Implications of Return Predictability across Horizons for Asset Pricing Models[*]

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Abstract

We analyze predictors-based variance bounds, i.e bounds on the variance of the stochastic discount factors (SDFs) that price a given set of returns conditional on the information contained in a vector of return predictors. For an asset pricing model identified by its state variables, information structure and model SDF, we supply a sufficient condition under which our predictors-based bounds constitute legitimate lower bounds on the variance of the SDF of the model. Using our predictors-based bounds we analyze discount factors produced by the long-run risk, the habit and the rare dysasters models. We document that consumption-based asset pricing models such as long-run risk and habit models do not produce SDFs volatile enough at the one-year horizon. When we look at long-horizons our evidence shows that it is the habit model, not the long-run risk model, that satisfies our bounds. The rare dysasters model satisfies our predictors based bounds at each horizon. As a consequence, the investment horizon and the use of conditioning information emerge as fundamental ingredients that permit either to set models apart, or to select the common behavior among apparently different models.

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1 Introduction

"There is no way to predict the price of stocks and bonds over the next few days or weeks. But it is quite possible to foresee the broad course of these prices over longer periods, such as the next three to five years."

Press release of the The Royal Swedish Academy of Sciences for the 2013 Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel

If there is valuable information for predicting stock and bond prices over time, and the more so the longer the horizon, when and how can we use this information to discriminate among competing asset pricing models? The answer we give in this paper is both methodological and empirical. From a methodological point of view, we offer a simple condition under which a variance bound that incorporates conditioning information from a given a set of stock and bond predictors constitutes a legitimate lower bound on the variance of the Stochastic Discount Factor (SDF) of a given asset pricing model. From an empirical point of view, we examine three leading asset pricing models, the External Habit model of Campbell and Cochrane (1999), the Rare Disaster model of Nakamura, Steinsson, Barro, and Ursúa (2013), and the model of Long Run Risk (LRR) of Bansal, Kiku, and Yaron (2012a), we show that all these models satisfy the condition for the predictors-based bounds to be a legitimate lower bounds on the variance of their SDFs, and then we use these bounds to explore the role that short and long horizon predictability plays in the econometric evaluation of these models.

The importance of our methodological contribution is based on the fact that, in practice, the most successful predictors for stock and bond returns are not theory-driven, i.e. derived by solving specific asset pricing models. The predictive relation between the (log) dividend/price ratio and stock market returns, for instance, is derived by solving forward a linearized expression for returns (Campbell and Shiller (1988a)). Similarly, the predictive relation between the cay variable of Lettau and Ludvigson (2001) and stock market returns follows from a linearized version of the consumer's intertemporal budget constraint. Furthermore, successful bond market predictors such as the term spread and the deviation of (log) bond prices from a permanent component follow from either a linearization argument similar to the one that generates the dynamic dividend growth model for the stock market or from finding an empirically suitable combination of forward rates (see e.g. Fama and Bliss (1987) and Cochrane and Piazzesi (2005)). If the question is to see whether a given theoretical model is able to generate sufficient variability in the discount factor, one needs to bridge the information contained in a set non-theory-driven predictors with the asset pricing model under scrutiny. In fact, the variability of the discount factor of a given model is conditional on some model specific state variables that are in general different from the variables used in the predictive regressions. As a consequence, the possible discrepancy between the informational content of the predictors and that of the state variables may render the empirical evidence on predictability not informative to reject some models, on one side, and decisive to draw inference on other models on the other.

We illustrate in Proposition 1 our condition for the variance of the SDF of a given model to satisfy the variance bounds obtained by conditioning on the predictors. The condition is on the returns discounted by the SDF of a given asset pricing model, and it requires the predictability of these discounted returns not to increase when the information in the predictors is added to information in the state variables of the model. To enhance the intuition, the implication of our condition can also be: since discounted returns are unpredictable when a model's SDF satisfies the Euler equation, if that model's SDF fails to achieve the variance threshold dictated by our predictors-based bound, then the discounted returns on some assets must become predictable when the information in the state variables is augmented with the information in the predictors.

In the empirical part of the paper, we first provide robust evidence that long horizon predictability translates into tight higher bounds on the variance of the SDFs. We then test our condition for the habit-formation model of Campbell and Cochrane (1999), the rare disaster model of Nakamura et al. (2013), and the long run risk model of Bansal et al. (2012a), and show that the condition cannot be rejected for these three models: our predictors-based bounds are legitimate bounds on the variance of their SDFs. Recall now that these three models match closely both the historical unconditional annual real return on the risk-free bond and the equity market. Moreover, they both incorporate a low frequency component that should make asset pricing puzzles less pronounced at longer horizons. Consistent with these statements, the conclusion drawn from the standard unconditional Hansen and Jagannathan (1991) bounds are not surprising: all models satisfy these unconditional cups at medium and long horizons. The conclusions are different when we use our predictors-based bounds. Our bounds show that both the habit-formation and the long run risk model share a common feature at the one-year horizon: the variance of their SDFs is too small when compared with the lower bound computed from the data and summarized in the predictors-based bounds, while the SDF implied by the rare disaster model is sufficiently volatile to satisfy our bounds. Across horizons is where we are able to tell models apart: first of all, the rare disaster model is the only one which is capable to satisfy our predictors-based bounds across horizons; then between the long run risk and habit model, the LRR model has problem in fitting the 5-year predictors-based bound, whereas the habit is well within the admissible region. If one were to look only at the long horizon bounds with no conditioning information, the conclusion would be that the long run equity premium puzzle can be resolved as long as sufficient time-nonseparability is incorporated in the preferences. However, our predictors-based bounds highlight that time-nonseparable preferences are not the full story. First, we show that the long run risk model of Bansal et al. (2012a) does not generate returns that are predictable enough compared to the predictability that emerges from the data. The habit model of Campbell and Cochrane (1999) is able to generate a plausible magnitude of return predictability but it fails to have enough volatility in its model implied SDF which mainly depends on the dynamics of state variable, surplus consumption ratio in this case. In fact, it is the rare disaster model of Nakamura et al. (2013) that emerges as the most difficult model to be rejected using our predictors-based bounds.

These results show the importance of understanding when and how we can employ the information contained in a set of predictors. The dynamic asset pricing models under consideration are constructed from a mixture of assumptions about preferences (such as recursive utility or habit persistence, etc) and exposure to fundamental shocks (such as rare disasters). Our predictors-based bounds allow us to abstract from these model ingredients since they highlight the transitory and long run implications of these models. Our bounds are simultaneously able to detect the common feature across these models, i.e. the low variance of SDFs implied by long run risk and habit model at the 1-year horizon, and to tell the rare disaster model apart from the other two upon looking across horizons.

Our work is related to Kirby (1998), who provides an explicit link between linear predictability and the Hansen and Jagannathan (1991) bounds. Whereas Kirby (1998) investigates whether the ability of predictors to forecast a given set of return is correctly priced by *some* rational asset pricing model, in the sense that there exist SDFs that price correctly those dynamic strategies which condition on the predictors, our interest here is different: we want to exploit the informational content of *a given set of predictors* to investigate the potential of *a given asset pricing model* to price *a given set of returns.* Our work is also related to the recent literature which, using a decomposition of the model's dynamics into transient and permanent components, investigates the implications of these components for valuation (see Hansen and Scheinkman (2009) and Borovicka, Hansen, Hendricks, and Scheinkman (2011)). In particular we view our predictors-based bounds as a useful tool for understanding the high- and low-frequency components of such models. Finally, our work is related to the recent information-theoretic literature that uses entropy bounds to restrict the admissible regions for the SDF and its components (see Bakshi and Chabi-Yo (2012) and Ghosh, Julliard, and Taylor (2011)). In particular our conclusions are in line with Backus, Chernov, and Zin (2011b) who show that the entropy of a model should be sufficiently large to account for observed excess returns.

The rest of this paper is organized as follows. Sections 2 introduces our predictors-based variance bounds and provides the condition under which these bounds are indeed legitimate lower bounds on the variance of the SDF of a given asset pricing model. Section 3 documents the existence of significant predictable variation in stock and bond returns and shows how conditioning information plays an important role in the construction of our bounds at different horizons. We then assess whether various SDF specifications are consistent with our predictors-based bounds. Section 4 addresses two questions: which among the asset classes considered in the paper, stocks and bonds, is key to our results; and how the comparison of model-implied return predictability versus the historical one is connected to our predictors-based variance bounds. Section 5 concludes.

2 Variance Bounds, Predictability and Asset Pricing

In this section we first define our predictors-based variance bounds, which are bounds on the variance of the SDFs that price a given set of returns conditional on the information contained in a vector Z_t of return's predictors. Given then any asset pricing model with SDF m_{t+h}^X, we ask: under what conditions does a predictors-based bound constitute a legitimate lower bound on the variance of m_{t+h}^X? We answer this question by identifying in Proposition 1 a simple condition, under which the variance of m_{t+h}^X must indeed satisfy the bound obtained by conditioning of the predictors Z_t. In Proposition 2, moreover, we rephrase our sufficient condition in terms of an upper bound on the $\mathcal{R}^2$ from predictive regressions of future returns on the current values of the predictors.

2.1 Variance bounds when returns are predictable

We consider an environment with N random returns on a set of assets traded at a given time t. We denote the return on each asset by $R_{j,t+h}$, with $h = 1, 2...$ the investment horizon, and we let R_{t+h} denote the vector collecting these N returns. Alongside the returns we consider a vector Z_t of return's predictors, and we denote with $\mathcal{F}_t^Z$ the informational content of these predictors. By saying that Z_t predicts the return $R_{j,t+h}$ on some asset j we mean $Var\left[E\left(R_{j,t+h}\left|\mathcal{F}_t^Z\right.\right)\right] > 0$ over some holding period h.[1]

We denote with $\mathcal{M}^Z$ the set of SDFs that price returns conditionally on the realizations of the predictors Z_t, that is

$$\mathcal{M}^Z = \left\{m_{t+h} \;\middle|\; E(m_{t+h}^2) < \infty,\ E\left(m_{t+h}R_{t+h}\left|\mathcal{F}_t^Z\right.\right) = e\right\} \tag{1}$$

where e denotes the unit vector. We assume $\mathcal{M}^Z$ non-empty, which from the standpoint of

[1]We assume that all the random variables are defined over a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, so that $\mathcal{F}_t^Z$ is formally the $\sigma-$algebra generated by the vector of predictors Z_t, and hence $\mathcal{F}_t^Z \subset \mathcal{F}$. We also assume that returns have finite unconditional first and second moments, μ and $E^2 \triangleq E\left[R_{t+h}R'_{t+h}\right]$, with unconditional variance-covariance matrix $\Sigma = E^2 - \mu\mu'$. Moreover, we assume that the matrix $E_t^2 \triangleq E\left[R_{t+h}R'_{t+h}\left|\mathcal{F}_t^Z\right.\right]$ of second moments conditional on the predictors is (almost surely) positive-definite and hence invertible, i.e. returns are linearly independent conditionally on information in the predictors. Therefore, denoting with $\mu_t \triangleq E\left[R_{t+h}\left|\mathcal{F}_t^Z\right.\right]$ the vector of conditional expected returns, both the conditional variance-covariance matrix $\Sigma_t = E_t^2 - \mu_t\mu'_t$ and its unconditional counterpart Σ are positive-definite (and hence invertible) as well.

interpretation means that the Law of One Price holds in the linear space of payoffs obtained by managed portfolios that condition on the predictors' realization. In other words, we assume that the returns incorporate efficiently the information contained in the predictors Z_t.

Given the SDFs in $\mathcal{M}^Z$, we call predictors-based variance bound, denoted with $\sigma_Z^2(v)$, the lower envelope of the set of all variances of SDFs in $\mathcal{M}^Z$, that is the map

$$\sigma_Z^2(v) = \inf \left\{ Var\left(m_{t+h}\right) \mid m_{t+h} \in \mathcal{M}^Z,\ E\left(m\right) = v,\ v \in \Re \right\} \tag{2}$$

where the variable v assumes the interpretation of shadow price of a unit risk-free zero-coupon bond with maturity $t+h$. The parabolic function $\sigma_Z^2(v)$ represents an unconditional frontier for SDFs in the sense of Gallant, Hansen, and Tauchen (1990): since it considers all SDFs that price returns conditionally on Z_t it takes full advantage of the predictive power of the vector Z_t while maintaining the simplicity of concentrating on the unconditional moments of such SDFs.

As observed by Bekaert and Liu (2004), when the conditional moments of returns are not correctly specified the predictors-based bound $\sigma_Z^2(v)$may fail to be a valid lower bound for the volatility of SDFs in $\mathcal{M}^Z$. To obviate to this problem, we extend to this conditional setting the duality between mean-variance frontiers for SDFs and maximum Sharpe ratios first illustrated for the unconditional case by Hansen and Jagannathan (1991) in their seminal work. More specifically, define the following set of returns from managed portfolios

$$\mathcal{R}^Z = \left\{ R_{t+h}^w \;\middle|\; R_{t+h}^w = w_t' R_{t+h},\ w_t\ \mathcal{F}_t^Z - measurable \text{ s.t. } E\left(w_t' e\right) = 1 \right\} \tag{3}$$

This set collects all the payoffs that are generated by trading strategies that exploit the information contained in the predictors at time t. As long as $\nu \neq 0$ one can show (see also Abhyankar, Basu, and Stremme (2007), Peñaranda and Sentana (2013)) that[2]

$$\sigma_Z^2(v) = \nu^2 \sup_{R_{t+h}^w \in \mathcal{R}^Z} \left(\frac{E(R_{t+h}^w) - \nu^{-1}}{Var\left(R_{t+h}^w\right)} \right)^2 \tag{4}$$

[2]When $\nu = 0$ the sup coincides with the reciprocal of the global minimum portfolio variance over the set $\mathcal{R}^Z$. Moreover, the sup is always attained with the only exception of the case in which ν is set equal to the expected return on the global minimum variance portfolio, case in which the sup is attained by a return whose expected price is zero.

