

# Delayed Capital Reallocation

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## Abstract

Why is capital reallocation across firms procyclical and more volatile than investment? To answer the question, this paper develops and estimates a tractable dynamic general equilibrium model. In the model, firms face idiosyncratic productivity risks and are subject to partial capital irreversibility and financing constraints. Partial capital irreversibility generates selling delays. I show that financing constraints interact with irreversibility and prolong the delays. As a result, reallocation activity is slowed down during recessions when financing constraints are tighter.

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# 1 Introduction

The reallocation of input factors from less productive to more productive firms is necessary for the efficient usage of these factors in production, especially after an economy is hit by adverse shocks. Unfortunately, during the past three recessions, the level and the turnover activity of capital reallocation appear to have been lower than in normal times. In fact, reallocation of existing productive capital is highly procyclical (Figure 1).<sup>1</sup> This observation is in sharp contrast with [Schumpeter \(1942\)](#)’s “creative destruction” view<sup>2</sup>: more capital stock should be liquidated and more restructuring should take place in recessions.

While the business cycle literature has studied the role of aggregate productivity and financing constraints shocks in explaining some business cycle dynamics such as GDP and investment,<sup>3</sup> the effects of these shocks on capital reallocation are rarely explored. This is an important oversight because capital reallocation is of the same magnitude of investment,<sup>4</sup> and is actually more volatile (the ratio of standard deviations to that of GDP are 10.91 and 3.86 respectively). The dynamics of reallocation can potentially provide us information on the sources of business cycles.

I construct a tractable general equilibrium model in which firms face aggregate productivity and aggregate financing constraint shocks, in addition to persistent idiosyncratic productivity risks. Firms also face fixed costs of operating and partial capital irreversibility (assets will be sold at discount in liquidation).<sup>5,6</sup> The financing constraints interact with irreversibility and generate capital reallocation delays from unproductive to productive firms. After credit crunches, the delays are prolonged and the TFP dispersion among firms expands.

In this economy, idiosyncratic productivity shocks generate the benefits from reallocating capital stock. Productive firms expand by borrowing, but in the presence of collateral

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<sup>1</sup>Following [Eisfeldt and Rampini \(2006\)](#), capital reallocation includes sales of property, plants, and equipment and acquisitions from the COMPUSTAT database. [Jovanovic and Rousseau \(2002\)](#) also use this measure to study the purchase of used assets.

<sup>2</sup>In general, “creative destruction” is essential for long-run productivity growth (see e.g., [Caballero and Hammour \(1994\)](#) and the reference therein), its role in recessions are controversial. For example, restructuring activity typically declines during recessions as shown in [Caballero and Hammour \(2000, 2005\)](#).

<sup>3</sup>See, for example, [King and Rebelo \(1999\)](#) and [Nolan and Thoenissen \(2009\)](#).

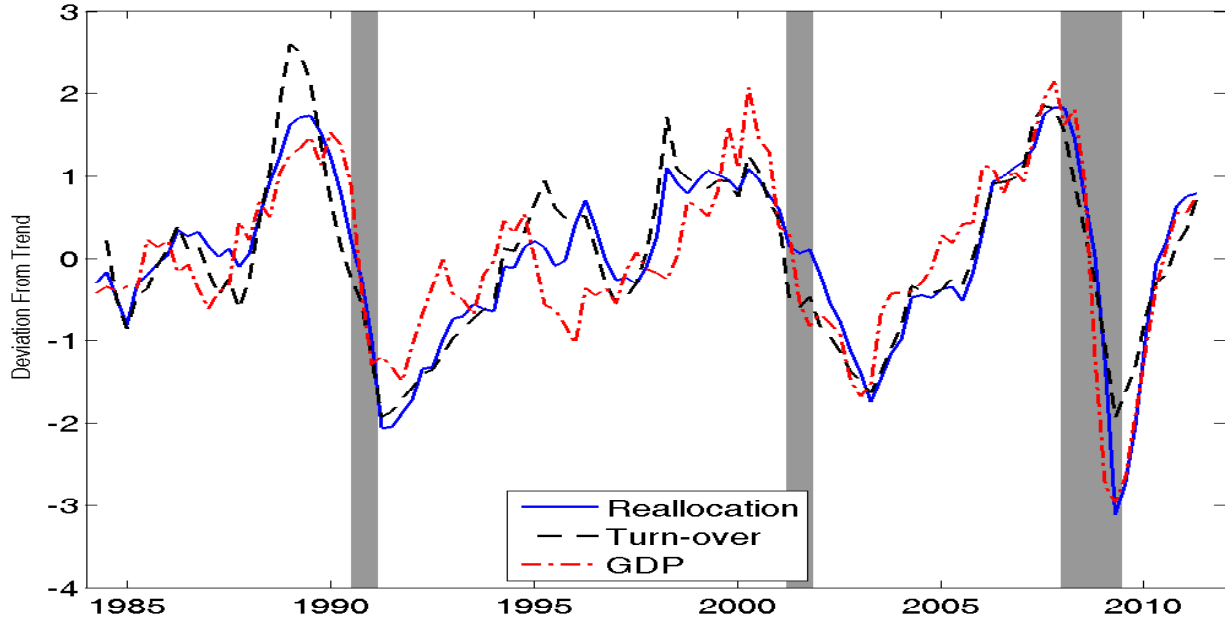
<sup>4</sup>In 2011, for example, the reallocation from COMPUSTAT is about \$0.65 trillion whereas the total U.S. fixed investment is about \$1.6 trillion. Non-listed firms probably buy more used assets according to [Eisfeldt and Rampini \(2007\)](#). In sum, capital reallocation is comparable to new investment.

<sup>5</sup>[Shleifer and Vishny \(1992\)](#) summarize two usual reasons for resale costs. First, when firms are liquidating, the potential buyers with the highest valuation are often those in the same industry who generally also have financial troubles. Assets may not go to the highest valuation users. Second, because of antitrust reasons, assets may need to be sold to industry outsiders, causing lower values for assets.

<sup>6</sup>[Ramey and Shapiro \(2001\)](#) provide empirical evidence of investment specificity and selling costs. They estimate the wedge between purchase price and resale price for different types of capital. Machine tools are sold at about a 69% discount off the purchase value, and structural equipment is sold at a 95% discount. These estimates suggest a large degree of specificity.

**Figure 1: Capital reallocation over cycles**

The series plotted are cyclical components of HP-filtered log data (quarterly frequency) normalized by standard deviations. Solid line represents seasonally adjusted reallocation. Following [Eisfeldt and Rampini \(2006\)](#) who use data with annual frequency, capital reallocation includes the sum of sales of property, plant, and equipment (SPPE) and acquisition (AQC) in 2005 dollars. Dashed line represents the turnover of reallocation as percentage of total assets. Dashed dotted line represents real GDP in 2005 dollars. Shaded regions denote NBER recessions. See Tables 3 and 4 in the Appendix for more detailed statistics. The ratio of standard deviation of capital reallocation to that of GDP is 10.91, which is much higher than the standard deviation of investment to that of GDP (3.86). Note that the standard deviation of GDP fluctuation is 1.42%.



constraints not all of the capital stock can necessarily be reallocated. Firms with falling productivity, are hesitant to sell assets because of the resale discount, and might gamble on the hope that they will regain higher productivity soon. But once these firms have low productivity today, they are likely to have low productivity tomorrow because of persistence. They thus face a cost of borrowing that is much higher than the expected return from capital. Hence, these firms let their capital stock depreciate while paying down existing debt by shrinking dividends (modelled as entrepreneurs' consumption). If these firms keep drawing low productivity, profitability stays low and the firm gradually shrink. Eventually, they give up capital when the option value of maintaining the depreciated capital is too low to compensate for the fixed costs of operation.

The key finding is that shocks to financing constraints are necessary to reduce capital reallocation in recessions, while aggregate TFP shocks are needed for the dynamics of other macro variables. A negative aggregate TFP shock reduces the value of staying in business for all firms and counterfactually increases liquidation i.e., capital reallocation. In contrast,

a negative credit shock that reduces firms' ability to borrow has a different impact on productive and unproductive firms. Productive firms would like to invest but can borrow less due to the credit crunch. Unproductive firms are not credit unconstrained but face reduced costs of borrowing and hiring labor due to less competition from productive ones (a general equilibrium effect on reducing wages and interest rates). Therefore, these unproductive firms further delay their liquidation decisions. Finally, only a mild correlation between the two aggregate shocks is required to match the aggregate dynamics.

Because capital reallocation slows down during recessions, the idiosyncratic TFP dispersion across firms expands and aggregate TFP declines with the tightened financing constraints, thus leading to a deeper recession. Therefore, capital irreversibility cause delay in selling and tightened financing constraints interact with this irreversibility and generate an even longer delay, which helps explain why capital reallocation slows down in spite of larger potential benefits to reallocate during recessions. A major credit crunch after a banking crisis, such as the one in the U.S. in 2008, exemplifies these interactions.<sup>7</sup>

Finally, I discuss the effect of uncertainty shocks in the spirit of [Bloom \(2009\)](#) which can be interpreted as an increase in cross-sectional idiosyncratic productivity dispersion. In my model, firms can save or borrow risk-free assets. The value after liquidation is endogenous because of the endogenous rate of return on bonds. Then, a mean preserving spread shock to idiosyncratic productivity will have an ambiguous impact on capital reallocation, as the value after liquidation can be either higher or lower. However, if uncertainty shocks in recessions also generate tighter financing constraints, capital reallocation will likely to decline. Recent work by [Gilchrist, Sim, and Zakrajsek \(2010\)](#) and [Christiano, Motto, and Rostagno \(2014\)](#) show how financing constraints could become tighter when dispersions among firms are larger using costly-state-verification contracts as in [Bernanke, Gertler, and Gilchrist \(1999\)](#).

The contribution is to consider the interaction of financing constraints and capital irreversibility. Without irreversibility, there will not be a delay of selling used capital. Without financing constraints, productive firms can borrow freely and push up both the interest rate and the wage rate, leaving a very small incentive for unproductive firms to delay selling.

The technical innovation might be of some independent interest. Solution to a macro model with capital irreversibility are usually complex and sometimes infeasible with aggregate shocks (not to mention estimation), since the evolving distribution of firms is a high dimension object.<sup>8</sup> I simplify individual firm's problem by deriving a closed-form portfolio

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<sup>7</sup>U.S. economy after 2008 experiences similar massive deleveraging in Japan after 1990, summarized in [Shirakawa \(2012\)](#). [Koo \(2011\)](#) calls this type of recessions "balance sheet recessions". Meanwhile, Japanese corporate sector has substantial less restructuring found by [Hoshi, Koibuchi, and Schaefer \(2011\)](#). Thus, linking financing constraints and capital reallocation sheds some light on corporate balance sheet adjustments.

<sup>8</sup>See, for example, [Bloom, Bond, and Reenen \(2007\)](#), [Bloom \(2009\)](#), and [Khan and Thomas \(2011\)](#), who use piece-wise functions to approximate individual value functions.

choice between bonds and capital stock with (real) “option values”, using finance portfolio choice theory as in [Campbell and Viceira \(2002\)](#).<sup>9</sup> Individuals’ decision rules are easily aggregated and the evolving distribution of firms are tractable, which leads to exact aggregation and avoids the approximation method such as the one in [Krusell and Smith \(1998\)](#).

**Literature Review.** [Dixit and Pindyck \(1994\)](#) and [Caballero and Engel \(1999\)](#) focus on the timing of irreversible investment. This paper focuses on asset selling. Since assets may turn to be productive, running unproductive firms has an option value which may exceed the resale value. I show how to directly quantify the option value which is history dependent and summarized in firms’ leverage ratios.

This paper shows how standard credit crunches can increase the dispersion endogenously through general equilibrium. Implication of shocks to the dispersion of firm-specific conditions can be found, for example, in [Bloom \(2009\)](#), [Gilchrist, Sim, and Zakrajsek \(2010\)](#), [Panousi and Papanikolaou \(2012\)](#), and [Christiano, Motto, and Rostagno \(2014\)](#). But [Bachmann and Bayer \(2012a,b\)](#) show that large dispersion shocks are difficult to reconcile with other observations such as the investment rate dispersion. Similarly, [Bachmann and Moscarini \(2011\)](#) study endogenous dispersion through firms’ risk-taking behaviors.

Further literature of macroeconomic implications of asset illiquidity and implications of financing constraints can be found in surveys by [Caballero \(1999\)](#) for capital illiquidity,<sup>10</sup> and [Brunnermeier, Eisenbach, and Sannikov \(2012\)](#) for financing constraints. Whether asset illiquidity or financing constraints can quantitatively amplify TFP and output losses is a matter of some debate.<sup>11,12</sup> [Buera and Moll \(2012\)](#) shows that how a credit crunch can distort the margin of efficiency, investment, and labor supply altogether.

Finally, innovation of this paper is to consider the interactions between capital irreversibility and financing constraints, which is related to [Kurlat \(2011\)](#) and [Khan and Thomas \(2011\)](#). [Kurlat \(2011\)](#) shows analytically why the secondary market for existing capital may

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<sup>9</sup>I follow and extend previous works by [Angeletos \(2007\)](#), [Kiyotaki and Moore \(2011\)](#), and [Buera and Moll \(2012\)](#). Under the class of CRRA preferences, if individual production functions feature constant returns to scale, the wealth spent on capital and bonds is simplified to a portfolio choice between the two.

<sup>10</sup>Partial irreversibility is important. Previous work on investment irreversibility focuses on zero resale value, or completely irreversible investment, such as in [Abel and Eberly \(1996, 1999\)](#) and [Thomas \(2002\)](#). With zero resale value, firms only consider when to buy instead of when to sell.

<sup>11</sup>[Thomas \(2002\)](#) and [Veracierto \(2002\)](#) argue that irreversibility is not important in general equilibrium since idiosyncratic adjustments will be smoothed out. However, [Kashyap and Gourio \(2007\)](#) show that whether lumpy investment is important in aggregate depends on production function of firms and the distribution of fixed costs. Recently, [Kiyotaki and Moore \(2011\)](#) study the illiquidity shocks and the amplification. [Eisfeldt \(2004\)](#) and [Kurlat \(2011\)](#) model the illiquidity through asymmetric information.

<sup>12</sup>See financial constraints’ impact on long-run output and TFP losses in [Buera, Kaboski, and Shin \(2011\)](#), [Moll \(2010\)](#), and [Midrigan and Xu \(2012\)](#). For example, [Midrigan and Xu \(2012\)](#) argue that financing frictions cannot generate the misallocation observed in [Hsieh and Klenow \(2009\)](#). [Moll \(2010\)](#) suggests that as firms have persistent idiosyncratic productivity shocks, they save enough to undo financial frictions. See also financing constraints’ effect on short-run output and TFP fluctuations in [Kocherlakota \(2000\)](#), [Cordoba and Ripoll \(2004\)](#) and more recently [Chen and Song \(2012\)](#).

shut down and its macroeconomic implications through adverse selection. He focuses on the resale prices by simplifying outside financing: entrepreneurs are not allowed to borrow. Instead, I focus on different degrees of borrowing constraints. [Khan and Thomas \(2011\)](#) quantitatively examine reallocation efficiency for given degrees of resale costs and financing frictions, focusing mainly on numerical aspects. I extensively use analytical methods (by focusing on a more specific process of idiosyncratic shocks) to explain the interaction of the two frictions on the capital reallocation delays through changes in interest rates and wages.<sup>13</sup> More importantly, in contrast to both papers, I look at the firms' capital structure through the portfolio choice perspective.<sup>14</sup>

## 2 The Model

Time is discrete and the horizon is infinite. There are two types of agents: entrepreneurs and households. Both are with measure 1. Households supply labor, consume, and save in bonds. Entrepreneurs own production technology and some of them run firms.

