Bargaining over a Divisible Good in the Market for Lemons*

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Abstract

A seller dynamically sells a divisible good to a buyer. Payoffs are interdependent as in Akerlof's market for lemons and it is common knowledge that there are gains from trade. The seller is informed about the quality of the good. The buyer makes an offer in every period and learns about the good's quality only through the seller's behavior. We characterize the limiting equilibrium outcome as bargaining frictions vanish and the good becomes arbitrarily divisible. When the gains from trade decrease in the number of units already traded, the gradual sale of high-quality goods arises as the main signaling device in the market. We also show that the limiting outcome is Coasean: the competition with his future selves drives the buyer's payoff to the lowest possible level.

KEYWORDS: bargaining, gradual sale, Coase conjecture, divisible objects, interdependent valuations, market for lemons.

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1 Introduction

This paper studies bargaining over a divisible good in the presence of informational asymmetries and interdependent values. Divisibility is a natural feature of several real-life negotiations. For example, in financial markets, negotiations typically regard both the price at which an asset is sold and the amount that will be traded. Divisibility opens new and potentially complex channels for a seller who holds a valuable asset to convey information about its quality. Accordingly, a buyer who worries about purchasing a low-quality asset will look at the entire history of trade and update his posterior about the quality of the asset before deciding the next trading offers.

To have a concrete example in mind, consider a bank negotiating a securitized asset (pool of mortgages, credit-card debts, automotive loans) with an institutional investor (e.g. a pension fund). These transactions involve dynamic negotiations and generally take place over-the-counter. Gains from trade arise as the pension fund is more interested in lending money to the final borrowers (e.g. homeowners) by owing the asset in exchange for the promise of future cash flows. These gains are typically decreasing in the amount negotiated as they are intimately linked to the pension's fund desire to diversify its portfolio. Moreover, the sooner the parties reach an agreement, the sooner the pension fund (the party most interested in lending) will become the creditor and, hence, the greater the gains from trade. Unfortunately, these relationships are often plagued by asymmetric information[1] which may lead to lower than optimal sales and/or inefficiently delayed transactions. The seller is directly involved in the process of securitization and hence has better information about the quality of these assets.[2] This information advantage is due to superior information about the pool of lenders which goes beyond the information revealed by the contract. For example, a CDO typically releases aggregate information such as the average FICO score but fails to release information about the distribution over FICO numbers,[3] the duration of residence of its homeowners, their average incomes, etc... As a result, the seller of subprime CDO could often better estimate the probability that the pool of homeowners from that asset will default in the future. Importantly, in all such situations the only information learnt by the buyer when a trade is realized is the seller´s willingness to negotiate that asset. Obviously sellers of better-quality assets are less reluctant to keep them in their

[1] See Ashcraft and Schuermann (2008), Downing et. al. (2009), Falting-Traeger and Mayer (2012) and Gorton and Metrick (2012) for a survey.

[2] Consider the classic example of the synthetic CDO Hudson Mezzanine. As explained in McLean and Nocera (2011), Goldman Sachs selected all the securities in that CDO, strived to sell it as fast as possible and simultaneously bet against that security by taking a short position.

[3] This issue is discussed with detail in Lewis (2010).

balence-sheets,[4] which generates an adverse-selection problem.

For a second example, consider a private equity firm that negotiates the control of a firm after restructuring. Often the private equity engages in negotiations to sell the firm to a big player in the market. Similarly to the example above, the sooner the transaction takes place the greater the gains from trade. In contrast to the example above, there are additional reasons to trade other than the allocation of the future cash flows. In particular, trade is often motivated by interest in better allocation of control rights. This suggests that gains from trade are often increasing in the volume negotiated. As we will see below, how gains from trade vary with the fraction of the firm traded will have a crucial impact in our model predictions.

Our goal is to understand how trade evolves when the parties are asymmetrically informed and negotiations are both over the quantity and the price. To this end, we extend the canonical bargaining model with incomplete information (Fudenberg, Levine and Tirole (1985), and Gul, Sonnenschein and Wilson (1986)) to account for interdependencies in values (Deneckere and Liang (2006) — DL henceforth) and divisibility. A buyer purchases a durable and divisible good from a seller who is informed about the good's quality, which may be either high or low. For each quality and quantity, there are positive gains from trade. These gains may be constant, increasing, or decreasing in the number of units already traded by the parties. Constant gains constitute a benchmark-case that allows us to analyze the effects of divisibility in the most parsimonious way. Decreasing gains arise when portfolio diversification is the main reason for trade and, as we will explain, lead to new insights into bargaining. Finally, increasing gains from trade are more likely to appear when the parties trade for control rights.

In our model, the good is divided into finitely many parts (or units). The buyer makes a take-it-or-leave-it offer in every period (however, as we discuss in Section 9, all our results extend to the case in which the buyer proposes menus of offers). The offer stipulates a price and a number of units to be traded. The bargaining process continues until the parties have traded all the available units.

The buyer learns about the good's quality only through the seller's behavior. In particular, owning a fraction of the good does not provide the buyer with additional information about its quality. In our leading example, an institutional investor does not learn about the probability that the future cash flows will realize by purchasing an additional unit of a securitized asset. Often, the buyer only learns this information in the future, when the asset cash flows materialize. More broadly, the game analyzed in this paper is both a benchmark model and a first step to investigate the effects of divisibility in bargaining.

[4] During the 2007-2008 financial crisis, a Goldman Sachs trader commented on a CDO: "It stinks... I don't want it in our books." (see McLean and Nocera (2011)).

Time on the market is the unique screening device when parties negotiate over the sale of an indivisible good. Therefore, the only way for a seller to convince a buyer that he has a high-quality good is by rejecting low offers in the initial phases of negotiations. Consistent with this observation, most of the theoretical literature[5] predicts similar trade patterns in dynamic markets under adverse selection: Buyers make low offers in the beginning of the relationship. These offers are accepted by some sellers with low-quality goods. However, sellers with high-quality goods reject these offers, which leads to delay and often periods of market freeze. When goods are divisible, the set of possible trade outcomes becomes considerably richer. Does adverse-selection lead to the sale of low-quality goods in the beginning of the relationship, followed by a market freeze and the subsequent sale of high-quality goods, as is found in indivisible-good models? Or does the gradual sale of high-quality goods arise as the main signaling device in such markets? Our paper is the first to address and answer these questions.

Our model yields clear predictions in terms of economic behavior. We show the existence of stationary equilibrium and characterize the generically unique (stationary) equilibrium outcome. First, in the benchmark case of constant gains from trade, the parties never have incentive to trade fractions of the good. Consequently, divisibility is irrelevant given constant gains. A similar pattern is obtained when gains from trade are increasing. Our model therefore predicts that when a private-equity fund sells the control of a particular business to another firm, timing on the market is the only signalling device.

On the other hand, as we explain below, a completely different behavior is obtained when gains from trade are decreasing in the amounted traded.

In equilibrium, the buyer employs only two types of offers: *cream-skimming* and *universal* offers. A *cream-skimming offer* is an offer to purchase all the remaining units of the good at a price that only the owner of the low-quality good would be willing to accept. A *universal offer* is an offer to purchase a fraction of the available good at a price that every type of seller will accept. Typically, the buyer starts by making *cream-skimming offers*. After several of these offers have been rejected, the buyer becomes more optimistic that the good is of high quality and decides to make a *universal offer* for a fraction of the good. Upon the acceptance of this offer, the buyer restarts the process of making *cream-skimming* offers for the remainder of the good.

