

Quid pro Quo: Dynamic Strategic Information Exchange*

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Abstract. *This paper studies a model in which two agents at any point in time have the opportunity to disclose information to each other. In equilibrium the incentive for doing so is provided by the implicit promise that information will be returned in exchange for information offered. It is shown that self-interested agents may engage in such information exchange even if only one of them will benefit from the information ex post. We characterize efficient symmetric equilibria by construction and establish that generally repeated alternating disclosure is necessary in order to achieve efficiency. We also show that, depending on the nature of information that may be disclosed, the environments that support equilibria with information exchange may differ drastically. (JEL Codes: D82, D83, C72, C73.)*

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1 Introduction

Can two decision makers both of whom have critical information concerning which of a number of possible actions is the correct one share that information when each has a preference for acting on it alone? This question naturally arises, for example, when multiple government agencies collect information intended to avert a terrorist attack or when separate researchers work on a common problem, as happened in the pursuit of a proof of Fermat's Last Theorem. While there may be a common benefit to making the correct decision, e.g., when a terrorist is arrested in the planning stage of an attack or a chosen proof strategy yields results, the desire to receive primary credit may stand in the way of information sharing.

Suppose, for example, that each of two rival intelligence agencies conducted an independent investigation on a crime and came up with a list of several suspects for the culprit. If they know that combining their information will identify the true culprit, would they voluntarily share their information when both are motivated to be the first to indict the

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true culprit? If they do, in what kind of manner will/should the information exchange take place? This paper provides some new insights to these questions by delineating the key factors that incentivise continued information exchange in such environments.

Proper incentive provision may counter the detrimental effects of the desire for primacy. In this paper we investigate the case where monetary incentives are not available but individuals are motivated by concerns for the future. Then there are two possible reasons for providing information, the *quid-pro-quo* reason, when there is the expectation that information will be disclosed by the other party if (and only if) information is first disclosed to that party, and the *screening* reason, that relies on the fact that information may be gleaned from others not acting on information provided. The quid-pro-quo reason is familiar from many dynamic environments, when in equilibrium individuals forgo short-term gain in the interest of future payoffs, and in particular is related to incremental exchange, incremental public goods provision and turn taking. The screening reason, as the name suggests, is reminiscent of dynamic screening settings where, for example, a seller extracts information about a buyer's valuation for an object by tempting the buyer with a sequence of price offers.

Our focus is on exchange of information via disclosure. We adopt the standard representation of information as a subset of some state space. Initially, two agents independently and privately learn a finite set, their information set, to which the true state of the world belongs. To avoid non-degeneracy, we assume that it is common knowledge that combining their information is useful in the sense that it reveals the state of the world without error, i.e., the true state of the world is the unique common element of both agents' information sets. Agents make choices in continuous time, with the proviso that a minimum amount of time has to pass once one agent has made a choice before either agent can make another choice.

Choices are of two kinds, actions and disclosures. For simplicity, we identify the space of actions with the state space. Each agent's objective is to take the action that corresponds to (is optimal in) the true state of the world. Having the other agent take the correct action is less desirable than no action being taken but not as damaging as taking a wrong action. If and when the correct action is taken, the game ends.

The other choice an agent has to make at any point in time is whether and how much information to disclose. First, we consider the case that agents may only disclose information on which states are in their information sets. A disclosure in this case amounts to picking a subset of the elements in one's information set that have not been disclosed yet. An agent does not have to disclose fully but must disclose truthfully. To highlight the role of disclosure, we shut down all avenues for communication other than disclosure. For this reason we assume that agents lack a common language for the undisclosed elements of the state space, so that the only property of a disclosed set that matters is its size, not the identity of its elements.

Note that disclosing agents risk losing the game as an immediate consequence of disclosure because the other agent would identify the true state of the world if he found the disclosed element in his information set as well. For any agent to disclose, therefore, there should be a prospect for him to be able identify the true state in the future, for instance, because the other agent is expected to disclose in return. However, this *quid-pro-quo* reason

is not enough to initiate information exchanges because there may only be a finite number of disclosures and the last disclosure cannot be motivated by this reason. The aforementioned screening reason comes to rescue here: If the one to disclose last disclosed all but one element in his information set, he retains the prospect of identifying the undisclosed element as the true state should the other agent not end the game after the last disclosure.

This reasoning illuminates some key equilibrium features: i) Each disclosure must be motivated by a future prospect of obtaining enough information in return to identify the true state of the world; ii) Once started, the agents take turns in disclosing information until the true state of the world is identified by one of the agents; iii) Since disclosing too much information at once is too risky, communication may necessarily be in the form of a prolonged conversation during which both agents become increasingly informed.

For the equilibrium analysis we focus on the case of a vanishing delay time between choices. We fully characterize the set of symmetric Markov equilibria when both agents start out with relatively accurate information, i.e. their information sets contain no more than three elements. For the general case, in which either agent's information set may contain any number of states, we construct a "focal" symmetric Markov equilibrium with maximum *quid-pro-quo* flavor. Our main results are for the case where agents have the same amount of information, as measured by the cardinality of their information sets. Here the focal equilibrium constructed for the general case induces an outcome in which both agents initially randomize over disclosing one or no state until there is a first disclosure after which agents start alternating in disclosing pairs of states until one of them can infer the true state. In the limit as the delay time between choices goes to zero the payoff from the focal equilibrium is an upper bound on the payoffs from all symmetric Markov equilibria, which is the efficient symmetric payoff for the game (asymptotically if the cardinality of the agents' information sets is an odd number). We also conjecture that the focal equilibrium is the robust one that is viable for the largest set of parameter values.

Next, we analyze the alternative case in which the agents may only disclose information on which states are *not* in their information sets. A disclosure in this case amounts to picking a subset of the elements in the complement of one's information set that have not been disclosed yet. In this case, an agent can identify the true state only if all but one state of his information set have been disclosed by his opponent. Thus, the agent to disclose last has no future prospect of obtaining useful information because no information is gleaned from his opponent not acting on the disclosed information, i.e., the screening reason is invalid. Consequently, we establish that no information exchanges is possible if the complement of either agent's information set is finite. On the other hand, we also show that potentially infinite *quid-pro-quo* exchange of information exists if both agents' information sets have infinite complements.

The distinct differences in the conditions for the existence of equilibrium with information exchange between the two alternative cases, lend potentially important implications on the optimal design of communication protocol since it would depend critically on the nature of information that may be disclosed. It may be worth noting, though, that in both cases the equilibrium exchanges of information share the common feature of continuing in *quid-pro-quo* fashion until the true state is identified.

Following the seminal papers by Crawford and Sobel (1982) and Green and Stokey

(1980), an extensive literature has developed on communication of information via cheap talk. However, information flows that arise in this literature are mostly either one-sided or one-off transmissions from each sender’s perspective, and when multi-round communication exists it relies heavily on the “jointly controlled lottery” of simultaneous exchanges of random messages. In particular, lacking is the study of sequential and interactive engagements in information exchange, which is closer to what we experience in the real world. As far as we are aware, this paper is one of the first ones that exhibit a prolonged, *quid-pro-quo* exchanges of information in equilibrium (albeit in verifiable information disclosure rather than in cheap talk).

One notable exception is Stein (2008) who provides an environment in which players with competing interests engage in continued exchanges of newly developed ideas. This equilibrium is sustained due to a strong form of complementarity: once a player comes up with a new idea, the next new idea may be discovered only by the other player only if the former player discloses his new idea to the latter, and vice versa. Furthermore, each idea benefits both players when discovered and disclosed. The current paper, having been developed independently, differs from Stein (2008) in an important respect: by disclosing information the players put themselves in immediate danger of losing the game, without any scope of increasing total welfare. In particular, the aforementioned kind of complementarity being absent, players effectively engage in a zero-sum game. Our model captures the “winner-takes-all” aspect of rivalry (as in R&D races of product development), while Stein (2008) focuses on the complementary aspect of knowledge advancement (as in cumulative, process innovations). Both aspects seem to be present in differing degrees in various situations of rivalry and thus, the two models are complementary.

In contexts of private financing of public projects, it has also been examined whether dynamic, incremental contribution may resolve the coordination problem. In a model where two agents alternate in a predetermined order to decide additional amounts of contribution toward completion of a project, Admati and Perry (1991) show that coordination failure prevails due to early contributors free-riding upon future ones. When the agents are allowed to contribute simultaneously, on the other hand, Marx and Matthews (2000) show that completion of the project can be achieved by incremental contributions, even in cases that completion would not be possible in static settings due to free-riding. The core idea is that incremental contributions gradually reduce the remaining cost of completion until the temptation to free-ride is waned enough to no longer stand in the way of completing the project even in static settings.

Although our paper shares the incremental feature of equilibrium disclosure, the nature of problem is different in our model: Disclosure of information does not provide a public good and thus, the problem stems from more inherent conflicts of interest rather than free-riding. Specifically, each disclosure carries a risk of immediate loss, while its potential future reward is possible only if this risk did not materialize. Due to this zero-sum aspect of conflict, simultaneous disclosure is untenable in our setting and consequently, incremental disclosure must take place in a *quid-pro-quo* manner.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 establishes some core properties common to all equilibria. Section 4 provides a full characterization of symmetric Markov equilibria when the agents information sets are small,

and Section 5 characterizes efficient equilibria for general information sets. Section 6 examines the alternative case that agents may only disclose states in the complement of their information sets. Section 7 contains deferred proofs.

2 Model

There is a finite set Ω of payoff relevant states. Two agents/players, 1 and 2, are interested in identifying the true state. At time $t = 0$, each agent i privately learns a subset of the state space, $S_i \in \Omega$, that contains the true state. For both i , $\#(S_i) = \nu_i > 1$ and $\#(S_1 \cap S_2) = 1$. Thus, agents can jointly but not individually identify the true state. Define

$$\mathcal{S}(\nu_1, \nu_2) := \{(R_1, R_2) \subset 2^\Omega \times 2^\Omega \mid \#(R_i) = \nu_i \text{ and } \#(R_1 \cap R_2) = 1\}.$$

The game begins with nature drawing a pair (S_1, S_2) from a uniform distribution on $\mathcal{S}(\nu_1, \nu_2)$. The lone element of $S_1 \cap S_2$ becomes the true state, which will be denoted by ω^* .

The timing of the game after both players have received their private signals is as follows. At any time $t \geq 0$ either agent, i , may make a move, which is either to “disclose” a nonempty subset D_i^t of S_i or “take an action” $a_i \in S_i$ [it shouldn’t matter if $a_i \in \Omega$]. Whenever there is a move by either player, say at $t' \geq 0$, the game pauses for a small length of time, $\Delta > 0$, in the sense that no move may be made during the time period $(t', t' + \Delta)$ which we call “pause period,” so that the earliest instant of time for a subsequent move is $t' + \Delta$. In general, at any time outside of pause periods either agent i may disclose a nonempty subset of elements of S_i that have not been disclosed already, or take an action in S_i . Each pause period may be interpreted as the minimal lapse of time between any two consecutive moves, which is a technically awkward notion to model in a continuous-time framework.¹ We focus on small $\Delta > 0$ and the limit results as $\Delta \rightarrow 0$.

A public history $h^t = (t, \{(D_1^\tau, D_2^\tau, a_1^\tau, a_2^\tau)\}_{\tau < t})$ keeps track of the current time and the history of disclosures and actions taken by both players. Player i ’s private history $h_i^t = (S_i, h^t)$ at time t combines the public history at the time with player i ’s private information about his possibility set S_i . The full history, which contains information about both players’ possibility sets, is given by $h_f^t = (S_1, S_2, h^t)$. Given any private history $h_i^t = (S_i, h^t)$, player i ’s information set is given by

$$I(h_i^t) := \{(R_1, R_2, h^t) \mid R_i = S_i, R_{-i} \supset \mathbb{D}_{-i}^t \cup A_{-i}^t\}$$

where $\mathbb{D}_j^t := \cup_{\tau < t} D_j^\tau$ and $A_j^t := \{a_j^\tau \mid \tau < t\}$, $j = 1, 2$,

with the understanding that $\#(R_j) = \nu_j$, $j = 1, 2$, and $\#(R_1 \cap R_2) = 1$.