### 2.1 Entrepreneurs

*Preferences.* At time  $t$ , a typical entrepreneur  $j$  has preferences over the consumption stream  $c_{jt}, c_{jt+1}, c_{jt+2}, \dots$ , and leisure stream  $(1 - h_{jt}), (1 - h_{jt+1}), (1 - h_{jt+2}), \dots$ , given by

$$\mathbb{E}_t \sum_{s=t}^{\infty} \beta^{s-t} [u(c_{js}) + \eta(1 - h_{js})] \quad (1)$$

where  $\beta \in (0, 1)$  is the discount factor,  $\mathbb{E}_t$  is the conditional expectation operator, and  $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$ .  $\sigma$  is the relative risk-aversion parameter. To simplify, I use  $\sigma = 1$  i.e.,  $u(c) = \log(c)$ . If  $j$  runs the firm,  $h_{jt} = 1$ ; if  $j$  does not run the firm,  $h_{jt} = 0$ , and there is  $\eta$  extra leisure utility. Though a utility costs,  $\eta$  represent the fixed costs to run the business. Modeling this way enable me to solve the exit conditions in closed-form.<sup>15</sup>

*Production.* In the beginning of time  $t$ ,  $j$ 's firm uses capital stock  $k_{jt}$  (installed in  $t - 1$ )

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<sup>13</sup>The interactions in the model occur through general equilibrium. Credit crunches reduce wage rates because of a frictionless labor market. Empirically, despite wage rigidities, real wage rates decline during recessions, as found by [Solon, Barsky, and Parker \(1994\)](#) and [Haefke, Sonntag, and Van Rens \(2012\)](#). The decline of real wages is a consequence of lower wages of newly hired workers, in spite of moderate wage rigidity for longer term employees. [Caggese and Cunat \(2008\)](#) show firms can substitute flexible employment contracts for permanent employment contracts to reduce efficiency wages.

<sup>14</sup>Recently, [Guerrieri and Lorenzoni \(2011\)](#) look at households' deleveraging and balance sheet after credit crunches. These households face durable consumption goods illiquidity and financing constraints.

<sup>15</sup>Alternatively, one can think of the case in which an entrepreneur's engagement produces output that are the fixed costs necessary for production.

and hire labor  $l_{jt}$  at a competitive wage rate  $w_t$ , to produce output:

$$y_{jt} = \tilde{z}_{jt} k_{jt}^\alpha (A_t l_{jt})^{1-\alpha} = (z_{jt} k_{jt})^\alpha (A_t l_{jt})^{1-\alpha},$$

where  $\alpha \in (0, 1)$ ,  $z_{jt}$  is the idiosyncratic productivity, and  $A_t$  is aggregate productivity. Both  $z_{jt}$  and  $A_t$  are realized at the beginning of  $t$ . For convenience,  $\tilde{z}^h = (z^h)^\alpha$  and  $\tilde{z}^l = (z^l)^\alpha$  denote the “measured” idiosyncratic productivity levels. The idiosyncratic productivity follows a two state Markov process where the transition probabilities are<sup>16</sup>

$$\text{Prob}(z_{jt+1} = z^l \mid z_{jt} = z^h) = p^{hl}, \quad \text{Prob}(z_{jt+1} = z^h \mid z_{jt} = z^l) = p^{lh}.$$

There is no insurance market for idiosyncratic productivity risks.

*Capital Accumulation.* Capital depreciates at a rate  $\delta$ . Firms can invest in new capital stock, buy, or sell existing assets. Inactive investment decisions are also allowed i.e.,  $j$  can choose to neither buy nor sell capital. One unit of efficient used assets, after being installed, is the same as one unit of new assets. Thus,  $j$ ’s capital stock evolves according to

$$k_{jt+1} = (1 - \delta)k_{jt} + i_{jt},$$

where  $i_{jt} > 0$ ,  $i_{jt} < 0$  and  $i_{jt} = 0$  denote buying, selling, and inaction in investment.

As in a neoclassical growth model, a buyer pays one unit of consumption goods for investment goods. But for each unit of used assets, only  $(1 - d)$  fraction is useful for other buyers. This transaction lost implies that sellers receive a payment of  $(1 - d)$  for each unit of asset sold from them.  $d$  represents the reallocation costs, the partial irreversibility of the capital stock due to capital specificity or adverse selection problems.

In sum, it costs 1 to invest (new or old capital) and  $(1 - d)$  to retire a unit of old capital. If the firm changes its quantity of capital from  $k$  to  $k'$ , the cost of doing so is

$$\psi(k', k) = \begin{cases} k' - (1 - \delta)k, & \text{if } k' > (1 - \delta)k \\ 0, & \text{if } k' = (1 - \delta)k \\ -(1 - d)[(1 - \delta)k - k'], & \text{if } k' < (1 - \delta)k \end{cases}$$

*Budget and Collateral Constraints.* Entrepreneur  $j$  has access to the financial market. Denote the bond position as  $b_{jt}$  at the beginning of  $t$  and the interest rate from  $t - 1$  to  $t$  as  $R_t$ . The budget constraint of  $j$  can be written as

$$c_{jt} + b_{jt+1} + \psi(k_{jt+1}, k_{jt}) = y_{jt} - w_t l_{jt} + R_t b_{jt}.$$

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<sup>16</sup>Note that,  $0 < p^{hl} < 1$ ,  $0 < p^{lh} < 1$ , and  $p^{hl} + p^{lh} < 1$ .



$j$  earns profits and interests, which are spent on dividends (consumption), new bonds, and paying the capital adjustment costs. Note that one can simplify profits further: because of a constant return to scale (CRS) production technology, the instantaneous profits of  $j$  are linear in  $k_{jt}$ <sup>17</sup>

$$\Pi(z_{jt}, k_{jt}; w_t) = \max_{l_{jt}} \{ (z_{jt}k_{jt})^\alpha (A_t l_{jt})^{1-\alpha} - w_t l_{jt} \} = (z_{jt}\pi_t)k_{jt},$$

where labor demand and aggregate profit rate is

$$l_{jt}^* = z_{jt}k_{jt} \left[ \frac{(1-\alpha)A_t^{1-\alpha}}{w_t} \right]^{1/\alpha}, \quad \pi_t = \alpha \left[ \frac{(1-\alpha)A_t}{w_t} \right]^{\frac{1-\alpha}{\alpha}}. \quad (2)$$

Thus, the budget constraint can be rewritten as

$$c_{jt} + b_{jt+1} + \psi(k_{jt+1}, k_{jt}) = z_{jt}\pi_t k_{jt} + R_t b_{jt}. \quad (3)$$

Entrepreneur  $j$  can short bonds (borrow), but not capital stock. Borrowing is bounded because  $j$  faces collateral constraints similar to those in [Kiyotaki and Moore \(1997\)](#) and [Hart and Moore \(1994\)](#).<sup>18</sup> The collateral constraint here includes resale frictions and an extra degree of financing frictions  $\theta_t$ :

$$R_{t+1}b_{jt+1} \geq -\theta_t(1-d)(1-\delta)k_{jt+1} \quad (4)$$

Collateral constraint (4) says that debt value cannot exceed  $\theta_t$  fraction of the resale value of the residual capital at  $t+1$ . In addition, (4) implies that the investing entrepreneur only needs to pay  $1 - \theta_t(1-d)(1-\delta)/R_{t+1}$  as down payment.  $\theta_t$  fluctuates and measures the external financing difficulties. For example, a permanently higher  $\theta_t$  represents a better financial development, whereas a temporary  $\theta_t$  decline indicates a sudden banking problem.

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<sup>17</sup>To see this, the first-order condition for labor is  $A_t^{1-\alpha}(z_{jt}k_{jt})^\alpha(1-\alpha)l_{jt}^{-\alpha} = w_t$ , so that the optimal labor demand is  $l_{jt}^* = z_{jt}k_{jt} \left[ \frac{(1-\alpha)A_t^{1-\alpha}}{w_t} \right]^{1/\alpha}$ , from which profits are

$$\begin{aligned} \Pi(z_{jt}, k_{jt}; w_t) &= (z_{jt}k_{jt})^\alpha (A_t l_{jt})^{1-\alpha} - w_t l_{jt} = A_t^{1-\alpha} z_{jt}k_{jt} \left[ \left[ \frac{(1-\alpha)A_t^{1-\alpha}}{w_t} \right]^{\frac{1-\alpha}{\alpha}} - w_t \left[ \frac{(1-\alpha)A_t^{1-\alpha}}{w_t} \right]^{\frac{1}{\alpha}} \right] \\ &= A_t^{\frac{1-\alpha}{\alpha}} z_{jt}k_{jt} \left[ \frac{(1-\alpha)}{w_t} \right]^{1/\alpha} \left[ \frac{w_t}{1-\alpha} - w_t \right] = z_{jt}\pi_t k_{jt}. \end{aligned}$$

The total output produced by entrepreneur  $j$  can be written as  $y_{jt} = \frac{z_{jt}\pi_t k_{jt}}{\alpha}$ . To interpret this result,  $\alpha$  fraction of the output becomes  $j$ 's profits while the  $1-\alpha$  fraction is paid through wages.

<sup>18</sup>This is a consequence of the fact that the human capital of the agent who is raising outside funds is inalienable. To ensure no “run away” default, the lender should be able to seize the tangible assets.



$\theta_t$  of (4) constrains capital stock allocation efficiency. Without (4),  $z^h$  owners can obtain any funds needed to invest in capital stock. The economy would reach the efficient production frontier, and as many entrepreneurs as possible can enjoy leisure.

*A Summary.* Each entrepreneur  $j$  maximizes (1) subject to (3) and (4), by choosing consumption  $c_{jt}$ , leisure  $h_{jt}$ , labor input  $l_{jt}$ , capital  $k_{jt+1}$ , and bonds  $b_{jt+1}$ , while taking the wage rate  $w_t$  and the interest rate  $R_{t+1}$  as given.

## 2.2 Households

A representative household has preferences over the consumption stream  $c_{ht}, c_{ht+1}, c_{ht+2}, \dots$ , and labor supply  $l_{ht}, l_{ht+1}, l_{ht+2}, \dots$ , given by

$$\mathbb{E}_t \sum_{s=t}^{\infty} \beta_h^{s-t} \left[ \frac{c_{h,s}^{1-\gamma} - 1}{1-\gamma} - \kappa \frac{(l_{h,s})^{1+\nu}}{1+\nu} \right],$$

where  $\beta_h \in (\beta, 1)$  is household's discount factor,  $\gamma$  is the relative risk-aversion,  $\nu$  is the inverse Frisch elasticity of labor supply, and  $0 \leq l_{ht} \leq 1$ . The household do not have production technology and therefore do not own physical capital. They can save in bonds so that the budget constraint is

$$c_{h,t} + b_{h,t} = w_t l_{h,t} + R_t b_{h,t}.$$

That is, labor income and return from bonds are used to finance new consumption and new bonds. The household problem is standard and the optimal solution is (assuming interior solution for labor supply)

$$\kappa c_{ht}^{\gamma} l_{ht}^{\nu} = w_t, \quad \mathbb{E}_t \frac{\beta_h (c_{ht+1})^{-\gamma}}{(c_{ht})^{-\gamma}} R_{t+1} = 1.$$

## 2.3 Recursive Equilibrium

I rewrite the entrepreneur's problem recursively and then define recursive equilibrium. Denote aggregate state as  $X = (\Gamma(k, b, z), \theta, A)$  where  $\Gamma(k, b, z)$  is the distribution of individuals' capital stock, bonds, and productivity at the beginning of each period. To emphasize, financial disturbances  $\theta$  and aggregate productivity fluctuations  $A$  are exogenous shocks.

Let  $V$  be the optimal value of an entrepreneur with  $k$ ,  $b$ , and  $z$ , given the aggregate state variable  $X$ . The value function  $V(k, b, z; X)$  satisfies the Bellman equation:

$$V(k, b, z; X) = \max\{W^1(k, b, z; X), W^0(k, b, z; X)\} \quad (5)$$

$$W^1(k, b, z; X) = \max_{\substack{k' > 0 \\ R'b' \geq -\theta(1-d)(1-\delta)k'}} \{u(z\pi k + Rb - \psi(k', k) - b') + \beta \mathbb{E}_{z,X}[V(k', b', z'; X')]\}$$

$$W^0(k, b, z; X) = \max_{b'} \{u(z\pi k + Rb + (1-\delta)(1-d)k - b') + \eta + \beta \mathbb{E}_{z,X}[V(0, b', z'; X')]\}$$

The first step maximization is over the two actions: (1) to run the firm and get  $W^1$  and (2) not to run the firm and get  $W^0$ . The second step is to choose the optimal consumption and savings (a portfolio of capital stock and bonds). Note that  $W^0$  has the leisure utility  $\eta$ , as an entrepreneur who gets  $W^0$  does not run the firm today (and there is no output tomorrow). Finally, I define the recursive equilibrium to close the model:

**Definition 1** (The First Recursive Equilibrium Definition):

The equilibrium consists of households' policy function  $\{c_h(X), l_h(X), b_h(X)\}$ , entrepreneurs' policy functions  $\{l(k, b, z; X), k'(k, b, z; X), b'(k, b, z; X)\}$ , a law of motion  $\Gamma(k, b, z) \rightarrow \Gamma(k', b', z')$ , and pricing functions  $\pi(X)$  and  $R'(X)$ , given an exogenous evolution of  $(\theta_{-1}, A_{-1}) \rightarrow (\theta, A)$  such that:

- (i)  $l, k'$  and  $b'$  solve the entrepreneur's problem in (5) given wages and interest rates.
- (ii)  $c_h, l_h$ , and  $b_h$  solve the household's problem i.e.,

$$\kappa c_h^\gamma l_h^\nu = w, \quad \mathbb{E}_X \frac{\beta_h (c'_h)^{-\gamma}}{(c_h)^{-\gamma}} R' = 1, \quad c_h + b_h = w l_h + R b_{h,-1}$$

(iii) Markets for labor and bonds clear

$$\int l_{jt} dj = l_h, \quad \int b'_j dj = 0$$

(iv) The distribution evolution  $\Gamma(k, b, z) \rightarrow \Gamma(k', b', z')$  is consistent with policy functions, given an initial condition.

Note that I chose to be brief in describing the evolution of firm distribution. The reason is that I will simplify the firm distribution and rewrite the equilibrium.

### 3 Equilibrium Characterization

I focus on the interesting equilibrium where there is an active secondary capital market. I begin by describing the decision rules of entrepreneurs, leaving the mathematical details for later. Doing so will give readers an idea of where the argument flow is and allow them to skip the details.

### 3.1 Entrepreneurs' Decision Rules

*A Quick Preview.* An individual entrepreneur's policy depends on idiosyncratic productivity and the leverage ratio i.e., capital stock over equity

$$\lambda = k/(k + b)$$

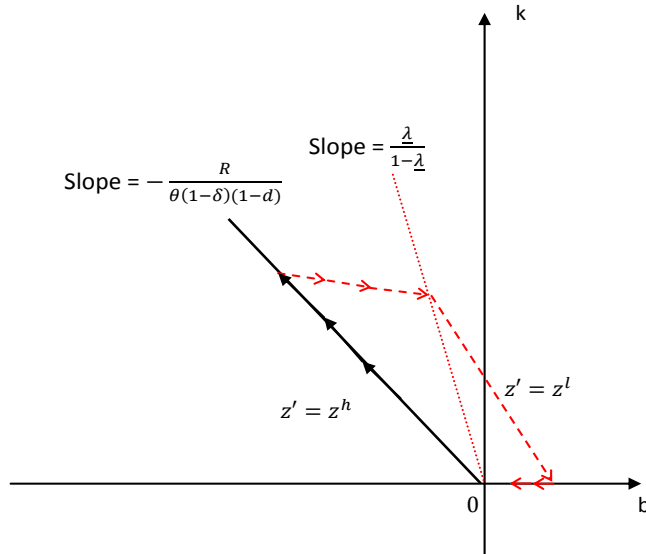
. The interesting equilibrium features that  $z^h$  owners buy capital while  $z^l$  owners hold on to it before liquidation. This type of equilibrium is the main concern because it has imperfect capital reallocation and possible binding financing constraints for productive firms.

In steady state, the optimal policy functions can be shown in the following way. Consider the dynamics of  $k$  and  $b$  (Figure 2).  $z^h$  owners always go to  $z' = z^h$  line or expand through the  $z' = z^h$  line such that the leverage is  $\bar{\lambda}$  and the slope  $k/b$  is  $\frac{\bar{\lambda}}{1-\bar{\lambda}}$ . For example, when  $\bar{\lambda}$  is the leverage ratio associated with the borrowing constraint,  $z^h$  owners reach the credit limit.  $z^l$  owners, on the other hand, let the capital depreciate and shrink their debt by reducing consumption until they reach leverage  $\underline{\lambda}$  (i.e.,  $k/b$  ratio is  $\frac{\underline{\lambda}}{1-\underline{\lambda}}$ ). Then, their firms are liquidated.

