) = \left[c^1_{s_t\left(h^{t-1}\right)}, 1\right],$$

where

$$s_t\left(h^{t-1}\right) = 2^{k_1-2}\left(2^{k_2}-1\right) + 1 + (1-2h_1)\,2^{k_1-3}\left(2^{k_2}-1\right) + ... + (1-2h_{t-1})\,2^{k_1-t-1}\left(2^{k_2}-1\right),$$

for $h^t = (h_1, h_2, ..., h_t)$.

After the first round is over one knows whether buyer 1's valuation is in $\left[c^1_{2^{k_2}}, 1\right]$, $\left[c^1_{2^{k_2+1}-1}, c^1_{2^{k_2}}\right]$,..., or $\left[0, c^1_{2^{k_1-1}\left(2^{k_2}-1\right)-2^{k_2}}\right]$. In each of these intervals there are $2^{k_2} - 1$ thresholds of buyer 2. For example $\left\{c^2_1, c^2_2, ..., c^2_{2^{k_2}-1}\right\} \subset \left[c^1_{2^{k_2}}, 1\right]$. Now one proceeds by bisection on the set of thresholds for buyer 2 belonging to the identified interval of buyer 1. Finally, let $\left[c^2_l, 1\right]$ be the last question for buyer 2 in the second round. If he answers with 1 then the question for buyer 1 in the last period is $\left[c^1_l, 1\right]$, otherwise $\left[c^1_{l+1}, 1\right]$.

We still need to show that $k_1 > k_2 \geq 2$ implies it is optimal to question buyer 1 in the second round. This is done by a simple computation. If buyer 1 is questioned in rounds 1 and 3 the highest welfare achievable is $w^{s*}_{2^{k_1-1}\left(2^{k_2}-1\right)+2,2^{k_1-1}\left(2^{k_2}-1\right)+1}$. If, on the other hand, buyer 2 is questioned in the second round the highest welfare achievable is $w^{s*}_{2^{k_2-1}\left(2^{k_1}-1\right)+1,2^{k_2-1}\left(2^{k_1}-1\right)+2}$. Clearly

$$2^{k_2-1}\left(2^{k_1}-1\right)+1>2^{k_1-1}\left(2^{k_2}-1\right)+1.$$

By Lemma 2 it is therefore optimal to question buyer 1 in the second round. It should also be noted that the constructed three round mechanism achieves higher welfare then any two round mechanism, which follows the reasoning of Lemma 5. If there are only two rounds the highest welfare is achieved by computing the expected value of the buyer questioned in the first round from the available information, and asking the buyer in the second round whether his valuation is above that value. All other questions for the buyer in the last round can then be productively used at the begining of questioning. Thus creating three rounds.

Finally, if $\min\{k_1,k_2\}=1$ one or both buyers are assigned only one question. By Lemma 5 we know that a single question suffices for the last round. Therefore it is optimal to question a buyer who was assigned a single question in the last round. ■

Proof ot Theorem 2. First, we assume $k^*\geq 3$. When $k^*=2$ there can be no more than two rounds by definition. Fix a K such that $k_1+k_2=k^*$. By the Theorem 1 informationally optimal mechanism for any fixed K never entails more than three rounds. Take any three round mechanism in which buyer 1 is questioned in the first round. It is without loss of generality to assume there are k_1-1 questions for buyer 1 in the first round followed by k_2 questions to buyer 2, while in the last round buyer 1 is asked a single question. This gives altogether $x=2^{k_1-1}\left(2^{k_2}-1\right)$ thresholds for buyer 2 and an upper bound on welfare of $w^{s*}_{x+2,x+1}$. By using first k_1-1 questions on buyer 2 rather than buyer 1, one could get $x'=2^{k_1+k_2-1}>x$ thresholds for buyer 2. This yields an upper bound on welfare of $w^{s*}_{x'+2,x'+1}$. This upper bound can be achieved by the strategies constructed similarly as in the proof of Theorem 1. One looks at the threshold strategies that achieve welfare $w^{s*}_{x'+2,x'+1}$ in the mechanism with simultaneous communication and embeds them into the asking questions setup.

We conclude that in every informationally optimal mechanism either buyer 1 is asked k^*-1 consecutive questions at the beginning followed by a single questions to buyer 2, or the roles of the players are reversed. ■