The game ends when either player takes the action ω^* . At each time $t \geq 0$ with history h^t , let $S_i^t := S_i \setminus \mathbb{D}_i^t$ denote the undisclosed elements for $i = 1, 2$, so long as the game has not ended. The set of “moves” available for player i at (non-pause) time t is

$$(1) \quad M(S_i^t) := (2^{S_i^t} \setminus \{\emptyset\}) \cup S_i$$

¹Mention Simon and Stinchcombe (1989) and Sakovicz (1993).

where $2^{S_i^t} \setminus \{\emptyset\}$ reflects all feasible disclosing strategies and S_i all feasible action strategy. The set of pure strategies of player i at (non-pause) time t is $2^{S_i^t} \cup S_i$ where $\emptyset \in 2^{S_i^t}$ denotes the strategy of “doing nothing.” To avoid confusion between disclosing $\{\omega\}$ and taking an action $\omega \in S_i$, we denote the latter as $\langle \omega \rangle$ in the sequel.

A strategy of player i at time t with a private history $h_i^t = (S_i^t, h^t)$ is represented by a collection of non-decreasing and right-continuous functions $\sigma_i(\mathbf{m}|h_i^t) : \mathbb{R}_+ \rightarrow [0, 1]$, $\mathbf{m} \in M(S_i^t)$, such that

$$(2) \quad \sigma_i(h_i^t)(\tau) := \sum_{\mathbf{m} \in M(S_i^t)} \sigma_i(\mathbf{m}|h_i^t)(\tau) \leq 1 \quad \text{for } \tau \geq 0, \text{ and}$$

$$(3) \quad \sigma_i(\mathbf{m}|h_i^{t'})(\tau) = \frac{\sigma_i(\mathbf{m}|h_i^t)(t' - t + \tau) - \lim_{k \uparrow t' - t} \sigma_i(\mathbf{m}|h_i^t)(k)}{1 - \lim_{k \uparrow t' - t} \sigma_i(h_i^t)(k)} \quad \forall t' > t \quad \forall \tau \geq 0,$$

so long as $1 - \lim_{k \uparrow t' - t} \sigma_i(h_i^t)(k) > 0$ where $h_i^{t'}$ is an extension of h_i^t with no additional moves made. The interpretation is: $\sigma_i(\mathbf{m}|h_i^t)(\tau)$ is the probability that player i will have taken the move $\mathbf{m} \in M(S_i^t)$ at or before $t + \tau$, conditional on no move having been taken by either player at or after t until \mathbf{m} is taken by player i . So, $\sigma_i(h_i^t)$ is a well-defined cdf by (2), and satisfies intertemporal consistency by (3).

For any public history h^t at (non-pause) time t , define the “public state” as

$$(4) \quad z(h^t) := \{\mathbb{D}_i^{t-\Delta}, A_i^{t-\Delta}, \iota(h^t) \mid i = 1, 2\} \quad \text{where } \iota(h^t) := \{D_i^{t-\Delta}, a_i^{t-\Delta} \mid i = 1, 2\}.$$

Here, $\iota(h^t)$ keeps track of the immediately preceding moves to which t is the first instance of time to make any move in response. For a private history $h_i^t = (S_i^t, h^t)$, define the “private state” as $z_i(h_i^t) := (S_i^t, z(h^t))$. Due to the “no common labeling” assumption to be introduced shortly, no additional information may be inferred between two points in time so long as the public (hence, private) state is unchanged, and consequently, the continuation game stays the same between such two points in time from the perspectives of both players. Thus, we focus on equilibria that use “Markov strategies” that depend only on private state which denoted by $z_i = (S_i^t, z(h^t))$.

To define Markov strategy, first consider a point in time t which is not the first moment after a pause period, i.e., $\iota(h^t) = \emptyset$, which we refer to as a “proactive moment/state” [look for a better term]. Then the private state at times t and $t + \tau > t$ are the same so long as no move has been taken in between. Hence, Markov property requires that player i 's strategy at t , $\sigma_i(z_i)$, is the same as her strategy at $t + \tau$ with no moves in between. If $\sigma_i(z_i)(0) = 1$, this means that $\sigma_i(z_i)$ can be represented by the “spot” probabilities $p(\mathbf{m}|z_i) \in [0, 1]$ with which player i takes \mathbf{m} at t , and also at any future point in time conditional on no move having been taken in between, such that $\sum_{\mathbf{m}} p(\mathbf{m}|z_i) = 1$.

If $\sigma_i(z_i)(0) < 1$, on the other hand, Markov property and (3) imply that

$$(5) \quad \sigma_i(\mathbf{m}|z_i)(\tau) = \frac{\sigma_i(\mathbf{m}|z_i)(\theta + \tau) - \sigma_i(\mathbf{m}|z_i)(\theta^-)}{1 - \sigma_i(z_i)(\theta^-)} \quad \forall \tau \geq 0 \quad \forall \theta > 0$$

and

$$(6) \quad \sigma_i(z_i)(\tau) = \frac{\sigma_i(z_i)(\theta + \tau) - \sigma_i(z_i)(\theta^-)}{1 - \sigma_i(z_i)(\theta^-)} \quad \forall \tau \geq 0 \quad \forall \theta > 0,$$

where $\sigma_i(\mathbf{m}|z_i)(\theta^-) = \lim_{\theta' \uparrow \theta} \sigma_i(\mathbf{m}|z_i)(\theta')$ and $\sigma_i(z_i)(\theta^-) = \lim_{\theta' \uparrow \theta} \sigma_i(z_i)(\theta')$. In particular, due to right-continuity, (6) implies that $\sigma_i(z_i)(0) = \sigma_i(z_i)(\theta + \tau) - \sigma_i(z_i)(\theta^-) = 0$ for all $\theta > 0$ and thus, $\sigma_i(z_i)$ is continuous everywhere. Differentiability presumed (see if this can be deduced), therefore, it follows from (6) that

$$(7) \quad \frac{d}{d\tau} \sigma_i(z_i)(\tau) \Big|_{\tau=0} = \frac{\frac{d}{d\tau} \sigma_i(z_i)(\theta + \tau) \Big|_{\tau=0}}{1 - \sigma_i(z_i)(\theta)} \quad \forall \theta > 0,$$

which in turn implies that $\sigma_i(z_i)$ is the cdf of exponential distribution, i.e.,

$$(8) \quad \frac{d}{d\tau} \sigma_i(z_i)(\tau) = Q e^{-Q\tau} \quad \text{and} \quad \sigma_i(z_i)(\tau) = 1 - e^{-Q\tau} \quad \forall \tau > 0,$$

where $Q > 0$ is the rate parameter [find reference]. Similarly, from (5) we deduce that

$$\begin{aligned} \frac{d}{d\tau} \sigma_i(\mathbf{m}|z_i)(0) &= \frac{d}{d\tau} \sigma_i(\mathbf{m}|z_i)(\tau) / e^{-Q\tau} \\ \implies \sigma_i(\mathbf{m}|z_i)(\tau) &= c_{\mathbf{m}} - q(\mathbf{m}|z_i) e^{-Q\tau} / Q \quad \text{where} \quad q(\mathbf{m}|z_i) = \frac{d}{d\tau} \sigma_i(\mathbf{m}|z_i)(0) \quad \forall \mathbf{m} \in M(S_i^t), \end{aligned}$$

where $c_{\mathbf{m}}$ is a constant. Since $\frac{\sigma_i(\mathbf{m}|z_i)(\infty)}{\sigma_i(z_i)(\mathbf{m}')(\infty)} = q(\mathbf{m}|z_i)/q(\mathbf{m}'|z_i)$ and $\sum_{\mathbf{m}} c_{\mathbf{m}} = 1$, it follows that

$$(9) \quad \sigma_i(\mathbf{m}|z_i)(\tau) = \frac{q(\mathbf{m}|z_i)}{Q} (1 - e^{-Q\tau}) \quad \text{with} \quad \sum_{\mathbf{m}} q(\mathbf{m}|z_i) = Q \quad \forall \mathbf{m} \in M(S_i^t).$$

Consequently, player i 's strategy $\sigma(z_i)$ at $z_i = (S_i^t, z(h^t))$ can be represented by ‘‘flow rates’’ $q(\mathbf{m}|z_i)_{\mathbf{m} \in M(S_i^t)} \in \mathbb{R}_+^{\#(M(S_i^t))}$ if t is a proactive moment, i.e., $\iota(h^t) = \emptyset$, and $\sigma_i(z_i)(0) \neq 1$.

Next consider a point in time t which is the first moment after a pause period, i.e., $\iota(h^t) \neq \emptyset$, which we refer to as a ‘‘reactive moment/state’’ [look for a better term]. Then the private state at any time $t + \tau > t$ is necessarily different from that at t , i.e., even if no move has been taken in between, whilst the private state stays the same for all $t + \tau > t$ so long as no move is taken. Hence, Markov property means that if no move is taken at t , the above description of strategy at proactive moments pertains to player i 's strategy for times $t + \tau > t$. At t , however, player i may take a move \mathbf{m} with a ‘‘spot’’ probability $p_{\mathbf{m}}^t$ subject to $\sum_{\mathbf{m}} p_{\mathbf{m}}^t \leq 1$. Note that, unlike at a proactive moment, the spot probabilities may sum less than 1. ‘‘Flow rates’’ are irrelevant at proactive moments because the private state is bound to change in the next instant. Thus, a strategy of player i at a reactive state $z_i = (S_i^t, \iota(h^t))$ is described by combining the spot probabilities, $p_{\mathbf{m}}^t$, with his strategy at the proactive state, say z'_i , that will immediately ensue if no move is taken at t :

$$\sigma_i(\mathbf{m}|z_i)(\tau) = p_{\mathbf{m}}^t + \left(1 - \sum_{\mathbf{m}} p_{\mathbf{m}}^t\right) \sigma_i(\mathbf{m}|z'_i)(\tau) \quad \forall \mathbf{m} \in M(S_i^t).$$

Therefore, if $\sigma_i(z'_i)(0) = 1$ then $\sigma_i(z_i)$ is represented by spot probabilities $p(\mathbf{m}|z_i) = p_{\mathbf{m}}^t + (1 - \sum_{\mathbf{m}} p_{\mathbf{m}}^t) p(\mathbf{m}|z'_i)$ that sum up to 1, i.e., $\sum_{\mathbf{m}} p(\mathbf{m}|z_i) = 1$; If $\sigma_i(z'_i)(0) < 1$, then $\sigma_i(z_i)$ is represented by both spot probabilities $p(\mathbf{m}|z_i) = p_{\mathbf{m}}^t$ and flow rates $q(\mathbf{m}|z_i) = q(\mathbf{m}|z'_i)$ as

$$(10) \quad \sigma_i(\mathbf{m}|z_i)(\tau) = p(\mathbf{m}|z_i) + \left(1 - \sum_{\mathbf{m}} p(\mathbf{m}|z_i)\right) \frac{q(\mathbf{m}|z_i)}{Q} (1 - e^{-Q\tau}) \quad \forall \mathbf{m} \in M(S_i^t).$$

Summarizing, a Markov strategy of player i is a collection of $(p_i(z_i), q_i(z_i))$ for all possible private states $z_i = (S_i^t, z(h^t))$ of player i , such that

$$[M1] \quad p_i(z_i) : M(S_i^t) \rightarrow [0, 1] \quad s.t. \quad p_i(z_i) \cdot \mathbf{1} \leq 1,$$

$$[M2] \quad q_i(z_i) : M(S_i^t) \rightarrow \mathbb{R}_+$$

$$[M3] \quad p_i(z_i) \cdot \mathbf{1} \in \{0, 1\} \text{ if } \iota(h^t) = \emptyset,$$

$$[M4] \quad q_i(z_i) = q_i(z'_i) \text{ if } \iota(h^t) = \emptyset \text{ where } z'_i \text{ is the private state of the history that extends } h^t \text{ with no further move taken.}$$

A Markov strategy $(p_i(z_i), q_i(z_i))$ induces a strategy represented by (10) where $p_i(z_i)(\mathbf{m})$, $q_i(z_i)(\mathbf{m})$ and $\sum_{\mathbf{m}} q_i(z_i)(\mathbf{m})$ playing the roles of $p(\mathbf{m}|z_i)$, $q(\mathbf{m}|z_i)$ and Q , respectively.