The region characterized by the two lines with slope  $\frac{\bar{\lambda}}{1-\bar{\lambda}}$  and  $\frac{\underline{\lambda}}{1-\underline{\lambda}}$  denote the inaction region. Inside the region, the reward for changing capital stock is insufficient. From outside the region (to the right of the  $\frac{\underline{\lambda}}{1-\underline{\lambda}}$  slope line), the optimal policies are such as to proceed

**Figure 2: Policy function illustration**

Dynamics of  $k$  and  $b$ . When entrepreneurs draw  $z^h$ , their firms expand (increase  $k$  while decrease  $b$ ) along the solid line. Whenever entrepreneurs draw  $z^l$ , they step on the dashed line (one specific path): let  $k$  depreciate while paying back existing debt (increase  $b$ ) until  $k/b = \frac{\underline{\lambda}}{1-\underline{\lambda}}$  when they liquidate the firm.



instantly to the  $k = 0$  line i.e., to liquidate in order to avoid the fixed costs.

The illustration above is in the steady state. When there are aggregate shocks,  $\bar{\lambda}$  and  $\underline{\lambda}$  will change in response. Now, I show the detail steps to derive the decision rules.

### 3.1.1 Policy Functions When $k' > 0$

First, I explore useful properties of the value function. The value function behaves normally, differentiable at  $k > 0$ , and has a “scale-invariant” property.

#### **Lemma 1** (Properties of the Value Function):

*The value function  $V$  has the following properties*

1.  $V(k, b, z; X)$  is increasing in  $k$ ,  $b$ , and  $z$ , and concave in  $(k, b)$ .

2.  $V$  satisfies

$$V(\gamma k, \gamma b, z; X) = V(k, b, z; X) + \frac{\log \gamma}{1 - \beta}. \quad (6)$$

3.  $V(k, b, z; X)$  is differentiable at  $k > 0$  and satisfies the envelope condition.

*Proof.* See the Appendix. □

One can prove properties 1 and 2 of Lemma 1 by contraction mapping, which maps the space of functions with properties 1 and 2 to itself. Let leverage of a firm be

$$\lambda = k/(k + b).$$

Lemma 1 says that value functions of entrepreneurs with the same pair of  $(\lambda, z)$  are affine transformations of each other. A firm with  $\gamma$  times of the size as another firm but the same  $(\lambda, z)$  is simply a scale up version of the latter. More importantly, target leverage of these entrepreneurs will be the same.<sup>19</sup> Notice that the fixed costs  $\eta$  affect the liquidation decision and the decision is based upon  $(\lambda, z)$  but not the level of  $k$  and  $b$ .

Because of potential inaction investment decisions, it is useful to work with the marginal value of capital. Let  $q(k, b, z; X)$  be the marginal value of capital that satisfies the envelope condition:

$$V_k(k, b, z; X) = u'(c(k, b, a; X))[z\pi + q(k, b, z; X)(1 - \delta)], \quad (7)$$

for  $k > 0$ .  $q$  measures the value of capital in consumption goods unit and shows how much entrepreneurs value their capital internally, particularly when the investment decision is

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<sup>19</sup>Their policies are  $(k', b')$  and  $(\gamma k', \gamma b')$  so the target leverages are  $k'/(k' + b')$  and  $\gamma k'/(\gamma k' + \gamma b')$ .

inaction. It turns out to be useful in solving policy functions because it only depends on leverage, keeping everything else fixed:

**Lemma 2** (Scale Invariance and Shadow Prices):

*The value function  $V$  and the shadow value  $q$  have the following properties*

1.  $V_k$  is homogeneous with degree  $-1$  in  $(k, b)$ .
2. For given  $z$  and  $X$ ,  $V_k/u'(c)$  depends only on  $k/(k+b)$ , but not on  $k$  or  $b$ .
3.  $q(k, b, z; X)$  can be simplified to  $q(\lambda, z; X)$ , where  $\lambda = \frac{k}{k+b}$ .

*Proof.* See the Appendix. □

$q$  is equivalent to the marginal reward to adjust capital. When the marginal reward to increase capital reaches 1, a firm buys capital. When the marginal reward to decrease capital reaches  $1 - d$ , the firm sells it. When there are no active purchases or sales, the marginal reward to increase or decrease capital is  $q$ , which should be less than 1 but greater than  $1 - d$ . Therefore, it is not optimal to adjust capital stock when

$$1 - d < q(\lambda, z; X) = \frac{V_k/u'(c) - z\pi}{1 - \delta} < 1.$$

Inside the inaction region,  $q$  is the option value of staying. Such characterization is similar to that in [Dixit \(1997\)](#). One may also interpret  $q(\lambda, z; X)$  as the “price” of each share of a firm with leverage  $\lambda = k/(k+b)$  and productivity pair  $z$ . When the firm is investing, each share of the stock is priced at 1. When sold, each share of the stock is priced at  $1 - d$ . When firms are inactive in investment, each share of the stock is  $q \in (1 - d, 1)$ .

The properties of the value function provide useful knowledge to solve entrepreneurs’ policy functions. To see this, let the (internal) rate of return on capital ( $k' > 0$ )  $r'$  be

$$r'(\lambda', z'; X' | \lambda, z; X) = \frac{z'\pi' + (1 - \delta)q(\lambda', z'; X')}{q(\lambda, z; X)},$$

let the net worth of an entrepreneur using the shadow value of capital be

$$n(k, b, z; X) = z\pi k + q(\lambda, z; X)(1 - \delta)k + Rb,$$

and let  $\phi$  denote the fraction of net worth spent on capital. We then have the following analytical solution

**Proposition 1** (Closed-form Policy Functions):

Consumption  $c = c(k, b, z; X)$ , capital  $k' = k'(k, b, z; X) > 0$ , and bonds  $b' = b'(k, b, z; X)$  can be expressed as

$$c = (1 - \beta)n(k, b, z; X), \quad k' = \frac{\phi}{q(\lambda, z; X)}\beta n(k, b, z; X), \quad b' = (1 - \phi)\beta n(k, b, z; X).$$

where  $\phi$  satisfies

$$\begin{cases} \mathbb{E}_{z,X} \left[ \frac{r' - R'}{\phi r' + (1 - \phi)R'} \right] = 0, & \text{if } \mathbb{E}_{z,X} \left[ \frac{r'}{\phi r' + (1 - \phi)R'} \right] = 1 \\ \phi = \frac{1}{1 - \theta(1 - \delta)(1 - d)/qR'}, & \text{if } \mathbb{E}_{z,X} \left[ \frac{r'}{\phi r' + (1 - \phi)R'} \right] < 1. \end{cases}$$

When entrepreneurs invest, they invest such that they target the same  $\phi$ .

*Proof.* See the Appendix. □

The above policy function implies the usual Euler equation used in asset pricing. To see this, notice that portfolio weight on capital stock is  $\phi = qk'/\beta n$  then

$$\begin{aligned} \frac{1}{\phi r' + (1 - \phi)R'} &= \beta \frac{n}{[z'\pi' + (1 - \delta)q']k' + R'b'} = \beta \frac{(1 - \beta)n}{(1 - \beta)n'} \\ &= \frac{\beta u'(c')}{u'(c)}, \end{aligned}$$

which is the stochastic discount factor. Then, when entrepreneurs are not financing constraint, the portfolio weight  $\phi$  solves

$$\mathbb{E}_{z,X} \left[ \frac{\beta u'(c')}{u'(c)} \frac{z'\pi' + (1 - \delta)q(\lambda', z'; X')}{q(\lambda, z; X)} \right] = 1,$$

the classic asset pricing formula “ $\mathbb{E}[\Lambda' r'] = 1$ ”, where “ $\Lambda$ ” is the stochastic discount factor.

In summary, a typical entrepreneur consumes  $(1 - \beta)$  fraction and saves the other  $\beta$  fraction of the net worth (that uses the option value of capital). She uses the savings to invest in a portfolio. The portfolio consists of risky assets (capital stock) and risk-free assets (bonds), allowing shorting on risk-free assets but not on risky ones. If she invests  $\phi$  fraction of a dollar in risky assets and the other  $1 - \phi$  fraction in risk-free assets, the next period’s rate of return is  $\phi r' + (1 - \phi)R'$ . The goal of portfolio choice is to maximize the expected log rate of return (i.e., the solution of  $\phi$ ).<sup>20</sup> Even though the saving rate is a constant  $\beta$  under log

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<sup>20</sup>Policy functions have closed-form expressions for any  $\sigma$  (available upon request). But under general CRRA utility, the saving rate (not necessarily  $\beta$ ) and portfolio weight  $\phi$  intertwine with each other. The

utility, different entrepreneurs save different fractions of the “accounting” net worth which is either  $z\pi k + (1 - \delta)k + Rb$  or  $z\pi k + (1 - \delta)(1 - d)k + Rb$ . Unlike the accounting net worth, the “economic” net worth evaluates capital at shadow prices, which varies across entrepreneurs when the investment decisions opt for inaction.

### 3.1.2 Inaction Regions and Liquidation

In equilibrium, there may or may not be inaction in investment. When there is, there exists at least an option value  $q \in (1 - d, 1)$ . To characterize the inaction region, one only needs to check how the shadow price  $q(\lambda, z; X)$  varies as  $\lambda$  and  $z$  change (for a given  $X$ ). Further, given that we have two productivity levels  $z^h$  and  $z^l$ ,  $z^h$  firms invest to the same leverage as shown in Proposition 1. The inaction region is thus the set of  $(\lambda, z^l)$  such that the shadow price is between  $1 - d$  and 1.

Therefore, some (and usually all)  $z^h$  owners invest and borrow. Because of the linear rate of return in individual level, they have the same target leverage  $\lambda' = k'/(k' + b')$  tomorrow regardless of their leverage today (Proposition 2).<sup>21</sup> For  $z^l$  owners, investment decisions are either to hold or to sell. It turns out that an entrepreneur who persistently draws  $z^l$  hold capital for finite periods. The shadow price during the holding process monotonically decreases until it reaches  $(1 - d)$  when the capital stock is liquidated. During the holding process, the leverage also decreases.

#### Proposition 2 (Leverage and Deleverage):

*In the neighborhood around steady state,*

1. *The shadow price (option value)  $q(\lambda, z; X)$  is an increasing function of  $\lambda$ .*
2.  *$z^h$  owners borrow and invest. Moreover, they have the same target leverage  $\frac{k'}{k' + b'} = \bar{\lambda}'$ .*
3. *Denote today's shadow price as  $q$  and tomorrow's shadow price as  $q'$ . Then,*

$$q' \begin{cases} = 1 & \text{if } z' = z^h \\ < q & \text{if } z' = z^l \end{cases} \quad \text{and} \quad \frac{k'}{k' + b'} \begin{cases} = \bar{\lambda} & \text{if } z' = z^h \\ < \frac{k}{k + b} & \text{if } z' = z^l \end{cases}$$

*Proof.* See the Appendix. □

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reason is that with general CRRA utility the income and substitution effect do not offset each other, for example illustrated in [Campbell and Viceira \(2002\)](#). The combination of the two effects are so-called “hedging demand” in the asset pricing literature. Depending on the investment opportunities in the long time frame, agents put different weights on capital and consume differently.

<sup>21</sup>  $k'/(k' + b')$  may or may not reach the leverage under credit limits, depending on the equilibrium.



The assumption  $\delta(1 - \delta) < \beta R$  is not hard to satisfied for usual quarterly data model i.e., when  $\delta$  is around 0.025 and  $\beta R$  is close to 1. In addition, the delveraging when being inactive in capital are intuitive. For  $z^l$  entrepreneurs, running business is not profitable compared to risk-free rate. Without resale costs, they will liquidate and repay all the debt immediately after turning from  $z^h$  to  $z^l$ . If they sell but become productive tomorrow, they can still buy back capital at the same price. With resale costs, in contrast, they have the same incentive to shrink the debt. But if they sell capital at a cost immediately, they will have to buy back at a higher price tomorrow if they turn productive again. Instead, dividends payments from the firm decrease to compensate debt payment, a painful process for the entrepreneurs.

Not surprisingly, capital is less and less valued during the inaction process. The fixed costs of running a business eventually force the  $z^l$  owners to liquidate. The intuition behind is that capital stock will eventually shrink to a very small amount. Even if turned to productive again tomorrow, the firm cannot generate much profits to compensate fixed costs. That is, there exists a stopping rule:

**Proposition 3 (Optimal Stopping Time):**

*For  $z^l$  owners, there exists an optimal capital liquidation rule (stopping-time rule or exit rule). Let  $n = z^l \pi + (1 - \delta)(1 - d) + R\lambda^{-1}(1 - \lambda)$  and suppose a finite  $\lambda \in [0, \bar{\lambda}]$  is a root of*

$$\begin{aligned} \eta &= \frac{\beta}{1 - \beta} p^{lh} \mathbb{E}_X \left[ \log \left( 1 + (1 - \delta) \frac{z^h \pi' + (1 - \delta) - (1 - d)R'}{\beta n R'} \right) \right] \\ &+ \frac{\beta}{1 - \beta} p^{ll} \mathbb{E}_X \left[ \log \left( 1 + (1 - \delta) \frac{z^l \pi' + (1 - \delta)(1 - d) - (1 - d)R'}{\beta n R'} \right) \right] \end{aligned} \quad (8)$$

1. When  $\frac{k}{k+b} > \lambda$ ,  $z^l$  owners are inactive in adjusting capital. When  $\frac{k}{k+b} < \lambda$ , they liquidate the whole firm. When  $\frac{k}{k+b} = \lambda$ , they are indifferent between holding or liquidating capital.

2. If no  $\lambda$  satisfies equation (8), then no  $z^l$  entrepreneur sells capital.

*Proof.* See the Appendix. □

The indifference condition (8) implies that the gains of liquidation (extra  $\eta$  utility) equals the expected discounted costs of not doing so (the right hand side, extra value of holding capital stock one more period). Note that, holding onto capital is similar to gambling for  $z^h$  draw in the future. The gambling is not worthwhile when the size of capital stock is so small that profits are not enough to compensate fixed costs, even when drawing  $z^h$  tomorrow.

### 3.2 Recursive Equilibrium, Revisited

Now, we are ready to simplify the distribution of firms and rewrite equilibrium definition. We know that entrepreneurs with the same leverage  $k/(k+b)$  and productivity put the same portfolio weights on  $k$  and  $b$ . Thus, I can define aggregate capital stock and aggregate bonds for a specific  $k/(k+b)$  ratio, given a productivity pair  $a$ , i.e.,

$$K(x, a) = \int_{\{(k,b): \frac{k}{k+b}=x\}} k\Gamma(dk, db, a), \quad B(x, a) = \int_{\{(k,b): \frac{k}{k+b}=x\}} b\Gamma(dk, db, a)$$

Equilibrium can be redefined as a mapping  $(K(x, a), B(x, a), \theta, A) \rightarrow (K'(x, a), B'(x, a), \theta', A')$ . Since drawing  $z^h$  always means investment,<sup>22</sup> keeping track of the firm distribution is equivalent to keeping track of firms with the time length of having been drawing  $z^l$ .

At the beginning of  $t$ , let  $s = 1, 2, \dots$  denote the *vintage* of entrepreneurs, who have been drawing  $s$  times of  $z^l$ . These firms did not invest in  $t-1$ . Let  $s = 0$  denote the state in which the entrepreneur just finished investing. That is, drawing  $z^h$  means that entrepreneurs will go to vintage  $s = 0$ , whereas drawing  $z^l$  means that they will go to the next vintage i.e., the vintage whose number equal 1 plus the number of current vintage.

Inside each vintage, the leverage ratio  $\lambda = k/(k+b)$  is the same, which allows me to replace  $q(\lambda, z; X)$  by vintage-specific price. When entrepreneurs decide to go from vintage  $s$  to  $s'$ , the shadow price of capital will be  $q_{s'}(X)$ , which is vintage-specific and corresponds to a specific  $\lambda' = k'/(k' + b')$ . When the secondary market is active, there exists an integer  $N_t < +\infty$  at time  $t$ , such that (1) vintage  $N_t + 1, N_t + 2, \dots$  entrepreneurs who draw  $z^l$  will hold no capital stock; (2) vintage  $0, 1, \dots, N-1$  entrepreneurs who draw  $z^l$  are from vintage will be inactive in capital; (3) vintage  $N$  entrepreneurs will be indifferent between staying or liquidating i.e., a fraction  $f_t$  of them stays while the other fraction  $1 - f_t$  liquidate.