We characterize the limiting equilibrium outcome (or simply limiting outcome) which arises as bargaining frictions vanish and the good becomes arbitrarily divisible (i.e., the number of units into which the good is divided goes to infinity). Our interest in perfectly

[5] Among the several papers which obtained this finding, see the important contributions of DL, Fuchs and Skrzypacz (2013 a,b), Camargo and Lester (2014), Fuchs et. al. (2014), Kim (2015) and Moreno and Wooders (2010, 2015).

divisible goods stems from the fact that in most applications of our model, the parties can essentially trade any fraction of the good. Moreover, we get the sharpest results when the size of each part of the good shrinks to zero.

Our main result shows a gradual sale of high-quality goods. That is, the buyer purchases the good from the "high seller" (i.e., the owner of the high-quality good) smoothly over time. At each point in time, the buyer also makes an offer for the remainder of the good at a price equal to his valuation when the quality is low. The "low seller" is indifferent between the two offers (cream-skimming and universal) and gradually reveals his identity. In other words, he sells the good smoothly (pooling with the high seller) until a certain random time, and then concedes by selling the remaining fraction of the good at once.

Let us now provide some intuition for the stark change in behavior led by the assumption of decreasing gains from trade. As mentioned above, divisibilities do not matter when the gains from trade are constant. The equilibrium involves a long delay after which the buyer breaks even purchasing the total number of units m at the price mc (where c denotes the seller's cost of a high-quality unit). Importantly, the buyer could not profitably deviate by making a universal offer for $k < m$ units at the price kc as he would also break even with this offer. This is no longer true when the gains from trade are decreasing. Indeed, we show that universal offers for a fraction of the good bring two advantages in comparison with a universal offer for the entire good: a direct payoff advantage and a strategic advantage. The direct payoff advantage is a result of the average value of the first k units being higher than the average value of all the m units. The strategic advantage is related to a higher value for the buyer in the continuation game following the acceptance of a universal offer for $k < m$ units. To understand the last point, first notice that the average value of the remaining units is smaller after the acceptance of a universal offer than after the rejection of a cream-skimming offer. This effect may force another delay in the future, after the universal offer is accepted. This delay, in turn, decreases the continuation value of low-type sellers, making them willing to sell at a price lower than the value assigned by the buyer to a low-quality good. Therefore the buyer profits from purchasing the good from a subset of low-type sellers at a low price. The combination of these two advantages makes partial offers relatively more valuable when gains from trade are decreasing. As a result, the high-quality good is sold gradually over time. In the limit, as the good becomes arbitrarily divisible, the number of new profitable trade opportunities increases without bound, leading to the smooth sale of the high-quality good.

It is well known that dynamic markets with adverse-selection may lead to impasses in bargaining (DL) and periods of illiquidity. Our model sheds light on a new economic force present in those markets — gradual trading. It shows that markets with ameliorated information asymmetries are achieved by a combination of gradual purchasing at high prices and large-purchase discounts. Applying these lessons to the main example of this paper, our theory predicts the slow and gradual trade of high-quality assets in markets in which two parties trade mainly for portfolio diversification.

Our second main result shows that when the gains from trade are decreasing, the limiting equilibrium outcome is Coasean. In the limit, the buyer breaks even at any point in time, irrespective of the offer (cream-skimming or universal) he opts to make. Consequently, his limit payoff is equal to zero. This result is reminiscent of the Coase conjecture and shows that the buyer's competition with his future selves drastically reduces his payoff. Our findings are in sharp contrast with the case in which the good is indivisible and the buyer's payoff remains positive in the limit as bargaining frictions vanish.

To provide some intuition, it is useful to compare our model with the indivisible-good version of our model analyzed in DL. When the good is indivisible, negotiations involve long impasses which severely reduce the price at which the buyer is able to purchase the good at the beginning of the bargaining process. In other words, the buyer benefits from long impasses and his initial payoff is strictly positive. Long impasses are sustained for two reasons. First, the buyer is not willing to make a generous offer unless he is sufficiently optimistic about the quality of the good. Second, the owner of the low-quality good is willing to wait and sell the good at a price much higher than his cost.

When the good is divisible and gains from trade are decreasing, new profitable trading opportunities are available to the buyer, beyond the possibility of purchasing all the remaining units at a high price. In particular, when this option yields a low payoff, the buyer strictly prefers to make universal offers and purchase a certain fraction of the remaining good. Compared to DL's model, this lowers the price that the owner of the low-quality good is able to charge in the middle of the negotiations. However, lower payments in the middle of the negotiations translate into shorter impasses and higher prices paid at the beginning of the bargaining process. Consequently, the buyer's initial payoff is drastically reduced and converges to zero in the limit.

Our analysis shows that when gains from trade are decreasing, the buyer's ability to screen the seller more finely using partial offers (for fractions of the good) takes away the commitment power that he otherwise gains from long impasses in bargaining environments with indivisible goods.

To sum up, our model generates novel and testable predictions for dynamic markets under adverse selection. In situations with increasing or constant gains from trade (as when a private equity firm negotiates the sale of another firm), time on the market is the main signaling device and the buyer keeps some of his bargaining power. On the other hand, when the gains from trade are decreasing (as when portfolio diversification is the main reason for trade), the high-quality good is slowly sold over time and, in the limit, the buyer loses all the bargaining power.

1.1 Related Literature

Bilateral bargaining with interdependent values (and indivisibility) has received considerable attention in the literature (Samuelson (1984), Evans (1989), Vincent (1989), DL, Fuchs and Skrzypacz (2013a), and Gerardi, Hörner and Maestri (2014)). The closest papers to ours are DL and Fuchs and Skrzypacz (2013a). DL solved the one-unit version of the model in this paper. Taking their construction as a stepping stone, we build an algorithm to extend their analysis to multiple units when there are two types of sellers. In DL, the gains from trade are bounded away from zero. Fuchs and Skrzypacz (2013a) bridged the gap between the value of the good to the buyer and the cost to the seller. We find that trade happens gradually over time when the good is very divisible. This finding is reminiscent of Fuchs and Skrzypacz (2013a) who show in a model with indivisibility that, as the gains from trade from the good of highest quality vanish, the bursts of trade found in DL disappear. Like Fuchs and Skrzypacz (2013a), in our model the buyer slowly learns the seller's type. Unlike Fuchs and Skrzypacz (2013a), however, in our model the buyer makes two kinds of offers as he learns the seller's type. On the one hand, he gradually makes universally accepted offers for small pieces of the good at large per-unit prices. On the other, he makes offers for all remaining units at large discounts. Finally, another important difference between the two papers is that in our model the gains from trade are bounded away from zero.

Our paper is also related to the burgeoning body of literature that studies the effects of adverse selection in dynamic markets. An important stream of this literature focuses on markets in which one of the players is short-run. Inderst (2005) and Moreno and Wooders (2010) pioneered the study of adverse-selection in decentralized dynamic markets. Camargo and Lester (2014) and Moreno and Wooders (2015) focus on the effect of policy interventions on liquidity in such markets. A question that has drawn much attention is how different transparency regimes affect the bargaining outcome (see Hörner and Vieille (2009) and Fuchs, Öry and Skrzypacz (2014) for a comparison of public and private offers, and Kim (2017) for the role of time-on-the-market information). Finally, Fuchs and Skrzypacz (2013b) characterize optimal market design policies. Beyond the issue of divisibility, our paper differs from the above studies by analyzing the strategic effects that arise when two long-run players bargain under adverse selection.