Payoffs are realized only when either player takes an action. If player i takes an action $a_i = \omega^*$ at time t , then i 's payoff is $\alpha > 0$ and $-i$'s payoff is $\beta < 0$; If both players take ω^* at the same time, we assume that each player's payoff is α and β with equal probabilities. If $a_i \neq \omega^*$, then i 's payoff equals $\gamma < 0$ and $-i$'s payoff is zero². At any point in time, each player tries to maximize the expected sum of his payoffs discounted at rate $r > 0$. We assume (and take granted) throughout the paper that

$$(11) \quad \frac{2\alpha + \gamma}{2} < \beta < 0 < \alpha + 2\beta.$$

The first inequality ensures that no player would want to preempt his opponent by taking an action if his posterior is uniform over a non-singleton set of states. The last inequality³ means that each player i prefers that the true state becomes known provided that the probability that he gets to act on the information before player $-i$ is at least $\frac{1}{3}$.

Definition 1 *A (Markov) equilibrium is a pair of Markov strategies of the two players that are mutual best responses of each other among all strategies.*

“No common labeling” (NCL) assumption

We assume that there is no common labeling of the elements of Ω . This means that for each player i his strategy σ_i is invariant under permutations of the elements of the state space. We will also require that this condition holds for completely mixed strategies that are used to justify off-the-equilibrium path beliefs.

Denoting a permutation of the state space by π and the set of all such permutations by Π_Ω , we have *no common labeling* if

$$\sigma_i(\mathbf{m}|h_i^t) = \sigma_i(\pi(\mathbf{m})|\pi(h_i^t)) \quad \forall \mathbf{m} \in M(S_i^t)$$

for all $h_i^t = (S_i, h^t) \in H_i$ and $\pi \in \Pi_\Omega$.

²Although set at 0 for expositional ease, this payoff is unimportant for our result because no player would take $a \neq \omega^*$ in equilibrium due to the assumption (11) below. [try to be more precise]

³Without this inequality little information exchange will take place when $\nu_1, \nu_2 \geq 3$. In an earlier version, we did not introduce this for $\nu_1 = \nu_2 = 2$, so the analysis was more complex.

The role of the no common labeling assumption is to emphasize the hard-information nature of our model: player i cannot indirectly communicate information about the elements in S_i ; all that player $-i$ learns about S_i from a disclosure D_i by player i is that $D_i \subset S_i$, and similarly for actions of player i . This also implies that at the disclosure stage player i 's only relevant decision concerns how many (further) elements of S_i to disclose.

The no-common labeling assumption constrains player i 's posterior belief on the equilibrium path in any equilibrium to be uniform over all sets S_{-i} that have not been ruled out by the history of play. To see this, recall that the prior distribution over the set $\mathcal{S}(\nu_1, \nu_2)$ is uniform. Suppose now that player i 's posterior belief following private history (S_i, h^t) is uniform over all S_{-i} that have not been ruled out by his private history (S_i, h^t) and i observes the disclosure of the set D_{-i} by his opponent at some time between t and $t' \geq t$ with no other moves taken between t and the disclosure of D_{-i} . Letting $[h_i^{t'}]$ denote the set of all private histories at t' that are consistent with (S_i, h^t) and disclosure of D_{-i} by t' , his posterior probability satisfies

$$\begin{aligned}
\text{Prob}(S_{-i}|[h^{t'}]) &= \frac{\text{Prob}(S_{-i}, [h^{t'}])}{\text{Prob}([h^{t'}])} \\
&= \frac{\text{Prob}([h^{t'}]|S_{-i}, S_i, h^t)\text{Prob}(S_{-i}, S_i, h^t)}{\text{Prob}([h^{t'}])} \\
&= \frac{\text{Prob}([h^{t'}]|S_{-i}, S_i, h^t)\text{Prob}(S_{-i}|S_i, h^t)\text{Prob}(S_i, h^t)}{\text{Prob}([h^{t'}])} \\
&= \frac{\sigma_{-i}(D_{-i}|S_{-i}, h^t)(t' - t)\text{Prob}(S_{-i}|S_i, h^t)\text{Prob}(S_i, h^t)}{\text{Prob}([h^{t'}])}
\end{aligned}$$

Let $\hat{\pi}$ be any permutation of Ω that leaves (the set) S_i , (the sets appearing in) h^t and (the set) D_{-i} fixed. Then

$$\begin{aligned}
\text{Prob}(\hat{\pi}(S_{-i})|[h^{t'}]) &= \frac{\sigma_{-i}(D_{-i}|\hat{\pi}(S_{-i}), h^t)(t' - t)\text{Prob}(\hat{\pi}(S_{-i})|S_i, h^t)\text{Prob}(S_i, h^t)}{\text{Prob}([h^{t'}])} \\
&= \frac{\sigma_{-i}(\hat{\pi}(D_{-i})|\hat{\pi}(S_{-i}), h^t)(t' - t)\text{Prob}(\hat{\pi}(S_{-i})|S_i, h^t)\text{Prob}(S_i, h^t)}{\text{Prob}([h^{t'}])}
\end{aligned}$$

We assumed that all S_{-i} that are not ruled out by i 's private history (S_i, h^t) have equal posterior probability. Formally this amounts to $\text{Prob}(\hat{\pi}(S_{-i})|S_i, h^t) = \text{Prob}(S_{-i}|S_i, h^t)$ for any S_{-i} with $\text{Prob}(S_{-i}|S_i, h^t) > 0$ and any permutation $\hat{\pi}$ of Ω that fixes h^t and S_i . Lack of a common labeling implies that $\sigma_{-i}(\hat{\pi}(D_{-i})|\hat{\pi}(S_{-i}), h^t) = \sigma_{-i}(D_{-i}|S_{-i}, h^t)$ for any permutation $\hat{\pi}$ of Ω . Therefore, if S_{-i} has positive probability given i 's private history (S_i, h^t) updated by his most recent D_{-i} , then any S'_{-i} that is not ruled by this information has the same posterior probability, i.e., $\text{Prob}(S_{-i}|h^{t'}) = \text{Prob}(\hat{\pi}(S_{-i})|h^{t'})$ for any S_{-i} with $\text{Prob}(S_{-i}|h^{t'}) > 0$ and any permutation $\hat{\pi}$ of Ω that fixes h^t , S_i , and D_{-i} , where $h^{t'}$ denotes the history extended from h^t by the disclosure of D_{-i} .

We justify off-the-equilibrium path beliefs by applying the no-common labeling condition to completely mixed strategies in the spirit of sequential equilibrium of Kreps and Wilson (1982). It then follows that both on and off the equilibrium path for player i at any

point in the game, any S_{-i} that has not been ruled out by history is equally likely. This implies in particular that any element of S_i that has not been ruled out by history is equally likely to be the true state of the world. (Maybe state this as a Lemma?) Furthermore, it also implies that disclosing $\mathbf{m} \subset S_i^t$ and $\mathbf{m}' \subset S_i^t$ are strategically equivalent for player i so long as $\#(\mathbf{m}) = \#(\mathbf{m}')$. Without loss of generality, therefore, we use $\sigma_i(\mathbf{m}|h_i^t)$ and $\sigma_i(\#(\mathbf{m})|h_i^t)$ interchangeably for $\mathbf{m} \subset S_i^t$ in the sequel.

3 General properties of equilibrium

In this section we establish some core properties that pertain to all equilibria. The following two classes of histories are of special interest.

$$\begin{aligned} H_i(\omega) &:= \{h_i^t \mid \{\omega\} = S_i \cap \mathbb{D}_{-i}^t\} \\ H'_i(\omega) &:= \{h_i^t \mid \{\omega\} = S_i^t\} \end{aligned}$$

The class $H_i(\omega)$ consists of all private histories of player i in which player $-i$ has disclosed a state of the world ω that player i initially considered possible. The class $H'_i(\omega)$ consists of all private histories of player i in which player i has disclosed every state of the world he initially considered possible with the exception of the state ω . Let $\delta := e^{-r\Delta}$ where $r > 0$ is the instantaneous discount rate.

Lemma 1 *In any equilibrium, $p_i(h_i^t)(\langle\omega\rangle) = 1 \forall h_i^t \in H_i(\omega)$.*

Proof: Obvious. □

Lemma 2 *In any equilibrium, $p_i(h_i^t)(\langle\omega\rangle) = q_i(h_i^t)(\langle\omega\rangle) = 0 \forall \omega \in S_i, \forall h_i^t \notin H_i(\omega) \cup H'_i(\omega)$.*

Proof: Given that the game is not yet over at t with any $h_i^t \notin H_i(\omega) \cup H'_i(\omega)$, player i 's payoff in the continuation game from taking any action $\omega \in S_i$ is bounded above by $\frac{(\#(S_i^t)-1)(\gamma+\delta\alpha)+\alpha}{\#(S_i^t)} < \frac{(\#(S_i^t)-1)\gamma+\#(S_i^t)\alpha}{\#(S_i^t)} \leq \frac{\gamma+2\alpha}{2}$, which is less than the lower bound, β , of the payoff from never making any move due to (11). □

Lemma 3 *In any equilibrium, if $h_i^t \in H'_i(\omega) \setminus H_i(\omega)$, then (i) $p_i(h_i^t)(\langle\omega\rangle) = 1$ if $D_i^{t-\Delta} \notin \iota(h_i^t)$, and (ii) $p_i(h_i^t)(\emptyset) = 1$ if $D_i^{t-\Delta} \in \iota(h_i^t)$ unless $\#(S_{-i}^t) = 1$ and $D_{-i}^{t-\Delta} \in \iota(h_i^t)$.*

Proof: It follows from Lemma 1 and Bayes' rule that after at time t with $h_i^t \in H'_i(\omega)$ player i assigns probability one to $\omega = \omega^*$ if $D_i^{t-\Delta} \notin \iota(h_i^t)$, whence taking action ω is the unique way of obtaining the maximum possible payoff, α , establishing (i).

Suppose $D_i^{t-\Delta} \in \iota(h_i^t)$. If $\#(S_{-i}^t) = 1$ and $D_{-i}^{t-\Delta} \notin \iota(h_i^t)$, player $-i$ would take the correct action due to Lemmas 1 and 3 (i) and consequently, $p_i(h_i^t)(\emptyset) = 1$ is optimal for player i . If $\#(S_{-i}^t) > 1$, by Lemma 1, agent i obtains an expected payoff of $(\#(D_i^{t-\Delta})\beta + \delta\alpha)/(1 + \#(D_i^{t-\Delta}))$ by making no move at t , which is greater than the expected payoff of taking $\langle\omega\rangle$, $(\#(D_i^{t-\Delta})\gamma + \alpha)/(1 + \#(D_i^{t-\Delta}))$. Since agent i does not take an action $a \notin S_i^t = \{\omega\}$ by Lemma 2 and disclosing $\{\omega\}$ at t is obviously suboptimal, (ii) is proved. □

The next lemma is an artifact of the pause period Δ . It is not critical for our main insight but is needed for our formal analysis.

Lemma 4 *In any equilibrium, $p_i(h_i^t)(2^{S_i^t} \setminus \{\emptyset\}) = 1$ if $h_i^t \in H'_{-i}(\omega') \setminus H_i(\omega)$, $D_{-i}^{t-\Delta} \in \iota(h_i^t)$ and $h_i^{t-\Delta} \notin H'_i(\omega)$.*

Proof: First consider the case that either $\#(S_i^t) > 1$ or $D_{-i}^{t-\Delta} \notin \iota(h_i^t)$. Then, player i does not take any action at t by Lemma 2. Since the game has not ended, there is a positive probability that $h_{-i}^t \notin H_{-i}(\omega)$ and agent $-i$ does not make any move at t by Lemma 3. In this contingency, player i benefits by disclosing a subset of S_i^t because that would delay the time for player $-i$ to take the correct action by Δ . In all other contingencies, player $-i$ would take the correct action at t anyway and thus, whether player i discloses or not is inconsequential.