For simplicity, I focus on small exogenous shocks around the steady state such that the equilibrium vintages do not change, i.e.,  $N_t = N$ . Note that  $N$  is an endogenous constant integer, as  $N$  itself varies in different steady states. In addition, I can leave the aggregate state  $X$  out and denote variables with vintage subscripts. For example, at time  $t$ , the shadow price of capital for those entrepreneurs who are going to vintage  $i$  is  $q_i$ . Then,  $q_0 = 1$  denotes the buying price and  $q_{N+1} = q_{N+2} = 1 - d$  denote the selling price.<sup>23</sup> In summary,

1. Entrepreneurs go to vintage 0 once they draw  $z^h$ .
2. For those vintage  $i$  entrepreneurs who draw  $z^l$ , they hold onto capital stock if they are

<sup>22</sup>For most parameters, every  $z^h$  entrepreneurs invest; for some parameters, some  $z^h$  entrepreneurs invest and others do not run firms (in which obviously  $z^l$  entrepreneurs do not run firms as well).

<sup>23</sup>“Shadow price” of capital of entrepreneurs who are going to invest and go to vintage 0.

from vintage 0, 1, ...,  $N$ . Their vintage number will be  $i + 1$  and they value capital at price  $q_{i+1}$ .

3. For those vintage  $N$  entrepreneurs who draw  $z^l$ , a fraction  $f_t$  of them stays in the business (going to vintage  $N + 1$ ) while  $1 - f_t$  liquidates their firms (going to vintage  $N + 2$ ). They value capital at price  $q_{N+1} = q_{N+2} = 1 - d$ .
4. Entrepreneurs in vintage  $N + 1, N + 2, \dots$ , do not run firms if they draw  $z^l$ .

Notice that  $q_i$  are the endogenous valuation of capital in each vintage that are to be determined. It will be the valuation such that the envelop conditions hold after entrepreneurs choose the desired consumption, capital stock, and bonds.

One can group vintages after  $N + 1$  together to be vintage  $N + 2$ , since entrepreneurs in vintages  $N + 2, N + 3, \dots$  only hold bonds. To express portfolio choices in each vintage, first define the transition probability and the associated productivity of each vintage as

$$p^{ih} = \begin{cases} p^{hh}, & \text{if } i = 0 \\ p^{lh}, & \text{if } i > 0 \end{cases}, \quad p^{il} = \begin{cases} p^{hl}, & \text{if } i = 0 \\ p^{ll}, & \text{if } i > 0 \end{cases}, \quad z_i = \begin{cases} z^h, & \text{if } i = 0 \\ z^l, & \text{if } i > 0 \end{cases}$$

Second, let the (internal) rate of return on capital from time  $t$  to time  $t + 1$  be  $r'_{ij}$  for those entrepreneurs who are going to vintage  $i$ , where  $i = 0, 1, \dots, N - 1$  and  $j \in h, l$  indicates drawing  $z^h$  and drawing  $z^l$  at time  $t + 1$ .<sup>24</sup> The vintage  $i$  specific rate of return on capital when  $z^h$  or  $z^l$  is realized can be written as

$$r'_{ih} = \frac{z_0\pi' + (1 - \delta)q'_0}{q_i}, \quad r'_{il} = \frac{z_{i+1}\pi' + (1 - \delta)q'_{i+1}}{q_i} \quad \text{for } i = 1, 2, \dots, N.$$

According to Proposition 1, the portfolio weight  $\phi$  on capital can be simplified to

**Corollary 1** (Vintage-specific Portfolio Choices):

Suppose  $\tilde{\phi}_i$  ( $i = 0, 1, 2, \dots, N$ ) is a solution to the following equation of  $\tilde{\phi}_i$

$$p^{ih}\mathbb{E}_X \left[ \frac{r'_{ih} - R'}{\tilde{\phi}_i(r'_{ih} - R') + R'} \right] + p^{il}\mathbb{E}_X \left[ \frac{r'_{il} - R'}{\tilde{\phi}_i(r'_{il} - R') + R'} \right] = 0$$

The capital weight  $\phi_i$  ( $i = 0, 1, 2, \dots, N$ ) for entrepreneurs who are going to vintage  $i$  solves

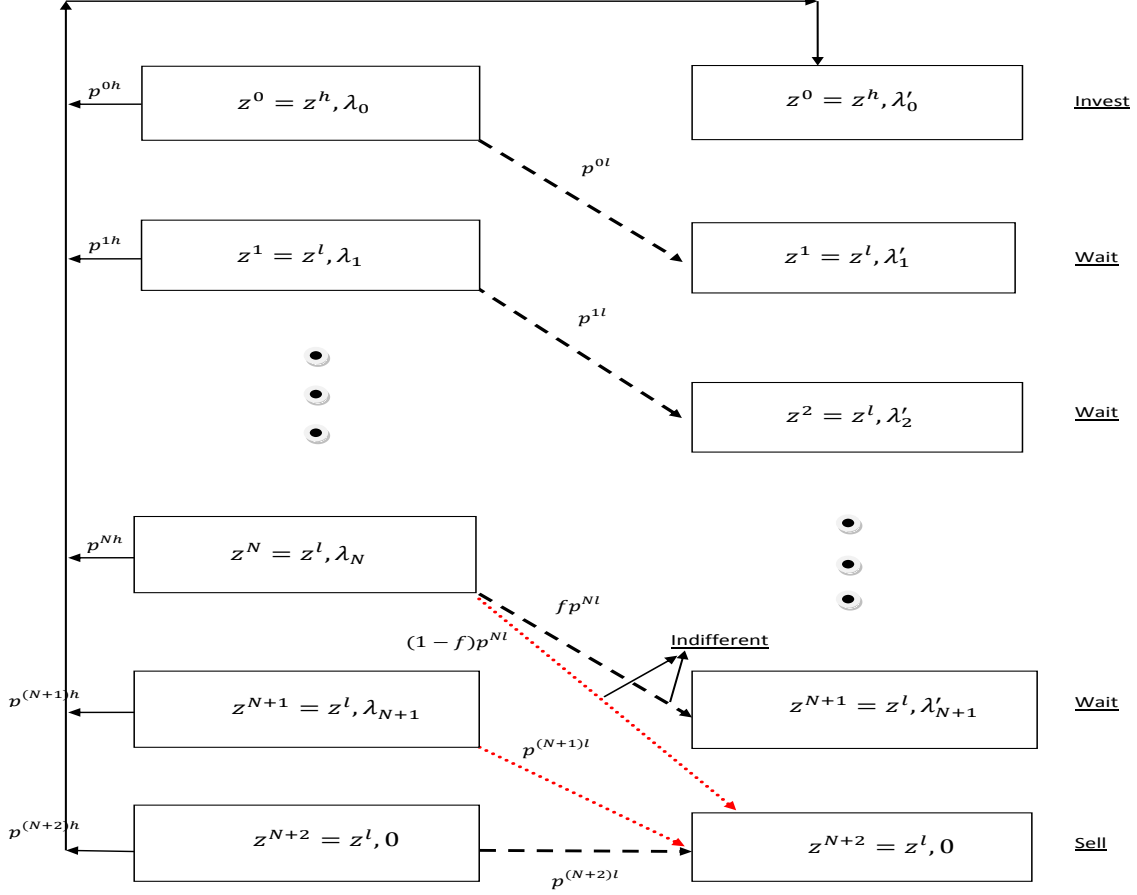
$$\phi_i = \min \left\{ \frac{1}{1 - \theta(1 - \delta)(1 - d)/R'}, \tilde{\phi}_i \right\}$$

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<sup>24</sup>For example, an entrepreneur going to vintage 3 in time  $t$  draws  $z^h$  at time  $t + 1$ , her rate of return on capital from  $t$  to  $t + 1$  is  $r'_{3h}$ .

**Figure 3: Evolution of the distribution**

Each box represents a vintage in which firms have the same  $\lambda = \frac{k}{k+b}$  leverage ratio. The vintage number is identical to how many periods an entrepreneur has been drawing  $z^l$ . Entrepreneurs who draw  $z^h$  invest and move to vintage 0 (black lines). Entrepreneurs who are from vintage 0 to  $N$  and draw  $z^l$  are possibly inactive (black dash lines). Entrepreneurs in vintage  $N+1$  or the last vintage  $N+2$  hold only bonds if drawing  $z^l$  (liquidate the firm or continuing holding only bonds).  $f_t$  denotes the fraction of vintage  $N$  entrepreneurs who draw  $z^l$  but choose to hold onto capital stock.  $1 - f_t$  then denotes the other fraction who liquidate their firms (red dotted lines).



Finally,  $\phi_{N+1}$  will be such that entrepreneurs choose to be inactive in capital stock and value the capital at price  $q_{N+1} = 1 - d$ .

Once we know  $\phi_i$  ( $i = 0, 1, 2, \dots, N+1$ ), one could back out the leverage ratio at the beginning of time  $t+1$  i.e.,

$$\lambda'_i = \frac{\phi_i}{\phi_i + q_i(1 - \phi_i)}. \quad (9)$$

I can fully characterize the firm distribution evolution from  $t$  to  $t+1$  in Figure 3 using the vintage formulation. In addition, I can simplify the equilibrium definition with the vintage

structure. To see this, Let  $K_{i,t}$  be the aggregate capital in each vintage  $i = 0, 1, \dots, N + 1$  and  $B_{N+2,t}$  be the aggregate bonds in vintage  $N + 2$ . Capital transition is characterized by aggregate capital in vintage  $0, 1, \dots, N + 1$ :

$$q_0 K'_0 = \phi_0 \sum_{i=0}^{N+1} p^{ih} \beta [z_0 \pi + q_0(1 - \delta) + R \frac{1 - \lambda_i}{\lambda_i}] K_i + \phi_0 p^{(N+2)h} \beta R B_{N+2}, \quad (10)$$

$$q_i K'_i = \phi_i p^{(i-1)l} \beta [z_i \pi + q_i(1 - \delta) + R \frac{1 - \lambda_{i-1}}{\lambda_{i-1}}] K_{i-1}, \quad i = 1, 2, \dots, N \quad (11)$$

$$q_{N+1} K'_{N+1} = \phi_{N+1} p^{(i-1)l} f \beta [z_{N+1} \pi + q_{N+1}(1 - \delta) + R \frac{1 - \lambda_N}{\lambda_N}] K_N$$

$$B'_{N+2} = p^{Nl} (1 - f) \beta [z_{N+2} \pi + q_{N+2}(1 - \delta) + R(1 - \lambda_N)/\lambda_N] K_N \\ + p^{(N+1)l} \beta [z_{N+2} \pi + q_{N+2}(1 - \delta) + R(1 - \lambda_{N+1})/\lambda_{N+1}] K_{N+1} + p^{(N+2)l} \beta R B_{N+2} \quad (12)$$

The aggregate capital in vintages  $i = 1, 2, \dots, N + 1$  satisfies

$$K'_i = \begin{cases} p^{(i-1)l} (1 - \delta) K_{i-1}, & \text{if } i = 0, 1, \dots, N \\ p^{(i-1)l} f (1 - \delta) K_{i-1} & \text{if } i = N + 1 \end{cases} \quad (13)$$

To pin-down  $f$ , the indifference condition is

$$\eta = \frac{\beta}{1 - \beta} p^{lh} \mathbb{E}_X \left[ \log \left( 1 + (1 - \delta) \frac{z^h \pi' + (1 - \delta) - (1 - d) R'}{\beta (z^N \pi + (1 - \delta)(1 - d) + R \frac{1 - \lambda_N}{\lambda_N}) R'} \right) \right] \\ + \frac{\beta}{1 - \beta} p^{ll} \mathbb{E}_X \left[ \log \left( 1 + (1 - \delta) \frac{z^l \pi' + (1 - \delta)(1 - d) - (1 - d) R'}{\beta (z^N \pi + (1 - \delta)(1 - d) + R \frac{1 - \lambda_N}{\lambda_N}) R'} \right) \right] \quad (14)$$

Market clearing conditions for credit and labor are

$$\sum_{i=0}^{N+1} \frac{(1 - \lambda_i) K_i}{\lambda_i} + B_{N+2} + b_h = 0, \quad \left( \frac{\pi}{\alpha} \right)^{\frac{1}{1-\alpha}} \left( \sum_{i=0}^{N+1} z_i K_i \right) = A l_h \quad (15)$$

where the labor market clearing condition is obtained from  $\left[ \frac{(1-\alpha)A^{1-\alpha}}{w} \right]^{1/\alpha} \left( \sum_{i=0}^{N+1} z_i K_i \right) = l_h$  together with the relationship between the real wage and the profit rate

$$\pi = \alpha \left[ \frac{(1 - \alpha) A}{w} \right]^{\frac{1-\alpha}{\alpha}} \quad (16)$$

**Definition 2** (The Second Recursive Equilibrium Definition):

The recursive competitive equilibrium is a function  $(\{\phi_i\}_{i=0}^{N+1}, \{q_i\}_{i=0}^{N+1}, \lambda'_{i=0}^{N+1}, \{K'_i\}_{i=0}^{N+1}, B'_{N+2}, f, \pi, R')$  of state variables  $(\{\lambda_i\}_{i=0}^{N+1}, \{K_i\}_{i=0}^{N+1}, B_{N+2}, R, \theta_{-1}, A_{-1})$  and a given initial condition, such that<sup>25</sup>

1.  $c_h, l_h$ , and  $b_h$  solve the household's problem i.e.,

$$\kappa c_h^\gamma l_h^\nu = w, \quad \mathbb{E}_X \frac{\beta_h (c'_h)^{-\gamma}}{(c_h)^{-\gamma}} R' = 1, \quad c_h + b_h = w l_h + R b_{h,-1}$$

2. equations (10) to (16) are satisfied

3.  $\{\phi_i\}_{i=0}^{N+1}$  solve the portfolio choice problems in Corollary 1 and  $\{\lambda_i\}_{i=0}^{N+1}$  solve (9)

4.  $q_0 = 1, q_{N+1} = 1 - d$ , and  $q_{N+2} = 1 - d$

5. together with the law of motion of  $(\theta, A)$

### 3.3 Delayed Capital Reallocation

The inaction region can be easily expressed by the set of leverage ratios and productivities such as in Figures 2:

$$\{(\frac{k}{k+b}, z) : \underline{\lambda} \leq \frac{k}{k+b} \leq \bar{\lambda} \quad \text{and} \quad z = z^l\}$$

where  $\underline{\lambda}$  is the lower bound and  $\bar{\lambda}$  is the upper bound. Any changes that lead to increase of  $\bar{\lambda} - \underline{\lambda}$  will expand the inaction region. One can directly check the stopping condition (8) to examine how does  $\underline{\lambda}$  change in response to changes in prices. For convenience, I repeat the (8).

$$\begin{aligned} \eta &= \frac{\beta}{1-\beta} p^{lh} \mathbb{E}_X \left[ \log \left( 1 + (1-\delta) \frac{z^h \pi' + (1-\delta) - (1-d)R'}{\beta(z^l \pi + (1-\delta)(1-d) + R^{\frac{1-\underline{\lambda}}{\underline{\lambda}}})R'} \right) \right] \\ &+ \frac{\beta}{1-\beta} p^{ll} \mathbb{E}_X \left[ \log \left( 1 + (1-\delta) \frac{z^l \pi' + (1-\delta)(1-d) - (1-d)R'}{\beta(z^l \pi + (1-\delta)(1-d) + R^{\frac{1-\underline{\lambda}}{\underline{\lambda}}})R'} \right) \right] \end{aligned} \quad (17)$$

Since the right hand side of (8) is a decreasing function of  $R'$  and an increasing function of  $\pi'$  and  $\underline{\lambda}$ , therefore a lower interest rate  $R'$  or a higher profit rate  $\pi'$  can lead to the decrease of  $\underline{\lambda}$  (in steady state). That is, the inaction region expands and low productive firms are less willing to liquidate.

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<sup>25</sup>The capital market clearing is embedded in the capital transition dynamics, and one can easily verify that the goods market clearing condition is satisfied (i.e., Walras' Law holds).

**Corollary 2** (Changes of Inaction Region):

*The inaction region expands when*

1.  $\pi'$  is higher
2.  $R'$  is lower

In steady state,  $A_t = 1$  and  $R'$  will be fixed as  $\beta_h^{-1}$ . Then, tighter financing constraints will limit the expansion of productive firms and thus create misallocation. Wages will tend to be lower and thus the profit rate  $\pi_t$  for entrepreneurs will be higher. Therefore, entrepreneurs with low idiosyncratic productivity are more willing to hold onto their assets.

However, we still have to be careful because tighter financing constraints at the same time decrease  $\bar{\lambda}$ , so the inaction region  $\bar{\lambda} - \underline{\lambda}$  might not increase. But if the response of  $\pi'$  to  $\theta$  changes are strong enough, the inaction region  $\bar{\lambda} - \underline{\lambda}$  will increase and so are the waiting periods (the number of vintages). I will show this result in numerical simulation.