Another important strand of the literature analyzes the effect of exogenous learning in the market for lemons. The pioneer work of Daley and Green (2012) analyzed a model in which noisy information about the value of a good is revealed to the market. Kaya and

Kim (2015) analyze a model in which the buyer observes a noisy and private signal about the quality of the good held by the seller. Daley and Green (2016) analyze the advent of exogenous news in a model in which two long-run players bargain over an indivisible good. Our model differs from these contributions as we assume that the good is divisible and abstract from exogenous learning.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 specifies the solution concept and proves the general existence and uniqueness (for generic parameters) of the equilibrium outcome. In Section 4, we define the limit (as the bargaining frictions vanish) of the equilibrium outcome. In Section 5, we analyze the benchmark case of constant gains from trade. Sections 6 and 7 are devoted to the analysis of bargaining with decreasing gains from trade and contain the main results. The case of increasing gains from trade is studied in Section 8. Finally, Section 9 analyzes an extension of the model and Section 10 concludes. Most proofs are relegated to a number of Appendices.

2 The Model

A buyer and a seller bargain over a divisible object. The value of each unit to each trader depends on the seller's private information (i.e., his type). The seller's type is low with probability $\hat{q} \in (0,1)$ and high with probability $1-\hat{q}$. Although the seller has only two types, we find it convenient (following several papers on bargaining with incomplete information) to assume that his type q is distributed uniformly over the unit interval.

Our goal is to analyze the effects of divisibility on the pattern of trade, with a particular emphasis on the limit case in which the good becomes perfectly divisible. To accomplish this, we start by examining a model with divisibility frictions in which a good of measure one is divided into $m > 1$ parts (or units). We then investigate how trade evolves when the number of units m grows large.

For each $k = 1, \ldots, m$, the buyer's valuation of the $(m-k+1)$-th part is equal to

$$v(k,q) = \alpha_k v(q),$$

where $\min\{\alpha_m, \ldots, \alpha_1\} = 1$,[6] and the function $v(\cdot)$ is equal to:

$$v(q) = \begin{cases} \underline{v} & \text{if } q \in [0,\hat{q}] \\ \bar{v} & \text{if } q \in (\hat{q},1]. \end{cases}$$

The seller's cost of each unit is equal to

[6] The fact that the minimum value of $\alpha_m, \ldots, \alpha_1$ is equal to one is a simple normalization. The relevant assumption is that $\alpha_m, \ldots, \alpha_1$ are all strictly positive.

$$c(q) = \begin{cases} 0 & \text{if } q \in [0, \hat{q}] \\ c & \text{if } q \in (\hat{q}, 1]. \end{cases}$$

We assume that $\bar{v} > c > \underline{v} > 0$, so that it is common knowledge that for each unit there are positive gains from trade. We refer to any $q \leqslant \hat{q}$ as a low type and to any $q > \hat{q}$ as a high type.

As we will see below, the pattern of trade crucially depends on whether the gains from trade are decreasing in the number of units already negotiated ($\alpha_m > \ldots > \alpha_1$), constant ($\alpha_m = \ldots = \alpha_1$) or increasing ($\alpha_m < \ldots < \alpha_1$). However, our initial results (in Section 3) hold independently of the shape of the gains from trade and we postpone the discussion of the three different cases to Sections 5-8 below.

In each period $t = 0, 1, \ldots,$ the buyer makes a proposal $\varphi_t = (k, p)$ to trade a certain number k of the remaining units in exchange for a transfer $p \geqslant 0$.[7] Then the seller decides whether to accept ($a_t = A$) or reject ($a_t = R$) the proposal φ_t. The game ends when all the m parts are traded. As discussed and motivated in the introduction, all learning is strategic. In particular, no information is revealed to the buyer when a transaction takes place.

The common discount factor is $\delta = e^{-r\Delta}$, where $r > 0$ is the rate at which the traders discount the future and Δ denotes the length of each period. Consider an outcome in which the seller has type q and accepts the offers $(k_1, p_1), \ldots, (k_M, p_M)$ in periods $t_1 < \ldots < t_M$, respectively. Then the payoffs are $\sum_{i=1}^{M} \delta^{t_i} [p_i - k_i c(q)]$ for the seller and $\sum_{i=1}^{M} \delta^{t_i} \left(\left(\alpha_{\bar{k}_i+1} + \ldots + \alpha_{\bar{k}_{i-1}} \right) v(q) - p_i \right)$ for the buyer, where $\bar{k}_0 = m$ and $\bar{k}_i = \bar{k}_{i-1} - k_i$ for $i = 1, \ldots, M$. Finally, both players' payoffs are equal to zero if all the offers are rejected.

3 Equilibrium

We let $h^0 = \emptyset$ denote the empty history, and, for each $t \geqslant 1$, we let $h^t = \left((\varphi_0, a_0), \ldots, (\varphi_{t-1}, a_{t-1}) \right)$ denote the (public) history of offers and acceptance decisions in periods $0, \ldots, t-1$. We also let (h^t, φ_t) denote the history that ends with the buyer's proposal in period t. The buyer's strategy σ_B assigns a proposal φ_t to every history h^t. The seller's strategy σ_S assigns an acceptance decision $a_t \in \{A, R\}$ to every type q and every history (h^t, φ_t). Finally, we let

[7] If the two players have already traded $k' = 1, \ldots, m-1$ at the end of period $t-1$, then the proposal φ_t is an element of the set $\{1, \ldots, m-k'\} \times \mathbb{R}_+$.

$\mu(h^t)$ and $\mu(h^t, \varphi_t)$ denote the buyer's beliefs over the seller's types after the histories h^t and (h^t, φ_t), respectively.[8] We use μ to denote the collection of beliefs.

Our solution concept is *stationary* perfect Bayesian equilibrium (or stationary equilibrium for brevity). In our context, the definition of perfect Bayesian equilibrium (see Fudenberg and Tirole (1991)) imposes the following two conditions on off-path beliefs:

i) $\mu(h^t, \varphi_t) = \mu(h^t)$ for every h^t and φ_t;

ii) Suppose that the action a_t has positive probability after the history (h^t, φ_t) given the beliefs $\mu(h^t, \varphi_t)$. Then the beliefs $\mu(h^t, (\varphi_t, a_t))$ are derived from $\mu(h^t, \varphi_t)$ using Bayes' rule.

The first condition captures the idea that the buyer cannot learn anything from his offer. The second condition forces the buyer to update his beliefs according to Bayes' rule when nothing "surprising" occurs.

The following observation will turn useful in the definition of the solution concept. A common feature of all perfect Bayesian equilibria of our game is that the seller will not be able to extract any rent once the buyer is convinced that the quality of the good is low.[9]

Lemma 1 *Consider a perfect Bayesian equilibrium $(\sigma_B, \sigma_S, \mu)$ and let h^t be a history at which the buyer assigns probability one to low types (i.e., $\mu([0, \hat{q}] | h^t) = 1$). The continuation payoff of type $q \leqslant \hat{q}$ at h^t is equal to zero.*

The proof of this lemma is standard and is relegated to Appendix A. Nonetheless, this result has important equilibrium implications. In particular, it implies that in any perfect Bayesian equilibrium, the low types are willing to accept any offer once the buyer discovers that the seller's type is low.