Next, consider the remaining case that $\#(S_i^t) = 1$ and $D_{-i}^{t-\Delta} \in \iota(h_i^t)$. It is clearly suboptimal for player i to take some action $\langle \omega \rangle$ because the maximum possible expected utility from doing so, $\frac{\gamma + \delta\alpha + \alpha}{2}$, is less than what he can guarantee by never making any move, β . Hence, $p_i(h_i^t)(\langle \omega \rangle) = 0$ for any $\omega \in S_i$. If $p_i(h_i^t)(2^{S_i^t} \setminus \{\emptyset\}) < 1$, therefore, Lemma ?? would imply that $p_i(h_i^t) \cdot \mathbf{1} = 0$ for all $h_i^t \succ h_i^t$, contradicting Lemma 3. Thus, $p_i(h_i^t)(2^{S_i^t} \setminus \{\emptyset\}) = 1$ should follow. Indeed, it is optimal for player i provided that player $-i$ behaves the same way when placed in the same situation. \square

Lemma 5 *In any equilibrium, $p_i(h_i^t)(S_i^t) = q_i(h_i^t)(S_i^t) = 0$ if $h_i^t \notin H'_{-i}(\omega)$.*

Proof: This is trivial if $h_i^t \in H_i(\omega)$ by Lemma 1. If $h_i^t \notin H_i(\omega)$, player i 's payoff from disclosing S_i^t is no higher than $\delta\beta$ by Lemma 1, while that from doing nothing forever is strictly greater than $\delta\beta$ because at time t player $-i$ assigns the same probability of being the true state to at least two elements of S_{-i} . \square

4 The set of equilibria when $\#(S_1)$ and $\#(S_2)$ are small

Consider the case that $\#(S_1) = \#(S_2) = 2$. Once player i discloses one element $\omega \in S_i$ at time t , agent $-i$ takes action ω at time $t + \Delta$ if $\omega \in S_{-i}$, or equivalently, if $\omega = \omega^*$; if $\omega \notin S_{-i}$, $-i$ does not take any action by Lemma 2 and discloses some element of S_{-i} to delay “losing” by Lemma 4. Thus, by first disclosing one element of S_i at time t , either agent obtains a payoff of

$$(12) \quad V_{2,2}(1) := \frac{\delta\beta + \delta^2\alpha}{2}.$$

Note that neither player would take an action without previous disclosure by Lemma 2. If one of the players, say i , never discloses without previous disclosure, therefore, the other agent finds it optimal to disclose one for a payoff of $V_{2,2}(1) > 0$, because he would get no more than 0 by disclosing none or all of S_i . Conversely, given that player i discloses one element of S_i with certainty, it is optimal for player $-i$ to disclose none to get an expected payoff of $\frac{\delta\alpha + \delta^2\beta}{2}$ because his payoff from disclosing one as well is $\frac{\delta}{4}(\frac{\alpha + \beta}{2} + \alpha + \beta + \delta\frac{\alpha + \beta}{2}) = \frac{\delta(3 + \delta)(\alpha + \beta)}{4} < \frac{\delta(\alpha + \beta)}{2} < \frac{\delta\alpha + \delta^2\beta}{2}$. Thus, we identified a Markov equilibrium: One player never discloses without previous moves and the other agent discloses one element at any $t \geq 0$

with no previous moves. Note that this is the only Markov equilibrium in which either player discloses with certainty or not disclose with certainty.

The remaining possibility is that both players randomize between disclosing and not. Since $p_i(h_i^t)(1) \in (0, 1)$ is not possible for history h_i^t with no previous moves by Lemmas ??, 2 and 5, this possibility means $p_i(h_i^t) \cdot \mathbf{1} = 0$ and $q_i(h_i^t)(1) > 0$ for both players. In light of Lemmas 1 and 3, the expected payoff of player $-i$ from waiting until $t > 0$ before disclosing one himself is

$$V_{2,2}(t|q_i) := \delta \int_0^t q_i e^{-(q_i+r)\tau} \frac{\alpha + \delta\beta}{2} d\tau + \delta e^{-(q_i+r)t} \frac{\beta + \delta\alpha}{2}$$

where $q_i = q_i(h_i^t)(1)$ with no previous moves: The first term is his expected payoff for the contingency that player i discloses one element of S_i before t , and the second term is that for the alternative contingency (so that he discloses one element of S_{-i} himself at t). The Markov equilibrium requires that player $-i$ is indifferent between disclosing one now and waiting a short period of time, i.e.,

$$(13) \quad \left. \frac{dV_{2,2}(t|q_i)}{dt} \right|_{t=0} = 0 \iff q_i = \frac{r(\beta + \delta\alpha)}{(1 - \delta)(\alpha - \beta)}.$$

Note that $q_i = q_{-i}$ by symmetry and $q_i > 0$ if $\beta + \delta\alpha > 0$, i.e., $r\Delta < \log(-\alpha/\beta)$, establishing a unique Markov equilibrium in flow strategies, which turns out to be symmetric.

Proposition 1 *When $\#(S_1) = \#(S_2) = 2$, there is a unique symmetric Markov equilibrium if $\Delta < \log(-\alpha/\beta)/r$. In this equilibrium, both players disclose one element of S_i with a flow probability q_i in (13). Once $\omega \in S_i$ is disclosed by player i at time t , player $-i$ takes ω at $t + \Delta$ if $\omega \in S_{-i}$; If $\omega \notin S_{-i}$, player $-i$ discloses a nonempty subset of S_{-i} at $t + \Delta$ and player i takes $\omega' \in S_i \setminus \{\omega\}$ at $t + 2\Delta$. The payoff in this equilibrium is $V_{2,2}^e := \delta \frac{\beta + \delta\alpha}{2}$. The only other Markov equilibrium is one in which one player discloses one element while the other discloses none at $t = 0$, followed by the same action-taking strategies as above.*

When $\max\{\#(S_1), \#(S_2)\} = 3$.

We say that a Markov equilibrium is *symmetric* if the $\sigma_1(h_1^t)$ and $\sigma_2(h_2^t)$ are identical if h_i^t , $i = 1, 2$, coincide in $\#(S_i^t)$, $\#(S_{-i}^t)$ and $\iota(h_i^t)$ from player i 's perspective.

In the case that $\#(S_i) = 3$ and $\#(S_{-i}) = 1$, the game ends by player $-i$ taking the correct action immediately due to Lemmas 5 and 3. Thus, the equilibrium payoffs are $V_{3,1}^e = \beta$ and $V_{1,3}^e = \alpha$ for players i and $-i$, respectively.

Lemma 6 *When $\#(S_1) = 3$ and $\#(S_2) = 2$, there are three ‘‘symmetric’’ Markov equilibrium outcomes for sufficiently small Δ :*

- (i) $p_1(h_1^t) \cdot \mathbf{1} = 0$ and $p_2(h_2^t)(1) = 1$ with no previous moves;
- (ii) $p_1(h_1^t)(2) = 1$ and $p_2(h_2^t) \cdot \mathbf{1} = 0$ with no previous moves;
- (iii) For h_i^t with no previous moves, $p_1(h_1^t) \cdot \mathbf{1} = p_2(h_2^t) \cdot \mathbf{1} = 0$ and

$$q_1(h_1^t)(1) = 0, \quad q_1(h_1^t)(2) = \frac{2r(2\beta + \delta\alpha)}{(3 - 2\delta)\alpha - (4 - 3\delta)\beta} \quad \text{and} \quad q_2(h_2^t)(1) = \frac{3r(\beta + \delta\alpha)}{(4 - 3\delta)\alpha - (3 - 2\delta)\beta}.$$

Proof: Suppose $p_2(h_2^t)(1) = 1$ with no previous moves. Then, it is optimal for player 1 to disclose none at t , i.e., $p_1(h_1^t) \cdot \mathbf{1} = 0$, because his expected payoff from doing so is $\delta \frac{\alpha + \delta\beta}{2}$ while that from disclosing one of S_1 is smaller since the probability of “winning” is lower (because agent 2 would take the correct action at $t + \Delta$ with probability $1/3$) and that from disclosing two is $\delta \frac{(\alpha + \beta)(1 + \delta/2) + 2\beta + \alpha}{6} < \delta \frac{\alpha + \beta}{2}$. Conversely, given $p_1(h_1^t) \cdot \mathbf{1} = 0$ and sufficiently low $q_1(h_1^t) \cdot \mathbf{1}$ with no previous moves, it is easily verified that $p_2(h_2^t)(1) = 1$ is optimal for player 2, establishing equilibrium (i).

Suppose $p_1(h_1^t)(2) = 1$ with no previous moves. Then, it is optimal for player 2 to disclose none at t , i.e., $p_2(h_2^t) \cdot \mathbf{1} = 0$, because his expected payoff from doing so is $\delta \frac{2\alpha + \delta\beta}{3}$ while that from disclosing one of S_1 is smaller since the probability of “winning” is lower (because agent 2 would take the correct action at $t + \Delta$ with probability $1/2$). Conversely, given $p_2(h_2^t) \cdot \mathbf{1} = 0$ and sufficiently low $q_2(h_2^t) \cdot \mathbf{1}$ with no previous moves, it is optimal for player 1 to disclose two of S_1 because his expected payoff from dosing so is $\delta \frac{2\beta + \delta\alpha}{3}$ while that from disclosing one of S_1 is smaller at $\delta \frac{\beta + 2\delta V_{2,2}^e}{3} = \delta \frac{\beta + \delta^2(\beta + \delta\alpha)}{3}$ by Proposition 1 (so long as $\beta + \delta\alpha > 0$), establishing equilibrium (ii).

Note from Lemma ?? that the discussion up to now also establishes that there is no other Markov equilibrium in which $p_i(h_i^t) \cdot \mathbf{1} > 0$ with no previous moves for either $i = 1, 2$.

It remains to consider the possibility that $p_i(h_i^t) \cdot \mathbf{1} = 0$ for both $i = 1, 2$. In this case, as before player 1’s expected payoff from disclosing two of S_1 , which is $\delta \frac{2\beta + \delta\alpha}{3}$, is larger than that from disclosing one. Denoting $q_2 = q_2(h_2^t)(1) > 0$ with no previous moves, his expected payoff from waiting for $t > 0$ before disclosing two is

$$V_{3,2}(t|q_2) := \delta \int_0^t q_2 e^{-(q_2+r)\tau} \frac{\alpha + \delta\beta}{2} d\tau + \delta e^{-(q_2+r)t} \frac{2\beta + \delta\alpha}{3}.$$

The equilibrium requires that player 1 is indifferent between disclosing two now and waiting a short period of time, i.e., $\left. \frac{dV_{3,2}(t|q_2)}{dt} \right|_{t=0} = 0$, from which we derive the value of $q_2(h_2^t)(1)$ in the lemma. Analogously, player 2’s expected payoff from waiting for $t > 0$ before disclosing one is

$$V_{2,3}(t|q_1) := \delta \int_0^t q_1 e^{-(q_1+r)\tau} \frac{2\alpha + \delta\beta}{3} d\tau + \delta e^{-(q_1+r)t} \frac{\beta + \delta\alpha}{2}$$

where $q_1 = q_1(h_1^t)(2) > 0$ with no previous moves. The equilibrium requires that player 2 is indifferent between disclosing one now and waiting a short period of time, i.e., $\left. \frac{dV_{2,3}(t|q_1)}{dt} \right|_{t=0} = 0$, from which we derive the value of $q_1(h_1^t)(2)$ in the lemma. Note that $q_1, q_2 > 0$ if $2\beta + \delta\alpha > 0$, i.e., $r\Delta < \log(-0.5\alpha/\beta)$, establishing a unique Markov equilibrium in flow strategies as specified in (iii). \square

Proposition 2 *There is $\hat{\delta} < 1$ such that, if $\delta > \hat{\delta}$, there are exactly the following two symmetric Markov equilibria when $\#(S_1) = \#(S_2) = 3$. In both, $p_i(h_i^t) \cdot \mathbf{1} = 0$ if $h_i^t = h_i^0$ or $h_i^t \succ h_i^0$ for $i = 1, 2$.*

(a) $q_i(h_i^t)(1) = r \frac{4\delta\alpha + (3+2\delta^2)\beta}{(\alpha-\beta)(3-4\delta+2\delta^2)}$ and $q_i(h_i^t)(2) = 0$ for $h_i^t \succ h_i^0$; and Lemma 6 (ii) represents the continuation equilibrium after disclosure by one player of one element $\omega \neq \omega^*$. The equilibrium payoff is $V_{3,3}^* := \delta \frac{\beta + 2\delta(2\alpha + \delta\beta)/3}{3}$.

(b) $q_i(h_i^t)(1) = 0$ and $q_i(h_i^t)(2) = r \frac{\delta\alpha + 2\beta}{(\alpha - \beta)(2 - \delta)}$ for $h_i^t \succ h_i^0$; and either Lemma 6 (i) or (iii) represents the continuation equilibrium after disclosure by one player of one element $\omega \neq \omega^*$. The equilibrium payoff is $V_{3,3} := \delta \frac{2\beta + \delta\alpha}{3} < V_{3,3}^*$.