In dynamics, tighter financing constraints lead to lower demand for borrowing and a drop of interest rate  $R'$ ; at the same time it creates misallocation and reduce wages which increase profit rate given a constant aggregate productivity  $A_t$ . In response,  $f_t$  increases as more  $z^l$  firms are willing to keep their assets. Note that we focus on the equilibrium in which the number of vintages  $N_t + 2$  does not change. Potentially the response to credit shocks could be large enough such that the number of vintages increase for a short period of times. In that case, there could be zero reallocation of capital for some time.

Reallocation is directly linked the economic efficiency. The longer the waiting periods, the more capital reallocation is delayed, and the less efficient is the economy (i.e., the lower is the aggregate TFP). To see this, the aggregate TFP can be measured by

$$TFP = \frac{Y}{K^\alpha L^{1-\alpha}}$$

where  $Y$  is the total output,  $K$  is the total capital stock, and  $L$  is the total labor hours used in production, including labor hours from households (i.e.,  $l_h$ ) and labor hours from entrepreneurs (i.e.,  $l_e$ )

$$L = l_h + l_e.$$

To compute  $l_e$  i.e., the measure of entrepreneurs who run firms, I use the transition probability of idiosyncratic technology and the fraction  $f_t$  of entrepreneurs who liquidate (with details in the appendix). Note that  $\alpha$  fraction of the output produced from a firm is the associated entrepreneur's profits. Output can be written as

$$Y = \frac{\pi}{\alpha}(z^h K_h + z^l K_l),$$



where  $K_h = K_0$  and  $K_l = \sum_{i=1}^{N+1} K_i$  denote the capital stock under  $z^h$  and  $z^l$  technology respectively. Together with the labor market clearing condition  $(\frac{\pi}{\alpha})^{\frac{1}{1-\alpha}} (z^h K_h + z^l K_l) = A l_h$ , TFP can be simplified to

$$TFP = \frac{\frac{\pi}{\alpha} (z^h K_h + z^l K_l)}{(K_h + K_l)^\alpha \left[ \left( \frac{\pi}{\alpha} \right)^{\frac{1}{1-\alpha}} (z^h K_h + z^l K_l) / A + l_e \right]^{1-\alpha}}. \quad (18)$$

When  $K_l \rightarrow 0$ , all capital is installed under  $z^h$  technology and a measure 0 entrepreneurs run firms, so that  $l_e \rightarrow 0$  and the TFP reaches the upper bound  $\tilde{z}^h = (z^h)^\alpha$ . When  $K_l > 0$ , aggregate TFP from (18) is

$$TFP = \frac{\frac{\pi}{\alpha} (z^h K_h / K_l + z^l)}{(K_h / K_l + 1)^\alpha \left[ \left( \frac{\pi}{\alpha} \right)^{\frac{1}{1-\alpha}} (z^h K_h / K_l + z^l) + l_e / K_l \right]^{1-\alpha}} < \left( \frac{z^h \frac{K_h}{K_l} + z^l}{\frac{K_h}{K_l} + 1} \right)^\alpha$$

Therefore, we know that the relative capital stock ratio  $\frac{K_h}{K_l}$  determines the economy efficiency. The smaller is  $\frac{K_h}{K_l}$  ratio, the lower is the upper bound for TFP since  $z^h > z^l$ . Intuitively, the longer the waiting period, the more capital is held by  $z^l$  firms which implies the smaller  $K_h / K_l$  ratio and thus a lower aggregate TFP. The quantitative effects of delayed reallocation and aggregate TFP losses are the main targets in the next section.

## 4 Numerical Examples

I calibrate the parameters to match the steady state result to several U.S. long-run economy characteristics in quarterly frequencies (Table 1). Following Veracierto (2002), the capital abstracts from components such as land, residential structure, and consumer durables. Thus, the capital corresponds to non-residential structures, plant, and equipment while the investment corresponds to the non-residential investment in the National Income and Product Accounts (NIPA). Meanwhile, the empirical counterpart for consumption should be non-durable goods and services consumption. Output is then defined as the sum of the consumption and the investment. The investment/output ratio is found to be 0.16 and the capital to output ratio is 6.0 which set targets for  $\alpha$  and  $\delta$ .  $\beta_h$  targets the interest rate from households' investment in financial assets. The risk-free interest rate is low but equity return is high (e.g. Mehra and Prescott (1985)). Thus  $R_t$  is commonly chosen to be 4% annually as a balance. Entrepreneurs' discount factor is set close to but smaller than households discount factor such that they are willing to accept 0.5% higher interest rate if they borrow. I set households' risk-aversion  $\gamma = 2$  and  $\nu = 0.33$  common in macro literature. Finally,  $l_h$  is set to 0.33 of total hours which calibrates  $\kappa$ .

Table 1: Baseline calibration

	Parameter	Value	Target/Source
<i>Preferences and Production Technology</i>			
Household discount factor	$\beta_h$	0.9900	annual interest rate 4%
Relative risk aversion	$\gamma$	2	exogenous
Inverse Frisch elasticity of labor supply	$\nu$	0.3300	exogenous
Utility weight on leisure	$\kappa$	8.9682	working time: 33%
<i>Production Technology</i>			
Depreciation rate of capital	$\delta$	0.0252	capital-to-GDP ratio: 6.0
Capital share of output	$\alpha$	0.2471	investment-to-GDP ratio: 16.0%
Entrepreneurs discount factor	$\beta$	0.9890	exogenous
Fixed costs	$\eta$	1.0590	waiting periods: 12.0
Transition probability 1	$p^{hh}$	0.9375	expected 4 year turn-over
Transition probability 2	$p^{ll}$	0.9375	expected 4 year turn-over
Idiosyncratic high productivity	$\Delta$	0.0570	cross-sectional std 5.70%
<i>Financial and Resale Frictions</i>			
Financing Constraint	$\theta$	0.4135	average debt/asset = 0.325
Resale Discount	$d$	0.0971	reallocation/capital expenditure = 0.35

Notes: The model is calibrated to quarterly frequency.

For the productivity transition matrix, one only needs  $p^{hl}$  and  $p^{lh}$ . I set them equal and to match 4 year (8 quarters) turn-over rate i.e., half of average 8 year business cycle similar to [Eisfeldt and Rampini \(2006\)](#). For the idiosyncratic productivity, I specify

$$\log(\tilde{z}_h) = \Delta, \quad \log(\tilde{z}_l) = -\Delta$$

such that the standard deviation of idiosyncratic productivity is  $\Delta$  in annual frequency.<sup>26</sup> I follow the cross-sectional standard deviation of productivity (5.7%) in [Basu, Fernald, and Kimball \(2006\)](#) and set  $\Delta = 0.057$ .

The parameters left are  $\eta$ ,  $\theta$ , and  $d$ . These three affect leverage, investment, and liquidation. The haircut  $\theta$  targets leverage where empirically, the debt-to-asset ratio is averaged to be 0.325 from flow of funds data. The degree of asset irreversibility  $d$  targets reallocation, where fraction of capital reallocation over total capital purchase is roughly 35%. Finally, the leisure utility  $\eta$  measures “fixed costs” and controls how long a persistently unproductive firm will hold the assets and deleveraging. I chose  $\eta$  such that there will be about 3 years (12 quarters) of waiting periods.

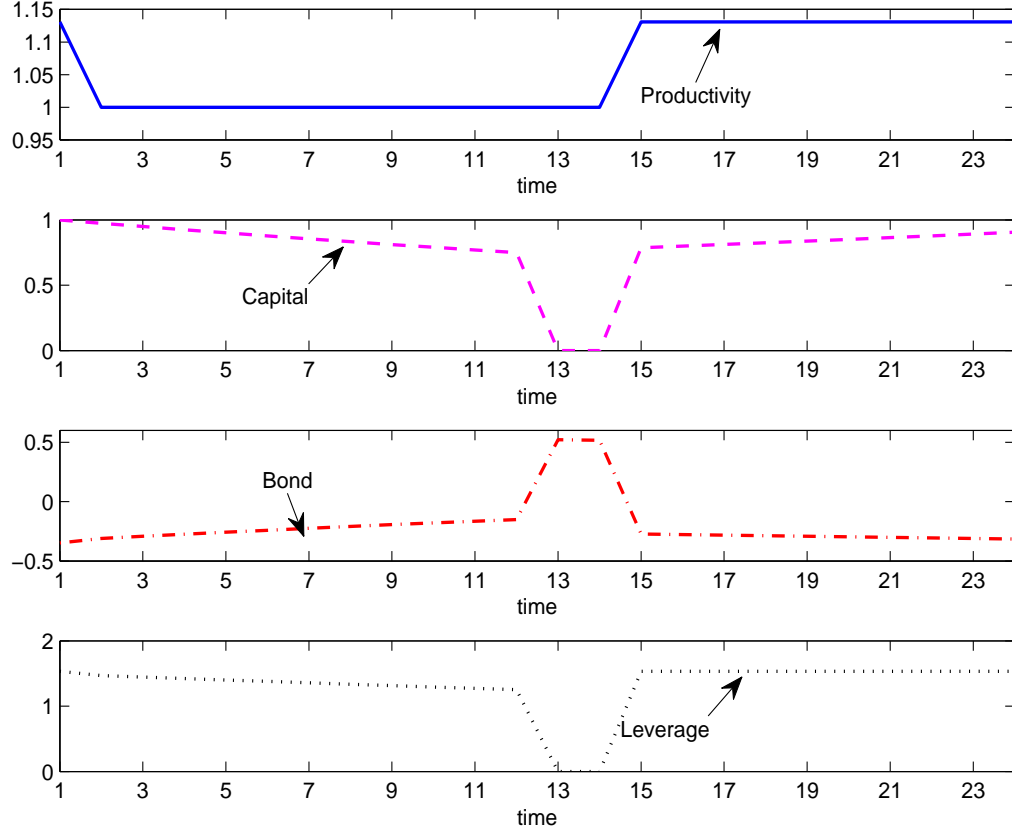
## 4.1 A Sample Path of Firm Dynamics

Under the calibrated parameters, there are 11 to 12 inactive quarters in the steady state. That is, entrepreneurs who turn from  $z^h$  to  $z^l$  and draw  $z^l$  for 11 quarters in a row neither buy nor sell capital during those 11 quarters. When they unfortunately draws the 12th  $z^l$ ,

<sup>26</sup>The standard deviation is  $2\Delta\sqrt{1 - (p^{hl})^4}$ .

Figure 4: **Capital, bond and leverage dynamics of a firm**

The firm's physical capital is normalized to be 1 at the end of period 1. Solid line: productivity draws normalizing the low productivity to be 1. Dash line: physical capital. Dash dotted line: bond. Dotted line: leverage ratio  $k/(k + b)$ .



one fraction of them sells the firm and saves in bonds while the other fraction decides to be inactive for another quarter. For those who still run firms but draw a 13th  $z^l$ , they liquidate the entire firm and save the revenue in bonds until they become productive again.

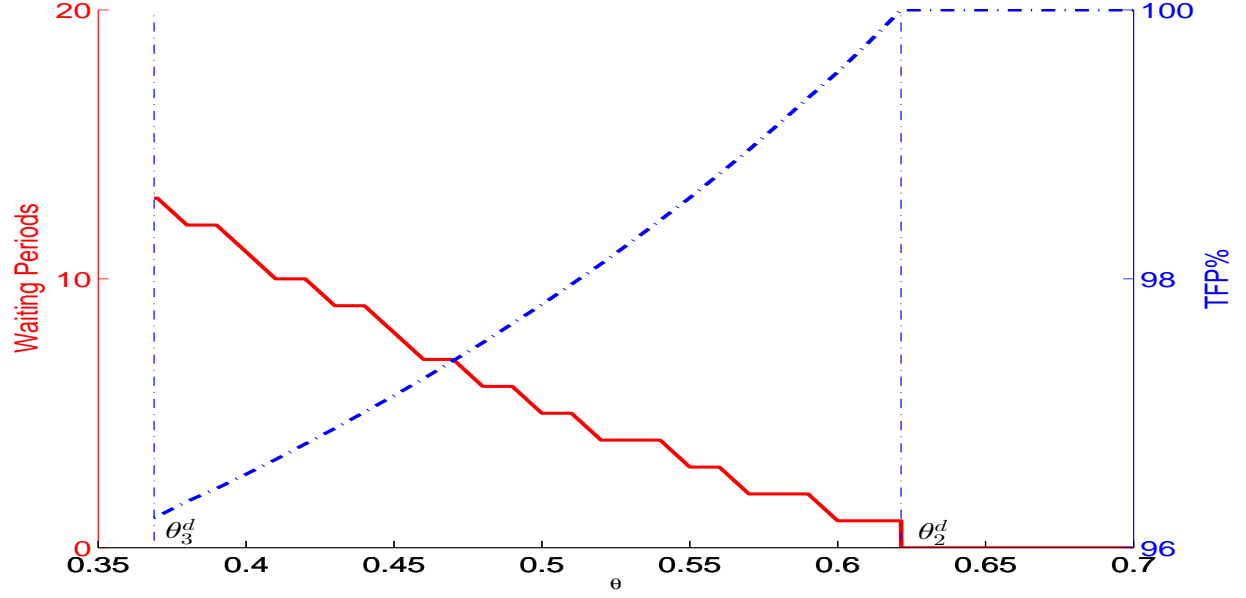
Consider a specific sample path of a firm. Suppose entrepreneur  $j$  has one unit of capital and was investing and borrowing before. Her bond position is  $-\theta(1 - \delta)(1 - d)/R$ . Then,  $j$  draws 12 quarters of  $z^l$  in a row from time  $t = 2$  on. In the 13th quarter,  $j$  draws  $z^l$  again and decides to liquidate the entire firm. After that,  $j$  keeps drawing two  $z^l$  for quarter 14 but draws  $z^h$  afterwards.

$j$  lets the capital depreciate in the first 12 quarters and liquidates it in the 13th quarter (firm dynamics in Figure 4) i.e., capital at the end of the 13th quarter is 0. During the inactive investment process, debt is being paid and leverage decreases. After liquidation,  $j$  saves only in bonds and consume  $(1 - \beta)$  of the bond value.

$j$  continues to hold bonds until drawing  $z^h$  again in the 15th quarter. Then, she uses

Figure 5: **Aggregate TFP Losses and Waiting Periods**

Steady state TFP and waiting periods as a function of  $\theta$  (when the steady state has capital reallocation). The red solid line denotes the waiting periods  $N$ . The blue dashed line denotes aggregate TFP as percentage of  $\bar{z}^h$ .



her net worth as a down payment to borrow and invest. Though she borrows to the limit, capital stock after investing is less than one, the amount  $j$  started with. The firm size is not as large as before because  $j$  does not have enough resources to expand. Her business was not profitable under  $z^l$  technology and capital was sold at a discount in quarter 13. If  $j$  keeps drawing  $z^h$ , she can continue investing and capital stock can gradually go back to one.