We extend the notion of stationary equilibrium (see Gul and Sonnenschein (1988), Ausubel and Deneckere (1992), and DL) to our setting. A natural way to define stationarity in our context is to require the acceptance decision of any seller's type to depend only on the number of units left for trade and on the buyer's proposal. However, as previously mentioned, in a perfect Bayesian equilibrium the reservation price of type $q \leqslant \hat{q}$ to sell any number of units must be equal to zero at any history h^t such that $\mu([0, \hat{q}] | h^t) = 1$. This immediately implies that in our model there do not exist perfect Bayesian equilibria that

[8] For any measurable subset Q of the unit interval, $\mu(Q|h^t)$ denotes the buyer's belief, at the history h^t, that the seller's type belongs to Q. The beliefs $\mu(h^t, \varphi_t)$ are defined in a similar way.

[9] In our game, there are (off-path) histories at which the buyer's equilibrium beliefs must assign probability one to the low types. For a concrete example, assume that $m = 2$ and $\delta < \frac{1}{2}$. Suppose that in the first period the buyer makes the offer $(1, p)$ with $p \in (2\delta c, c)$. In any stationary equilibrium, the high types must reject the offer, while the low types must accept it. This is because the buyer will never pay more than c for a unit (see below). Clearly, if the offer $(1, p)$ is accepted the buyer learns that the seller's type is low.

satisfy the "natural" definition of stationarity. The minimal departure from it is to let the acceptance decisions of the low types depend on whether or not the buyer is convinced that the seller's type is low. This leads us to the following definition of stationary perfect Bayesian equilibrium.

Definition 1 *A perfect Bayesian equilibrium* $(\sigma_B, \sigma_S, \mu)$ *is stationary if there exists a (measurable) function* $P(\cdot,\cdot,\cdot;\delta) : \{1, ..., m\}^2 \times [0,1] \to \mathbb{R}_+$ *satisfying the following conditions:*

i) Suppose that k *units are left for trade. For each* $q > \hat{q}$*, an offer* (k', p), $k' \leqslant k$*, is accepted by type* q *if and only if* $p \geqslant P(k, k', q; \delta)$*;*

ii) Consider a history h^t *such that* $\mu([0, \hat{q}] | h^t) < 1$ *and suppose that* k *units are left for trade. For each* $q \leqslant \hat{q}$*, an offer* (k', p), $k' \leqslant k$*, is accepted by type* q *if and only if* $p \geqslant P(k, k', q; \delta)$*.*

In the next section, we establish the existence and (essentially) the uniqueness of stationary equilibria and characterize the equilibrium behavior. For brevity, when there is no ambiguity, we suppress the dependence of the reservation price function $P(k, k', \cdot; \delta)$ on δ and write $P(k, k', \cdot)$.

3.1 Existence and Uniqueness of Stationary Equilibrium

We start our analysis with a result that holds in all stationary equilibria. The high types' equilibrium behavior is rather simple. In fact, they behave myopically and accept an offer if and only if the profits generated by that offer are weakly positive.

Lemma 2 *Let* $(\sigma_B, \sigma_S, \mu)$ *be a stationary equilibrium. Then,* $P(k, k', q) = k'c$ *for every* k*,* k' *and* $q > \hat{q}$*.*

The statement of the lemma is an immediate consequence of the fact that at any history, the equilibrium continuation payoff of the high types is zero. The proof of this result (in Appendix A) is by induction on the number of units left for trade. The first step of the argument (i.e., the high types' continuation payoff is zero when there is only one unit left for trade) was proved by DL. We now provide a sketch of the proof of the induction argument. Suppose that in every stationary equilibrium the high types get a payoff equal to zero when there are k or fewer units for trade. Let $\bar{u}_H$ denote the highest continuation payoff that the high types get across all histories at which there are $k+1$ units for trade.[10]

[10] For simplicity, here we assume that the maximum payoff does exist. The formal proof dispenses with this assumption.

By contradiction, assume that $\bar{u}_H > 0$. Our assumptions imply that, at some point, the buyer makes an offer to buy a certain number k' of the $k+1$ remaining units at the price $k'c + \bar{u}_H$. Clearly, this offer is accepted by the high types (if they reject it, they will receive at most $\delta\bar{u}_H$) as well as by the low types (if they reject the offer, their identity will be revealed and their continuation payoff will be zero). However, the same argument shows that both the high and the low types are willing to sell the k' units even at a price slightly lower than $k'c + \bar{u}_H$. We conclude that in equilibrium the buyer will never make the offer $(k', k'c + \bar{u}_H)$.

Our next result addresses the issue of the existence of stationary equilibria.

Proposition 1 *There exists a stationary equilibrium.*

The proof of Proposition 1 is in Appendix A where we construct the equilibrium strategy profile and the buyer's beliefs, and show that unilateral deviations are not profitable. Here we illustrate the equilibrium on-path behavior. In particular, the buyer employs at most two types of offers.[11] Suppose there are k units left for trade. The first type of offer is to purchase all the remaining k units at some price $p \leqslant kc$. The offer (k, p), in turn, can be accepted by all the types or only by (some of) the low types. Therefore, the rejection of (k, p) increases the buyer's confidence that the quality of the good is high. Because of this, we refer to an offer of the form (k, p) (when k is the number of remaining units) as a *cream-skimming offer*.

The second type of offer is to purchase some (but not all) of the remaining units, say $k' < k$, at the price $k'c$ at which the high types break even. Notice that the offer $(k', k'c)$ is accepted by all the types (recall that $P(k, k', q) = k'c$ for $q > \hat{q}$ and Lemma 1). Thus, we refer to an offer of the form $(k', k'c)$ as a *universal offer*.

Why does not the buyer make use of other types of offers? First, in equilibrium, any offer of the form (k, p) with $p \geqslant kc$ is accepted by all the types (if the low types reject such an offer, their payoff is equal to zero). Thus, the buyer will never pay more than c for any unit of the good.

It remains to show that it is not optimal for the buyer to purchase some of the remaining units only from the low types. We start with the following observation. In our equilibrium, the low types' continuation payoff depends only on the number of remaining units and the

[11] As we will see below, the number of types of offers employed by the buyer depends on the shape of the gains from trade. When the gains are decreasing, the buyer uses both types of offers (see Sections 6 and 7). In contrast, when the gains are constant or increasing, only the first type of offers is used in equilibrium (see Sections 5 and 8).

buyer's belief. Also, for any number k of remaining units, the low types' payoff is (weakly) increasing in the probability assigned to the high types by the buyer's beliefs.[12]

Suppose that in equilibrium, when there are k units the buyer makes the offer (k', p), with $k' < k$ and $p < k'c$, and this offer is accepted by (some of) the low types. By accepting this offer, these types reveal their identity and will obtain a continuation payoff equal to zero. This, together with the observation above on the low types' continuation payoff, immediately implies that, for any $\varepsilon > 0$, the offer $(k, p+\varepsilon)$ to purchase all the remaining units at the price $p+\varepsilon$ is (weakly) more likely to be accepted than the original offer (k', p). It is then easy to show that, because of discounting, the buyer is strictly better off by making the offer $(k, p+\varepsilon)$ (for some small ε) than by making the offer (k', p). Intuitively, the buyer is able to speed up trade and expedite the consumption of the good without conceding too much to the seller.

The reservation price functions $P(1,1,\cdot), P(2,2,\cdot), \ldots, P(m,m,\cdot)$ play an important role in our analysis. To simplify notation, for the remainder of this paper we will use $P(k,\cdot)$ to indicate $P(k,k,\cdot)$.