Proof: Consider an arbitrary symmetric Markov equilibrium when $\#(S_1) = \#(S_2) = 3$. By Lemma ??, $p_i(h_i^t) \cdot \mathbf{1}$ is either 0 or 1 with no previous move. If it is 1, it is straightforwardly verified from Lemma 6 that the expected payoff from waiting is at least $\delta(\alpha + 2\delta \frac{2\beta + \delta\alpha}{3})/3$ which is larger than those from disclosing one or two elements, a contradiction. Thus, we deduce that $p_i(h_i^t) \cdot \mathbf{1} = 0$.

Consider the continuation equilibrium starting from $t + \Delta$ when one player, say 2, disclosed one element $\omega \in S_2$ at t . In light of Lemmas 1 and 2, both players know that the game ends by player 1 taking the correct action at $t + \Delta$ if $\omega = \omega^*$. Hence, the strategy of player 2 in the continuation game from $t + \Delta$ is entirely for the contingency that $\omega \neq \omega^*$. In the contingency that $\omega \neq \omega^*$, therefore, the continuation game is strategically equivalent to the game considered in Lemma 6.

First, consider the case that the equilibrium of this continuation game is Lemma 6 (ii). Then, the expected payoff of a player who discloses one element first (in the original game) is $\delta(\beta + 2\delta \frac{2\alpha + \delta\beta}{3})/3$ while that of disclosing two elements is lower at $\delta(2\beta + \delta\alpha)/3$ for sufficiently large δ . Thus, in equilibrium $q_i(h_i^t)(2) = 0$ and the expected payoff from waiting for $t > 0$ before disclosing one is

$$V_{3,3}^*(t|q^*) := \delta \int_0^t q^* e^{-(q^*+r)\tau} \frac{\alpha + 2\delta(2\beta + \delta\alpha)/3}{3} d\tau + \delta e^{-(q^*+r)t} \frac{\beta + 2\delta(2\alpha + \delta\beta)/3}{3}$$

where $q^* = q_i(h_i^t)(1) > 0$ with no previous moves. The equilibrium requires that $\frac{dV_{3,3}^*(t|q^*)}{dt} \Big|_{t=0} = 0$, from which we derive the value of $q_i(h_i^t)(1)$ in Proposition 2 (a).

Next, consider the case that Lemma 6 (i) or (iii) represents the continuation equilibrium after disclosure by one player of one element $\omega \neq \omega^*$. Then, the expected payoff of a player who discloses one element first (in the original game) is $\delta(\beta + 2\delta \frac{\beta + \delta\alpha}{2})/3$ while that of disclosing two elements is greater at $\delta(2\beta + \delta\alpha)/3$ for sufficiently large δ . Thus, in equilibrium $q_i(h_i^t)(1) = 0$ and the expected payoff from waiting for $t > 0$ before disclosing two is

$$V_{3,3}(t|q^*) := \delta \int_0^t q^* e^{-(q^*+r)\tau} \frac{2\alpha + \delta\beta}{3} d\tau + \delta e^{-(q^*+r)t} \frac{2\beta + \delta\alpha}{3}$$

where $q^* = q_i(h_i^t)(2) > 0$ with no previous moves. The equilibrium requires that $\frac{dV_{3,3}(t|q^*)}{dt} \Big|_{t=0} = 0$, from which we derive the value of $q_i(h_i^t)(2)$ in Proposition 2 (b). \square

5 Symmetric equilibrium for any $\#(S_1)$ and $\#(S_2)$

When the players start with more than three elements, a larger variety of equilibria are possible [briefly explain why]. Nevertheless, in Theorem 2 we characterize an efficient symmetric Markov equilibrium for the cases that the two players start with the same amount of information: They keep on disclosing information in turns, initially mixing

between disclosing one element and none until the first disclosure, after which they alternate disclosing two elements at a time until ω^* is identified. We report these findings formally in this section. In the sequel an $n \times n$ game means our game when $\#(S_1) = \#(S_2) = n \geq 2$.

First, we construct a Markov strategy, denoted by σ^* , that applies to any $\#(S_1)$ and $\#(S_2)$, i.e., even when $\#(S_1) \neq \#(S_2)$. Then, we verify that it indeed constitutes a Markov equilibrium in Theorem 1.

In order to describe σ^* it proves useful to recursively define the payoff $\phi(n)$ that a player receives when n elements remain in his opponent's set, $n - 1$ remain in his set, and starting from his opponent the players alternate disclosing two elements until ω^* is taken by one player either because it was disclosed by the other player or because he disclosed all but one element yet the other player did not take the correct action subsequently. Similarly, we define the payoff $\psi(n)$ that a player receives when n elements remain in his set, $n - 1$ elements remain in his opponent's set, and starting from himself the players alternate disclosing two elements until ω^* is taken by one player as explained above. Thus,

$$\begin{aligned}\phi(3) &:= \delta \left(\frac{2}{3}\alpha + \frac{\delta}{3}\beta \right), & \psi(3) &:= \delta \left(\frac{2}{3}\beta + \frac{\delta}{3}\alpha \right); \\ \phi(n) &:= \delta \left(\frac{2}{n}\alpha + \delta \frac{n-2}{n}\psi(n-1) \right), & \psi(n) &:= \delta \left(\frac{2}{n}\beta + \delta \frac{n-2}{n}\phi(n-1) \right), \quad n \geq 4.\end{aligned}$$

In $n \times n$ game for any $n \geq 3$, the expected payoff of a player from waiting until t before disclosing one element himself conditional on the other player disclosing one element at a flow rate $q_{n,n}^* > 0$ is:

$$V_{n,n}(t|q_{n,n}^*) := \delta \int_0^t q_{n,n}^* e^{-(q_{n,n}^*+r)\tau} \frac{\alpha + \delta(n-1)\psi(n)}{n} d\tau + \delta e^{-(q_{n,n}^*+r)t} \frac{\beta + \delta(n-1)\phi(n)}{n}.$$

The equilibrium requires that

$$(14) \quad \left. \frac{dV_{n,n}(t|q_{n,n}^*)}{dt} \right|_{t=0} = 0 \iff q_{n,n}^* = r \frac{\beta + \delta(n-1)\phi(n)}{\alpha + \delta(n-1)\psi(n)} > 0,$$

where the inequality follows because $\frac{\beta}{n} + \delta \frac{n-1}{n}\phi(n)$ is the payoff of a player in an $n \times n$ game when he first discloses one element and then the two players alternate in disclosing two elements each until the game ends, while $\frac{\alpha}{n} + \frac{n-1}{n}\delta\psi(n)$ is that when the other player starts by disclosing one element, which is larger than the former.⁴ Let $q_{2,2}^* = q_i$ in (13).

Using the functions ϕ , ψ and $q_{n,n}^*$, we now define the strategy $\sigma^* = (p^*, q^*)$ as below:

$$p^*(h_i^t)(\omega) = 1 \quad \text{if} \quad h_i^t \in H_i(\omega) \text{ or } h_i^t \in H'_i(\omega) \text{ and } D_i^{t-\Delta} \notin \iota(h_i^t);$$

⁴One can verify this by writing out the two values—they are slight modifications of $\bar{V}_{n,n}^*$ (2 pages later).

and for all other $h_i^t \in H_i$, denoting $n = \#(S_i^t)$ and $k = \#(S_{-i}^t)$,

$$\left\{ \begin{array}{ll} p^*(h_i^t)(2^{S_i^t} \setminus \{\emptyset\}) = 1 & \text{if } k = 1 \text{ and } D_{-i}^t \in \iota(h_i^t), \\ q^*(h_i^t)(1) = q_{2,2}^* & \text{if } n = k = 2, \\ q^*(h_i^t)(1) = q_{n,n}^* & \text{if } n = k > 2 \\ p^*(h_i^t)(2) = 1 & \text{if } n = k + 1 > 2 \\ p^*(h_i^t)(n-1) = 1 & \text{if } n > k = 2 \text{ and } \delta\alpha + (n-1)\beta > 0 \\ p^*(h_i^t) \cdot \mathbf{1} = q^*(h_i^t) \cdot \mathbf{1} = 0 & \text{if } n > k = 2 \text{ and } \delta\alpha + (n-1)\beta \leq 0 \\ p^*(h_i^t) \cdot \mathbf{1} = q^*(h_i^t) \cdot \mathbf{1} = 0 & \text{if } n = 2 < k \text{ and } \delta\alpha + (k-1)\beta > 0 \\ p^*(h_i^t)(1) = 1 & \text{if } n = 2 < k \text{ and } \delta\alpha + (k-1)\beta \leq 0 \\ p^*(h_i^t)(n-k+1) = 1 & \text{if } n > k > 2 \text{ and } (n-k+1)\beta + \delta(k-1)\phi(k) > 0 \\ p^*(h_i^t) \cdot \mathbf{1} = q^*(h_i^t) \cdot \mathbf{1} = 0 & \text{if } n > k > 2 \text{ and } (n-k+1)\beta + \delta(k-1)\phi(k) \leq 0 \\ p^*(h_i^t) \cdot \mathbf{1} = q^*(h_i^t) \cdot \mathbf{1} = 0 & \text{if } k > n > 2 \text{ and } (k-n+1)\beta + \delta(n-1)\phi(n) > 0 \\ p^*(h_i^t)(n-1) = 1 & \text{if } k > n > 2 \text{ and } (k-n+1)\beta + \delta(n-1)\phi(n) \leq 0 \\ & \text{and } \delta\alpha + (n-1)\beta > 0 \\ p^*(h_i^t) \cdot \mathbf{1} = q^*(h_i^t) \cdot \mathbf{1} = 0 & \text{if } k > n > 2 \text{ and } (k-n+1)\beta + \delta(n-1)\phi(n) \leq 0 \\ & \text{and } \delta\alpha + (n-1)\beta \leq 0, \end{array} \right.$$

with the understanding that $p^*(h_i^t) \cdot \mathbf{1} = 0$ if $q^*(h_i^t) \cdot \mathbf{1} > 0$.

Theorem 1 For any $\nu \geq 2$, there is $\Delta(\nu) > 0$ such that the strategy σ^* constitute a Markov equilibrium if $\max\{\#(S_1), \#(S_2)\} \leq \nu$ and $\Delta < \Delta(\nu)$.

Proof: In Appendix. □

So long as the expected payoff in the continuation game is positive in each period, the equilibrium payoff of σ^* at any state increases in δ . Let $\bar{V}_{n,k}^*$ denote the limit of the continuation payoff of player i after history h_i^t with $\#(S_i^t) = n$ and $\#(S_{-i}^t) = k$ as $\delta \rightarrow 1$, conditional on the game does not end at time t . Then, the limit value of the equilibrium payoff of σ^* is calculated as

$$(15) \quad \bar{V}_{N,N}^* = \frac{\alpha + \beta}{2} \text{ if } N \text{ is even; } \bar{V}_{N,N}^* = \frac{(N^2 - 1)\alpha + (N^2 + 1)\beta}{2N^2} \text{ if } N \text{ is odd.}$$

To verify this, note that if N is an even number, then

$$\begin{aligned} \bar{V}_{N,N}^* &= \frac{\beta}{N} + \frac{N-1}{N} \left(\frac{2\alpha}{N} + \frac{N-2}{N} \left(\frac{2\beta}{N-1} + \frac{N-3}{N-1} \left(\frac{2\alpha}{N-2} + \frac{N-4}{N-2} \left(\frac{2\beta}{N-3} + \dots \right. \right. \right. \right. \\ &\quad \left. \left. \left. \left. \dots + \frac{4}{6} \left(\frac{2\beta}{5} + \frac{3}{5} \left(\frac{2\alpha}{4} + \frac{2}{4} \left(\frac{\alpha + 2\beta}{3} \right) \right) \right) \right) \right) \right), \end{aligned}$$

so that the coefficients of α and β are

$$\begin{aligned} \alpha &: \frac{(N-1)2}{N^2} + \frac{(N-1)(N-2)(N-3)2}{N^2(N-1)(N-2)} + \dots + \frac{(N-1)\dots 4 \cdot 3 \cdot 2}{N^2(N-1)\dots 5 \cdot 4} + \frac{(N-1)\dots 4 \cdot 3 \cdot 2}{N^2(N-1)\dots 4 \cdot 3} \\ &= 2 \frac{(N-1) + (N-3) + (N-5) + \dots + 3 + 1}{N^2} = \frac{1}{2}; \\ \beta &: \frac{1}{N} + \frac{(N-1)(N-2)2}{N^2(N-1)} + \dots + \frac{(N-1)\dots 5 \cdot 4 \cdot 2}{N^2(N-1)\dots 6 \cdot 5} + \frac{(N-1)\dots 4 \cdot 3 \cdot 2 \cdot 2}{N^2(N-1)\dots 5 \cdot 4 \cdot 3} \\ &= \frac{N + 2(N-2) + 2(N-4) + 2(N-6) + \dots + 2 \cdot 4 + 2 \cdot 2}{N^2} = \frac{1}{2}. \end{aligned}$$