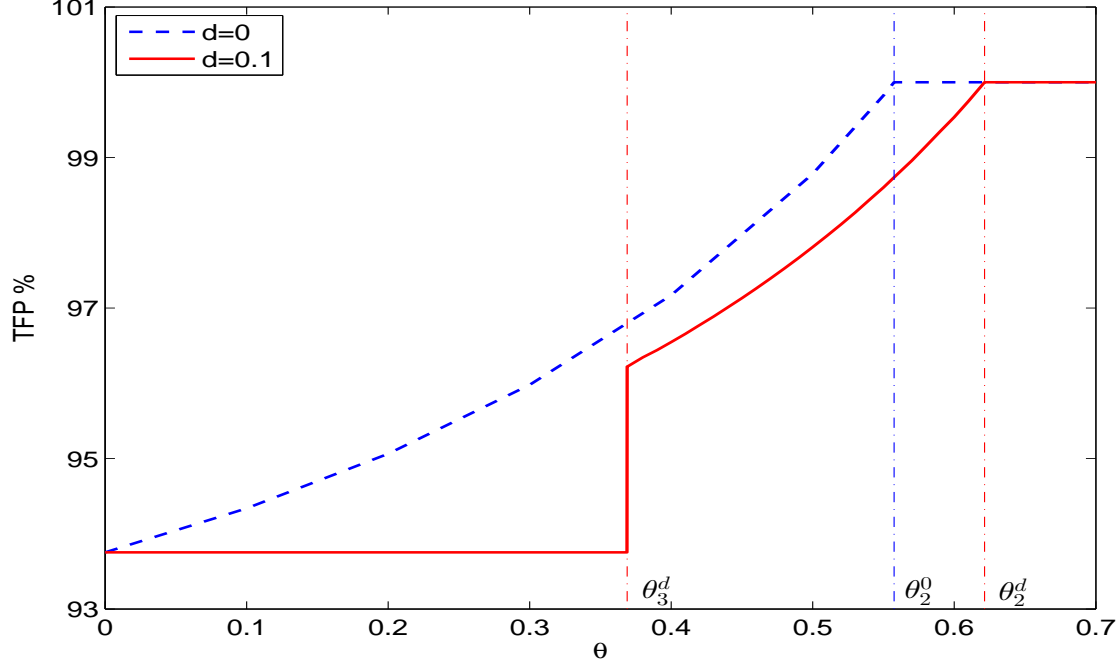
## 4.2 Comparative Statics

To examine the interactions of financial friction and capital partially irreversibility, I vary  $\theta$  to see the changes of aggregate total factor productivity (TFP) and the fraction of entrepreneurs who liquidate their firms. Let  $\theta$  decrease from  $+\infty$  to 0. The economy features no borrowing constraint when  $\theta > \theta_1^d = 0.6540$ .  $z^h$  firms have enough credit to reallocate all available capital from  $z^l$  firms. For the calibrated  $d = 0.1$ , which is not too large compared to some empirical evidence, every  $z^l$  owners liquidate their firms when  $\theta$  is above  $\theta_1^d$ . Capital stock is fully under  $z^h$  technology and thus aggregate TFP equals  $\bar{z}^h$ . Notice that if  $d$  is large enough,  $z^l$  owners may not sell their capital regardless of the level of  $\theta$ .

When  $\theta$  reaches  $\theta_2^d = 0.6214$ , some previous  $z^h$  entrepreneurs who just drew  $z^l$  start to hold capital for one period (Figure 5).  $z^h$  owners invest and borrow to the limit. As  $\theta$  becomes even smaller, the inaction region starts to expand and waiting periods increase

Figure 6: **Aggregate TFP Losses in  $d > 0$  and  $d = 0$  economy**

Aggregate TFP as percentage of  $\bar{z}^h$  in the steady state, when only  $\theta$  changes. The red solid line:  $d > 0$  economy. The blue dashed line:  $d = 0$  economy.



i.e., persistently unlucky  $z^l$  owners wait longer and longer before selling capital. Capital reallocation is less which reduces aggregate TFP.

When  $\theta = \theta_3^d = 0.3689$ , the secondary market shuts down so that no single  $z^l$  owner sells capital (Figure 6). All entrepreneurs effectively save only through running firms. The reason is that a larger degree of financial frictions further limit borrowing and reallocation and wage rate drop further lower. Then,  $\pi$  will tend to be very large while  $R = \beta_h - 1$  is still fixed in steady state. When  $\theta < \theta_3^d$ , the condition  $R' \geq z^l \pi' + (1 - \delta)$  under which  $z^l$  owners do not invest is no longer satisfied. Therefore,  $z^l$  owners always find investing in capital stock better than saving in bonds. The economy is thus characterized by no productivity risk-sharing, in contrast to some degree of risk-sharing through financial market. As a comparison, I plot the simple economy with  $d = 0$ .

The important message is that both markets can shut down together if the two frictions interact.  $\theta \in [0, \theta_3^d]$  is an extreme interaction between asset irreversibility and financial constraints. Asset illiquidity delays liquidation and tighter borrowing constraints prolong the delay. Once the profit rate is high enough and the interest rate is low enough due to large financial frictions, no liquidation takes place and the credit market effectively shuts down. Then, the economy is as if in the  $d = 0$  economy with  $\theta = 0$ , even though  $\theta_3^d$  is still far from zero (which exemplifies the interaction).

Finally, notice that such TFP losses are large and significant compared to the literature on financial frictions' impact on capital misallocation.<sup>27</sup> Given a degree of financial frictions, capital partially irreversibility can add losses about 0.5% to 1.5% of the efficient economy aggregate TFP ( $\tilde{z}^h$ ). In the extreme case, there is about 2.5% more losses when borrowing is still allowed but both credit market and secondary market are effectively shut down. The studies in the literature are thus sensitive to the introduction of capital irreversibility, a common phenomenon in the secondary market.

### 4.3 Equilibrium Response to Shocks

For estimating the shocks and their persistence, I use output and capital reallocation (both after HP-filtered) as the observations. The unobservable shocks are financial shocks and aggregate productivity shocks. I use maximum likelihood methods to back out the information of the shocks, conditional on the observations. Specifically, I assume:

$$\log \theta_t = (1 - \rho_\theta) \log \theta + \rho_\theta \log \theta_{t-1} + \epsilon_t^\theta,$$

$$\log A_t = \rho_A \log A_{t-1} + \epsilon_t^A.$$

Innovation process  $\epsilon_t = [\epsilon_t^\theta, \epsilon_t^A]^T$  is Gaussian with  $E[\epsilon_t] = 0$ ,  $E[\epsilon_t \epsilon_s'] = 0$ ,  $E[\epsilon_t \epsilon_t'] = \Sigma_\epsilon$  and

$$\Sigma_\epsilon = \begin{bmatrix} \sigma_\theta^2 & \sigma_{\theta A} \\ \sigma_{A\theta} & \sigma_A^2 \end{bmatrix}.$$

I use the HP-filtered cyclical components of real reallocation and real GDP data from 1984Q1 to 2011Q4 to estimate the standard deviation and the persistence parameters. The correlation in the variance-covariance matrix represents channels that are not modeled. For example, one rationale could be that adverse aggregate TFP shocks are from less capital utilization resulting from less funding resources (so that  $\sigma_{\theta A} > 0$ ).

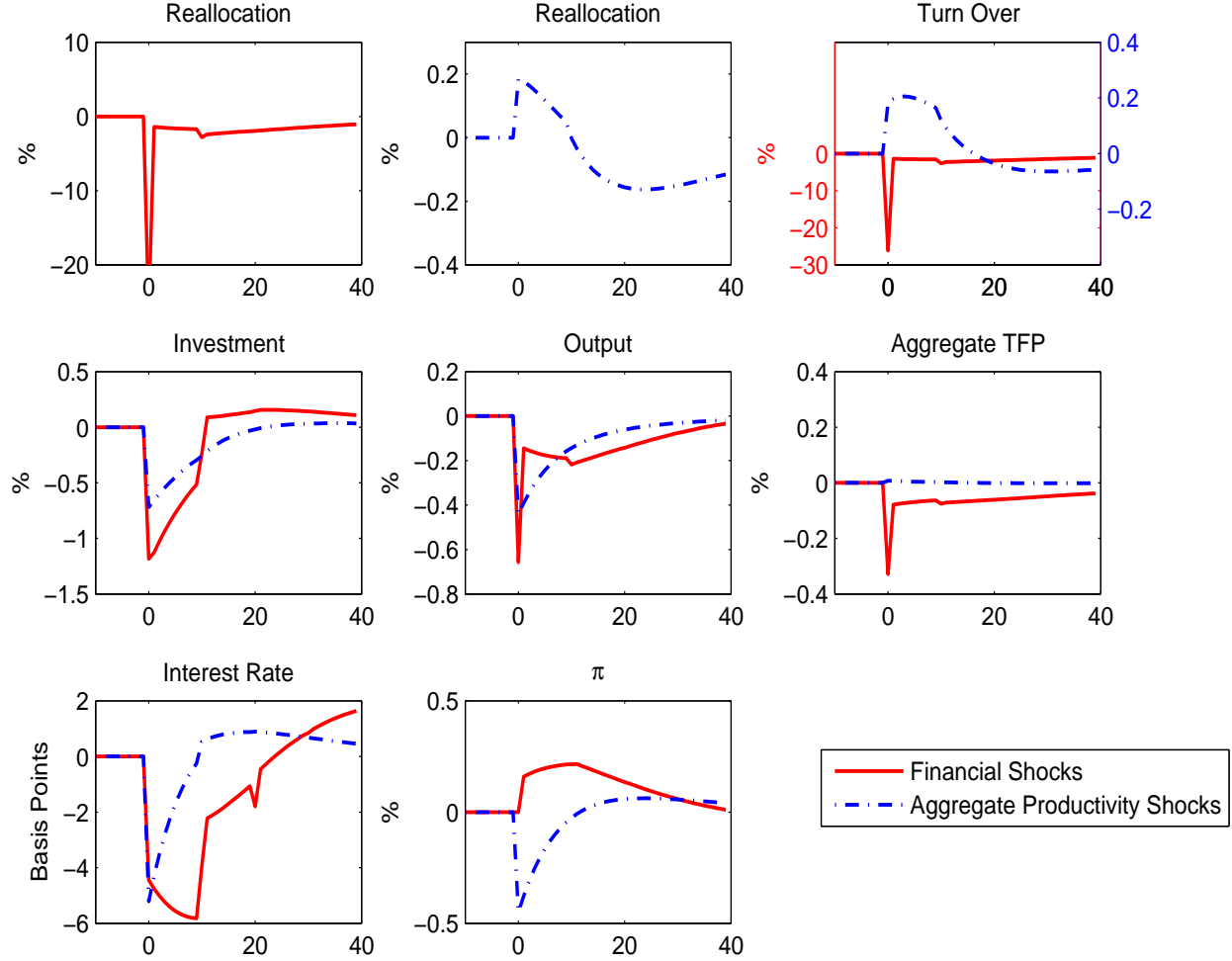
I experiment with standard aggregate TFP shocks and financial shocks. With large aggregate shocks, the model becomes intractable because the number of vintages changes after large shocks, leaving complex dynamics to solve. Instead, I focus on small aggregate shocks such that the equilibrium vintages do not change. I solve the dynamics around the steady state using first-order perturbation methods. Then I verify that the shocks are small enough through the response of the fraction ( $f_t$ ) of entrepreneurs that stay in vintage  $N = 12$ . If  $f_t$  is still less than 1, the vintages do not change.

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<sup>27</sup>For example, [Midrigan and Xu \(2012\)](#) found that misallocation results in TFP losses of only about 0.3% in the benchmark calibrated economy and at most 5% when the credit market completely shuts down. In [Moll \(2010\)](#) the magnitude of TFP losses depends on the persistence of idiosyncratic productivity shocks.

### Figure 7: Experiment: Responses to two types of shocks

Responses to one standard deviation of negative financial shocks (shocks to  $\theta$ ) and negative aggregate productivity shocks (shocks to  $A$ ). Reallocation: capital reallocation. Turn-over: capital reallocation as percentage of total assets. Aggregate TFP: the Solow residuals after adjusted by  $A$  changes. The solid line denotes the response to financial shocks while the dashed line denotes the response to aggregate productivity shocks.



The standard deviation of aggregate TFP shocks (shocks to  $A$ ) is 0.45%, which is close to other estimation results found in the literature such as in [Thomas \(2002\)](#) (with 0.53%). Second, the size of the credit shocks (about 1.15%) is larger than aggregate TFP shocks (0.45%). Finally, credit shocks ( $\rho_\theta = 0.9701$ ) are more persistent than TFP shocks ( $\rho_A = 0.8721$ ). The correlation between financial shocks and aggregate TFP shocks is 0.27. Even though I only use the two observed series (output and reallocation) for estimation (to avoid stochastic singularity issues because I focus on two shocks), the estimated aggregate TFP shocks and financing shocks generate key business cycle statistics that are close to the data (Table 5 in the Appendix). Figure 7 show the impulses to a one standard deviation (1.15%) credit shocks and a one standard deviation (0.45%) aggregate productivity shocks, by neglecting the correlation of the two shocks.



In response to credit shocks, tightened financing constraints largely reduce the investment from  $z^h$  firms. Because high productive firms are constrained, aggregate TFP drops which reduces real interest rate for saving i.e., the benefits after liquidation are smaller. In addition, demand for labor shrinks and real wage rate declines in equilibrium. Running firms now has lower labor input costs (i.e.,  $\pi$  increases). Therefore, since both the benefits after liquidation and the costs of keeping running business are smaller, more  $z^l$  firms delay selling assets. These selling delays lead to less reallocation and thus a larger further drop of aggregate TFP and output.

Notice that investment drops because productive firms are more constrained. Purchase of used assets will drop at the same time. But the selling margin intensifies the drop of used assets. Reallocation thus appears to be more volatile than new investment.

Shocks to aggregate productivity, however, generate completely different dynamics. First, capital reallocation is more initially and the turn-over of capital will be high for about 3 years. Since aggregate productivity drops, the profit rate of investing in capital declines (see  $\pi_t$  responses). The  $z^l$  owners thus have less incentive to hold capital, and more capital is liquidated. Second, compared to the economy before the shocks, fewer  $z^l$  owners stay to operate firms such that the measured TFP dispersion will be smaller. In fact, aggregate TFP (correcting the drop of  $A_t$ ) increases slightly after the shocks.

To better illustrate, when productivity  $A_t$  is lower, the first order effect is that profit rate is instantaneously lower since

$$\pi_t = \alpha \left[ \frac{(1 - \alpha)A_t}{w_t} \right]^{\frac{1-\alpha}{\alpha}}. \quad (19)$$

Though wage rate  $w_t$  will also drop, it will decrease less than  $A_t$  because demand for labor will not drop equally with  $A_t$ . In addition, (saving) interest rate  $R_t$  will change slightly because households' savings make it stable. In steady state,  $R_t$  is  $\beta_h^{-1}$  regardless of aggregate productivity. Then,  $z^l$  firms find keeping assets very unattractive as the return generated from business is low. Reallocation from  $z^l$  firms thus increase after negative  $A_t$  shocks, similar to “creative destruction” conventional wisdom.

## 4.4 Full Simulation

Whether aggregate shocks can generate less reallocation in recessions depends on whether shocks can delay  $z^l$  firms in selling assets. To examine more thoroughly the reallocation-output co-movement, I simulate the model i.e., when financial shocks or aggregate productivity shocks repeatedly hit the economy. For example, Table 2 shows the simulation using one type of shocks only. Note that the estimates are from previous exercise.

Table 2: **Only One Type of Shocks**

	Volatility		Co-movement	
	Standard deviation	Standard deviation to that of output	Correlation with output	
	Output	Reallocation	Reallocation	Reallocation Turn-over
Data:	1.42%	10.91	0.85	0.79
Model:				
Only financial shocks	1.38%	11.03	0.83	0.71
Only aggregate TFP shocks	1.31%	1.77	0.18	-0.33

First, reallocation is more volatile in the economy with only financial shocks. From the impulse responses, aggregate TFP shocks have the opposite effects on reallocation. That is why we should observe a more volatile reallocation in responses to only financial shocks.

Second, aggregate TFP shocks still generate a slightly positive correlation between reallocation and output. This fact is because after one-time aggregate TFP shock, eventually capital available for reallocation will be less, as in the impulse responses in Figure 7. To clearly see the delay of capital reallocation, the reallocation turnover is negatively correlated with output if aggregate productivity shocks are the only driving force.

In summary, one needs both aggregate TFP shocks and credit shocks to generate consumption, investment, and output dynamics as in Table 5; however, to capture both procyclical reallocation, financial shocks are necessary. The dynamics of capital reallocation thus provide some useful information of the source(s) of business cycles.

## 5 Discussion

Without irreversibility, there is no inactive investment decisions such that there is no waiting periods. Without borrowing constraints,  $z^h$  firms can borrow as much as possible to reallocate assets. The number of waiting periods is small and in fact is zero in our calibrated economy. In order to generate prolonged capital reallocation delay during recessions, the interactions between the two frictions are the key ingredients.

What about shocks to the cross-sectional dispersion of idiosyncratic productivities, such as in Bloom (2009)? Uncertainty shocks increase the real option value of holding assets. However, there is no definite answer in this paper. For example, when the gap of  $z^h$  and  $z^l$  increases while the unconditional mean are kept the same, it is ambiguous whether the right-hand side of (8) is larger or smaller, keeping everything else equal. Such “uncertainty

shock” might not lead to the decrease of  $\underline{\lambda}$  i.e., the inaction region might not expand. Intuitively, when  $z^l$  becomes smaller, the scenario of drawing  $z^l$  again tomorrow could be very unattractive given the interest rate and the fixed costs. At the same time, the dynamics of profit rate  $\pi$  and interest rate  $R'$  are ambiguous, which further adds to the “uncertain” response after uncertainty shocks. Note that the key reason for this ambiguous result is that the benefits after liquidation is not trivially determined, once we introduce credit markets.