In words, for any given level of the risk-free rate the predictors-based bound is proportional to the square of the maximum Sharpe ratio that can be generated by managed portfolios that exploit the information contained in the predictors Z_t. Observing that a mis-specification of the conditional expected returns and variances introduces a duality gap in (4), Bekaert and Liu (2004) suggest to always use the right-hand side to actually compute a variance bound that incorporates conditioning information since, by the very own definition of sup, this right hand side will always constitute a valid lower bound on the variance of the SDFs in $\mathcal{M}^Z$ (albeit, not necessarily the highest lower bound if mis-specification of the first two conditional moments of returns actually occurs). In the empirical part of this paper we follow the lead of Bekaert and Liu (2004), and estimate our predictors-based bounds using the solution to the left-hand side of (4) supplied by Bekaert and Liu (2004).

Before proceeding, we remark that from the horizon perspective our predictors-based bounds are conservative since they are based on those SDFs that price trading strategies which, although they take full advantage of the information in the predictors Z_t, still are required to be "buy and hold" over the horizon h. The predictors-based bounds that would be obtained from those SDFs that price the (truly) dynamic trading strategies that are allowed to be rebalanced at the intermediate dates $t+1$, $t+2, ...t+h$ would clearly impose a much harder yardstick. In fact, allowing for intertemporal rebalancing would expand the set $\mathcal{R}^Z$ of managed returns and, via the duality in (4), that would yield a much higher bound $\sigma_Z^2(v)$. In this paper, however, we concentrate on "buy and hold" strategies and leave the more general framework to future research.

2.2 Predictors-based bounds and asset pricing modelling

Let's consider now the asset pricing modelling side of our argument. Our main interest is to understand when and how we can employ information contained in the set of predictors Z_t, and synthesized in the predictors-based bound $\sigma_Z^2(v)$, to evaluate a given asset pricing model. To formalize our discussion, we identify any given asset pricing model with the triple $\left(X_t, \mathcal{F}_t^X, m_{t+h}^X\right)$ where X_t denotes the set of state variables of the model, $\mathcal{F}_t^X$ denotes the informational content in state variables X_t,[3] and m^X_{t+h} denotes the SDF of the given asset pricing model. Since from the standpoint of a given asset pricing model agents maximize their utility based on the information contained in the state variables X_t, the SDF m^X_{t+h} together with the returns R_{t+h} must satisfy the first order condition

$$E\left(m^X_{t+h} R_{t+h} \middle| \mathcal{F}^X_t \right) = e \tag{5}$$

More generally, the SDF m^X_{t+h} must price all the managed portfolio that condition on the state variables X_t of the given asset pricing model.

To help the intuition it is useful to exemplify this general framework with the three asset pricing models that we analyze in the empirical part. The first example is the Bansal et al. (2012a) model of long run risk, where the state variables are the first two conditional moments of log consumption growth g_t, that is $X_t = (x_t, \sigma^2_t)$, the information $\mathcal{F}^X_t$ is generated by the innovations in these first two conditional moments, and the SDF takes the form

$$\ln\left(m^X_{t+h}\right) = A + Bg_{t+h} + Cr_{a,t+h}$$

where $r_{a,t+h}$ denotes the (continuously compounded) return on an asset that delivers a dividend equal to aggregate consumption, and A, B, C are functions of the subjective discount factor, risk-aversion coefficient and intertemporal elasticity of substitution of the representative investor. A second example is the External Habit model of Campbell and Cochrane (1999), where the state variable is log surplus consumption ratio s_t, so that in this case $X_t = s_t$, the information $\mathcal{F}^X_t$ is generated by the innovations in surplus consumption ratio, and the SDFs takes the form

$$\ln\left(m^X_{t+h}\right) = A' + B'\left(g_{t+h} + s_{t+h}\right))$$

with A' and B' functions of the subjective discount factor and of the risk-aversion coefficient. The last example is the Rare Disaster model of Nakamura et al. (2013), where the state variables are I_t, the indicator of disaster occurrence at time t, and z_t, the amount by which consumption differs from potential due to current and past disasters. Hence, in this case $X_t = (I_t, z_t)$, the

[3]Formally, $\mathcal{F}^X_t$ is the $\sigma-$algebra generated by the vector of state variables X_t, and hence $\mathcal{F}^X_t \subset \mathcal{F}$ where $\mathcal{F}$ is the $\sigma-$algebra of the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ over which all the random variables are defined.

information F_t^X is generated by state variables, and the SDF takes the same functional form as in the long run risk model.

The question we want to address is: under what conditions does the predictors-based bound $\sigma_Z^2(v)$ constitutes a *legitimate* lower bound on the variance of the SDF of a given asset pricing model? To address this question, given an asset pricing model $\left(X_t, \mathcal{F}_t^X, m_{t+h}^X\right)$ we denote with $\mathcal{F}_t^{X,Z}$ the information set obtained by adjoining to the vector of state variables X_t the vector Z_t of predictors.[4] Moreover, we let $\nu^X \equiv E\left(m_{t+h}^X\right)$ denote the price assigned by the SDF m_{t+h}^X to a unit zero-coupon bond with maturity $t+h$. With this notation at hand we are now able to state the following sufficient condition for the predictors-based bound σ_Z^2 to constitute a legitimate volatility bound for m_{t+h}^X.

Proposition 1. *Suppose that the asset pricing model* $\left(X_t, \mathcal{F}_t^X, m_{t+h}^X\right)$ *satisfies*

$$E\left(m_{t+h}^X R_{t+h}\middle|\, \mathcal{F}_t^X\right) = E\left(m_{t+h}^X R_{t+h}\middle|\, \mathcal{F}_t^{X,Z}\right) \tag{*}$$

Then the predictors-based frontier for SDFs $\sigma_Z^2(v)$ *constitutes a legitimate lower bound for the volatility of* m_{t+h}^X*, in the sense that*

$$Var\left(m_{t+h}^X\right) \geqslant \sigma_Z^2\left(\nu^X\right)$$

is a necessary condition for (5) to hold.

Proof. The iterative property of conditional expectation implies that

$$E\left(m_{t+h}^X R_{t+h}\middle|\, \mathcal{F}_t^Z\right) = E\left[E\left(m_{t+h}^X R_{t+h}\middle|\, \mathcal{F}_t^{X,Z}\right)\middle|\, \mathcal{F}_t^Z\right]$$

This, together with the orthogonality condition $E\left[m_{t+h}^X R_{t+h}\middle|\, \mathcal{F}_t^X\right] = e$ and (*) implies that

$$E\left(m_{t+h}^X R_{t+h}\middle|\, \mathcal{F}_t^Z\right) = e$$

[4]Formally, $\mathcal{F}_t^{X,Z} = \sigma\left(\mathcal{F}_t^X \cup \mathcal{F}_t^Z\right)$, i.e. it is the smallest $\sigma-$algebra that contains all the information in X_t and Z_t, therefore $\mathcal{F}_t^{X,Z} \subset \mathcal{F}$.

that is $m^X_{t+h} \in \mathcal{M}^Z$, from which

$$Var\left(m^X_{t+h}\right) \geqslant \sigma^2_Z\left(\nu^X\right)$$

follows readily■

To better place our result in the vast literature on predictability and asset pricing observe that, from Kirby (1998) on, it is standard in that literature to assume that the predictors belong to a general information set $\mathcal{F}^I_t$ which investors condition on when pricing assets. More formally, in the literature it is customary to concentrate on those SDFs m_{t+h} which satisfy

$$E\left(m_{t+h}R_{t+h}\middle|\,\mathcal{F}^I_t\right) = e \tag{6}$$

for some information set $\mathcal{F}^I_t$ such that $\mathcal{F}^I_t \supset \mathcal{F}^Z_t$. This perspective is clearly useful to investigate if the ability of Z_t to predict a given set of return is correctly priced by *some* rational asset pricing model, since whenever $\mathcal{F}^I_t \supset \mathcal{F}^Z_t$ then any SDF that satisfies (6) must also price those dynamic strategies that condition on the predictors Z_t. If, however, one wants to exploit the informational content of a *given set of predictors* to investigate the potential of *a given asset pricing model* to price *a given set of returns*, then the information sets $\mathcal{F}^Z_t$ and $\mathcal{F}^X_t$ must be taken as given, there is no guarantee that $\mathcal{F}^Z_t \subset \mathcal{F}^X_t$, and this is where our condition (*) finds its bite.

Whenever condition (*) holds, therefore, $\sigma^2_Z(v)$ is a legitimate lower bound on the volatility of the SDF of the given asset pricing model. If $Var\left(m^X_{t+h}\right) < \sigma^2_Z\left(\nu^X\right)$ but condition (*) fails, however, we cannot reject out of hand the asset pricing model $\left(X_t, \mathcal{F}^X_t, m^X_{t+h}\right)$, since in that case the orthogonality condition $E\left(m^X_{t+h}R_{t+h}\middle|\,\mathcal{F}^X_t\right) = e$ is in principle compatible with a volatility level lower that the one dictated by conditioning on the predictors Z_t. An alternative, but logically equivalent, way to express the implication in Proposition 1 is contained in the next result.

Corollary 1. *If, given the predictors-based bound $\sigma^2_Z(\nu)$, an asset pricing model $\left(X_t, \mathcal{F}^X_t, m^X_{t+h}\right)$*

satisfies $E\left(m_{t+h}^{X} R_{t+h} \middle| \mathcal{F}_t^X\right) = e$ *and* $Var\left(m_{t+h}^{X}\right) < \sigma_Z^2\left(\nu^X\right)$, *then for some return* $R_{j,t+h}$

$$Var\left[E\left(m_{t+h}^{X} R_{j,t+h} \middle| \mathcal{F}_t^{X,Z}\right)\right] > 0$$

Proof. By Proposition 1, if $E\left(m_{t+h}^{X} R_{t+h} \middle| \mathcal{F}_t^X\right) = e$ and $Var\left(m_{t+h}^{X}\right) < \sigma_Z^2\left(\nu^X\right)$ then $E\left[m_{t+h}^{X} R_{j,t+h} \middle| \mathcal{F}_t^{X,Z}\right] \neq 1$ for some return $R_{j,t+h}$, hence

$$Var\left(E\left[m_{t+h}^{X} R_{j,t+h} \middle| \mathcal{F}_t^{X,Z}\right]\right) = Var\left(E\left[m_{t+h}^{X} R_{t+h} \middle| \mathcal{F}_t^{X,Z}\right] - 1\right) > 0$$

■

Corollary 1 supplies a dual interpretation of Proposition 1 in terms of predictability of discounted returns. Given an asset pricing model $\left(X_t, \mathcal{F}_t^X, m_{t+h}^X\right)$, the discounted returns $m_{t+h}^{X} R_{t+h}$ cannot be predicted by the state variables X_t alone if the model satisfies the Euler equation. If the SDF m_{t+h}^X satisfies the conditional Euler equation and yet it fails to achieve the variance threshold dictated by the predictors-based bound $\sigma_Z^2\left(\nu^X\right)$, however, then the discounted return of some asset becomes predictable upon augmenting the state variables X_t with the predictors Z_t.

2.3 Predictors-based bounds and predictive $\mathcal{R}^2 s$

We show now that the predictors-based bound $\sigma_Z^2(\nu)$ generates also an upper bound for the $\mathcal{R}^2 s$ from regressions of the returns R_{t+h} on the predictors Z_t. When taken together with Proposition 1, this implies that the variance of the SDF of any asset pricing model $\left(X_t, \mathcal{F}_t^X, m_{t+h}^X\right)$ that satisfies condition (*) bounds from above these predictive $\mathcal{R}^2 s$ as well. We establish these facts in the next Proposition, under the assumption that a risk-free return $R_{f,t+h}$ is available to the investors.

Proposition 2. *Given an asset pricing model* $\left(X_t, \mathcal{F}_t^X, m_{t+h}^X\right)$ *and the return* $R_{j,t+h}$ *on a traded asset, suppose that* $Var\left(R_{j,t+h} \middle| \mathcal{F}_t^Z\right)$ *is constant and condition (*) holds. Then*

$$\mathcal{R}_j^2 \equiv \frac{Var\left[E\left(R_{j,t+h} \middle| \mathcal{F}_t^Z\right)\right]}{Var\left(R_{j,t+h}\right)} \leqslant R_{f,t+h}^2 \sigma_Z^2\left(\nu_{\min}\right) \leqslant R_{f,t+h}^2 Var\left(m_{t+h}^X\right) \tag{7}$$

where $\sigma_Z^2(\nu)$ is the global minimum variance over all SDFs in $\mathcal{M}^Z$.