If N is an odd number, similarly we have

$$\bar{V}_{N,N}^* = \frac{\beta}{N} + \frac{N-1}{N} \left(\frac{2\alpha}{N} + \frac{N-2}{N} \left(\frac{2\beta}{N-1} + \frac{N-3}{N-1} \left(\frac{2\alpha}{N-2} + \frac{N-4}{N-2} \left(\frac{2\beta}{N-3} + \dots \right. \right. \right. \right. \\ \left. \left. \left. \left. \dots + \frac{4}{6} \left(\frac{2\alpha}{5} + \frac{3}{5} \left(\frac{2\beta}{4} + \frac{2}{4} \left(\frac{2\alpha+\beta}{3} \right) \right) \right) \right) \right) \right),$$

so that the coefficients of α and β are

$$\alpha : \frac{(N-1)2}{N^2} + \frac{(N-1)(N-2)(N-3)2}{N^2(N-1)(N-2)} + \dots + \frac{(N-1)\dots 5 \cdot 4 \cdot 2}{N^2(N-1)\dots 6 \cdot 5} + \frac{(N-1)\dots 4 \cdot 3 \cdot 2 \cdot 2}{N^2(N-1)\dots 5 \cdot 4 \cdot 3} \\ = \frac{2(N-1) + (N-3) + (N-5) + \dots + 4 + 2}{N^2} = \frac{N^2 - 1}{2N^2}; \\ \beta : \frac{1}{N} + \frac{(N-1)(N-2)2}{N^2(N-1)} + \dots + \frac{(N-1)\dots 4 \cdot 3 \cdot 2}{N^2(N-1)\dots 6 \cdot 5 \cdot 4} + \frac{(N-1)\dots 4 \cdot 3 \cdot 2 \cdot 1}{N^2(N-1)\dots 5 \cdot 4 \cdot 3} \\ = \frac{N + 2(N-2) + 2(N-4) + 2(N-6) + \dots + 2 \cdot 3 + 2 \cdot 1}{N^2} = \frac{N^2 + 1}{2N^2}.$$

Theorem 2 *In any symmetric Markov equilibrium of an $N \times N$ game, $V_{N,N}^e \leq \bar{V}_{N,N}^*$.*

Proof: In Appendix. □

Fix an odd $N \geq 5$. Consider the following non-Markov symmetric strategy of $N \times N$ game: Both players disclose one at $t = 0$ with probability 1; If neither disclosed ω^* , they play σ^* in the continuation $(N-1) \times (N-1)$ game from $t = \Delta$ onwards. This equilibrium is supported by the belief that if one player deviates by not disclosing at $t = 0$, then in the continuation equilibrium, the deviator discloses $k \geq 3$ at $t = \Delta$ with certainty, followed by the other disclosing k , then σ^* if the game continues. We call this the “penalty- k ” strategy.

Theorem 3 *For any odd $N \geq 5$, there is $\rho_N > 0$ and $k \in \{3, 4, \dots, N-1\}$ such that the penalty- k strategy is a symmetric equilibrium of $N \times N$ game for sufficiently large $\delta < 1$ if $|\alpha/\beta| > \rho_N$. As $\delta \rightarrow 1$, the corresponding equilibrium payoff tends to $\delta \frac{\alpha+\beta}{2}$.*

Proof: Fix odd $N \geq 5$. Consider the continuation penalty- k strategy when $\nu_1 = N$ and $\nu_2 = N-1$ where $k = N-1$: The payoff of player 1 from disclosing $N-1$ is clearly $\delta \frac{(N-1)\beta + \delta\alpha}{N}$ which is positive if $|\alpha/\beta| > N-1$. Thus, the penalty- k strategy when $\nu_1 = N$ and $\nu_2 = N-1$ is a continuation equilibrium, supported by the belief that player 2 never discloses any and player 1 discloses all but one whenever $\nu_1 \leq N$ and $\nu_2 = N-1$. Given this, the payoff of deviating by disclosing none at $t = 0$ of $N \times N$ game is $\frac{\delta}{N}(\alpha + (N-1)\delta \frac{(N-1)\beta + \delta\alpha}{N}) = \frac{\delta}{N^2}((N + (N-1)\delta^2)\alpha + (N-1)^2\delta\beta)$, which is clearly smaller than $\delta \frac{\alpha+\beta}{2}$ for sufficiently large $\delta < 1$. Since the equilibrium payoff obviously converges to $\delta \frac{\alpha+\beta}{2}$ as $\delta \rightarrow 1$, the penalty- k strategy is verified to be an equilibrium for sufficiently large $\delta < 1$ if $|\alpha/\beta| > \rho_N = N-1$, completing the proof.

It is worth noting that the bound ρ_N may be improved by considering other values of $k < N-1$. The continuation payoff of player 1 from disclosing k when $\nu_1 = N$ and $\nu_2 = N-1$ is $v_1(k) = \frac{\delta}{N}(k\beta + (N-k)\frac{k\alpha + \delta\psi(N-k)}{N-1})$. In the subsequent continuation game

with $\nu_1 = N - k$ and $\nu_2 = N - 1$, the continuation payoff of player 2 from disclosing k is $v_2(k) = \frac{\delta}{N-1}(k\beta + (N - k - 1)\delta\phi(N - k))$. It is straightforward to verify that $v_2(k) < v_1(k)$. Hence, the penalty- k strategy when $\nu_1 = N$ and $\nu_2 = N - 1$ is a continuation equilibrium if $v_2(k) \geq 0$, which generates a lower bound $\rho_N(k)$ on $|\alpha/\beta|$. If, in addition, the payoff of deviating by disclosing none at $t = 0$ in the $N \times N$ game, $\frac{\delta}{N}(\alpha + (N - 1)v_1(k))$ is smaller than $\delta\frac{\alpha+\beta}{2}$ for sufficiently large $\delta < 1$ if k is large enough. Find all such k and set $\rho_N = \min_{k \leq N-1} \rho_N(k)$. \square

6 When only elements of $\Omega \setminus S_i$ may be disclosed

Modify the model such that agent i may only send a message in $2^{\Omega \setminus S_i}$, i.e., agent i may only disclose elements in the complement of S_i , but not elements in S_i . Suppose that S_i 's are finite, $(S_1 \cup S_2) \subset \Omega$ and $S_1 \cap S_2 = \{\omega^*\}$.

Suppose there exists an equilibrium in which the agents disclose information, i.e., elements in $\Omega \setminus S_i$, such that there is a positive probability that either agent i discloses all elements of $S_{-i} \setminus \{\omega^*\}$ so that the other agent may identify ω^* . Then, because there are only a finite number of pieces of information to disclose, there is a history, \bar{h}_i^t , along the equilibrium path at which either agent, say 1, may make the “last” disclosure with positive probability, in the sense that if agent 1 indeed discloses then there may be no further disclosure by either player. However, by making the last disclosure, agent 1 may let agent 2 win with positive probability (this happens if, with the newly disclosed info, all but one element of S_2 have been revealed to be in $\Omega \setminus S_1$ by agent 1's disclosure), but agent 1 himself will not be helped by it because no further disclosure would be forthcoming. This contradicts the supposed equilibrium. Consequently, we conclude that no information disclosure takes place that has any payoff-relevant consequences in the considered environment.

Theorem 4 *If $\#(\Omega) < \infty$ and only elements of $\Omega \setminus S_i$ may be disclosed, then in any equilibrium no information disclosure takes place that leads to identification of ω^* with a non-zero probability.*

Note that the situation was different in the previous section when only elements in S_i may be disclosed: Then, the agent making the last disclosure still can win if the other does not identify the true state, because the fact that the other cannot identify conveys useful information for the disclosing agent.

As such, the key insight is that for there to be any welfare-enhancing disclosure of information, any player at any stage of disclosure should be able to maintain future prospect of useful disclosure by the other party in return. We demonstrate below, by an example, that this is possible even when only elements of $\Omega \setminus S_i$ may be disclosed if Ω is an infinite set.

Suppose that $\#(\Omega) = \#(\mathbb{N})$ where \mathbb{N} is the set of all natural numbers, and that $\#(S_1) = \#(S_2) = 2$ and $S_1 \cap S_2 = \{\omega^*\}$. Consider the following equilibrium:

- (i) Starting with player 1, the two players take turns in disclosing, without delay, a fixed fraction, say f_i for player i , of the elements in $\Omega \setminus S_i$ that have not been disclosed by either player yet.

- (ii) If either deviates by disclosing less than the equilibrium fraction, then there is no further disclosure by either player (which is clearly a continuation equilibrium). If either deviates by disclosing more, then the game continues as per (i).

As soon as the elements disclosed by a player i contains one element of S_{-i} , player $-i$ identifies the other element of S_{-i} to be ω^* and thus, takes the correct action after the lapse of Δ from the disclosure, ending the game. To verify the equilibrium, we need to show that at any point of disclosure along the equilibrium path, i.e., at time $t = n\Delta$ where $n = 0, 1, 2, \dots$, the player i who is to disclose then ($i = 1$ if n is even and $i = 2$ if odd) has no incentive to deviate.

First, note that deviation by disclosing more than the equilibrium fraction would increase the probability of player i losing as a result of the disclosure; however, conditional on such a disclosure not ending the game, it would not change the future prospects of winning and losing by either player according to (i) above. Therefore, disclosing more than the equilibrium fraction is not a beneficial deviation.

Next, if player i discloses less than the equilibrium fraction, then the expected payoff of player i is at most zero (because he has no chance of winning due to (ii) above), in particular, it is zero if he discloses none. Note that the equilibrium payoff of player i at every point in time at which he is to disclose, V_i^e , must satisfy $V_i^e = f_i\beta\delta + (1 - f_i)f_{-i}\alpha\delta^2 + (1 - f_i)(1 - f_{-i})V^e\delta^2$, i.e.,

$$(16) \quad V_i^e = \frac{f_i\beta\delta + (1 - f_i)f_{-i}\alpha\delta^2}{1 - (1 - f_i)(1 - f_{-i})\delta^2}.$$

Therefore, the strategy specified as (i) and (ii) above indeed constitutes an equilibrium if and only if $V^e \geq 0$, which is the case for sufficiently large $\delta < 1$ if $f_i\beta + (1 - f_i)f_{-i}\alpha > 0$, for instance, when $f_i = f_{-i} = 1/2$ due to (11). In this equilibrium, ω^* is identified with a probability arbitrarily close to 1 within a finite time (but not with certainty within any fixed finite time).

7 APPENDIX

Proof of Theorem 1:

The following notation proves useful: $V_{n,k}^*$ and $W_{n,k}^*$ denote the expected payoffs of agents i and $-i$, respectively, when $\#(S_i) = n$, $\#(S_{-i}) = k$, and they play σ^* . Thus, in particular,

$$(17) \quad \phi(n) = W_{n,n-1}^* \quad \text{and} \quad \psi(n) = V_{n,n-1}^* \quad \text{for} \quad n \geq 3.$$

By the NCL assumption, we have the following result which we take granted in the sequel: For any private history h_i^t , player i assigns a posterior probability of $1/\#(S_i^t)$ to $\omega \in S^{-i}$ for all $\omega \in S_i^t$.

The optimality of the part of strategy σ^* that involves taking an action or not is straightforward from Lemmas 1–4. In the sequel, therefore, we verify the optimality of the part of

strategy σ^* that involves disclosure. The following lemma, proved at the end of Appendix, is useful.