In our equilibrium, for every $k = 1, \ldots, m$, the function $P(k,\cdot)$ is increasing and left-continuous (also recall that $P(k,q) = kc$ for $q > \hat{q}$). Therefore, at any point in time along the equilibrium path, the set of types who have not sold all their units is of the form $[q, 1]$ for some $q \in [0, 1)$. With some abuse of notation, we use (k, q) to denote an arbitrary state of the economy. In state (k, q), there are still k units left for trade, and the buyer believes that the seller's type is uniformly distributed over the set $[q, 1]$.

Let $W(k, q)$ denote the buyer's *ex-ante* expected payoff when the state is (k, q).[13] Recall that, in equilibrium, only cream-skimming and universal offers are used. Thus, for any state (k, q), $W(k, q)$ satisfies:

$$W(k,q) = \max\left\{ \left(\max_{q' \in [q,1]} \int_q^{q'} \left[(\alpha_k + \ldots + \alpha_1) v(s) - P(k, q') \right] ds + \delta W(k, q') \right), \right.$$
$$\left. \left\{ \int_q^1 \left[(\alpha_k + \ldots + \alpha_{k'+1}) v(s) - (k - k') c \right] ds + \delta W(k', q) \right\}_{k'=1,\ldots,k-1} \right\}. \tag{1}$$

The right-hand side of the first line of equation (1) represents the buyer's expected payoff if he chooses the optimal cream-skimming offer. The second line of the equation describes the payoffs associated with the universal offers.

On the equilibrium path, the players' behavior is rather simple. Suppose that the state is (k, q). Then the buyer makes the offer (either cream-skimming or universal) that

[12] This property holds generically for all stationary equilibria (see Proposition 2).

[13] Of course, the buyer's payoff $W(k,\cdot)$ depends on the discount factor. As for the reservation prices, we suppress the dependence of $W(k,\cdot)$ on δ when there is no ambiguity.

maximizes $W(k,q)$. The seller of type q' accepts any universal offer. Furthermore, he accepts a cream-skimming offer provided that the price that the buyer is willing to pay is above his reservation price $P(k,q')$.[14]

At a first glance, it may seem as if we have limited the scope of our analysis by restricting attention to the equilibrium above. Nevertheless, as Proposition 2 shows, nothing is lost inasmuch as one considers stationary equilibria.[15] Two equilibria are outcome equivalent if, conditional on each quality of the good (low or high), they induce the same probability distribution over histories of accepted offers.

Proposition 2 *For generic parameters, all stationary equilibria are outcome equivalent.*

The proof of Proposition 2 (in Appendix B) is divided into two parts. The first part shows the uniqueness of the equilibrium outcome under the assumption that the reservation price functions $P(1,\cdot),\ldots,P(m,\cdot)$ are increasing. The proof is by induction on the number of units. DL show that the equilibrium price function $P(1,\cdot)$ is unique. We assume that $P(1,\cdot),\ldots,P(k-1,\cdot)$ are uniquely determined and demonstrate that the same result holds for $P(k,\cdot)$.

In a stationary equilibrium, the reservation prices of the seller's types are pinned down by their continuation payoffs. These, in turn, are determined by the buyer's strategy. We show that in a stationary equilibrium the buyer's offers are either cream-skimming or universal. As mentioned above, it is not optimal for the buyer to purchase some (but not all) of the remaining units only from the low types (the buyer could get all the units at the same price). When the state is (k,q) and q is sufficiently close to $\hat{q}$, the buyer's best response is unique and equal to (k,kc). In words, when the buyer is sufficiently optimistic about the quality of the good, he prefers to purchase all the the remaining units from all the types. This shows that the reservation price $P(k,q)$ is uniquely determined for any $q>\hat{q}-\varepsilon$, for some small ε. We then extend the reservation price $P(k,\cdot)$ to the left of $\hat{q}-\varepsilon$. For generic values of the parameters, the buyer's optimal offer (in the class of cream-skimming or universal offers) is unique. Thus, the extension of $P(k,\cdot)$ to the left of $\hat{q}-\varepsilon$ is generically unique.

The second part of the proof relaxes the assumption that the reservation price functions are increasing. Of course, given an equilibrium, it is also possible to construct another (outcome equivalent) equilibrium by permuting the low types. We show that, for generic

[14] Notice that along the equilibrium path the low types are indifferent between accepting and rejecting a cream-skimming offer (k,p) with $p<kc$.

[15] It is an open question whether there is a perfect Bayesian equilibrium that is not outcome equivalent to a stationary equilibrium.

values of the parameters, the equilibrium described above is unique up to a permutation of the low types.

Throughout the rest of the paper, we focus on the generic case of a unique equilibrium outcome.

4 Frequent Offers

As is often the case in bargaining games, it is rather intractable to describe the equilibrium behavior for an arbitrary value of the discount factor δ. Therefore, we investigate how trade evolves when the bargaining frictions vanish. We fix the total number of units m and characterize the equilibrium outcome in the limit, as the length of each period Δ_n shrinks to zero (and the discount factor $\delta_n = e^{-r\Delta_n}$ converges to one). We refer to this as the m-limitimg equilibrium outcome (we simply write limiting equilibrium outcome when the number of units m is unambiguous).

Consider a sequence $\{\Delta_n\}$ converging to zero and the corresponding sequence $\{\delta_n\} = \left\{e^{-r\Delta_n}\right\}$ converging to one. For each δ_n, we consider the stationary equilibrium described in Section 3.1, and construct the sequence $\{P_n(k,\cdot;\delta_n), W_n(k,\cdot;\delta_n)\}_{k=1,\ldots,m}$ of reservation prices and continuation payoffs. The limit of this sequence is well defined (see below) and we denote it by $\{P(k,\cdot), W(k,\cdot)\}_{k=1,\ldots,m}$.

For any pair (k,q), we let

$$P(k,q^-) = \lim_{q'\uparrow q} P(k,q')$$
$$P(k,q^+) = \lim_{q'\downarrow q} P(k,q')$$

denote the limit from the left and from the right, respectively, of the function $P(k,\cdot)$ at q.

As we will see below, the limiting equilibrium outcome alternates between phases of no delay and impasses. The distinction between these two phases concerns the discounted time that it takes for the buyer to purchase the remaining units from a set of low types with positive measure (i.e., to purchase the remaining units of the low-quality good with positive probability). We say that there is no delay between the states (k,q) and (k',q') (with $k \geqslant k'$ and $q < q'$) if the discounted time taken in equilibrium to transition from the state (k,q) to the state (k',q') converges to zero as bargaining frictions vanish. In contrast, we say that bargaining reaches an impasse at the state (k,q) if for every $q' > q$, the discounted time taken in equilibrium to transition from the state (k,q) to the state (k,q') remains bounded away from zero when the discount factor converges to one. Finally, suppose that there is an impasse at (k,q). We measure the size of the impasse in terms of the real time during which the equilibrium stays around the state (k,q). Formally, these concepts are defined in Definition 2 and Definition 3.

Fix the discount factor δ and consider the corresponding equilibrium of the game. Let h_δ denote the on-path history in which all the offers of the form (k, p) with $p < kc$ are rejected. Of course, h_δ is a finite history which ends when the buyer makes an offer for all the remaining units at the price at which the high types break even. Let T_δ denote the length of the history h_δ. For each $t < T_\delta$, we let h_δ^t denote the first $t+1$ elements of h_δ (i.e., we truncate h_δ at period t).