Proof. Since under condition (*) the second inequality follows readily from Proposition 1, we only need to establish the first inequality. To this end, denoting with $R^e_{j,t+h} = R_{j,t+h} - R_{f,t+h}$ the excess return on asset j, for any $m_{t+h} \in \mathcal{M}^Z$ we have $E\left(m_{t+h} R^e_{j,t+h} \middle| \mathcal{F}_t^Z \right) = 0$, that is

$$E\left(R^e_{j,t+h} \middle| \mathcal{F}_t^Z \right) = -R_{f,t+h} cov\left(m_{t+h}, R^e_{j,t+h} \middle| \mathcal{F}_t^Z \right)$$

Squaring both sides up, exploiting the conditional Cauchy-Schwarz inequality, taking expectations and exploiting the fact that the variance cannot exceed the second moment, we have

$$\begin{aligned}
Var\left[E\left(R_{j,t+h} \middle| \mathcal{F}_t^Z \right)\right] &= Var\left[E\left(R^e_{j,t+h} \middle| \mathcal{F}_t^Z \right)\right] \\
&\leq R^2_{f,t+h} E\left[Var\left(R_{j,t+h} \middle| \mathcal{F}_t^Z \right) Var\left(m_{t+h} \middle| \mathcal{F}_t^Z \right)\right] \\
&\leq R^2_{f,t+h} Var\left(R_{j,t+h}\right) E\left[Var\left(m_{t+h} \middle| \mathcal{F}_t^Z \right)\right] \\
&\leq R^2_{f,t+h} Var\left(R_{j,t+h}\right) Var\left(m_{t+h}\right)
\end{aligned}$$

where the second inequality follows from the assumption of constant conditional variance and the last inequality follows from decomposing the total variance of m_{t+h} into the sum of average conditional variance plus variance of the conditional expectation. Therefore

$$\mathcal{R}_j^2 \equiv \frac{Var\left(E\left[R_{j,t+h} \middle| \mathcal{F}_t^Z \right]\right)}{Var\left(R_{j,t+h}\right)} \leqslant R^2_{f,t+h} Var\left(m_{t+h}\right), \qquad \forall\ m_{t+h} \in \mathcal{M}^Z$$

from which the first inequality in (7) follows from the definition of $\sigma_Z^2(\nu)$ in (2)$\blacksquare$

It is useful to compare this result with the literature, in particular with Proposition 5 in Ross (2005) (see also Zhou (2010)). In line with our general approach of allowing the information in the predictors to be not necessarily included in the information in the state variables, a first contribution of our Proposition 2 is to show that, if condition (*) is violated, the $\mathcal{R}^2$ from a predictive regression is not constrained to be below the volatility of the SDF of a given pricing model, that is, $R^2_{f,t+h} Var\left(m^X_{t+h}\right) < \mathcal{R}_j^2$ is potentially compatible with the model Euler equation $E\left(m^X_{t+h} R_{t+h} \middle| \mathcal{F}_t^X \right) = e$. This fact can not emerge from Proposition 5 in Ross (2005), since there agents are assumed to price conditionally on a generic information set $\mathcal{F}_t^I$ which is implicitly assumed to satisfy $\mathcal{F}_t^I \supset \mathcal{F}_t^Z$, i.e. to contain all the information in the predictors. Our proof, moreover, highlights the importance of assuming returns to have constant conditional variance, which implies

$$E\left[Var\left(R_{j,t+h}|\,\mathcal{F}_t^Z\right)Var\left(m_{t+h}|\,\mathcal{F}_t^Z\right)\right] \leq Var\left(R_{j,t+h}\right)Var\left(m_{t+h}\right)$$

from which the bound on $\mathcal{R}_j^2$ obtained in Proposition 2 follows. Without constant conditional second moments, that is, in the case of stochastic volatility, it is not at all obvious that a bound as the one in (7) can be established at all.

3 Empirical Results

In this section we put the theoretical framework introduced in the previous section to work. We first introduce the linear predictive model for returns and use it throughout the empirical part to compute the predictors-based bound $\sigma_Z^2(\nu)$ for different sets of assets across horizons. We then test condition (*) for the long run risk model, the external habit model and the rare disaster model. Since this condition is never rejected, we go ahead and compare the volatilities of the model implied SDFs with our predictors-based bounds. We complement our analysis with a parameters' sensitivity analysis, we reinterpret our findings in terms of upper bounds on the R^2s from predictive regressions, and finally we analyze the robustness of our results to the possibility of misspecification of the model for the conditional moments of returns.

3.1 The predictive model

The empirical analysis is based on the (gross) return on the constant maturity bonds, and on the (gross) return on the value weighted portfolio of all stocks traded in the NYSE, the AMEX, and NASDAQ.[5]. Table 1 presents full-sample statistics of the 5-year constant maturity bond and stock returns for the common sample period (1952Q2 to 2012Q4). Over this sample period, the mean nominal return on stocks was 11.45% per annum, the mean nominal return on bonds was

[5]For a detailed description of data construction see the Appendix.

6.31% per annum, and the mean short-term interest rate - not shown in the table - was about 5.65% per annum. The standard deviation of stock nominal returns was 16.68% per annum, and the standard deviation of bond log returns was 5.77% per annum.

[Insert Table 1]

We consider two different set of assets:

1. SET A: risk-free bond, 5-year Treasury constant maturity and equity market returns
2. SET B: risk-free bond, 5-, 7-, 10- and 30-year Treasury constant maturity government bonds and equity market returns.

These two sets correspond to a universe of equity and bond portfolios whose return properties are the subject of much scrutiny in the empirical asset pricing research.

In contrast to the simple random walk view, stock and bond returns do seem predictable, and markedly more so the longer the return horizon. We use a typical specification that regresses rates of return on lagged predictors to review this claim. In particular we consider the following linear predictive system:

$$R^i_{t+h} = \beta^i_{0,h} + \beta^i_{1,h} z^i_t + u^i_{t+h} \tag{8}$$

where $i = S, B$ stands for stocks and bonds, respectively, and $Z_t = \left(z^S_t, z^B_t\right)$ denotes the vector of returns' predictors, potentially different for stock and bonds. We let the holding period range from one quarter to five years, i.e. $h = 1, 4, 20$ quarters. Table 2-Panel A presents regressions of the real stock returns R^s_{t+h} on the dividend-price ratio pd_t and the consumption-wealth ratio cay_t. The choice of these two stock market predictors is motivated by the present value logic, see Campbell and Shiller (1988b), and a linearization of the accumulation equation for aggregate wealth in a representative agent economy, see Campbell and Mankiw (1989) and Lettau and Ludvigson (2001). They are both "noisy" predictors of future asset returns. Although the $R^2 = 4\%$ at quarterly horizon does not look that impressive, the R^2 rises with horizon, reaching a value of about 50%, at the 5 years horizon.[6] Each variable has an important impact on forecasting long horizon returns: using the dividend yield as the sole forecasting variable, for

[6] Using excess returns yields similar results.

instance, would lead to an R^2 of "only" 22% at the 5-years horizons. Table 2-Panel B presents regressions of the 5-year maturity bond log return onto the lagged short-term nominal interest rate y_t, the lagged yield spread spr_t and the Cochrane and Piazzesi (2005) CP_t factor. The results show that our predictive system is able to capture fluctuations in bond excess returns at all horizons.[7] These results are consistent with much of the recent empirical research on the predictability of stock and bond returns (see Fama and French (1989), Campbell (1987) and Cochrane (2001; 2008) among others).

[Insert Table 2]

We conclude this section with three observations. First, our predictive model (8) is based on predictors that are not a direct consequence of any specific asset pricing models. In our notation, this means that $\mathcal{F}_t^Z$ is different from $\mathcal{F}_t^X$. We apply our methodological framework to assess whether the evidence of predictability at different horizon based on $\mathcal{F}_t^Z$ and reviewed above can in fact be used to construct legitimate bounds for an asset pricing model in which expectations are taken conditioning on the information $\mathcal{F}_t^X$ generated by the model's state variables. Second, we are interested in understanding the link between predictors, horizons and bounds, and not in finding the best predictive model. For this reason we consider a set of traditional and widely used predictors. Our list of potential return predictors is not exhaustive and, in this sense, our bounds provide a conservative estimation of the minimum variability required by any valid SDF. In fact one could expand the set of predictors to make our predictors-based bounds even tighter. For instance, the degree of predictability could be improved both at short- and long-horizons, by using the variance risk premium (see Bollerslev and Zhou, 2009) or the long-run past market variance (see Bandi and Perron, 2008), respectively. Future research might want to consider how to select the best (in terms of fit, measured by the R^2) subset of predictors to build even tighter bounds. Third, one might think that our conclusions are weakened by the Goyal and Welch (2003; 2008) results that return forecasts based on dividend yields and a number of other variables do not work out of sample. However for our findings to be valid we only require

[7] Our results are not driven by the inclusion of the CP factor. In a system where the CP factor is not included the Newey-West corrected t-statistic on the yield spread is above 2.2 at all horizons and the predictive system achieves a high R^2 value of 44.5% at the five year horizon.

forecastable returns. As shown by Cochrane (2008), the out-of-sample R^2 is important for the practical usefulness of return forecasts in forming aggressive real-time market-timing portfolios, but is not a *test* of forecastable returns. Within our setting, this means that one can find bad out-of-sample performance even when the model actually posits that conditional returns do vary with predictors, i.e. for models where $Var\left[E\left(R_{j,t+h}\left|\mathcal{F}_t^Z\right.\right)\right] > 0$ over some holding period h.[8]

3.2 Predictors-based bounds across horizons

It seems apparent from Table 2 that expected returns vary over time. To examine the ability of the predictors to improve the usefulness of the variance bounds as a diagnostic tool, in this section we compare the predictors-based bounds to the classical, unconditional HJ bounds. Along with the predictive versus unconditional dimension, we also analyze the effect of altering the investment horizon. Whereas Cochrane and Hansen (1992) were the first to carry out this exercise on the unconditional variance bounds, our analysis extends their results and highlights the interaction of conditioning information with the horizon dimension.

To compute the predictors-based bounds we use the solution to the left-hand side of Eq. (4). Bekaert and Liu (2004) show that the optimal trading strategy w_t that incorporates information as the following expression:

$$w_t = \left(\mu_t \mu_t^{\intercal} + \Sigma_t\right)^{-1} (1 - w\mu_t)$$

where μ_t and Σ_t are the vector of conditional expected returns and the conditional variance-covariance matrix, respectively, and $w = (\nu - b)/(1 - d)$, $\nu = E[M_{t+1}]$, $b = E\left[e' \left(\mu_t \mu_t^{\intercal} + \Sigma_t\right)^{-1} \mu_t\right]$ and $d = E\left[\mu_t' \left(\mu_t \mu_t^{\intercal} + \Sigma_t\right)^{-1} \mu_t\right]$. To compute the first and second conditional moments of asset returns, μ_t and Σ_t, we use the linear predictive model in equation (8).[9] For simplicity, we assume the conditional covariance matrix for returns to be constant, and estimate it has the residual in the forecasting regressions.

Figure 3 presents our results for SET A. We consider investment horizons of 1 quarter, 1 year, and 5 years. The shortest investment horizon coincides with the sampling interval of return-

[8] For instance Cochrane (2008) sets up a null in which return forecasts account for all dividend-yield volatility, and finds out-of-sample performance as bad or worse than that in the data 40% of the time.

[9] We investigate the effect of potential model misspecification in our regressions on the construction of the bounds in Section 3.3.5.

s. In each panel we report the efficient bounds generated with conditioning information (solid lines) along with the unconditional HJ bounds (dashed lines) that make no use of conditioning information. Similar to Cochrane and Hansen (1992), Figure 3 shows that the bottom of the mean standard deviation frontier shifts up and to the left as we increase the investment horizon. At the same time, although the lower bound for volatility increases, it does so slowly. Importantly, the picture shows that the strong predictability at long horizons documented in Table 2 translates into a tight lower bound on the variance of the SDF. In particular Figure 3 imparts two conclusions. First, the estimates of our predictor-based bounds are sharper relative to the unconditional ones. In Figure 3, for instance, the minimum point of the frontier at the 5-year horizon obtained using conditioning information is about 1.6 times sharper than the unconditional lower bound, thereby substantiating the incremental value of conditioning information in asset pricing applications. The difference between the bounds with and without conditioning information at the 5-year horizon reflects the considerable long run predictability documented in Table 2. Second, the implications of using conditioning information strongly depend on the holding period: while conditioning information adds little at short horizons, the volatility implications of the long horizon returns are more dramatic and reveal the fundamental role played by conditioning information over time. Consistently with the results on the predictive regressions, therefore, the role of the information contained in the predictors becomes more apparent as we lengthen the investment horizon.

Figure 5 presents the same analysis for set B. Comparing Figures 3 with 5, we observe that expanding the number of assets, i.e. moving from SET A to SET B, leads to a bound that is intrinsically tighter.

Taken together figure 3 and 5 highlight the two effects that are at work simultaneously: the conditioning information embedded in the conditional moments of returns and the horizon at which this information becomes relevant. The tightening of the volatility bounds is the combination of these two forces at work simultaneously.

3.3 Predictors-based bounds and asset pricing models

We compare now our predictors-based bounds to the long run risk models of Bansal et al. (2012a), the external habit-formation model of Campbell and Cochrane (1999) and the rare disaster model of Nakamura et al. (2013).[10] There are two main reasons for concentrating our attention on these three models. First, they embed different preferences and specify the long run and short run risk in distinct ways (see also Hansen (2009)). Second, to assess the ability of a model to produce realistic SDF dynamics we need to compute its first and second unconditional moments. While the conditional variances are amenable to closed-form characterization, the unconditional variances are tractable only via simulations.[11] Therefore we rely on a statistic generated from a simulation procedure and we look for models for which the solution methods are well established. To simulate the series of m_{t+h}^{X} we use either calibrated or estimated values for the parameters. Tables 7, 8 and 9 report the complete specification of the parameter values for preferences and exogenous dynamics, along with their standard errors in case the parameters are estimated. The calibrated parameters for the long run risk and habit models are taken from Bansal et al. (2012a) and Campbell and Cochrane (1999), respectively. The estimated parameter values are taken from Bansal, Kiku, and Yaron (2012b), Aldrich and Gallant (2011) and Nakamura et al. (2013).[12, 13, 14]

[Insert Tables 7, 8 and 9 about here]

[10] In a previous version of this paper we also consider the rare event of Backus, Chernov, and Martin (2011a) and an affine 3-factors model suggested by Koijen, Lustig, and Nieuwerburgh (2012) to explain the cross section of bond and stock returns.

[11] There are some exception, see for instance the rare disaster model given i.i.d. uncertainties.

[12] Aldrich and Gallant (2011) present for each parameter the posterior mean and posterior standard deviation. We refer to the posterior mean of each parameter as our point estimate for that parameter.

[13] We consider a specification of the habit model where the preference parameter b that determines the behavior of the risk-free rate is set to zero. Wachter (2006) allows b to differ from zero to match the upward-sloping yield curve for nominal Treasury bonds. We leave for future research the sensitivity analysis of the parameter b on the volatility of the habit-implied SDF.

[14] Nakamura et al. (2013) estimates their model using data for 24 countries over more than 100 years. To be consistent with other studies in our paper, we only adopt the estimations of US for county specific parameters. For each parameter, the posterior mean and posterior standard deviation are presented. We refer to the posterior mean of each parameter as our point estimate for that parameter.

3.3.1 Testing for condition (*)

We have seen that a violation of condition (*) would prevent us from using the predictors-based bounds to test the validity of the SDF of a given model. Therefore, to make sure that it is sensible to apply our predictors-based bound to the LRR, External Habit and Rare Disaster models, we first check condition (*) for each asset class and horizon.