Lemma 7 *Suppose $\#(S_1) = n \geq \#(S_2) = k > 2$. For $d = 1, \dots, k - 1$, let $\Psi(d)$ be agent 1's payoff when he discloses $n - k + d$ elements first, after which both agents behave according to σ^* . Then, there exist $\delta(n, k) < 1$ such that $\Psi(1) > \Psi(d)$ for $d = 2, \dots, k - 1$ if $\delta(n, k) < \delta < 1$.*

Consider the case that $\nu \geq \#(S_1) = n > \#(S_2)$ at time $t \geq 0$. If $\#(S_2) = 1$, foreseeing that agent 2 will take the correct action anyway by Lemma 3, agent 1 finds it optimal to disclose none (out of indifference) unless some element if $D_{-i}^{t-\Delta} \neq \emptyset$, in which case it is optimal to disclose some element to delay “losing” the game.

Suppose $\#(S_2) = 2$. It is clearly not optimal for agent 1 to disclose all because then he will surely lose. If agent 1 discloses $n - 1$ elements, then either agent 2 takes the correct action (with probability $\frac{n-1}{n}$) or else agent 1 does subsequently. Hence, agent 1's payoff from disclosing $n - 1$ is

$$(18) \quad \delta \frac{(n-1)\beta + \delta\alpha}{n}$$

If agent 1 discloses $d < n - 1$ elements (so that $n - d \geq 2$), either agent 2 takes the correct action in the case that ω^* has been disclosed, or else subsequently agent 2 discloses zero or one element according to σ^* which is 0 if $d > d^* := \max\{d \mid \delta\alpha + (n - d - 1)\beta \leq 0\}$ and 1 otherwise.

Specifically, if agent 1 discloses $d > d^*$ elements, σ^* prescribes that subsequently agent 2 would disclose none but agent 1 would disclose $n - d - 1$ elements. Hence, agent 1's payoff from initially disclosing $d > d^*$ elements is

$$\frac{d}{n}\delta\beta + \frac{n-d}{n}\delta^2 V_{n-d,2}^* = \frac{d}{n}\delta\beta + \frac{n-d}{n}\delta^2 \frac{\delta\alpha + (n-d-1)\beta}{n-d} = \frac{\delta^2(\delta\alpha + (n-1)\beta) + d(\delta - \delta^2)\beta}{n}.$$

Observe that agent 1 can do better by initially disclosing $n - 1$ so long as this payoff is non-negative; Otherwise, agent 1 would be better off by disclosing none in the initial period (and continue to do so as long as there has been no disclosure by either agent), obtaining a payoff of 0.

If agent 1 initially discloses $d \leq d^*$ elements, on the other hand, agent 2 would subsequently disclose one element according to σ^* and consequently, either agent 1 then takes the correct action or else agent 2 does subsequently. Hence, agent 1's payoff from initially disclosing $d \leq d^*$ elements is

$$\delta \frac{d}{n}\beta + \frac{n-d}{n}\delta^2 \frac{\delta\alpha + \beta}{2}$$

which assumes its maximum, $\delta(\alpha + \beta)/2 > 0$, at $d = 0$. However, this is feasible if and only if $d^* \geq 0$, i.e., $\delta\alpha + (n - 1)\beta \leq 0$.

Combining the previous arguments, if $\delta\alpha + (n - 1)\beta \leq 0$ it is optimal for agent 1 to disclose none initially (followed by agent 2 disclosing one); if $\delta\alpha + (n - 1)\beta > 0$ it is optimal

for agent 1 to initially disclose $n - 1$. This confirms optimality of σ^* for $\#(S_1) = n > \#(S_2) = 2$.

Next, suppose $\nu \geq \#(S_1) = n > \#(S_2) = k > 2$. By Lemma 7, agent 1 initially disclosing $d > n - k + 1$ elements is dominated by disclosing $d = n - k + 1$ elements.

If agent 1 initially discloses $d < n - k$ elements (so that $n - d > k$), then subsequently agent 2 would either disclose $k - 1$ or none according to σ^* . If agent 2 discloses none, agent 1 then either disclose more element or none according to σ^* : in the former case agent 1 would have done better by initially disclosing $n - k + 1$ due to discounting, while in the latter initially disclosing d is obviously not optimal because the expected payoff is negative. If agent 2 discloses $k - 1$ in the next period, on the other hand, agent 1 is better off by disclosing none initially because then agent 2 would disclose $k - 1$ immediately as well according to σ^* . Hence, agent 1 initially disclosing $d \in \{1, \dots, n - k - 1\}$ elements is not optimal.

To determine the optimal disclosing strategy, therefore, we need to compare disclosing none, $n - k + 1$ elements, and $n - k$ elements. The payoff from the second is non-negative then that from the third is worse due to discounting, in which case disclosing $n - k + 1$ elements is optimal because the payoff from disclosing none is worse at 0; Otherwise, disclosing none is optimal because player 1 can guarantee a non-negative payoff by never disclosing any. This confirms optimality of σ^* for all $k < n$. Analogous arguments, which we omit here, verify optimality of σ^* for all $1 < \#(S_1) = n < \#(S_2) = k \leq \nu$.

Finally, to verify optimality of σ^* for $n = k > 2$, note that disclosing one and none are equivalent by definition of $\pi(n)$. That the payoff from disclosing $d > 1$ elements is lower can be verified straightforwardly from the observation above that disclosing none is optimal for a player when he has one less element than the other player. This complete the proof of Theorem 1.

Proof of Lemma 7. The Lemma is clear for $n = 3$, so we assume $n > 3$ below. Observe that

$$(19) \quad \Psi(1) = \frac{n - k + 1}{n} \delta \beta + \frac{k - 1}{n} \delta^2 \phi(k) \quad \text{and} \quad \Psi(d) = \frac{n - k + d}{n} \delta \beta + \frac{k - d}{n} \delta^2 W_{k-d,k}^*.$$

Consider $d \geq 2$ such that agent 2 would disclose none subsequently. If agent 1 would disclose none subsequent as well, then $W_{k-d,k}^* = 0$; If agent 1 would disclose all but one subsequently, then $W_{k-d,k}^* = \delta \left(\frac{k-d-1}{k-d} \beta + \frac{1}{k-d} \delta \alpha \right) \leq \phi(k-d+1) < \phi(k)$ where the last two inequalities can be verified by routine calculations. In either case, $\Psi(1) > \Psi(d)$ follows from (19) because $\phi(k) > 0$.

Next, consider $d \geq 2$ such that agent 2 would disclose $d + 1$ subsequently. This would imply that if player 1 disclosed one less element then agent 2 would disclose d elements subsequently, because $(d+1)\beta + \delta(k-d-1)\phi(k-d) > 0$ implies $d\beta + \delta(k-d)\phi(k-d+1) > 0$ if $d < k - 2$, or $(k-1)\beta + \alpha > 0$ implies $(k-2)\beta + \delta 2\phi(3) > 0$ if $d = k - 2$, as is easily verified by routine calculations. Hence, it suffices to show that $\Psi(d') > \Psi(d)$ where $d' = d - 1$. If $(k - d')$ is an even number, the exante winning probabilities for agent 1 when he initially

discloses d' and d elements are, respectively,

$$\begin{aligned}\rho(d') &= \frac{k-d'}{n} \cdot \frac{d'+1}{k} + \frac{k-d'-2}{n} \cdot \frac{2}{k} + \cdots + \frac{4}{n} \cdot \frac{2}{k} + \frac{2}{n} \cdot \frac{2}{k} \\ &= \frac{2(2+4+\cdots+(k-d')) + (k-d')(d'-1)}{nk}\end{aligned}$$

and

$$\begin{aligned}\rho(d) &= \frac{k-d'-1}{n} \cdot \frac{d'+2}{k} + \frac{k-d'-3}{n} \cdot \frac{2}{k} + \cdots + \frac{3}{n} \cdot \frac{2}{k} + \frac{1}{n} \cdot \frac{2}{k} \\ &= \frac{2(1+3+\cdots+(k-d'-1)) + (k-d'-1)d'}{nk} \\ &= \frac{2(2+4+\cdots+(k-d') - \frac{k-d'}{2}) + (k-d')(d'-1) + k - 2d'}{nk} \\ &= \frac{2(2+4+\cdots+(k-d')) + (k-d')(d'-1) - d'}{nk} < \rho(d').\end{aligned}$$

Since one of the agents should win,

$$\Psi(d') \rightarrow \rho(d')\alpha + (1 - \rho(d'))\beta \quad \text{and} \quad \Psi(d) \rightarrow \rho(d)\alpha + (1 - \rho(d))\beta \quad \text{as} \quad \delta \rightarrow 1.$$

As desired, therefore, $\Psi(d') > \Psi(d)$ for sufficiently large δ because $\rho(d') > \rho(d)$.

Analogously, if $(k-d')$ is an odd number, the ex ante winning probabilities for agent 1 when he initially discloses d' and d elements are, respectively,

$$\begin{aligned}\rho(d') &= \frac{k-d'}{n} \cdot \frac{d'+1}{k} + \frac{k-d'-2}{n} \cdot \frac{2}{k} + \cdots + \frac{3}{n} \cdot \frac{2}{k} + \frac{1}{n} \cdot \frac{2}{k} \\ &= \frac{2(1+3+\cdots+(k-d')) + (k-d')(d'-1)}{nk} \\ &= \frac{2(2+4+\cdots+(k-d'+1) - \frac{k-d'+1}{2}) + (k-d')(d'-1)}{nk} \\ &= \frac{2(2+4+\cdots+(k-d'-1)) + (k-d')d' + 1}{nk}\end{aligned}$$

and

$$\begin{aligned}\rho(d) &= \frac{k-d'-1}{n} \cdot \frac{d'+2}{k} + \frac{k-d'-3}{n} \cdot \frac{2}{k} + \cdots + \frac{4}{n} \cdot \frac{2}{k} + \frac{2}{n} \cdot \frac{2}{k} \\ &= \frac{2(2+4+\cdots+(k-d'-1)) + (k-d'-1)d'}{nk} < \rho(d').\end{aligned}$$

Since one of the agents should win,

$$\Psi(d') \rightarrow \rho(d')\alpha + (1 - \rho(d'))\beta \quad \text{and} \quad \Psi(d) \rightarrow \rho(d)\alpha + (1 - \rho(d))\beta \quad \text{as} \quad \delta \rightarrow 1$$

and, therefore, $\Psi(d') > \Psi(d)$ for sufficiently large δ as desired. \square

Proof of Theorem 2:

For any even N , $V_{N,N}^e \leq \bar{V}_{N,N}^* = \frac{\alpha+\beta}{2}$ follows because the total surplus is bounded above by $\alpha + \beta$.

For odd N , first we establish that

[A] $p_i(h_i^t) \cdot \mathbf{1} > 0$ cannot hold for $i = 1, 2$ simultaneously in any symmetric Markov equilibrium.

To reach a contradiction, consider an h^t such that $p_i(h_i^t) \cdot \mathbf{1} > 0$ for $i = 1, 2$ with “minimal” $\#(S_i^t) = \nu_i$, i.e., $p_i(h_i^t) \cdot \mathbf{1} = 0$ for at least one $i = 1, 2$ for any history h^t such that $\#(S_i^t) \leq \nu_i$ for $i = 1, 2$ with at least one strict inequality. Suppose wlog that $p_1(h_1^t) \cdot \mathbf{1} \leq p_2(h_2^t) \cdot \mathbf{1}$. Below, we compare player 1’s expected payoff from disclosing none at t with that of equilibrium payoff for three alternative contingencies.

(i) Consider the (equilibrium) contingency that player 2 disclosed none at t . Then, $p_2(h_2^t) \cdot \mathbf{1} < 1$. By Lemma ??, therefore, having disclosed none at t , player 1 can take any move m such that $p_1(h_1^t)(m) > 0$ “immediately after t ” and thereby, guarantee a payoff arbitrarily close to the expected payoff he would get by following the equilibrium strategy conditional on player 2 not disclosing any element at t .

(ii) In the contingency that player 2 disclosed ω^* , player 1 would be clearly strictly better off by disclosing none at t than by following the equilibrium strategy conditional on player 2 disclosing ω^* at t .

(iii) Consider the contingency that player 2 discloses a subset of elements not containing ω^* . For concreteness, let k ($< v_2$) denote the number of elements that player 2 has remaining at $t + \Delta$, so that the continuation game from $t + \Delta$ is the subgame in which player 1 starts with v_1 elements and 2 with k elements, which we denote by $\Gamma_{v_1,k}$. If player 2 were not to take a “spot” move at the beginning of $\Gamma_{v_1,k}$, player 1 can take any move m such that $p_1(h_1^t)(m) > 0$ and thereby, guarantee a payoff arbitrarily close to the expected payoff he would get by following the equilibrium strategy conditional on player 2 having disclosed $v_2 - k$ elements not containing ω^* .