Definition 2 *Fix a sequence of discount factors* $\{\delta_n\}_{n=1}^{\infty}$ *converging to one. Consider two states,* (k^1, q^1) *and* (k^2, q^2) *, with* $k^1 \geqslant k^2$ *and* $q^1 < q^2$*. We say that the real time to reach* (k^2, q^2) *from* (k^1, q^1) *converges to zero if the following holds. For every* $\varepsilon > 0$*, there exists* $\bar{n}$ *such that for every* $n > \bar{n}$ *we can find* $t_1 < t_2 < T_{\delta_n}$ *satisfying:*

(i) for $j = 1, 2$*, the state associated to the history* $h_{\delta_n}^{t_j}$ *is* $(k^j, \tilde{q}^j)$ *for some* $\tilde{q}^j \in (q^j - \varepsilon, q^j + \varepsilon)$ *;*

(ii) $\delta_n^{t_2 - t_1} > 1 - \varepsilon$.

Definition 3 *Fix a sequence of discount factors* $\{\delta_n\}_{n=1}^{\infty}$ *converging to one. We say that there is an impasse at the state* (k, q) *if there exists* $\gamma \in (0, 1)$ *for which the following holds. For every* $\varepsilon > 0$*, there exists* $\bar{n}$ *such that for every* $n > \bar{n}$ *we can find* $t_1 < t_2 < T_{\delta_n}$ *satisfying:*

(i) for $j = 1, 2$*, the state associated to the history* $h_{\delta_n}^{t_j}$ *is* (k, q^j) *for some* $q^j \in (q - \varepsilon, q + \varepsilon)$ *;*

(ii) $\delta_n^{t_2 - t_1} < \gamma$.

We say that the impasse is of real time $1 - \xi$ *if* ξ *is the infimum over all* γ *leading to an impasse at* (k, q) *.*

In the following sections, we investigate how divisibility, together with the shape of the gains from trade, affects the pattern of trade when the bargaining frictions vanish. To make the analysis interesting, we assume that $c > \hat{q}\underline{v} + (1 - \hat{q})\bar{v}$ (recall that we normalize the smallest α_k to one). In fact, it is easy to show that if $c \leqslant \hat{q}\underline{v} + (1 - \hat{q})\bar{v}$, then the equilibrium outcome of our bargaining game converges, as the discount factor δ goes to one, to the first best and the Coase conjecture obtains.[16,17]

[16] Recall that there are positive gains from trade for every unit and every type of the seller. Therefore, the first best is achieved when all the m units are traded without delay. According to the Coase conjecture, as the length of the period between consecutive offers vanishes, the real time that it takes to trade the whole good of either quality converges to zero.

[17] It is also straightforward to check that in a model with private values and positive gains from trade ($c \leqslant \underline{v} = \bar{v}$ in our context), the Coase conjecture continues to hold when the object is divisible.

We also point out that all our results for fixed values of δ (the results in Section 3, Proposition 3, and Proposition 6 below) hold when values are private ($c \leqslant \underline{v} = \bar{v}$).

5 Constant Gains from Trade

In this section, we briefly examine the benchmark case of constant gains from trade: $\alpha_m = \ldots = \alpha_1 = 1$. We show that when all the units are equally valuable, divisibility does not play any role.

We begin with a result which holds for any discount factor δ. In addition to being interesting on its own, this result will provide an immediate characterization of the limiting equilibrium outcome.

In equilibrium, the buyer only makes offers for the entire good and the outcome is exactly as in DL. The linearity of the buyer's utility (in the number of units) is crucial for this result. To provide some intuition, we divide the argument into two steps. First, we assume that the buyer is forced to make cream-skimming offers and investigate how the equilibrium outcome depends on the number of units available for trade. In particular, we guess and verify that the seller's reservation price $P(k,\cdot)$ to sell the last k units is linear in k. In equilibrium, the reservation prices of the different types are pinned down by the speed with which the buyer screens the seller. When both the buyer's utility and the reservation price are linear in k, the optimal way to screen the seller is independent of the number of units left for trade (the buyer faces exactly the same problem for every k). This verifies the linearity of $P(k,\cdot)$ in k.

This, in turn, implies that when the buyer is required to make cream-skimming offers, his continuation payoff is linear in the number of units available. It remains to check whether the buyer can benefit from making a universal offer for $k' = 1, \ldots, k-1$ of the last k units. The answer is no; with linear utility and linear continuation payoff, the optimal fraction of the good to purchase with universal offers can only be zero or one.

Below we state and provide a formal proof of our result.

Proposition 3 *Suppose that the gains from trade are constant: $\alpha_m = \ldots = \alpha_1 = 1$. Then all equilibrium offers are cream-skimming.*

Proof. For every $q < 1$ let $t(q)$ be such that

$$W(1,q) = \int_q^{t(q)} [v(s) - P(1,t(q))]\,ds + \delta W(1,t(q)) = \max_{q' \geqslant q} \int_q^{q'} [v(s) - P(1,q')]\,ds + \delta W(1,q').$$

Assume that for every $k = 2, \ldots, m$, and for every q

$$\begin{aligned} P(k,q) &= kP(1,q) \\ W(k,q) = kW(1,q) &= \textstyle\int_q^{t(q)} [kv(s) - P(k,t(q))]\,ds + \delta W(k,t(q)). \end{aligned} \tag{2}$$

It is immediate to see that for every $k > 1$, and for every q

$$W(k, q) = \max_{q' \geqslant q} \int_q^{q'} [kv(s) - P(k, q')] \, ds + \delta W(k, q').$$

We claim that it is impossible to find $k' < k$ and q such that

$$W(k, q) \leqslant (k - k') \int_q^1 (v(s) - c) \, ds + \delta W(k', q). \tag{3}$$

This establishes the suboptimality of universal offers and proves our result.

By contradiction, let (k', k, q) be a triple that satisfies inequality (3), with k being the smallest number of (remaining) units for which the inequality can be satisfied. Then, it follows from the definition of $W(\cdot, \cdot)$ in equation (2) that (recall that $\delta < 1$ and $W(1, q) > 0$ for every q - see DL)

$$kW(1, q) \leqslant (k - k') \int_q^1 (v(s) - c) \, ds + \delta k' W(1, q) < (k - k') \int_q^1 (v(s) - c) \, ds + k' W(1, q),$$

and, thus,

$$W(1, q) < \int_q^1 (v(s) - c) \, ds,$$

which leads to a contradiction. ■

We now illustrate the equilibrium outcome of the game when the length of the period shrinks to zero. To simplify the exposition, here and throughout the rest of the paper we adopt the following convention. Suppose that the buyer believes that the seller's type is uniformly distributed over the set $[\tilde{q}, 1]$ for some $\tilde{q} \in [0, 1]$ (recall that along the equilibrium path, the beliefs take this form). Then we say that the buyer's belief is $\tilde{q}$.

We let $\bar{q}_1 \in (0, \hat{q})$ denote the belief at which the buyer breaks even if he purchases one unit of the good at the price c. Thus, $\bar{q}_1$ is implicitly defined by

$$\int_{\bar{q}_1}^1 (v(s) - c) \, ds = 0. \tag{4}$$

Recall that when the gains from trade are constant ($\alpha_m = \ldots = \alpha_1 = 1$), all the offers are cream-skimming. Therefore, the characterization of the limiting equilibrium outcome follows directly from DL. Suppose that the number of remaining units is $k = 1, \ldots, m$ (of course, on the equilibrium path there are always m units left for trade). When the belief is larger than $\bar{q}_1$, the buyer's continuation payoff is strictly positive since it is bounded below by the payoff of the offer (k, kc). Therefore, the buyer has an incentive to speed up trade.