To test condition (*) we run the following two regressions:

$$\begin{aligned} m_{t,t+h}^{X} R_{t,t+h}^{i} &= \alpha_{1,h} + \beta_{1,h} X_t + \varepsilon_{1,t+h} \\ m_{t,t+h}^{X} R_{t,t+h}^{i} &= \alpha_{2,h} + \beta_{2,h} X_t + \gamma_h Z_t + \varepsilon_{2,t+h} \end{aligned}$$

with $i = S, B$ the asset class, and we compute the differences of the fitted values

$$\widehat{m_{t+h}^{X} R_{t+h}^{i}}^{X,Z} - \widehat{m_{t+h}^{X} R_{t+h}^{i}}^{X}$$

Condition (*) is satisfied as long as zero is included within the 90% confidence interval of the difference between these two distributions.

With the test assets, the set of candidate predictors and the model-implied stochastic discount factors at hand, we are now ready to test condition (*). Figure 1 displays the results for the LRR (Panel A), the habit (Panel B) and rare disaster models (Panel C) when the test asset is the Equity Market. Empirically, there is no horizon at which we reject condition (*), and this is true for all models. We obtain analogous results when we consider the yield on the Treasury Bill and the constant maturity coupon bond returns as test assets.[15] In summary, we can conclude that the predictors that we employ in our linear prediction model are indeed useful in sharpening the unconditional variance bounds, since the predictors-based bounds that we derive are indeed legitimate lower bounds for the volatility of the SDF of both models.

[Insert Figure 1 about here]

To conclude this section we remark that, although an analysis of the power properties of our test procedure is not an objective of this paper, still we are able to show that the condition is in

[15] Results are available upon requests from the authors.

fact rejected in cases in which a rejection would be the expected outcome. To see this, consider the simplest possible consumption-based asset pricing model, i.e. the model with a representative consumer with CCRA utility and whose endowment/consumption process exhibits i.i.d. growth. Figure 2 shows that in this case our procedure does reject condition (*) soundly, as one would definitely expect.

[**Insert Figure 2 about here**]

3.3.2 Model-implied SDFs and predictors-based bounds

We compare now the SDFs implied by three asset pricing models under scrutiny at the light of the information contained in the returns' predictors. To do so, since condition (*) is satisfied we can use our predictors-based bounds as a legitimate differentiating diagnostic, that is, it makes sense to check whether the variance of the h-period SDFs implied by each of the three models satisfies the predictors-based variance bounds at the different horizons.

Figures 3, 4, 5, and 6 compare the volatility of the SDFs generated by the three competing models with the predictors-based bounds at different horizons and for different sets of testing assets. The (blue) stars and (red) triangles represent population values using calibrated and estimated parameters, respectively. In particular for Bansal et al. (2012a) and Campbell and Cochrane (1999)'s model, we use the dynamics of consumption growth and of the state variables to simulate 600,000 monthly observations (50,000 years) of the model-implied SDF, while for Nakamura et al. (2013)'s model, we simulate 50,000 annual observations.[16] From this long time series we then calculate the SDF unconditional moments.[17] Since the volatility bound itself is estimated from the data, it is random. Moreover, the computation of the mean and standard deviation of the SDF using a specific utility function relies on estimates of the moments of the consumption process, and so it is random as well. To account for the first source of uncertainty, we construct confidence intervals for the bounds with conditioning information based on 50,000 random samples of size 234 from the data and a block bootstrap. The bounds along with their

[16] Our parameterizations of estimated values of rare disaster model are taken from Nakamura et al. (2013), in which all parameters are estimated at the annual frequency.

[17] Using a single simulation run to infer the population values for the entities of interest is consistent with, among others, the approach of Campbell and Cochrane (1999) and Beeler and Campbell (2009).

lowest 90% confidence interval are reported in Figure 4 and 6.[18] To account for the second source of uncertainty, we follow an approach similar to Cecchetti, Lam, and Mark (1994) and Burnside (1994). We show the uncertainty (one standard deviation) in calculation of the mean and standard deviation of the model implied SDFs by using estimated parameters. The average mean and standard deviation from 10 simulations with estimated parameters (red triangles) and one standard deviation confidence area (magenta ellipses) are reported in Figures 3, 4, 5, and 6. So far, we only take account uncertainty in the parameter values for the dynamics.

Figures 3 to 6 provide a visual representation of the importance of jointly considering conditioning information and horizons. Starting with Figure 3, at the yearly horizon the SDFs of the LRR and habit models computed with calibrated parameters and the rare disaster model computed with estimated parameters all satisfy the unconditional HJ bounds. In fact, this is not surprising for the first two models, since they are calibrated closely to offer conformity with the historical unconditional annual real return on risk-free bond and the equity market. However, the conclusions are different when we incorporate conditioning information. In this case the LRR model falls largely below the bounds and hence cannot be considered a valid SDF. Going on to Figure 4, we see that even accounting for the uncertainty that arises in the comparison of the mean and standard deviation of the SDF implied by a particular model of preferences with the bound that is computed from asset returns data, the conclusion does not change: it is problematic for the LRR model to satisfy the 1-year predictors-based bound. The habit model, on the other side, fares much better and falls within the lowest 90% confidence interval. In the meantime, the performance of model implied SDF with estimated parameters for both of the models is slightly smaller than the one using calibrated parameters. However, SDF of the rare disaster model falls fully into the predictors-based bound even after accounting for estimation uncertainty.

Recent theoretical and empirical research in macro-finance has highlighted the importance of capturing low frequencies components for an asset pricing model to be successful. One would then expect this low frequency component to make asset pricing puzzles less pronounced at longer horizons. With the visual aid of Figure 3 we can see that this statement holds true when

[18] We select the optimal block length for the bootstrap according to Politis and White (2001). We thank Andrew Patton for making the code available on his website.

no conditioning information is incorporated: the 5-year unconditional HJ bounds are satisfied by all models. However, this conclusion changes significantly when we look at the 5-year predictors-based bound: in this case the habit and rare disaster models satisfy the predictors-based bound with a good margin, while the LRR models clearly struggles. As expected, using SET B and hence expanding the set of assets, exacerbates this results: Figure 6 shows that, after accounting for uncertainty, while the habit and rare disaster model still satisfy the bound at long horizons, the long run risk models generates instead an SDF not volatile enough at long horizons. Hence as the investment horizon increases, it is quite possible that the equity premium puzzle does not vanish, but becomes instead more pronounced. This shows why an asset pricing model may reproduce some (unconditional) asset market phenomena at short horizon, while finding it onerous to satisfy bounds at longer horizons.

It is important to stress that if we considered the long-horizon bounds with no conditioning information, we would have concluded that the long run equity premium puzzle can be resolved as long as sufficient time-nonseparability is incorporated in preferences. However our predictors-based bounds highlight how time-nonseparable preferences are not the full story: we show that the return predictability, particularly at low frequency, requires high volatility of model generated SDFs, that can not be matched by the LRR and habit models. These empirical results are relevant as far as condition (*) is satisfied.

The results discussed so far are summarized in Table 3. The last two columns report the minimum value of the predictors-based bound for SET A and SET B, respectively. The remaining columns report the unconditional variance of the model-implied SDF, using estimated or calibrated parameters. We compare the minimum point estimate of the predictors-based cup with the population value of the volatility of the SDF. Although the unconditional variance of the SDF should be compared with the point on the parabola corresponding to the unconditional mean of the model-implied SDF, we can safely conclude that an SDF is not a valid one if its variance is below the minimum point. The Table reveals that at the yearly horizon the standard deviation implied by the Bansal et al. (2012a), the Campbell and Cochrane (1999) and the Nakamura et al. (2013) models using estimated parameters are 0.47, 0.74 and 1.29 respectively. The LRR value is lower than 0.73, the minimum volatility suggested by the predictors-based bound constructed from SET A. For the habit-model the value is close to the minimum point, although as we have seen in Figure 4 the bounds are still not satisfied. Finally, the rare disaster value is significantly larger then the minimum volatility of predictor-based bound, also suggested by Figure 4, it locates into the bounds. Hence when we consider SET A we observe that, although both LRR and habit models satisfy the unconditional bounds, none of them is able to satisfy the 1-year predictors-based bounds. When we look at long horizons, the SDF from the long run risk model has a volatility of 1.53 using estimated parameters. Although this value is pronounced, it is still lower than the minimum volatility obtained from the SET B, namely 2.44. On the other hand the habit and rare disaster models, with the sound values 4.29 and 6.18 using estimated parameters, satisfy the bound comfortably.

In sum, this section shows that our predictors-based bounds incorporating conditioning information from a well-established set of stock and bond predictors are a useful tool to assess the performance of candidate asset pricing models at multiple horizons. It is noteworthy that each asset pricing model parametrization approximates quite reasonably the (annual) unconditional equity premium and the real risk-free return, while simultaneously calibrating closely to the first two moments of consumption growth. However our evidence reveals that the variance of the SDF from LRR and habit models fail to meet the restrictions imposed by the predictor-based HJ bounds at the $1-$ year horizon. Moreover at a long 5-years investment horizon, and using SET B, the standard deviation of the SDF implied by the LRR never approaches the bounds; it is instead the habit-model that turn out to be able to generate enough volatility at long horizons to satisfy the long run equity premium puzzle; meanwhile, the rare disaster model is always capable to satisfy the predictors-based bounds at each horizon when we consider both SET A and SET B.

As a final remark, note that the conclusions in this section are indeed conservative. As discussed at the end of Section 2.1, our predictors-based bounds could be tightened even more by enlarging the set of assets, the set of predictors, or in general by considering truly dynamic investment strategies with rebalancing. We leave the analysis of these extensions to future research.

3.3.3 Parameters sensitivity analysis

Table 3 reports the population variance of the simulated SDFs for the long run risk and external habit formation models,[19] obtained using both calibrated and estimated parameters. This exercise shows that the population values for the volatility of the SDF implied by the long run risk model and external habit model would be slightly greater when using calibrated values. For the LRR model this is mainly driven by the lower persistence in the consumption growth volatility: compared to the benchmark calibration, where the half-life is essentially infinite (58-year), the estimated value implies a half-life slightly over 33-year. For the habit model the results are due to the lower value for the γ parameter which enters the coefficient of relative risk aversion. At the same time if we look once again at Figures 4 and 6 we see that our conclusions are largely unaffected by using either calibrated or estimated parameters: the evidence reveals that the variance of the SDF from each model fails to meet the lower bound restriction at the 1-year horizon for both SET A and SET B, and only the Campbell and Cochrane (1999) model is able to resolve the long horizon equity premium.

[Insert Table 3 about here]

3.3.4 Bounds, models and R^2

We now evaluate the asset pricing models scrutinized so far through the lenses of the $R^2 s$ of the predictive model (8) that underlies our predictive-based bounds. In Table 2 we compare the variance, scaled by the squared gross risk-free rate, of the SDFs of the LRR, habit and rare disaster models with the R^2s of predictive regressions for stock and 5-year constant maturity government bond real gross returns. The table, moreover, reports the minimum variance of both the unconditional HJ bounds and of our predictors-based bounds, both scaled by the squared gross risk-free rate as well.

[Insert Table 4 about here]

[19] In this sebsection, we do not consider the rare disaster model, since there is no calibrated values of model parameters.

The implication of Proposition 2 in Section 2.3 is that, as long as Condition (*) holds, for any asset class the minimum scaled variance of the predictors-based bounds $R^2_{f,t+h}\sigma^2_Z(\nu_{\min})$ should be intermediate between the predictive R^2 and the scaled variance of the models SDFs. After the analysis carried on in the previous section, it is not surprising that this implication is challenged by the data at different horizons for the LRR and the habit models, while it is not challenged at all for the rare disaster model. What is more interesting here is to observe that at shorter horizons the scaled variance of model implied SDFs by the LRR and the habit models are very close to, and in certain cases outright below, the predictive R^2s. This happens, in particular, for the case of the LRR model and the R^2 of the 5-year constant maturity government bond real gross returns, where the variance of the scaled SDF is well below the predictive R^2 at both the 1-quarter and the 1-year horizons (although, to be fair, these R^2 still belong to the confidence interval around the scaled variance of the SDF). When compared to the scaled minimum variance of the predictors-based bounds, however, the predictive R^2s line up nicely below the minimum values at all horizons.

The findings in this section are interesting from two points of view. First, the fact that the scaled variance of the LRR model falls below the R^2 of the 5-year constant maturity government bond real gross returns at the 1-quarter and 1-year horizon implies a short horizon challenge to the model that complements the long horizon challenge discussed above. Second, this inversion between the scaled variance of a model SDF and a predictive R^2, while on the other side the R^2 falls nicely below the scaled minimum variance of the predictors-based bound, reinforces the point made in Proposition 2 that the difference between the information sets associated with the predictors and that posited by a given model must be duly accounted for when employing predictive $\mathcal{R}^2$ as model diagnostics.

3.3.5 Predictability, model mis-specification and variance bounds

We conclude this section by investigating the performance of our bounds along two further dimensions: robustness and efficiency. Recall that the results presented so far are obtained under the assumptions of a time-invariant variance-covariance matrix for returns and a linear model for their conditional mean. To investigate possible mis-specification of the conditional moments and the efficiency of our bound we plot in Figure 9 alternative implementations of the variance bounds: specifically, the optimal bounds of Ferson and Siegel (2003, 2009) (FS), Bekaert and Liu (2004) (BL) and Gallant et al. (1990)(GHT).