Importantly, the illustration does *not* suggest that aggregate productivity shocks or uncertainty shocks are not important. Instead, it shows that aggregate productivity shocks alone have difficulties in explaining the selling margin, while uncertainty shocks alone may give ambiguous direction of capital reallocation. The key message is that a tougher outside financing condition seem to be necessary, even if aggregate productivity or uncertainty shocks are the driving force. While I consider the reallocation margin, [Gilchrist, Sim, and Zakrajsek \(2010\)](#) and [Christiano, Motto, and Rostagno \(2014\)](#) illustrate the similar issues with the consideration of default over business cycles.

Finally, this paper does not model changes of illiquidity. The first reason is that if illiquidity comes from asymmetric information, good quality assets might be forced to be liquidated in recessions and mitigate the information problem as in [Eisfeldt \(2004\)](#). Second, if the increases of illiquidity are all because of fire-sale of real assets as in [Shleifer and Vishny \(1992\)](#), the larger TFP dispersion during recessions is hard to be justified. Fire-sale theories suggest the most efficient firms of using the assets are also in financial troubles, which should lead to a smaller TFP dispersion. The last and probably the most important one is that if the illiquidity can be amplified, then this paper proposes one cause for the *initial* drop of asset liquidity: a credit crunch can reduce the number of buyers and sellers simultaneously and lead to endogenous change of capital irreversibility. Recently, [Lanteri \(2014\)](#) shows the change of irreversibility generated from the endogenous choices of new and used capital stock.

## 6 Final Remark

This paper shows that inefficient firms might want to hold onto capital stock in recessions and delay the reallocation process, in contrast to the “creative destruction” conventional wisdom. The reason is that lower rates of return from saving in risk-free assets and lower real wage rates in recessions will make keeping capital stock with risky returns more attractive to unproductive firms. Therefore, lowering interest rate as a policy response in recessions might delay capital reallocation further, though it can help productive firms to expand. The trade-offs are thus worthy of careful consideration.

The challenge to link asset liquidation from individual firms and aggregate capital reallocation is the evolving distribution of firms. I model the selling decision as a stopping-time problem that turns out to simplify the aggregate distribution dramatically. Meanwhile, the real option value of capital stock before liquidation shed some light on how firms price their assets internally.

One future prospect is how the resale costs endogenously interact with the depth of asset markets. The reallocation costs, in that case, come from matching between buyers and sellers. Sellers may find it costly to search potential buyers, especially during downturns. In contrast, asset markets are generally deeper in economic booms. The resale discounts are smaller in boom times and delayed selling by inefficient firms is shortened. A better allocation of assets will deepen asset markets further, and labor market conditions will improve too. Therefore, policy targeted the resale market depth may have a large effect by improving the efficiency of asset allocation. This channel may also shed light on unemployment issues.

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# Appendices

## A Data Description

For capital reallocation, the quarterly COMPUSTAT contains useful information for ownership changes of productive assets from 1984Q1. Following [Eisfeldt and Rampini \(2006\)](#), who use annual COMPUSTAT data from 1971, I measure capital reallocation by sales of property, plant and equipment (SPPE, data item 107 with combined data code entries excluded), plus acquisitions (AQC, data item 129 with combined data code entries excluded). The measure captures transactions after which the capital is used by a new firm and new productivity is thus applied. The advantage of using quarterly data compared to annual data is more observations. However, quarterly data is shown in the “cash flow statement” and there is a substantial seasonal pattern. Therefore, I apply seasonal adjustment to the data.

For aggregate consumption, investment, and GDP, I obtain the data from FRED, a macroeconomic dataset managed by Federal Reserve Bank at St. Louis. Note that I exclude residential investment, consumer durables, government expenditure, and net export because the model abstract from these components.

## B Extra Tables

Table 3: **Summary Statistics for COMPUSTAT Capital Reallocation**

Level variables are in millions of 2005 dollars for a given calendar quarter. “PP&E” stands for property, plant and equipment, “CapEx” for capital expenditures, “Reallocation” is the sum of acquisitions plus sales of PP&E, and “Investment” is defined as the capital expenditure plus acquisition. Total Reallocation/Total Previous PP&E ratio is computed as the sample mean of the numerator over the sample mean of the denominator to avoid the problem of firms with extremely large assets.

Variable	Mean	Median	Std. Dev.
Assets	2435.11	129.94	15712.73
PP&E	602.24	17.16	3851.315
CapEx	20.12	1.23	101.23
Acquisitions	6.12	0.00	45.67
Sales of PP&E	3.51	0.00	18.50
Total Sales of PP&E/Total Reallocation	30.71%		
Total Reallocation/Total Investment	32.1%		
Total Reallocation/Total Previous PP&E	1.44%		



Table 4: **Capital reallocation**

Correlation of real GDP and the various definitions of capital reallocation, after taking natural log and then HP filtered. Numbers in the bracket are the standard deviation after correcting heteroscedasticity and autocorrelation. Acquisition: COMPUSTAT data items 129. SPPE: sales of property, plant and equipment, COMPUSTAT data item 107. AQC turnover: acquisition divided by total asset (item 6) last period. SPPE turnover: SPPE divided by total property, plant and equipment (item 8) last period. Total Reallocation is the sum of acquisition and SPPE. GDP is real GDP in 2005 dollars. All series are seasonal adjusted and “\*\*\*” denotes 1% significance level.

Correlation	Acquisition	SPPE	Reallocation	SPPE turnover	AQC turnover
Corr with GDP	0.840*** (0.064)	0.430*** (0.148)	0.854*** (0.057)	0.411*** (0.128)	0.786*** (0.071)

Table 5: **Key statistics in the data and in the model**

Data are cyclical components of HP filtered series from 1984Q1 to 2011Q4. Standard deviations denote the standard deviations of percentage deviations from trends.

	Volatility				Co-movement			
	Standard deviation	Standard deviation to that of output			Correlation with Output			
		Output	Consumption	Investment	Reallocation	Consumption	Investment	Reallocation
Data:	1.42%	0.55	3.86	10.91	0.91	0.96	0.85	-0.42
Model:	1.35%	0.61	4.01	11.05	0.88	0.91	0.61	-0.37

## C Proofs

### C.1 Lemma 1

First, I prove (i) and (ii) of Lemma 1. Define the Bellman operator  $\mathcal{T}$  as

$$\mathcal{T}V(k, b, z; X) = \max \{W^1(k, b, z; X), W^0(k, b, z; X)\}$$

$$W^1(k, b, z; X) = \max_{k' > 0, R'b' \geq -\theta(1-d)(1-\delta)k'} u(z\pi k + Rb - \psi(k', k) - b') - \eta + \beta \mathbb{E}_{z,X}[V(k', b', z'; X')]$$

$$W^0(k, b, z; X) = \max_{b'} \{u(z\pi k + Rb + (1-\delta)(1-d)k - b') + \beta \mathbb{E}_{z,X}[V(0, b', a'; X')]\}$$

The value function is the fixed point of the contraction mapping in some closed space  $\mathcal{V}_1$  of functions (see [Stokey, Lucas, and Prescott \(1989\)](#)). I will show that  $\mathcal{V}_1$  includes the properties (1) and (2) in the Lemma. To simplify notation, let

$$w^1(k, b, k', b', z; X) = u(k, b, k', b', z; X) - \eta + \beta \mathbb{E}_{z,X}[V(k', b', z'; X')]$$

$$w^0(k, b, k', b', z; X) = u(k, b, 0, b', z; X) + \beta \mathbb{E}_{z,X}[V(0, b', a'; X')]$$

with slight abuse of notation of utility function  $u(\cdot)$ .

(1) *Increasing in  $z$ ,  $k$  and  $b$ , and concavity*

See [Stokey, Lucas, and Prescott \(1989\)](#).

(2)  *$V$  satisfies the following property*

$$V(\gamma k, \gamma b, z; X) = V(k, b, z; X) + \frac{\log \gamma}{1 - \beta}$$

I will prove the contraction mapping  $\mathcal{TV}$  satisfies the same property (2) if  $V$  satisfies (2). Then, the unique fixed point  $V$  satisfies (2).

Consider an agent with state  $(k, b, a)$  with  $(k', b')$  as the optimal policy. Now, consider another agent with  $(\gamma k, \gamma b, a)$ , where  $\gamma > 0$ . First notice that the policy  $(\gamma k', \gamma b')$  is feasible, i.e., it satisfies budget and borrowing constraints. Second, given a consistent choice  $h \in \{0, 1\}$ ,

$$\begin{aligned} \mathcal{TV}(\gamma k, \gamma b, z; X) &\geq w^h(\gamma k, \gamma b, \gamma k', \gamma b', z; X) \\ &= \log(\gamma(z\pi k + Rb - \psi(k', k) - b') - \eta h + \beta \mathbb{E}_{z, X}[V(k', b', z'; X')]) + \frac{\beta \log \gamma}{1 - \beta} \\ &= \log(z\pi k + Rb - \psi(k', k) - b') - \eta h + \beta \mathbb{E}_{z, X}[V(k', b', z'; X')] + \frac{\log \gamma}{1 - \beta} \end{aligned}$$

or

$$\mathcal{TV}(\gamma k, \gamma b, z; X) \geq \mathcal{TV}(k, b, z; X) + \frac{\log \gamma}{1 - \beta}.$$

Conversely, starting at  $(\gamma k, \gamma b, z)$ , scaling by  $1/\gamma$ , and following similar procedure above, one has

$$\mathcal{TV}(k, b, z; X) \geq \mathcal{TV}(\gamma k, \gamma b, z; X) - \frac{\log \gamma}{1 - \beta}.$$

Combining the two gives

$$\mathcal{TV}(\gamma k, \gamma b, z; X) = \mathcal{TV}(k, b, z; X) + \frac{\log \gamma}{1 - \beta}.$$

Therefore, the mapping  $\mathcal{TV}$  has the same property. Because  $V$  is the unique fixed point,  $V(\gamma k, \gamma b, z; X) = V(k, b, z; X) + \frac{\log \gamma}{1 - \beta}$ .

(iii) *Differentiability*

The differentiability of  $V(k, b, \eta, a; X)$  when  $k' \geq (1 - \delta)k$  is standard, which relies on the differentiability of standard dynamic programming problem as proved by [Benveniste and Scheinkman \(1979\)](#) (or see [Stokey, Lucas, and Prescott \(1989\)](#)). When  $k' = (1 - \delta)k$ , I follow methods from [Clausen and Strub \(2012\)](#) in Banach space (the space of  $k$  and  $b$ ) and adjust to the dynamic programming problem of my model. The general idea is that the value function is the upper envelop of value function of buying, inactive and selling. It is therefore super-differentiable. At the same time, it has potential downward kink (sub-differentiable) because of  $\psi(k', k)$  function. Therefore, the value function will be both super-differentiable and sub-differentiable, and thus differentiable. The detail derivation is long and tedious but available upon request.

## C.2 Lemma 2

I will use the results in the previous Lemma.

(i) To save notation, I abstract from aggregate state variable  $X$ . From Lemma 1,

$$V(\gamma(k+e), \gamma b, z) = V(k+e, b, z) + \frac{\log \gamma}{1-\beta}$$

Take a derivative with respect to  $e$  and evaluate it at  $e = 0$ ; one has  $\gamma V_k(\gamma k, \gamma b, z) = V_k(k, b, z)$ . Divide  $\gamma$  on both sides and one can prove that  $V_k$  is homogeneous with degree  $-1$ .

(ii) Consider two entrepreneurs with  $(k_0, b_0, z)$  and  $(\gamma k_0, \gamma b_0, z)$ . Using equation (6) of Lemma 1, the targeted capital stock and bonds are scaled up by  $\gamma$  and thus the optimal consumption choices are  $c_0$  and  $\gamma c_0$  from the budget constraints. Therefore, using property (i) of this Lemma,  $V_k/u'(c)$  is the same for the two entrepreneurs. More generally,  $V_k/u'(c)$  depends only on  $k/(k+b)$ .

(iii) By definition,  $q(k, b, z) = (V_k/u'(c) - z\pi)(1-\delta)^{-1}$ . Using (ii), we know that  $q(k, b, z)$  can be written as  $q(\lambda, z)$  where  $\lambda = k/(k+b)$ .

### C.3 Proposition 1: Investment and Disinvestment

#### C.3.1 When $k = 0$

Notice that

$$V(0, b, z; X) = \max\{W^1(0, b, z; X), W^0(0, b, z; X)\}$$

$$\begin{aligned} W^1(0, b, z; X) &= \max_{k', b'} \{\log(Rb - k' - b') - \eta + \beta \mathbb{E}_{z, X} [V(k', b', z'; X')]\} \\ &= \max_{k', b'} \{\log(Rb - k' - b') - \eta + \frac{\beta \log(k' + b')}{1-\beta} + \beta \mathbb{E}_{z, X} [V(\lambda', 1 - \lambda', z'; X')]\} \\ W^0(0, b, z; X) &= \max_{b'} \{\log(Rb - b') + \frac{\beta \log b'}{1-\beta} + \beta \mathbb{E}_{z, X} [V(0, 1, z'; X)]\} \end{aligned}$$

For  $W^1$ , one can maximize out  $k' + b'$  with optimal solution  $k' + b' = \beta R$  and

$$W^1(0, b, z; X) = \log(1-\beta) + \frac{\beta \log \beta}{1-\beta} + \frac{\log Rb}{1-\beta} - \eta + \max_{\lambda'} \left\{ \frac{\beta \log \lambda'}{1-\beta} + \beta \mathbb{E}_{z, X} \left[ V\left(1, \frac{1-\lambda'}{\lambda'}, z'; X'\right) \right] \right\}$$

which means that investing entrepreneur will pick a common target leverage  $\lambda' = \bar{\lambda}'$  to maximize the expressions in the bracket. Also, the consumption is

$$c = (1-\beta)Rb$$

For  $W^0$ , the optimal solution is  $b' = \beta R$  and therefore one has

$$W^0(0, b, z; X) = \log(1-\beta) + \frac{\beta \log \beta}{1-\beta} + \frac{\log Rb}{1-\beta} + \beta \mathbb{E}_{z, X} [V(0, 1, z'; X)]$$

The consumption is again

$$c = (1-\beta)Rb$$

### C.3.2 When $k > 0$ .

(1) if the entrepreneur decides to invest,  $W^1(k, b, z; X)$  can be rewritten as

$$\begin{aligned} & W^1(k, b, z; X) \\ &= \max_{k', b'} \left\{ \log(z\pi k + Rb - \psi(k', 1) - b') - \eta + \beta \mathbb{E}_{z, X} [V(k', b', z'; X')] \right\} \\ &= \max_{\lambda', \bar{k}'} \left\{ \log(z\pi k + (1 - \delta)k + Rb - k' - b') - \eta + \frac{\beta \log(k' + b')}{1 - \beta} + \beta \mathbb{E}_{z, X} [V(\lambda', 1 - \lambda', z'; X')] \right\} \end{aligned}$$

One can maximize out  $k' + b'$  with optimal solution  $k' + b' = \beta [z\pi k + (1 - \delta)k + Rb]$  and

$$W^1(k, b, z; X) = \log(1 - \beta) + \frac{\beta \log \beta}{1 - \beta} + \frac{\log(z\pi k + (1 - \delta)k + Rb)}{1 - \beta} - \eta + \max_{\lambda'} \left\{ \frac{\beta \log \lambda'}{1 - \beta} + \beta \mathbb{E}_{z, X} \left[ V\left(1, \frac{1 - \lambda'}{\lambda'}, z'; X'\right) \right] \right\}$$

which means again that investing entrepreneur will pick a common target leverage  $\lambda' = \bar{\lambda}'$  and is the same as those investing entrepreneurs without firms. The consumption function is

$$c = (1 - \beta)(z\pi k + (1 - \delta)k + Rb)$$

(2) If the entrepreneur decides to sell.

$$\begin{aligned} & W^0(k, b, a; X) \\ &= \max_{b'} \left\{ \log(z\pi k + (1 - \delta)(1 - d)k + Rb - b') + \beta \mathbb{E}_{z, X} [V(0, b', z'; X')] \right\} \\ &= \max_{b'} \left\{ \log(z\pi k + (1 - \delta)(1 - d)k + Rb - b') + \beta \mathbb{E}_{z, X} [V(0, 1, z'; X')] + \frac{\beta \log(b')}{1 - \beta} \right\}. \end{aligned}$$