Using standard Coasean arguments, DL show that in the limit (as δ goes to one), all the types above $\bar{q}_1$ sell the k remaining units *without delay* at the price kc.

Suppose now that the buyer's belief is below $\bar{q}_1$. First, it is immediate to see that there must be delay before the buyer reaches an agreement with the high types. Without delay, the low types would have an incentive to pool with the high types and sell their units at the price kc. But this would yield a loss to the buyer.

One of the main breakthroughs of DL was to show that delays are resolved by impasses and to figure out the exact size of the impasses. When δ is close to one, the reservation price of the types below $\bar{q}_1$ must be smaller than $k\underline{v}$, otherwise the buyer's continuation payoff would be negative (in the limit, the continuation payoff from trading with all the types larger than $\bar{q}_1$ approaches zero). Thus, the seller's reservation price crosses the buyer's low valuation $k\underline{v}$ when the belief is close to $\bar{q}_1$. DL use this fact to show that (in the limit), the time it takes to go from state $(k, \bar{q}_1 - \varepsilon)$ to state $(k, \bar{q}_1)$ coincides with the time that it takes to go from $(k, \bar{q}_1)$ to $(k, \bar{q}_1 + \varepsilon)$ (i.e., there is "double delay" at the state $(k, \bar{q}_1)$). Putting these findings together, it is easy to see that the impasse at the state $(k, \bar{q}_1)$ is of real time $1 - \left(\frac{\underline{v}}{c}\right)^2$.

We conclude that for every $k = 1, \ldots, m$, the limit reservation price and continuation payoff are:

$$P(k,q) := \begin{cases} \frac{k\underline{v}^2}{c} & \text{if } q < \bar{q}_1 \\ kc & \text{if } q > \bar{q}_1, \end{cases} \tag{5}$$

and

$$W(k,q) := \begin{cases} \int_q^{\bar{q}_1} \left(kv(s) - P\left(k, \bar{q}_1^-\right)\right) ds & \text{if } q \leqslant \bar{q}_1 \\ \int_q^1 k\left(v(s) - c\right) ds & \text{if } q > \bar{q}_1. \end{cases} \tag{6}$$

where $P\left(k, \bar{q}_1^-\right) = \lim_{q \uparrow \bar{q}_1} P(k,q)$.

To sum up, the limiting equilibrium outcome of the model with constant gains from trade is as follows. The buyer purchases all the m units from the types smaller than $\bar{q}_1$ without delay. Then an impasse occurs. During the impasse, the buyer keeps increasing the price but the offers are accepted with probability close to zero. Once the impasse is resolved, the buyer reaches an immediate agreement with all the types above $\bar{q}_1$.

The analysis in this section shows that when the gains from trade are constant, divisibility is of no consequence as timing on the market is the only signaling device. As we will see below, this is in contrast with the case of decreasing gains from trade, in which divisibility plays a crucial role, and the size of the good still to be traded emerges as a new signaling component.

6 Decreasing Gains from Trade

We now analyze the case of decreasing gains from trade and assume that $\alpha_m > \ldots > \alpha_1 = 1$. It turns out that this case is significantly different from the case of constant gains. In fact, indivisibility is not an innocuous assumption when the gains from trade are decreasing, and the pattern of trade is greatly affected by the possibility of purchasing fractions of the good.

In this section, we derive some preliminary properties of the limiting equilibrium outcome. Notice first that in the case in which one unit is left for trade, the equilibrium outcome is exactly as in DL. In the limit, the buyer trades without delay with all the types below $\bar{q}_1$ (defined in equation (4)). Then there is an impasse at the state $(1, \bar{q}_1)$. Once the impasse is resolved there is immediate trading with all the remaining types. The sequence of reservation price $\{P_n(1,\cdot)\}$ converges to $P(1,\cdot)$ defined in equation (5), while the sequence of continuation payoff $\{W_n(1,\cdot)\}$ converges to $W(1,\cdot)$ defined in equation (6).

We establish similar convergence results for any number of units.

Lemma 3 *For every $k = 2, \ldots, m$, there exist $W(k,\cdot) : [0,1] \to \mathbb{R}_+$ and $P(k,\cdot) : [0,1] \to \mathbb{R}_+$ such that $\{W_n(k,\cdot)\}$ has a subsequence converging uniformly to $W(k,\cdot)$, and $\{P_n(k,\cdot)\}$ has a subsequence converging pointwise to $P(k,\cdot)$.*

Proof. Notice that $W_n(k,\cdot)$ are equicontinuous functions with Lipschitz coefficient $(\alpha_k + \ldots + \alpha_1)\bar{v}$. They are also uniformly bounded by $(\alpha_k + \ldots + \alpha_1)\bar{v}$. Therefore, the conclusion follows from Arzelà-Ascoli Theorem. The functions $P_n(k,\cdot)$ are monotonic (and hence have bounded variation) and are clearly uniformly bounded. The conclusion follows from Helly´s First Theorem (Theorem 6.1.18 in Kannan and Krueger, 1996). ■

As will become evident below, a result stronger than Lemma 3 actually holds. The algorithm we construct in Appendix D shows that (for generic values of the parameters) all the convergent sequences $\{W_n(k,\cdot), P_n(k,\cdot)\}$, $k = 2, \ldots, m$, have the same limit which we denote by $(W(k,\cdot), P(k,\cdot))$.

Our next result confirms the existence of an impasse at $(1, \bar{q}_1)$, analogous to the case in which the buyer and the seller trade a single unit, and further establishes $(1, \bar{q}_1)$ as the final state at which an impasse occurs.

Proposition 4 *In the limit, the last impasse is at $(1, \bar{q}_1)$ and is of real time $1 - \left(\frac{\alpha_1 \underline{v}}{c}\right)^2$.*

The proof of Proposition 4 is in Appendix D. Recall that the gains per unit are decreasing and that $\bar{q}_1$ is the type at which the buyer breaks even with the last unit if he pays the price

c. Thus, for any number $k \geqslant 2$ of units left for trade and for any $q \geqslant \bar{q}_1$, the buyer's expected payoff is bounded away from zero when the state is (k, q). The usual Coasean argument implies that, in the limit, there cannot be any delay once the state (k, q) is reached. This implies that if there is an impasse at (k, q') with $k \geqslant 2$, then it must necessarily be the case that $q' < \bar{q}_1$. Now, suppose that the last impasse is at (k, q') for some $k \geqslant 2$. Notice that once the impasse is resolved, the buyer will end up purchasing the remaining k units at the price kc. However, in the limit, the buyer can increase his payoff at (k, q') by purchasing $k-1$ units from all the types at the price $(k-1)c$ and then the last unit at the price $P\left(1, \bar{q}_1^-\right)$ (respectively c) from the types smaller (respectively larger) than $\bar{q}_1$. Clearly, the second course of action is more profitable than the first one since the buyer pays $P\left(1, \bar{q}_1^-\right)$ instead of c to purchase the last unit from the types in $(q', \bar{q}_1)$. This implies that for δ_n sufficiently close to one, the buyer would have a profitable deviation. A similar contradiction can be easily derived if one assumes that, in the limit, there are no impasses.

Once we have established the last impasse, we can move on to characterize the rest of the m-limiting equilibrium outcome. To do so, we construct an algorithm which pins down the entire sequence of impasses. We provide a formal description of the algorithm in Appendix D. In what follows, we illustrate how to identify the penultimate impasse and describe the main properties of the m-limiting equilibrium outcome (for arbitrary values of m). This is less demanding in terms of notation and allows us to focus on the intuition behind our results.