[**Insert Figure 9 about here**]

Bekaert and Liu (2004) show that their bound, obtained by maximizing the Sharpe ratio over all returns obtained from portfolios that condition on Z_t and that cost 1 on average (see (4) in Section 2.1), must be a parabola under the null of correct moments specification. Figure 9 shows that in our case we obtain a smooth parabola indeed. The figure, moreover, shows that the GHT bound, obtained via the inf in (4), and the BL bound are virtually on top of each another, i.e. there is no duality gap. This suggests that the BL bound closely approximate the efficient use of conditioning information. Overall the three alternative implementations of the variance bounds that incorporate information from the predictors Z_t generate similar bounds with no visible misspecification. The FS is the lowest bound, see also Table 5: this is readily understood by observing that the FS bound collects all those payoffs that are generated by trading strategies that reflect the information available at time t, and that have unit price almost surely equal to one, and not just on average as for the BL case.[20] Although the FS approach yields the most conservative bounds, the variance bounds do not shift enough to change our conclusions. The evidence suggests that misspecification of the conditional moments does not seem to play a major role to change the results.

[20]More formally, the FS bound (see Ferson and Siegel (2003)) is defined as

$$\sigma^2_{FS}(v) = \nu^2 \sup_{R^w_{t+h} \in \mathcal{R}^{FS}} \left(\frac{E(R^w_{t+h}) - \nu^{-1}}{Var\left(R^w_{t+h}\right)} \right)^2$$

where

$$\mathcal{R}^{FS} = \left\{ R^w_{t+h} \in \mathcal{R}^Z \;\middle|\; w'_t e = 1 \; almost\ surely \right\}$$

i.e. the FS variance bound follows from maximizing the Sharpe ratio over the set of returns from portfolios that, while conditioning on Z_t, are required to have unit price almost surely, and not just on average. Therefore, it is evident that $\sigma^2_{FS}(v) \leq \sigma^2_Z(v)$.

4 Extensions

In Section 3.1 we have shown that incorporating predictability of asset returns does make the variance bounds tighter. In this Section we answer the following two questions: first, the predictability of which asset class, stocks or bonds, contributes the most to the sharper variance bounds exhibited in the previous section? Second, does the failure of an asset pricing models relate to the comparison of model-implied returns predictability versus historical data predictability? To conclude, we provide some possible interpretations for our empirical results.

4.1 Stock-based versus bonds-based variance bounds

To check the relative importance of different asset classes for predictability, and hence for sharpening the bounds, we consider the following experiment. We build the variance bounds according to different scenarios. Each scenario imposes some restrictions on the predictive system in (8). In the first case (restriction I) stock returns are unpredictable. In the second case (restriction II), treasury government bonds returns are unpredictable.[21] In both cases, we maintain the assumption that the risk-free rate is predictable.

Figure 7 displays the results for both SET A (see Panel A) and SET B (see Panel B): the variance bounds without information (dashed black line), with conditioning information (solid red line), with conditioning information together with restriction I (solid blue line) and with conditioning information together with restriction II (solid green line).

[Insert Figure 7 about here]

From Figure 7 we can draw two main conclusions. First, it is the predictability in stock returns that really tightens the variance bounds, particularly so at long horizon. In particular, under Restriction I the minimum point of the frontier for the volatility of SDF at the 5-year horizon based on the return properties of SET A (SET B) is about 1.54 (1.16) times less than the one with conditioning information and otherwise unrestricted. Second, the additional tightening due to the predictability of treasury government bond is instead marginal. Figure 7(c) shows that

[21] For SET A, this implies that the 5-year maturity treasury government bond return is assumed to be unpredictable. For SET B, we shut down the predictability of 5-, 7-, 10-, 20- and 30-year maturity treasury government bonds returns simultaneously.

even when we shut down the predictability of all the treasury government bonds simultaneously, the minimum value of the variance bound at the 5-year horizon is still 0.93 times that of the benchmark case (i.e. the variance bound generated with conditioning information, red solid line).

Summing up, the shape of variance bounds on SDFs essentially depends on the model we choose for predicting stock returns, whereas the predictability of bond returns plays a somehow more marginal role.

4.2 Historical versus model-implied predictability

In this section we investigate whether the asset pricing models analyzed in this paper can capture the return predictability in the data. The following empirical evidence concentrates exclusively on the stock market and uses the log of price-dividend ratio as a sole predictor.

Table 6 displays results for the predictive regressions of future gross real returns over the 1, 3, 5 and 8 years horizons, with both the actual data and the model simulated data. Model-implied predictive regression results are listed in Column 2 (Bansal et al. (2012a)), Column 3 (Campbell and Cochrane (1999)) and Column 4 (Nakamura et al. (2013)); in these cases the regressions are run on simulated data, and the results are the median of 1000 simulations with associated 95% confidence interval. In the data, the predictability of gross returns increases as the horizon goes up, the R^2 rising, from 7.9% at the 1-year horizon to about 32.8% at the 8-year horizon. When we look at the model-implied results, we observe that the long run risk model by Bansal et al. (2012a) and the rare disaster model by Nakamura et al. (2013) feature modest predictability, with the median R^2 in the range of of 0.0%-7.5% and 0.0% - 1.5%, respectively. The estimated slope coefficients of the long run risk model are close to those obtained by using real data across all the horizons, while for the rare disaster model they too low at the short horizon, -0.03 at the one-year, and too high at the longer horizons, -1.11 at the eight-year. On the other hand, for Campbell and Cochrane (1999)'s model, the median R^2 starts low, 9%, then rise to 37.5% at the 8-year horizon which is even higher than the one generated from real data, 32.8%. As shown in Table 6, however, the very high R^2s are paired with a very high magnitude of slope coefficients. For instance, the model implied slope coefficients are -0.29

at one-year horizon and -1.31 at eight-year horizon while the corresponding point estimates for the historical data are -0.13 and -0.63, respectively. The predictive power for the price-dividend ratio is amplified by a factor of almost two. Table 6 also presents the 95% confidence interval of model-implied R^2s. The evidence suggests that these three models can match the return predictability observed in the data, as the data's regression R^2s are well within the 95% model-based confidence intervals.

[**Insert Tables 6 about here**]

Figure 8 provides a visual representation of these results.

[**Insert Figure 8 about here**]

Our results show that the ability of a model to replicate the return predictability observed in real data is not a crucial criteria for model selection. However, the return predictability can be used to construct the variance bounds of SDFs, provided condition (*) is satisfied. In conclusion, our predictors-based bounds emerge as a criteria more effective than model-based predictability to discriminate among asset pricing models.

5 Conclusions

We analyze predictors-based variance bounds, i.e. bounds on the variance of those SDFs that price a given set of returns conditional on the information contained in a vector of returns predictors. We identify a simple sufficient condition under which the predictors-based bounds constitute legitimate lower bounds on the variance of the SDF of a given asset pricing model. We use our predictors-based bounds to assess the performance of three leading consumption-based asset pricing models: the long run risk model of Bansal et al. (2012a), the habit-formation model of Campbell and Cochrane (1999) and the rare disaster model of Nakamura et al. (2013).

Our results point to the importance of jointly considering conditioning information and horizons. The asset pricing models we consider reasonably mimic the annual *unconditional* equity premium and the real risk-free return. However our evidence reveals that the variance of model-implied SDFs by Bansal et al. (2012a) and Campbell and Cochrane (1999) fails to meet the lower bound restriction at 1-year horizon once conditioning information is allowed for. Consistent with the idea that asset pricing puzzles are less pronounced at longer horizons, the 5-year *unconditional* HJ bounds are always satisfied. However, this conclusion is not robust when conditioning information is considered: the habit model solves the equity premium at long horizons whereas the long run risk model produces an SDF that is not volatile enough. As a consequence, it is possible that the equity premium puzzle does not vanish, but rather gets more pronounced as the investment horizon increases. On the other hand, the rare disaster model of Nakamura et al. (2013) satisfies our predictors-based bound at each horizon with all predictors considered.

Our predictors-based bounds represent a convenient tool for researchers. First, they are informative of the dynamics that an admissible SDF should have at short-, medium- and long-term horizons. Second, our bounds yield a graphical and intuitive comparison of the performance of asset pricing models by isolating the common feature behind them at a specific horizon. In fact, the dynamic asset pricing models under consideration are constructed from a mixture of assumptions about preferences (such as recursive utility or habit persistence, etc) and exposure to fundamental shocks. The predictors-based bounds allow one to highlight the transitory and long run implications of these economic models. Consistent with the idea that all models are approximations of reality and as such likely to be mis-specified along some dimensions, our predictors-based bounds use the investment horizon and conditioning information as the fundamental ingredients that allow researchers to set models apart models (as, for us, at long horizons), or to identify the common behavior among apparently different models (as, in our case, at the 1-year horizon). In conclusion, our predictors-based bounds emerge as a criteria more effective than model-based predictability to discriminate among asset pricing models.

Data

We consider a set of quarterly equity and bond returns over the period 1952Q2 to 2012Q4. Our choice of the start date is dictated by the availability of data for our predictors. Real returns are computed by deflating nominal returns by the Consumer Price Index inflation. We obtain the time series of bond and stock returns using monthly daily returns on stocks and bonds. Quarterly returns are constructed by compounding their monthly counterparts. The h-horizon continuously compounded excess return is calculated as $r_{t,t+h} = r^e_{t+1} + \ldots + r^e_{t+h}$ where $r^e_{t+j} = \ln(R_{t+j}) - \ln(R_{f,t+j})$ is the 1-year excess log stock return between dates $t+j-1$ and $t+j$; R_{t+j} is the simple gross return; and $R_{f,t+j}$ is the gross risk-free rate (3-month Treasury bill) at the beginning of period $t+j$.

1. Stock returns: Return data on the value-market index are obtained from the Center for Research in Security Prices (CRSP) at the University of Chicago. We use the NYSE/Amex value-weighted index with dividends as our market proxy, R_{t+1}.
2. Bond returns: Returns on bonds are extracted from the US Treasuries and Inflation Indices File and the Stock Indices File of the Center of Research in Security Prices (CRSP) at the University of Chicago. The CRSP US Treasuries and Inflation Indices File provides returns on constant maturity coupon bonds, with maturities ranging from 1 year to 30 years, starting on January, 1942. The nominal short-term rate ($R_{f,t+1}$) is the annualized yield on the 3-month Treasury bill taken from the CRSP treasury files.
3. Predictors: price-dividend ratio, cay. The single CP factor is constructed as in Cochrane and Piazzesi (2005) from contemporaneous forward rates.
4. Inflation: we use the seasonally unadjusted CPI from the Bureau of Labor Statistics. Quarterly inflation is the log growth rate in the CPI.

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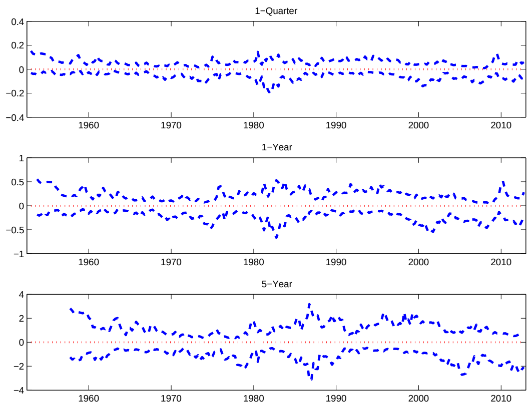
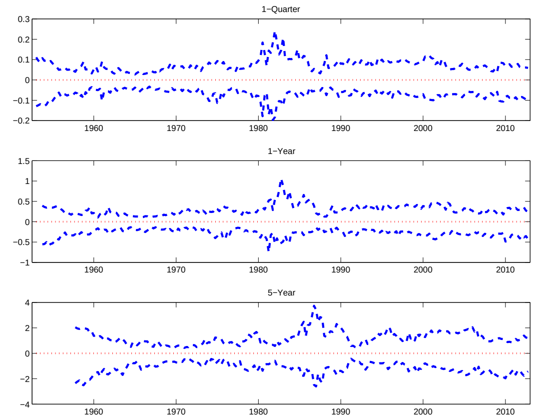
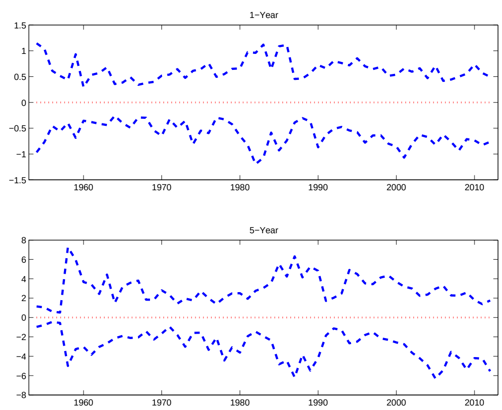


(a) Long Run Risk Model



(b) External Habit Model



(c) Rare Disaster

Figure 1 Empirical verification of Condition(*) - Market returns. Dashed blue lines give the 90% confidence interval of the differences between the estimated values of discounted returns with and without using predictors, at horizon 1-Quarter, 1-Year , 5-Year. The discounted returns are the product of model generated SDFs and real market index returns. Dotted red lines locate the benchmark of zero value. We do 1000 times simulations over 241 quarters for both long run risk model by Bansal et al. (2012a) (Panel A), external habit model by Campbell and Cochrane (1999) (Panel B) and rare disaster model by Nakamura et al. (2013) (Panel C). We use *estimated* parameters as given in Tables 7) and 8). We regress the discounted returns on model generated state variables and model generated state variables plus predictors, respectively. Finally we plot the 90% confidence interval of the difference between the fitted discount returns from the two regressions. Sample: 1952Q2 - 2012Q3.