Notice that  $\mathbb{E}_{z, X} [V(0, 1, z; X)]$  does not depend on the choice of  $b'$ . Therefore, given  $b$  and  $k$  the optimal solution  $b'$  is  $b' = \beta [z\pi k + (1 - \delta)(1 - d)k + Rb]$  and we have

$$W^0(k, b, z; X) = \log(1 - \beta) + \frac{\beta \log \beta}{1 - \beta} + \frac{\log(z\pi k + (1 - \delta)(1 - d)k + Rb)}{1 - \beta} + \beta \mathbb{E}_{z, X} [V(0, 1, z'; X')].$$

The consumption function is thus

$$c = (1 - \beta)(z\pi k + (1 - \delta)(1 - d)k + Rb)$$

(3) If the entrepreneur decides to be inactive,  $W^1(k, b, z; X)$  can be rewritten as

$$\begin{aligned} & W^1(k, b, z; X) \\ &= \max_{k', b'} \left\{ \log(z\pi k + Rb - \psi(k', 1) - b') - \eta + \beta \mathbb{E}_{z, X} [V(k', b', z'; X')] \right\} \\ &= \max_{\lambda', \bar{k}'} \left\{ \log(z\pi k + q(1 - \delta)k + Rb - qk' - b') - \eta + \frac{\beta \log(qk' + b')}{1 - \beta} + \beta \mathbb{E}_{z, X} \left[ V\left(\frac{1}{q + \frac{1 - \lambda'}{\lambda'}}, \frac{1}{q \frac{\lambda'}{1 - \lambda'} + 1}, z'; X'\right) \right] \right\} \end{aligned}$$

One can maximize out  $qk' + b'$  with optimal solution and  $k' + b' = \beta [z\pi k + (1 - \delta)qk + Rb]$

$$\begin{aligned} W^1(k, b, z; X) &= \log(1 - \beta) + \frac{\beta \log \beta}{1 - \beta} + \frac{\log(z\pi k + (1 - \delta)qk + Rb)}{1 - \beta} - \eta \\ &+ \max_{\lambda'} \left\{ \frac{\beta \log \left( \frac{\lambda'}{q\lambda' + 1 - \lambda'} \right)}{1 - \beta} + \beta \mathbb{E}_{z, X} \left[ V\left(1, \frac{1 - \lambda'}{\lambda'}, z'; X'\right) \right] \right\} \end{aligned}$$

which means again that investing entrepreneur will pick a common target leverage  $\lambda' = \bar{\lambda}'$  and is the same as those investing entrepreneurs without firms. The consumption function is

$$c = (1 - \beta) [z\pi k + (1 - \delta)qk + Rb]$$

Thus far, we know that consumption function has the above algebraic form, regardless of whether the entrepreneur needs to invest, sell or be inactive. One can replace  $q = 1$  and  $q = 1 - d$  for those who invest and who sell. The remaining question is to solve

$$\max_{\lambda'} \left\{ \frac{\beta \log \left( \frac{\lambda'}{q\lambda' + 1 - \lambda'} \right)}{1 - \beta} + \beta \mathbb{E}_{z, X} \left[ V\left(1, \frac{1 - \lambda'}{\lambda'}, z'; X'\right) \right] \right\}$$

where the first-order condition is

$$\frac{1}{(1 - \beta) \left( q + \frac{1 - \lambda'}{\lambda'} \right)} = \mathbb{E}_{z, X} V_b\left(1, \frac{1 - \lambda'}{\lambda'}\right)$$

Using the envelop condition and using  $c = (1 - \beta) [z\pi k + (1 - \delta)qk + Rb]$ , one has

$$\frac{1}{q + \frac{1 - \lambda'}{\lambda'}} = \mathbb{E}_{z, X} \left[ \frac{R'}{z'\pi' + (1 - \delta)q' + R' \frac{1 - \lambda'}{\lambda'}} \right]$$

For convenience, denote  $\phi$  such that  $qk' = \phi\beta [z\pi k + (1 - \delta)qk + Rb]$  and  $b' = (1 - \phi)\beta [z\pi k + (1 - \delta)qk + Rb]$ , then the above equation can be simplified to

$$\mathbb{E}_{z, X} \left[ \frac{R'}{\phi \frac{z'\pi' + (1 - \delta)q'}{q} + (1 - \phi)R'} \right] = 1$$

Notice that  $\frac{V_b}{R} = \frac{V_k}{z\pi + (1 - \delta)q}$ , one also has

$$\mathbb{E}_{z, X} \left[ \frac{\frac{z'\pi' + (1 - \delta)q'}{q}}{\phi \frac{z'\pi' + (1 - \delta)q'}{q} + (1 - \phi)R'} \right] = 1$$

## C.4 Proposition 2 : Leverage and Deleverage

Consider an entrepreneur with leverage  $\lambda$ . Normalize capital stock by  $k = 1$ , the state variable is  $(1, \frac{1 - \lambda}{\lambda}, z; X)$ . Consumption is can be expressed as  $c = (1 - \beta) [z\pi + (1 - \delta) + R\lambda^{-1}(1 - \lambda)]$ . In

addition, from the budget constraint  $c = z\pi + R\frac{1-\lambda}{\lambda} - (1-\delta)\frac{1-\lambda'}{\lambda'}$  such that

$$z\pi + R\lambda - (1-\delta)\frac{1-\lambda'}{\lambda'} = (1-\beta) \left[ z\pi + (1-\delta)q + R\frac{1-\lambda}{\lambda} \right],$$

where both  $q$  and  $\lambda'$  are functions of  $(\lambda, z; X)$ . Then,

$$q(\lambda, z; X) = \frac{\beta \left[ z\pi + R\frac{1-\lambda}{\lambda} \right] - (1-\delta)\frac{1-\lambda'}{\lambda'}}{(1-\beta)(1-\delta)}.$$

For convenience, let me do change of variable  $\tilde{\lambda} = b/k = (1-\lambda)/\lambda$ . Since  $\tilde{\lambda}$  and  $\lambda$  has one to one mapping, I can express

$$q(\tilde{\lambda}, z; X) = \frac{\beta \left[ z\pi + R\tilde{\lambda} \right] - (1-\delta)\tilde{\lambda}}{(1-\beta)(1-\delta)}.$$

The goal is to prove that  $q$  is an increasing function of  $\lambda$  or a decreasing function of  $\tilde{\lambda}$ . Take derivative w.r.t.  $\tilde{\lambda}$ , one has

$$\frac{\partial q}{\partial \tilde{\lambda}} = \frac{\beta R - (1-\delta)\frac{\partial \tilde{\lambda}'}{\partial \tilde{\lambda}}}{(1-\beta)(1-\delta)}. \quad (20)$$

Further, if being inactive in investment  $k' = (1-\delta)$  and  $\psi(k, 1) = 0$ , the envelop condition gives

$$V_b(1, \tilde{\lambda}, z; X) = \frac{R}{z\pi + R\tilde{\lambda} - \psi(k', 1) - k'(\tilde{\lambda})'} = \frac{R}{z\pi + R\tilde{\lambda} - (1-\delta)\tilde{\lambda}'}$$

Therefore

$$\frac{\partial V_b(1, \tilde{\lambda}, z; X)}{\partial \tilde{\lambda}} = -\frac{R \left[ R - (1-\delta)\frac{\partial \tilde{\lambda}'}{\partial \tilde{\lambda}} \right]}{\left[ z\pi + R\tilde{\lambda} - (1-\delta)\tilde{\lambda}' \right]^2}.$$

In addition, one can use the expression for consumption such that

$$V_b(1, \tilde{\lambda}, z; X) = \frac{R}{(1-\beta) \left[ z\pi + (1-\delta)q(\tilde{\lambda}, z; X) + R\tilde{\lambda} \right]}$$

and therefore

$$\frac{\partial V_b(1, \tilde{\lambda}, z; X)}{\partial \tilde{\lambda}} = -\frac{R \left[ (1-\delta)\frac{\partial q}{\partial \tilde{\lambda}} + R \right]}{(1-\beta) \left[ z\pi + (1-\delta)q(\tilde{\lambda}, z; X) + R\tilde{\lambda} \right]^2} = -\frac{R \left[ \frac{\beta R - (1-\delta)\frac{\partial \tilde{\lambda}'}{\partial \tilde{\lambda}}}{(1-\beta)(1-\delta)} + R \right]}{(1-\beta) \left[ z\pi + (1-\delta)q(\tilde{\lambda}, z; X) + R\tilde{\lambda} \right]^2}.$$

I equate the above two expressions for  $\frac{\partial V_b(1, \tilde{\lambda}, z; X)}{\partial \tilde{\lambda}}$  and obtain

$$\frac{\partial \tilde{\lambda}'}{\partial \tilde{\lambda}} = \frac{\beta R}{\delta(1-\delta)}.$$

Using this result, equation (20) then becomes

$$\frac{\partial q}{\partial \tilde{\lambda}} = -\frac{\beta R}{\delta^2} < 0.$$

which proves that  $q$  is a decreasing function of  $\tilde{\lambda}$  and thus an increasing function of  $\lambda$ .

Notice that  $\delta$  is close to 0, while  $\beta R$  is close to 1. Then,  $\tilde{\lambda}' > \tilde{\lambda}$  because  $\frac{\partial \tilde{\lambda}'}{\partial \lambda} > 1$ . The option value  $q(\tilde{\lambda}', z^l; X') < q(\tilde{\lambda}, z^l; X)$ , in the neighbourhood around steady state i.e., when  $X'$  is not very different from  $X$ .

## C.5 Proposition 3: Liquidation Decisions

Consider an entrepreneur with a firm, according to the previous proof, the three values associated with investing, being inactive, and selling are

$$\begin{aligned} W^1(k, b, z; X) &= \log(1 - \beta) + \frac{\beta \log \beta}{1 - \beta} + \frac{\log(z\pi k + (1 - \delta)k + Rb)}{1 - \beta} - \eta \\ &+ \max_{\lambda'} \left\{ \frac{\beta \log \lambda'}{1 - \beta} + \beta \mathbb{E}_{z, X} \left[ V\left(1, \frac{1 - \lambda'}{\lambda'}, z'; X'\right) \right] \right\} \end{aligned}$$

$$\begin{aligned} W^1(k, b, z; X) &= \log(1 - \beta) + \frac{\beta \log \beta}{1 - \beta} + \frac{\log(z\pi k + (1 - \delta)qk + Rb)}{1 - \beta} - \eta \\ &+ \max_x \left\{ \frac{\beta \log \left( \frac{1}{q + x - 1} \right)}{1 - \beta} + \beta \mathbb{E}_{z, X} \left[ V(1, x - 1, z'; X') \right] \right\} \end{aligned}$$

$$W^0(k, b, z; X) = \log(1 - \beta) + \frac{\beta \log \beta}{1 - \beta} + \frac{\log(z\pi k + (1 - \delta)(1 - d)k + Rb)}{1 - \beta} + \beta \mathbb{E}_{z, X} [V(0, 1, z'; X')]$$

The algebraic form of being inactive is the same as investing by replacing  $q = 1$ .

$$W^1 - W^0 = \log \left( 1 + \frac{(1 - \delta)[q(\lambda) - (1 - d)]}{z\pi + (1 - \delta)(1 - d) + R\lambda^{-1}(1 - \lambda)} \right) + \max_x \left\{ \frac{\beta \log \left( \frac{1}{q(\lambda) + x - 1} \right)}{1 - \beta} + \beta \mathbb{E}_{z, X} [V(1, x - 1, z'; X')] \right\}$$

The last two components are constant while the first two components are increasing function of  $\lambda$ . Therefore,  $W^1 - W^0$  will be a increasing function of  $\lambda$  and there will be single cross with zero if there is crossing such that there exist a  $\underline{\lambda}$  below which  $W^0$  dominate and above which  $W^1$  dominates. That is,  $\underline{\lambda}$  is the liquidation threshold.

From the above discussion, one knows that the value function can be written as

$$V(k, b, z; X) = J_b(X) + \frac{\log(z\pi k + (1 - \delta)k + Rb)}{1 - \beta}$$

if entrepreneurs decide to invest,

$$V(k, b, z; X) = J(\lambda, z; X) + \frac{\log(z\pi k + (1 - \delta)qk + Rb)}{1 - \beta}$$

if entrepreneurs decide to being inactive, and

$$V(k, b, z; X) = J_s(X) + \frac{\log(z\pi k + (1 - \delta)(1 - d)k + Rb)}{1 - \beta}$$

if entrepreneurs decide to sell existing business or keep out of business. Importantly,  $J_b$ ,  $J$ , and

$J_s$  are independent of  $k$  and  $b$  levels. To compute the leverage with which entrepreneurs liquidate the firms, I use no one-shot deviation strategy. Consider an entrepreneur who is about to liquidate business and who is with leverage  $\lambda$  and net-worth  $n = z\pi + (1 - \delta)(1 - d) + R\lambda^{-1}(1 - \lambda)$ . Again, I normalize capital stock to be 1. If liquidate, the value of doing so is

$$\log((1 - \beta)n) + p^l \mathbb{E}_X \left[ J_b(X') + \frac{\log(R\beta n)}{1 - \beta} \right] + p^{lh} \mathbb{E}_X \left[ J_s(X') + \frac{\log(R\beta n)}{1 - \beta} \right]$$

If keeping running for one more period but liquidating next period if still drawing  $z^l$ , the value of doing so is

$$\begin{aligned} & \log((1 - \beta)n) - \eta + p^l \mathbb{E}_X \left[ J_b(X') + \frac{\log(z^h \pi' (1 - \delta) + (1 - \delta)(1 - \delta) + R'b')}{1 - \beta} \right] \\ & + p^{lh} \mathbb{E}_X \left[ J_s(X') + \frac{\log(z^l \pi' (1 - \delta) + (1 - \delta)(1 - \delta)(1 - d) + R'b')}{1 - \beta} \right] \end{aligned}$$

where  $b' = \beta n - (1 - d)(1 - \delta)$ . Then, the leverage cut-off  $\underline{\lambda}$  should make this two expression the same i.e.,

$$\begin{aligned} \tilde{\eta} &= \frac{\beta}{1 - \beta} p^{lh} \mathbb{E}_X \left[ \log \left( 1 + (1 - \delta) \frac{z^h \pi' + (1 - \delta) - (1 - d)R'}{\beta n R'} \right) \right] \\ &+ \frac{\beta}{1 - \beta} p^l \mathbb{E}_X \left[ \log \left( 1 + (1 - \delta) \frac{z^l \pi' + (1 - \delta)(1 - d) - (1 - d)R'}{\beta n R'} \right) \right] \end{aligned}$$

## C.6 Aggregate TFP

We know that aggregate TFP can be measure by

$$TFP = \frac{\frac{\pi}{\alpha} (z^h K_h + z^l K_l)}{(K_h + K_l)^\alpha \left[ \left( \frac{\pi}{\alpha} \right)^{\frac{1}{1-\alpha}} (z^h K_h + z^l K_l) / A + l_e \right]^{1-\alpha}}. \quad (21)$$

We know information of all variables except  $l_e$ . To compute  $l_e$ , we first compute the measure of entrepreneurs  $m_i$  in each vintage  $i$ . For vintage 0 entrepreneurs, the evolution satisfies

$$m'_0 = p^{hh} m_0 + p^{lh} \sum_{i=1}^{N+2} m_i$$

For vintage  $i = 1, 2, \dots, N$  entrepreneurs,

$$m'_i = p^{i-1l} m_{i-1}$$

For vintage  $i = N + 1$ ,

$$m'_{N+1} = p^{Nl} f m_N$$

For vintage  $i = N + 2$ ,

$$m'_{N+2} = p^{Nl} (1 - f) m_N + p^{(N+2)l} m_{N+2}$$



Therefore, if we stack the measure as  $m = [m_1, m_2, \dots, m_{N+1}, m_{N+2}]^T$ , we have

$$m' = Pm$$

where

$$P = \begin{bmatrix} p^{hh} & p^{hl} & & & & \\ p^{lh} & & p^{ll} & & & \\ p^{lh} & & & p^{ll} & & \\ \dots & & & \dots & & \\ p^{lh} & & & & fp^{ll} & (1-f)p^{ll} \\ p^{lh} & & & & & p^{ll} \\ p^{lh} & & & & & p^{ll} \end{bmatrix}_{(N+2) \times (N+2)}$$

In steady state, once  $f$  is determined, the right eigenvector (after normalization) of  $P^T$  associated with eigenvalue one is the stationary population of entrepreneurs in each vintage. For a detail description, see [Ljungqvist and Sargent \(2004\)](#). Further, we can use the relationship between  $m'$  and  $m$  to compute  $m'$  and then  $l'_e$

$$l'_e = 1 - m_{N+2}$$