Proof ot Theorem 3. From the Theorem 2 we know an informationally optimal mechanism has at most two rounds. Suppose buyer 1 is questioned in the first round - the other case is handled the same way. $k^* - 1$ questions for buyer 1 correspond to $x = 2^{k^*-1} - 1$ thresholds and the upper bound on welfare of $w^{s*}_{x+1,x+2}$. From Kos (2009) we know there exists an equilibrium of some simultaneous mechanism that achieves $w^{s*}_{x+1,x+2}$. Let the buyer 1's thresholds in such an equilibrium be $c^1 = \left(c^1_1, c^1_2, ..., c^1_{2^{k*-1}-1}\right)$ and buyer 2's $c^2 = \left(c^2_1, c^2_2, ..., c^2_{2^{k*}-1}\right)$. We use these thresholds to construct a mechanism with sequential questioning and an equilibrium of it that achieves the same welfare. Let

$$\begin{aligned}
\eta_1 &= \left[c^1_{2^{k^*-2}}, 1\right], \\
\eta_2(1) &= \left[c^1_{2^{k^*-3}}, 1\right], \eta_2(0) = \left[c^1_{2^{k^*-2}+2^{k^*-3}}, 1\right] \\
\eta_3(1,1) &= \left[c^1_{2^{k^*-4}}, 1\right], \eta_3(1,0) = \left[c^1_{2^{k^*-3}+2^{k^*-4}}, 1\right],
\end{aligned}$$

etc. In general one obtains

$$\eta_t(h_1, h_2, ..., h_{t-1}) = \left[c^1_{s_t(h^{t-1})}, 1\right],$$

for $t = 1, ..., k^* - 1$, where

$$s_t\left(h^{t-1}\right) = 2^{k^*-2} + (1 - 2h_1)\, 2^{k^*-3} + (1 - 2h_2)\, 2^{k^*-4} + ... + (1 - 2h_{t-1})\, 2^{k^*-t}.$$

The thresholds of buyer 1 are $\eta_{k^*}\left(h^{k^*-1}\right) = \left[c^2_{z_t\left(h^{k^*-1}\right)}, 1\right]$, where

$$z_t\left(h^{k^*-1}\right) = 1 + (1 - h_{k^*-1})\, 2^0 + (1 - h_{k^*-2})\, 2^1 + ... + (1 - h_1)\, 2^{k^*-2}.$$

We define allocation rule Q^* by

$$Q^*_1\left(h^{k^*}\right) = 1 - Q^*_2\left(h^{k^*}\right) = \left\{ \begin{array}{c} 1 \text{ if } h_{k^*} = 0 \\ 0 \text{ if } h_{k^*} = 1 \end{array} \right\}.$$

It is easy to see that Q^* is the welfare maximizing allocation rule for the specified structure of questioning when buyers answer truthfully. To complete the picture we need to specify the transfers to support truthful reporting as an equilibrium. This is easily done for buyer 2; namely, if he reports that his valuation is above the threshold specified by the history h^{k^*-1} he pays the amount equal to the threshold:

$$M_2\left(h^{k^*}\right)=\left\{\begin{array}{c} 0 \text{ if } h_{k^*}=0 \\ c^2_{z_t\left(h^{k^*-1}\right)} \text{ if } h_{k^*}=1 \end{array}\right\}.$$

The case of buyer 1 is a bit trickier. Let us first point out that each history uniquely defines an interval to which buyer 1's valuation belongs if he reported truthfully:

$$I\left(h^{k^*}\right)=\cap_{j=1}^{k^*-1}\left(\eta_j\left(h^{j-1}\right)\mathbf{1}_{[h_j=1]}\cup\left(\eta_j\left(h^{j-1}\right)\right)^C\mathbf{1}_{[h_j=0]}\right),$$

where we use the notation $A\mathbf{1}_p$ to denote set A if p is true and an empty set otherwise. Even more, it is easy to see that such an interval is of the form $[c^1_j, c^1_{j-1})$ for some $j=1,...,2^{k^*-1}$. The transfer from buyer 1 when he loses the object, i.e. $h_{k^*}=1$, is 0. If he wins, and the history is h^{k^*}, therefore $I\left(h^{k^*}\right)=[c^1_j, c^1_{j-1})$ for some j, the transfer he pays is

$$M_1\left(h^{k^*}\right)=\sum_{i=j}^{2^{k^*-1}}\frac{F_2\left(c^2_i\right)-F_2\left(c^2_{i+1}\right)}{F_2\left(c^2_j\right)}c^1_i.$$

Finally, we need to verify that reporting truthfully for both buyers is a Perfect Bayesian Equilibrium of this model. For buyer 2 it is optimal to be truthful no matter what buyer 1 did. On the other hand, the expected payoff of buyer 1 when answering truthfully with valuation $v_1\in[c^1_j, c^1_{j-1})$, for some j, when buyer 2 also reports truthfully is $F_2\left(c^2_j\right)\left[v_1-\sum_{i=j}^{2^{k^*-1}}\frac{F_2\left(c^2_i\right)-F_2\left(c^2_{i+1}\right)}{F_2\left(c^2_j\right)}c^1_i\right]$. From here on it is easily verified that buyer 1 does not want to deviate. He can achieve only two things by deviating. Either the history is such that the threshold which is used for buyer 2 in the last period is higher than c^2_j or lower than c^2_j. In both cases the expected payoff of buyer 1 decreases. Notice that the buyer questioned in the first round does not update his belief about the other buyer's type at all, while the beliefs of the buyer in the second round are irrelevant. ■