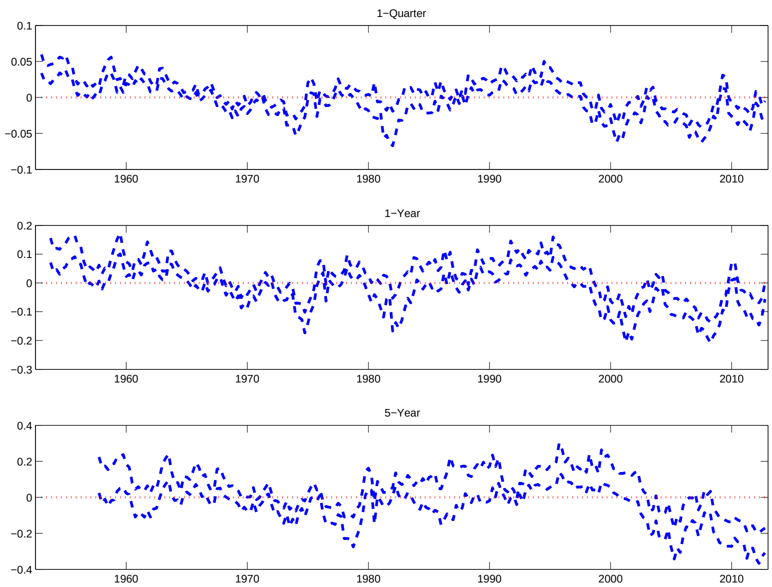
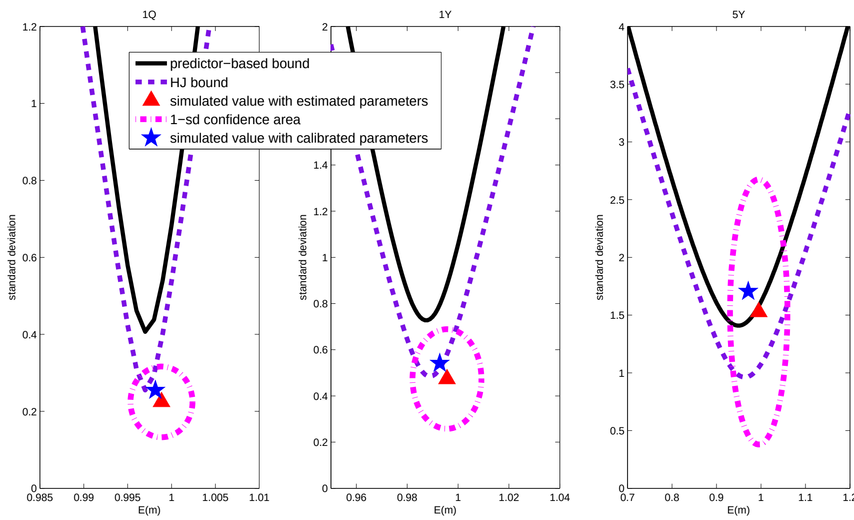
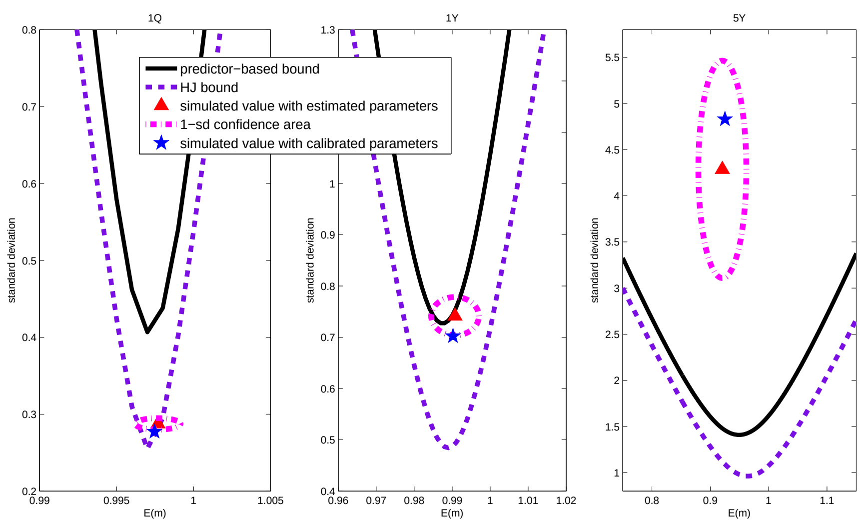


Figure 2 An analytical example in which Condition(*) fails - Market returns. Dashed blue lines give the 90% confidence interval of the differences between the estimated values of discounted returns with and without using predictors, at horizon 1-Quarter, 1-Year , 5-Year. The discounted returns are the product of model generated SDFs and real market index returns. Dotted red lines locate the benchmark of zero value. We do 1000 times simulations over 241 quarters of a consumption-based model with i.i.d consumption growth and CRRA utility. In this example, the risk aversion coefficient equals 7.42 and the subjective discounted factor equals 0.9989. Then we regress the discounted returns on model generated state variables and model generated state variables plus predictors, respectively. Finally we plot the 90% confidence interval of the difference between the fitted discount returns from the two regressions. Sample: 1952Q2 - 2012Q3.

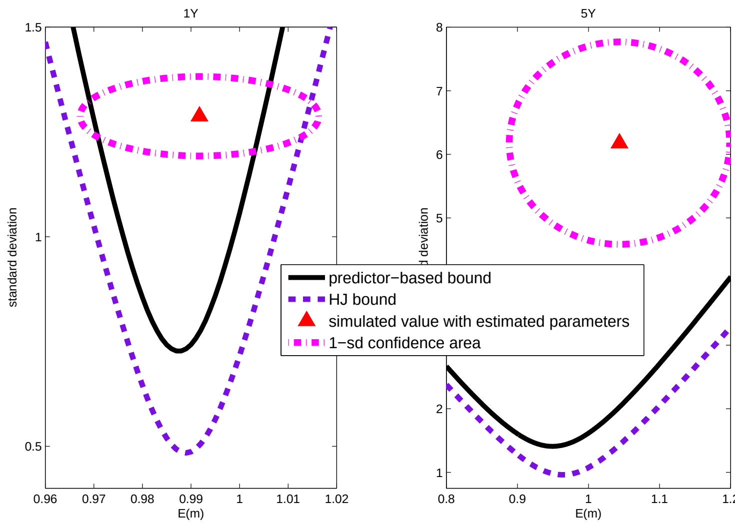


(a) Long Run Risk Model



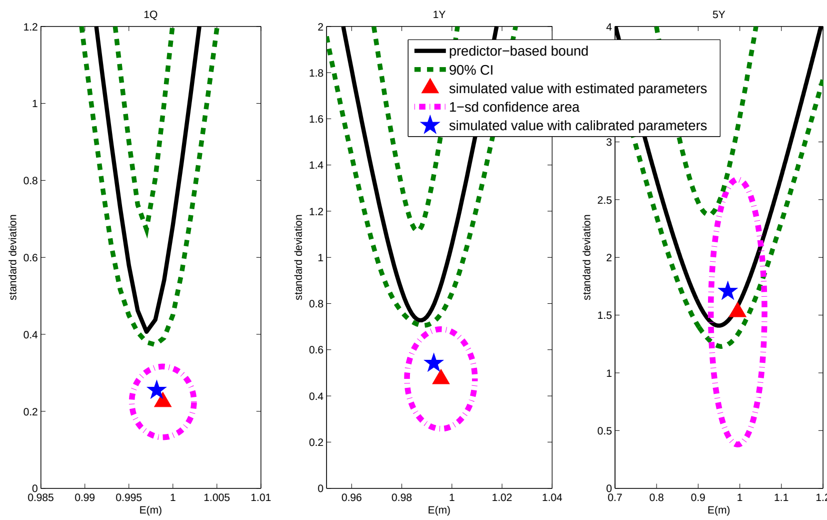
(b) External Habit Model

Figure 3 Volatility bounds on stochastic discount factors for different investment horizons, SET A.

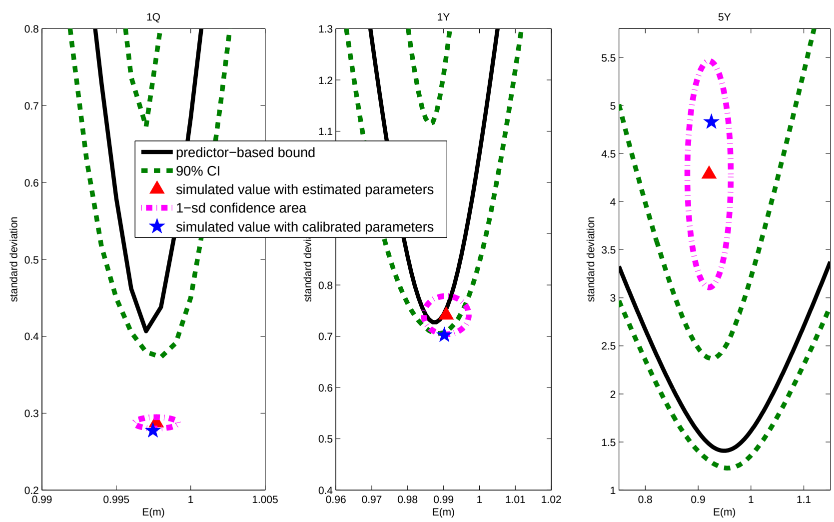


(a) Rare Disaster

Figure 3 Volatility bounds on stochastic discount factors for different investment horizons, SET A. Dashed line gives the volatility bound when no conditional information is used. Solid line gives the volatility bound using conditional information based on Bekaert and Liu's (2004) specification. The red triangle reports average mean and standard deviation values from 10 simulations run of 600,000 months with *estimated* parameters. The blue star reports the same objects computed with calibrated parameters. The ellipses (dashed dotted area) show the uncertainty (one standard deviation) in the calculation of the mean and standard deviation of the model implied SDFs. We only take account uncertainty in the parameter values for the state dynamics (i.e. we take as given parameters that characterize the preferences). Long horizon returns are computed by compounding quarterly returns. Sample: 1952Q2 - 2012Q3.

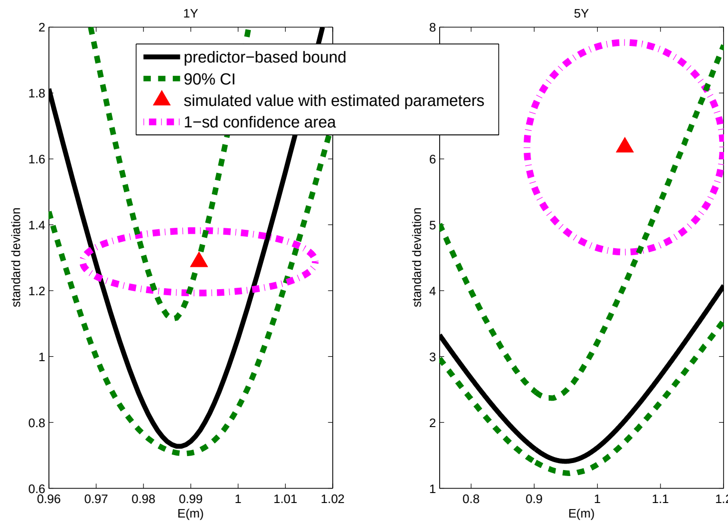


(b) Long Run Risk Model



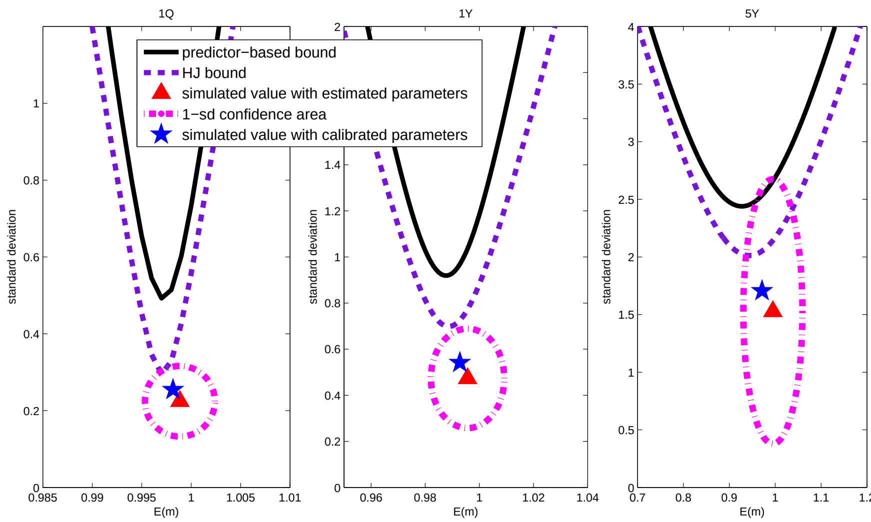
(c) External Habit Model

Figure 4 Volatility bounds on stochastic discount factors for different investment horizons, SET A.

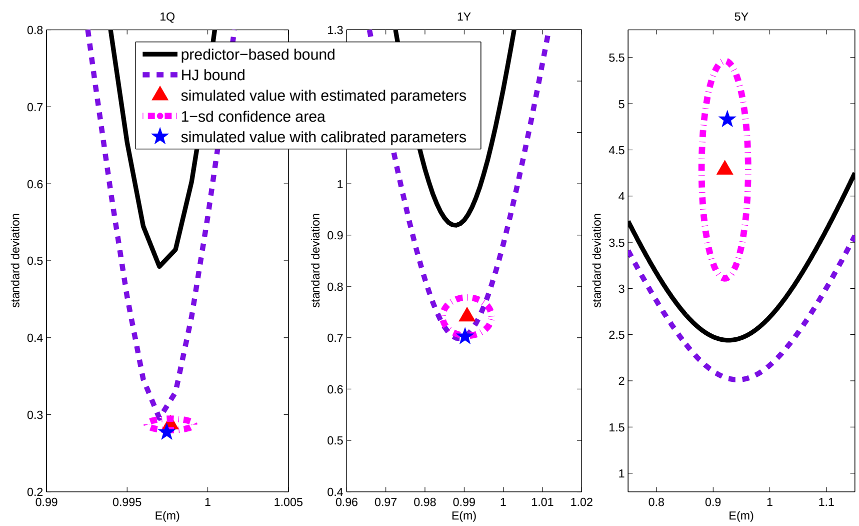


(a) Rare Disaster

Figure 4 Volatility bounds on stochastic discount factors for different investment horizons, SET A. Solid line gives the volatility bound using conditional information based on Bekaert and Liu's(2004) specification. Dashed line gives the bootstrapped 90% confidence interval for corresponding variance bound. The red trangle reports average mean and standard deviation values from 10 simulations run of 600,000 months with estimated parameters. While the five angle star reports the same objects computed with calibrated paramters. The dashed dotted area are the one standard deviation confidence ellipses for simulated point with estimated parameters. The ellipses show the sensitivity of simulated SDFs to the estimated value of dynamics parameters. Long horizon returns are computed by compounding quarterly returns. Sample: 1952Q2 - 2012Q3.