If player 2 were to take a spot move at the beginning of $\Gamma_{v_1,k}$, i.e., he is suppose to disclose some elements “immediately after $t + \Delta$ ” with probability 1. Then, player 1 is not to take a spot move “immediately after $t + \Delta$ ” due to h^t being “minimal”. Let $a > 0$ denote the number of elements that player 2 would disclose immediately in $\Gamma_{v_1,k}$.

Consider the contingency that player 1 discloses ℓ elements at time t as per the supposed equilibrium, in which case the continuation subgame would be $\Gamma_{v_1-\ell,k}$, conditional on the game not having ended.

First, consider the case that player 2 may disclose $b > 0$ elements immediately in the continuation equilibrium of $\Gamma_{v_1-\ell,k}$, either by spot or flow strategy, which implies that $V_{v_1-\ell,k}^2(a) \leq V_{v_1-\ell,k}^2(b)$ where $V_{v_1-\ell,k}^i(a)$ is the continuation equilibrium payoff of player $i = 1, 2$, in the subgame $\Gamma_{v_1-\ell,k}$ when player 2 start the subgame by disclosing a elements, and similarly for $V_{v_1-\ell,k}^i(b)$. Since subsequently in $\Gamma_{v_1-\ell,k}$ players alternate in disclosing until ω^* is identified, i.e., no flow strategy will be employed (to be elaborated; this also means that

selections of a and b (when player 2 mixes) are inconsequential because the payoffs are the same), it further follows that $V_{v_1-\ell,k}^1(a) \geq V_{v_1-\ell,k}^1(b)$. Note that, for the contingency considered in (iii), player 1's payoff from disclosing ℓ elements at t is at most $\frac{\ell}{v_1}\beta + \frac{v_1-\ell}{v_1}\delta V_{v_1-\ell,k}^1(b)$, while that from disclosing none at t is no lower than $\delta(\frac{\ell}{v_1}\beta + \frac{v_1-\ell}{v_1}V_{v_1-\ell,k}^1(a))$ because subsequent to disclosure of a elements by player 2 at $t + \Delta$, player 1 can disclose ℓ more elements than he is supposed to do in $\Gamma_{v_1-\ell,k}$ subsequent to the same event (i.e., player 2 disclosing a of k elements). Note that $\delta(\frac{\ell}{v_1}\beta + \frac{v_1-\ell}{v_1}V_{v_1-\ell,k}^1(a)) > \frac{\ell}{v_1}\beta + \frac{v_1-\ell}{v_1}\delta V_{v_1-\ell,k}^1(b)$, i.e., player 1 is better off by disclosing none at t .

Second, consider the case that player 2 does not disclose immediately in the continuation equilibrium of $\Gamma_{v_1-\ell,k}$, either by spot or flow strategy. Then, player 1 should disclose with a spot probability, say ℓ' elements, after which the continuation game $\Gamma_{v_1-\ell-\ell',k}$ starts with player 2 disclosing some elements with a spot probability. Therefore, the same argument applies as before (i.e., with $\ell + \ell'$ assuming the role played by ℓ above) to establish the same conclusion.

Combining (i)–(iii), we have reached a contradiction to $p_1(h_1^t) \cdot \mathbf{1} > 0$ and thus, have proved [A]. For any symmetric Markov equilibrium σ , therefore, we can construct a specific equilibrium path as below, bearing in mind that neither player would take any action unless the other player has disclosed ω^* or he has disclosed all but one element yet the other player has not taken ω^* subsequently according to Lemmas ??–??.

i) At $t = 0$, since $p_i(h_i^0) \cdot \mathbf{1} = 0$ by [A], let $d_1 \in \text{supp}(q_i(h_i^0))$ such that $\frac{d_1}{N}\delta\alpha + \delta^2 V_{N,N-d_1}^e \geq V_{N,N}^e$, where $V_{n,k}^e$ is the continuation payoff of the player associated with σ in the continuation subgame in which he has n elements and the other player has k elements left undisclosed, conditional on ω^* has not been disclosed yet. Consider the path in which player 1 discloses d_1 at $t = 0$.

ii) At $t = \Delta$, note that $p_1(h_1^\Delta) = 0$ because otherwise player 1 would have been better off by disclosing $d_1 + d$ at $t = 0$ due to discounting where $d \in \text{supp}(p_1(h_1^\Delta))$. Conditional on $h_2^\Delta \notin H_2(\omega)$, if $p_2(h_2^\Delta) \cdot \mathbf{1} = 1$ then let $d_2 \in \text{supp}(p_2(h_2^\Delta))$ such that $\frac{d_2}{N}\delta\alpha + \delta^2 V_{N-d_1,N-d_2}^e \geq V_{N-d_1,N}^e$; if $p_2(h_2^\Delta) \cdot \mathbf{1} < 1$ then let $d_2 \in \text{supp}(p_2(h_2^\Delta)) \cup \text{supp}(q_2(h_2^\Delta))$ such that $\frac{d_2}{N}\delta\alpha + \delta^2 V_{N-d_1,N-d_2}^e \geq V_{N-d_1,N}^e$. Consider the path in which player 2 discloses d_2 at $t = \Delta$.

iii) Analogously, for any odd $n = 3, 5, \dots$, conditional on $h_1^{(n-1)\Delta} \notin H_1(\omega)$, let $d_n \in \text{supp}(p_1(h_1^{(n-1)\Delta}))$ such that $\frac{d_n}{S_1^{(n-1)\Delta}}\delta\alpha + \delta^2 V_{S_2^{(n-1)\Delta}, S_1^{(n-1)\Delta}-d_n}^e \geq V_{S_2^{(n-1)\Delta}, S_1^{(n-1)\Delta}}^e$ if $p_1(h_1^{(n-1)\Delta}) \cdot \mathbf{1} = 1$; and let $d_n \in \text{supp}(p_1(h_1^{(n-1)\Delta})) \cup \text{supp}(q_1(h_1^{(n-1)\Delta}))$ such that $\frac{d_n}{S_1^{(n-1)\Delta}}\delta\alpha + \delta^2 V_{S_2^{(n-1)\Delta}, S_1^{(n-1)\Delta}-d_n}^e \geq V_{S_2^{(n-1)\Delta}, S_1^{(n-1)\Delta}}^e$ if $p_1(h_1^{(n-1)\Delta}) \cdot \mathbf{1} < 1$.

iv) Likewise, for any even $n = 4, 6, \dots$, conditional on $h_2^{(n-1)\Delta} \notin H_2(\omega)$, let $d_n \in \text{supp}(p_2(h_2^{(n-1)\Delta}))$ such that $\frac{d_n}{S_2^{(n-1)\Delta}}\delta\alpha + \delta^2 V_{S_1^{(n-1)\Delta}, S_2^{(n-1)\Delta}-d_n}^e \geq V_{S_1^{(n-1)\Delta}, S_2^{(n-1)\Delta}}^e$ if $p_2(h_2^{(n-1)\Delta}) \cdot \mathbf{1} = 1$; and let $d_n \in \text{supp}(p_2(h_2^{(n-1)\Delta})) \cup \text{supp}(q_2(h_2^{(n-1)\Delta}))$ such that $\frac{d_n}{S_2^{(n-1)\Delta}}\delta\alpha + \delta^2 V_{S_1^{(n-1)\Delta}, S_2^{(n-1)\Delta}-d_n}^e \geq V_{S_1^{(n-1)\Delta}, S_2^{(n-1)\Delta}}^e$ if $p_2(h_2^{(n-1)\Delta}) \cdot \mathbf{1} < 1$.

Let $\hat{\sigma}$ denote the path constructed as above. By finiteness, there is $T < \infty$ such that d_T is the last of potential disclosures. Note that $d_T = S_i^{(T-1)\Delta} - 1$ where $i = 1$ if T is odd and $i = 2$ otherwise, because, given that there will be no further disclosure in the future,

disclosing less than $S_i^{(T-1)\Delta} - 1$ elements at $t = (T - 1)\Delta$ would only enhance the chance of the other player winning without any chance for player i to win and thus, would be suboptimal for player i . Thus, by construction, player 1's expected payoff from $\hat{\sigma}$ is

$$\begin{aligned}
\hat{V}^1 &= \frac{d_1\beta}{N}\delta + \frac{N-d_1}{N}\left(\frac{d_2\alpha}{N}\delta^2 + \frac{N-d_2}{N}\left(\frac{d_3\beta}{S_1^{2\Delta}}\delta^3 + \frac{S_1^{2\Delta}-d_3}{S_1^{2\Delta}}\left(\frac{d_4\alpha}{S_2^{3\Delta}}\delta^4 + \frac{S_2^{3\Delta}-d_4}{S_2^{3\Delta}}\left(\frac{d_5\beta}{S_1^{4\Delta}}\delta^5 + \dots\right.\right.\right.\right. \\
&\quad \left.\left.\left.\left.\dots + \frac{S_i^{(T-3)\Delta}-d_{T-2}}{S_i^{(T-3)\Delta}}\left(\frac{d_{T-1}\beta}{S_{-i}^{(T-2)\Delta}}\delta^{T-1} + \frac{S_{-i}^{(T-2)\Delta}-d_{T-1}}{S_{-i}^{(T-2)\Delta}}\left(\frac{d_T\beta\delta^T + \alpha\delta^{T+1}}{S_i^{(T-1)\Delta}}\right)\right)\right)\right)\dots\right) \\
&= \frac{d_1\beta}{N}\delta + \frac{S_1^{2\Delta}}{N}\left(\frac{d_2\alpha}{N}\delta^2 + \frac{S_2^{3\Delta}}{N}\left(\frac{d_3\beta}{S_1^{2\Delta}}\delta^3 + \frac{S_1^{4\Delta}}{S_1^{2\Delta}}\left(\frac{d_4\alpha}{S_2^{3\Delta}}\delta^4 + \frac{S_2^{5\Delta}}{S_2^{3\Delta}}\left(\frac{d_5\beta}{S_1^{4\Delta}}\delta^5 + \dots\right.\right.\right.\right. \\
&\quad \left.\left.\left.\left.\dots + \frac{S_i^{(T-1)\Delta}}{S_i^{(T-3)\Delta}}\left(\frac{d_{T-1}\beta}{S_{-i}^{(T-2)\Delta}}\delta^{T-1} + \frac{S_{-i}^{(T-2)\Delta}-d_{T-1}}{S_{-i}^{(T-2)\Delta}}\left(\frac{d_T\beta\delta^T + \alpha\delta^{T+1}}{S_i^{(T-1)\Delta}}\right)\right)\right)\right)\dots\right) \\
&\rightarrow \frac{c_\alpha\alpha + c_\beta\beta}{N^2} \quad \text{as } \delta \rightarrow 1
\end{aligned}$$

where c_α and c_β are nonnegative integers. Furthermore, $c_\alpha + c_\beta = N^2$ because exactly one player will have won by end of the game, which also implies that $\lim_{\delta \rightarrow 1} \hat{V}^2 = \alpha + \beta - \lim_{\delta \rightarrow 1} \hat{V}^1 = \frac{c_\beta\alpha + c_\alpha\beta}{N^2}$. In addition, note that $\hat{V}^1 \geq V_{N,N}^e$ and $\hat{V}^2 \geq V_{N,N}^e$ from construction.

To reach a contradiction, suppose that $V_{N,N}^e > \bar{V}_{N,N}^* = \frac{(N^2-1)\alpha + (N^2+1)\beta}{2N^2}$. Then, $\lim_{\delta \rightarrow 1} \hat{V}^1 \geq \frac{(N^2+1)\alpha + (N^2-1)\beta}{2N^2}$ because $\hat{V}^1 \geq V_{N,N}^e$ and c_α and c_β are integers such that $c_\alpha + c_\beta = N^2$. This would imply that $\lim_{\delta \rightarrow 1} \hat{V}^2 = \alpha + \beta - \lim_{\delta \rightarrow 1} \hat{V}^1 \leq \frac{(N^2-1)\alpha + (N^2+1)\beta}{2N^2} = \bar{V}_{N,N}^* < V_{N,N}^e$, a contradiction. This completes the proof. \square

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