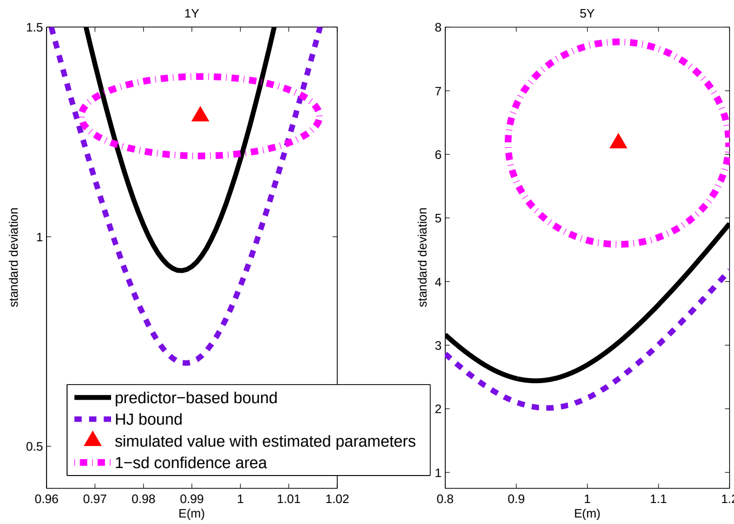


(b) Long Run Risk Model



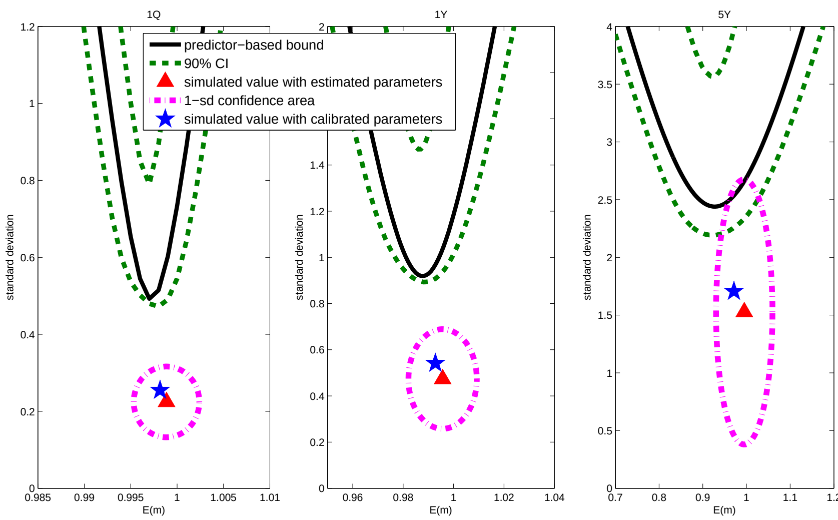
(c) External Habit Model

Figure 5 Volatility bounds on stochastic discount factors for different investment horizons, SET B.

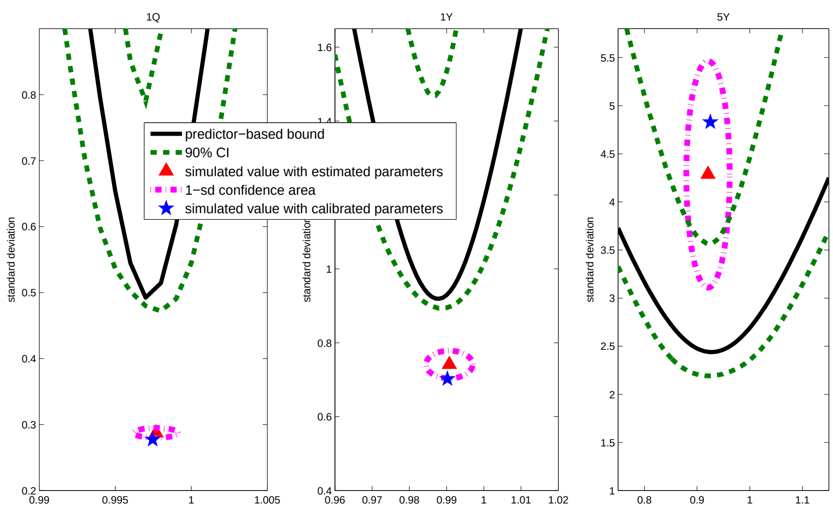


(a) Rare Disaster

Figure 5 Volatility bounds on stochastic discount factors for different investment horizons, SET B. Dashed line gives the volatility bound when no conditional information is used. Solid line gives the volatility bound using conditional information based on Bekaert and Liu's (2004) specification. The red triangle reports average mean and standard deviation values from 10 simulations run of 600,000 months with *estimated* parameters. The blue star reports the same objects computed with calibrated parameters. The ellipses (dashed dotted area) show the uncertainty (one standard deviation) in the calculation of the mean and standard deviation of the model implied SDFs. We only take account uncertainty in the parameter values for the state dynamics (i.e. we take as given parameters that characterize the preferences). Long horizon returns are computed by compounding quarterly returns. Sample: 1952Q2 - 2012Q3.

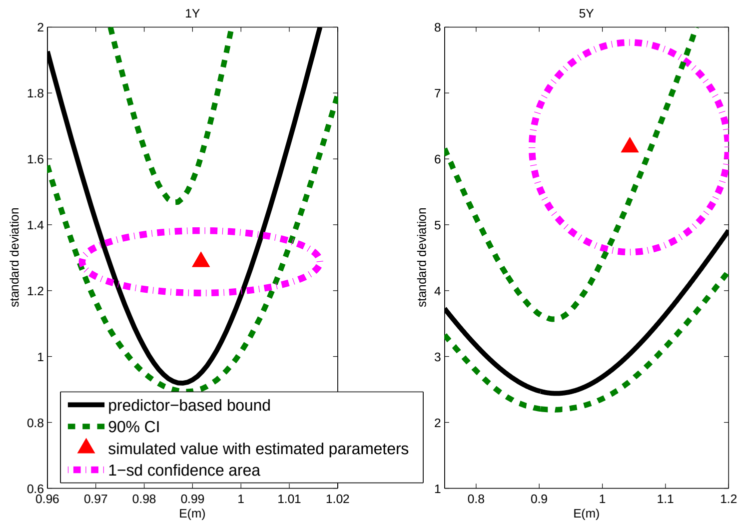


(b) Long Run Risk Model



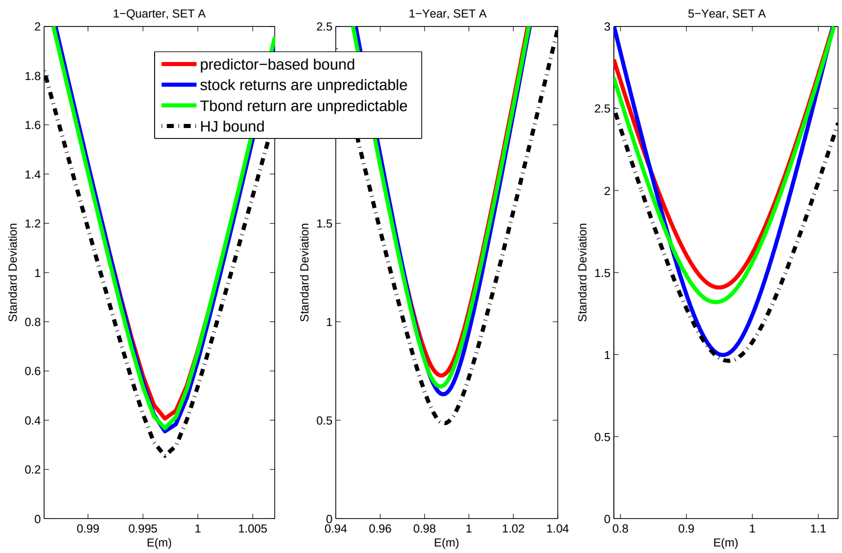
(c) External Habit Model

Figure 6 Volatility bounds on stochastic discount factors for different investment horizons, SET B.

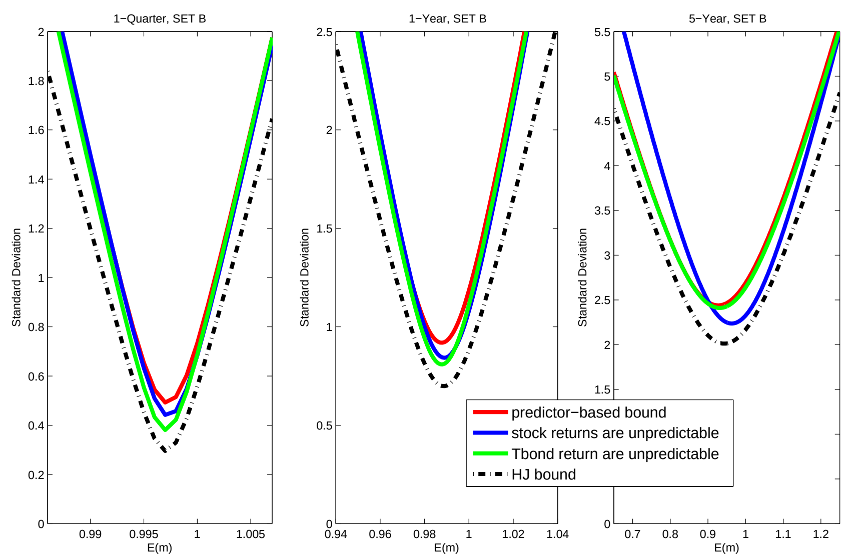


(a) Rare Disaster

Figure 6 Volatility bounds on stochastic discount factors for different investment horizons, SET B. Solid line gives the volatility bound using conditional information based on Bekaert and Liu's(2004) specification. Dashed line gives the bootstrapped 90% confidence interval for corresponding variance bound. The red trangle reports average mean and standard deviation values from 10 simulations run of 600,000 months with estimated parameters. While the five angle star reports the same objects computed with calibrated paramters. The dashed dotted area are the one standard deviation confidence ellipses for simulated point with estimated parameters. The ellipses show the sensitivity of simulated SDFs to the estimated value of dynamics parameters. Long horizon returns are computed by compounding quarterly returns. Sample: 1952Q2 - 2012Q3.



(b) SET A



(c) SET B

Figure 7 Volatility bounds on stochastic discount factors for different investment horizons. Black Dashed line gives the volatility bound when no conditional information is used. Solid red line gives the volatility bound using conditional information by Bekaert and Liu(2004). Solid blue line gives the volatility bound with restriction which implies stock returns are unpredictable. Solid green line gives the volatility bound with restriction that Treasury bonds bonds returns are unpredictable (for SET A is 5-year treasury government bond and for SET B are 5-, 7- ,10-, 20- and 30-year maturity treasury government bonds). The bounds are generated using SET A (see Panel A) and SET B (see Panel B). Long horizon returns are computed by compounding quarterly returns. Sample: 1952Q2 - 2012Q3.

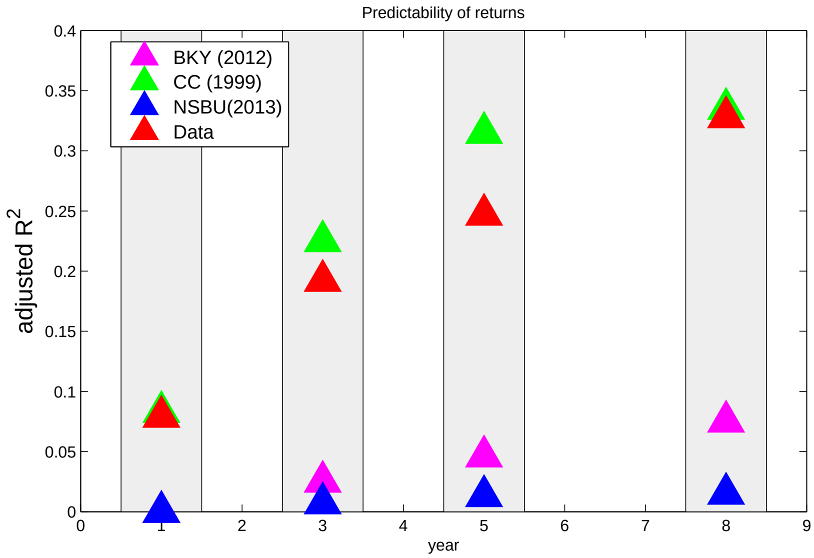


Figure 8 Predictability of excess return by PD-ratio: this figure presents R2s from projecting model simulated one-, three-, five- and eight-year gross return of the aggregate stock market portfolio onto lagged model simulated price-dividend ratio. Red triangle stands for the results by using annulized real data from 1952Q2 to 2012Q3. Blue, magenta, green and blue triangles presents the results by using model generated simulated data.

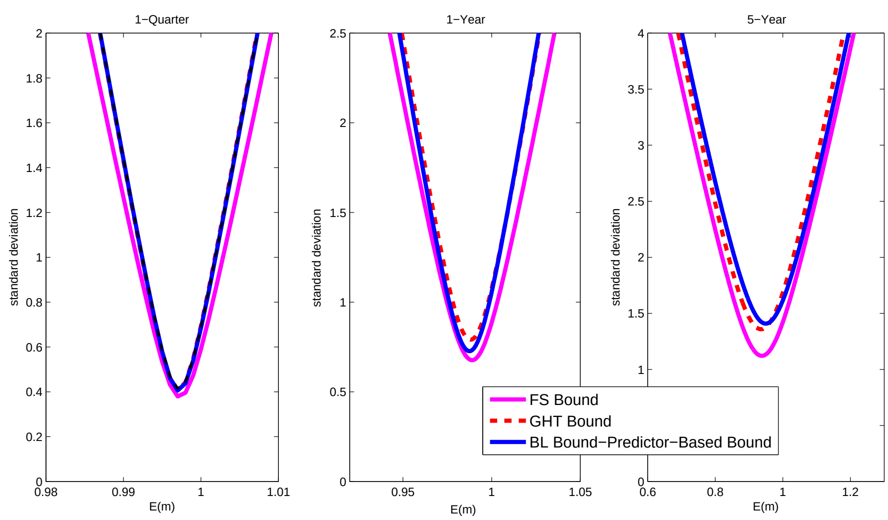


Figure 9 Alternative implementation of the HJ bounds: optimal bounds of Ferson and Siegel (2001,2002) (FS), Bekaert and Liu (2004) (BL) and Gallant, Hansen, and Tauchen (1990) (GHT) for SET A. Sample: 1952Q2 - 2012Q3.

	Asset	
	Stocks	Bonds
Mean return (% p.a.)	11.45	6.31
Standard deviation (% p.a.)	16.68	5.77

Table 1 This table reports sample statistics of quarterly nominal stock and bond total returns. Stock returns are nominal returns on the stock total returns on the value weighted portfolio of all stocks traded in the NYSE, the AMEX, and NASDAQ from CRSP. Bond returns are nominal returns on the 5-year constant maturity bond from the CRSP Fixed Term Indices File. Sample: 1947Q2: 2012Q3.

Panel A: Predictive regressions for stock returns

Horizon h (in quarters)	pd_t [t-stat]	cay_t [t-stat]	$R^2(\%)$
1	-0.03	0.85	4.9
	(-1.99)	(2.93)	
4	-0.13	3.24	17.4
	(-2.37)	(2.75)	
20	-0.60	16.23	50.0
	(-4.70)	(4.17)	

Panel B: Predictive regressions for bond returns

Horizon h (in quarters)	spr_t	y_t	CP_t	$R^2(\%)$
	[t-stat]	[t-stat]	[t-stat]	
1	7.92	0.76	-0.01	7.1
	(2.97)	(1.60)	(-0.06)	
4	7.68	2.32	1.41	29.1
	(1.66)	(2.24)	(4.48)	
20	12.40	17.97	2.16	44.9
	(0.68)	(4.20)	(2.42)	

Table 2 Panel A reports quarterly overlapping regressions of multiple horizon real gross stock returns onto a constant, the log price-dividend ratio and cay_t. Panel B reports monthly overlapping regressions of multiple horizon real gross return on a 5-year constant maturity coupon bond from CRSP onto a constant, the log short rate $y(t)$, the yield spread $spr(t)$ and Cochrane-Piazzesi factor $CP(t)$. The short rate is the log yield on the 30-day Treasury Bill from CRSP, and the spread is the difference between the log yield on a 5-year artificial zero-coupon bond from the CRSP Fama-Bliss Discount Bond File, and the log yield on the Treasury Bill (T-bill). The table reports coefficient estimates, the R^2 of the regression, and, in brackets, the Newey-West corrected t-statistics. Sample: 1947Q2: 2012Q3.

	Long-Run Risk Bansal-Kiku-Yaron		External Habit Campbell-Cochrane		Rare Disaster Nakamura et al.	Lower Bound $\sigma_{\min}$ Data	
	(calibration)	(estimation)	(calibration)	(estimation)			
						SET A	SET B
1-quarter	0.3861	0.2244	0.2775	0.2874		0.4066	0.4925
						[0.380 0.63]	[0.480 0.790]
1-year	0.8900	0.4733	0.7058	0.7449	1.2875	0.7275	0.9191
						[0.706 1.135]	[0.893 1.481]
5-year	2.5507	1.5275	4.0077	4.2860	6.1765	1.4088	2.4395
						[1.232 2.471]	[2.193 3.566]

Table 3 BL Bounds with conditioning information. We compute the variance of the SDFs at different holding horizons, via simulations, respectively, for the models that incorporate long-run risk and external habit persistence with both calibrated and estimated parameters which are based on model parameters tabulated in Table Appendix I - III. The reported values are the average values of standard deviation from 10 single simulations run of 600,000 month. Then we construct the related 1-quarter, 1-year and 5-year SDFs. We construct Hansen-Jagannathan Bounds at different horizons, using the optimally scaled bounds of Bekaert and Liu (2004), based on the return predictive system (see equation(8)). The quarterly data used in the construction of σ is from 1952Q2 to 2012Q3, with the 90% confidence intervals. To compute the confidence intervals, we create 50,000 random samples of sample size from the data, where the sampling in the block bootstrap is based on the optimal block length we calculated for each asset return regression residuals. Real returns are computed by deflating the nominal returns by the Consumer Price Index inflation. 1-year, 5-year holding returns are computed by compounding related quarterly returns of each asset.

Panel A. Variance bound-based upper bound on R^2						
	HJ bounds		predictor-based bounds		regression R^2	
Horizon quarter	SET A	SET B	SET A	SET B	Stock Returns	Bond Returns
1	0.065	0.088	0.166	0.243	0.049	0.071
			[0.145 0.454]	[0.231 0.626]		
4	0.238	0.494	0.536	0.855	0.174	0.291
			[0.505 1.304]	[0.807 2.221]		
20	0.993	4.342	2.129	6.383	0.500	0.449
			[1.628 6.549]	[5.158 13.563]		

Panel B. Molde-based upper bound on R^2			
Horizon quarter	Bansal-Kiku-Yaron	Campbell-Cochrane	Rare Disaster Nakamura et al.
1	0.051	0.083	
	[0.018 0.100]	[0.079 0.087]	
4	0.227	0.562	1.678
	[0.067 0.481]	[0.502 0.604]	[1.440 1.935]
20	2.503	19.703	40.945
	[0.154 7.677]	[10.364 32.016]	[22.560 64.767]

Table 4 Upper bound on the R^2 of return predictive regressions. This table presents the upper bound on R^2 of returns' predictive regressions. We compute the upper bound $R^2_{f,t+h}\sigma^2_Z(v_{\min})$ of R^2, together with the corresponding 90% confidence intervals for SETA and SETB. Panel A contains the results based on variance bounds of SDFs. For the unconditional HJ bounds, in the first two coloums of, we reported their minimum value at each horizon, 1-quater, 1-year and 5-year. For the predictor-based bounds, we report $\sigma^2_Z(v_{\min})$, i.e. the minimum value of the predictor-based bounds, computed using Bekaert and Liu (2004) specification, again for each horizon. The brackets below the columns of the predictirs-based bounds report the values of the 90% bootstraped confidence intervals. For the scaled variance of the models $R^2_{f,t+h}Var(m^X_{t+h})$, we report our the values for the Bansal et al. (2012a) model, the Campbell and Cochrane (1999) model and Nakamura et al. (2013) model (see Panel B). The 90% confidence interval of these scaled variances are reported in the brackets. The last two coloums report, from Table 2, the predictive regression R^2 of stock returns and 5-year constant maturity govenment bond returns. For the risk free rate, we adopt the mean of the 3-month Tbill returns. 1-year, 5-year holding returns computed compounding related quarterly returns of each asset.Sample: 1952Q2-2012Q3.

	Lower Bound $\sigma_{\min}$	
	SET A	SET B
1-quarter	0.3798	0.4925
	[0.367 0.608]	[0.461 0.740]
1-year	0.6764	0.9191
	[0.662 0.977]	[0.929 1.435]
5-year	1.1215	2.4395
	[1.044 1.945]	[2.095 3.198]

Table 5 FS Bounds with conditioning information. We construct Hansen-Jagannathan Bounds at different horizon, using Ferson and Siegel (2003) approach, based on the return predictive system (see equation(8)). The quarterly data used in the construction of σ is from 1952Q2 to 2012Q3, with the 90% confidence intervals. To compute the confidence intervals, we create 50,000 random samples of sample size from the data, where the sampling in the block bootstrap is based on the optimal block length we calculated for each asset return regression residuals. Real returns are computed by deflating the nominal returns by the Consumer Price Index inflation. 1-year, 5-year holding returns are computed by compounding related quarterly returns of each asset.

	Data		Bansal-Kiku-Yaron		Campbell-Cochrane		Rare Disaster Nakamura et al.	
Horizon year	β	R^2	β	R^2	β	R^2	β	R^2
1	−0.127 (0.053)	0.079	−0.080 (0.092)	0.000 [−0.016 0.118]	−0.291 (0.118)	0.089 [0.002 0.214]	−0.027 (0.304)	0.000 [−0.017 0.252]
3	−0.312 (0.084)	0.192	−0.274 (0.230)	0.025 [−0.017 0.288]	−0.732 (0.206)	0.238 [0.022 0.488]	−0.344 (0.495)	0.007 [−0.018 0.286]
5	−0.433 (0.075)	0.247	−0.523 (0.365)	0.046 [−0.017 0.392]	−1.020 (0.245)	0.327 [0.014 0.614]	−0.694 (0.596)	0.013 [−0.019 0.332]
8	−0.631 (0.101)	0.328	−0.929 (0.525)	0.075 [−0.018 0.477]	−1.3101 (0.264)	0.375 [0.000 0.734]	−1.107 (0.715)	0.015 [−0.020 0.311]

Table 6 Predictability of Gross Returns, and Price–Dividend Ratios. This table presents adjusted R^2s and slope coefficients from projecting model simulated one-, three-, five- and eight-year gross return of the aggregate stock market portfolio onto lagged model simulated price-dividend ratio. The results correspond to regressing $r_{t+1} + r_{t+2} \cdots + r_{t+h} = \alpha(h) + \beta(h) \log (P_t/D_t) + \varepsilon_{t+h}$, where r^e_{t+1} is the excess return, and h denotes the forecast horizon in year. Data statistics are reported in the "Data" column, sample 1952Q2-2012Q3.The "Bansal-Kiku-Yaron" column presents predictability evidence implied by the Bansal Kiku and Yaron's (2012) model. The "Campbell-Cochrane" column presents predictability evidence implied by Campbell and Cochrane's (1999) model with estimated parameters reported in 8. "Nakamura et al." column presents predictability evidence implied by Nakamura et al.(2013) model. The entries for the models are based on 1,000 simulations each with 724 monthly observations that are time-aggregated to an annual frequency. The reported value for each model is the median of simulated slope and mean of simulated adjusted R^2, along with 95% confidence interval in the square brackets. Standard errors which reported in parentheses are Newey-West corrected.

	Parameter	BKY2012a *calibration*	BKY2012b *estimation*
Preferences			
Time preference	δ	0.9989	0.9989 (0.0010)
Risk aversion	γ	7.5	7.42 (1.55)
EIS	ψ	1.5	2.05 (0.84)
Consumption growth dynamics, g_t			
Mean	μ	0.0015	0.0012 (0.0007)
Dividends growth dynamics, $g_{d,t}$			
Mean	μ_d	0.0015	0.0020 (0.0017)
Persistence	ϕ	2.5	4.45 (1.63)
Volatility parameter	φ_d	5.96	5.00 (1.39)
Comsumption exposure	π	2.6	0
Correlation between innovations	ρ_{dc}		0.49 (0.33)
Long-run risk, x_t			
Persistence	ρ	0.975	0.9812 (0.0086)
Volatility parameter	φ_e	0.038	0.0306 (0.0160)
Consumption growth volatility, σ_t			
Mean	$\overline{\sigma}$	0.0072	0.0073 (0.0015)
Persistence	v	0.999	0.9983 (0.0021)
Volatility parameter	σ_w	0.28×10^{-5}	2.62×10^{-5} (3.10×10^{-6})

Table 7 Appendix-I: Parametrization of asset pricing models incorporating long-run risk. Our parameterizations of calibrated values are taken from Bansal et al. (2012a) Table 1. The estimated parameters are from (Bansal et al., 2012b) Table II and standard errors of estimation are reported in parentheses. The models are simulated at the monthly frequency.

	Parameter	Campbell-Cochrane	
		calibration	*estimation*
Preference			
Time preference	δ	0.9909	0.9903 (0.0004)
Risk aversion	γ	2	0.9756 (0.0772)
Consumption growth dynamics, g_t			
Mean	$\overline{g}$	0.0016	0.0017 (0.00007)
Volatility parameter	σ	0.0043	0.0050 (0.00019)
Dividend growth dynamics, Δd_t			
Volatility parameter	σ_w	0.0323	0.0319 (0.0014)
Corr between innovitions	ρ_{dc}	0.2	0.1945 (0.0093)
Steady state surplus consumption ratio	$\overline{S}$	0.0570	0.0637
Persistence in consumption surplus ratio	ϕ	0.9884	0.9877 (0.0003)
Log of risk-free rate	$r^f \times 10^2$	0.0783	0.0854 (0.0365)

Table 8 Appendix-II: Parametrization of asset pricing models incorporating external habit persistence. Our parameterizations of calibrated values are taken from Campbell and Cochrane (1999). The estimated values for Campbell-Cochrane model are from (Aldrich and Gallant, 2011) and standard errors of estimation are reported in parentheses. The models are simulated at the monthly frequency.

	Parameter	Annual
Preferences		
Time preference	δ	0.967
Risk aversion	γ	6.4
Elasticity of intertemporal substitution	ψ	2
Potential consumption dynamics, g_t, only for US		
Mean of potential consumption growth, x_t	μ	0.022 (0.003)
Volatility parameter	σ_ϵ	0.003 (0.002)
Volatility parameter	σ_η	0.018 (0.002)
Disaster parameters		
Probabilities of		
a world-wide disaster	p_W	0.037 (0.016)
a country will enter a disaster when a world disaster begins	p_{CbW}	0.623 (0.076)
a country will enter a disaster "on its own."	p_{CbI}	0.006 (0.003)
a country will stay at the disaster state	$1 - p_{Ce}$	0.835 (0.027)
Disaster gap process, z_t		
Persistence	ρ_z	0.500 (0.034)
a temporary drop in consumption caused by shock, ϕ_t		
Mean	ϕ	−0.111 (0.008)
Volatility parameter	σ_θ	0.121 (0.015)
a permanent shift in consumption caused by shock, θ_t		
Mean	θ	−0.025 (0.007)
Volatility parameter	σ_ϕ	0.083 (0.006)

Table 9 Appendix-III: Parametrization of asset pricing models incorporating rare disasters. Our parameterizations of estimated values are taken from Nakamura et al. (2013) and standard errors of estimation are reported in parentheses. The models are simulated at the annual frequency.