For every $k \leqslant m$, we denote by $\bar{q}_k$ the type at which the buyer breaks even if he trades the $(m-k+1)$-th unit at the price c. Formally, $\bar{q}_k$ is implicitly defined by

$$\int_{\bar{q}_k}^{1} \left(\alpha_k v(s) - c\right) ds = 0, \tag{7}$$

provided that the solution to the above equation exists and is positive. The cases in which the solution is negative or does not exist are irrelevant to our construction, as an impasse cannot occur given either circumstance when there are k units left for trade. This is because the buyer can guarantee a strictly positive payoff by making a universal offer for one unit. For concreteness, in these cases we set $\bar{q}_k$ equal to $-k$. Notice that $\bar{q}_m < \ldots < \bar{q}_2 < \bar{q}_1$ since $\alpha_m > \ldots > \alpha_1$.

For any $k \geqslant 2$ and $q < \bar{q}_1$, consider the strategy of acquiring $(k-1)$ units at the price $(k-1)c$ from all the types $[q, 1]$ and then the last unit from the types in $(q, \bar{q}_1)$ at the price $P\left(1, \bar{q}_1^-\right)$. Let $\hat{q}_k$ denote the type at which the buyer breaks even. To simplify the exposition, let us assume that $\hat{q}_k$ is well defined and positive.[18] Formally, $\hat{q}_k$ is the solution

[18] Of course, in the formal statement of the result and in its proof, we consider the general case.

to the following equation:

$$\int_{\hat{q}_k}^{1} \left(\left(\alpha_k + \ldots + \alpha_2 \right) v\left(s\right) - \left(k-1\right) c \right) ds + \int_{\hat{q}_k}^{\bar{q}_1} \left(\alpha_1 \underline{v} - P\left(1, \bar{q}_1^-\right) \right) ds = 0. \tag{8}$$

Notice that $\hat{q}_k > 0$ implies that

$$\left(\alpha_k + \ldots + \alpha_1\right) \underline{v} < \left(k-1\right) c + P\left(1, \bar{q}_1^-\right).$$

6.1 Two Units

It is worthwhile to devote some attention to the limiting equilibrium outcome when the good is divided into two units. The case $m = 2$ provides a simple explanation of why the high-quality good is sold gradually over time and why divisibility is disadvantageous to the buyer.

We know from the previous section that the bargaining process ends immediately when there are at least two units left for trade and the buyer's belief is $q \geqslant \bar{q}_1$. In fact, the buyer can guarantee a strictly positive payoff by paying c for each of the remaining units. This immediately implies that, in the limit, the reservation price for two units of all the types larger than $\bar{q}_1$ is equal to $2c$.

Suppose now that the state is $(2, q)$ with $q \in (\hat{q}_2, \bar{q}_1)$. It follows from Proposition 4 that the limiting equilibrium outcome must reach the state $(1, \bar{q}_1)$. Does this happen immediately or with some delay (in the limit)? The fact that $q \in (\hat{q}_2, \bar{q}_1)$ implies that the buyer can guarantee a strictly positive payoff by purchasing the first unit from all the types above q at the price c, and the second unit from the types in $(q, \bar{q}_1)$ at the price $P\left(1, \bar{q}_1^-\right)$. The usual Coasean forces allow us to conclude that when the bargaining frictions vanish the transition from the state $(2, q)$ to the state $(1, \bar{q}_1)$ is instantaneous. This, in turn, implies that for $q \in (\hat{q}_2, \bar{q}_1)$, the reservation price $P(2, q)$ is equal to

$$P\left(2, \hat{q}_2^+\right) = c + P\left(1, \bar{q}_1^-\right) > \left(\alpha_2 + \alpha_1\right) \underline{v}.$$

We conclude that as q approaches $\hat{q}_2$ from above, the buyer's expected payoff at $(2, q)$ converges to zero.

Consider now the state $(2, q)$ with $q < \hat{q}_2$. If the buyer purchases one unit from all the types at the price c and the second unit from the types in $(q, \bar{q}_1)$ at the price $P\left(1, \bar{q}_1^-\right)$, then his payoff will be negative. Also, for δ_n sufficiently close to one, the reservation price $P_n(2, q)$ must be bounded away from $P\left(2, \hat{q}_2^+\right)$, otherwise, the buyer's payoff would be negative. This implies that if we start from $(2, q)$ with $q < \hat{q}_2$, then we reach an impasse at $(2, \hat{q}_2)$. Using an argument similar to DL, it is easy to show that the "double delay" result holds at every impasse. Thus, the impasse at $(2, \hat{q}_2)$ is of real time $1 - \left(\frac{(\alpha_2+\alpha_1)\underline{v}}{c+P\left(1,\bar{q}_1^-\right)}\right)^2$, and that for $q < \hat{q}_2$, the reservation price $P(2, q)$ is equal to

$$P\left(2, \hat{q}_2^-\right) = \frac{\left((\alpha_2 + \alpha_1)\,\underline{v}\right)^2}{c + P\left(1, \bar{q}_1^-\right)}.$$

To sum up, when $m = 2$ the limiting equilibrium outcome is as follows. The buyer purchases both units from the types smaller than $\hat{q}_2$ at the price $\frac{((\alpha_2+\alpha_1)\underline{v})^2}{c+P\left(1,\bar{q}_1^-\right)}$, and the bargaining process reaches the state $(2, \hat{q}_2)$ without delay. At that point a first impasse occurs (the impasse is of real time $1 - \left(\frac{((\alpha_2+\alpha_1)\underline{v})}{c+P\left(1,\bar{q}_1^-\right)}\right)^2$) . After the impasse is resolved, the buyer purchases (without delay) the first unit from all the types above $\hat{q}_2$ and the second unit from the types in the interval $(\hat{q}_2, \bar{q}_1)$. The second and final impasse occurs at the state $(1, \bar{q}_1)$. The bargaining process ends as soon as the impasse is resolved, as the buyer proposes to pay c to get the second unit of the good.

Notice that the buyer's continuation payoff is equal to zero when the bargaining process reaches an impasse. Thus, the buyer's limiting equilibrium payoff is equal to

$$W(2,0) = \int_0^{\hat{q}_2} \left((\alpha_2 + \alpha_1)\, v(s) - P(2,s)\right) ds = \hat{q}_2 (\alpha_2 + \alpha_1)\,\underline{v}\left(1 - \frac{((\alpha_2 + \alpha_1)\,\underline{v})}{c + P\left(1, \bar{q}_1^-\right)}\right),$$

where $\hat{q}_2 < \bar{q}_2$ (this follows immediately from the definitions of $\hat{q}_2$ and $\bar{q}_2$).

We now compare this case with the case in which the good is indivisible. In the context of our model, indivisibility can be simply described as a restriction on the type of admissible offers. In particular, we assume that the buyer can only make offers of the form $(2, p)$ for $p \geqslant 0$. Let $\bar{q}_{DL}$ denote the type at which the buyer breaks even if he makes the offer $(2, 2c)$. Formally, $\bar{q}_{DL}$ is the solution to the following equation

$$\int_{\bar{q}_{DL}}^{1} \left((\alpha_2 + \alpha_1)\, v(s) - 2c\right) ds = 0.$$

Notice that $\bar{q}_{DL} \in (\bar{q}_2, \bar{q}_1)$ since $\alpha_2 > \alpha_1$. It follows from DL that in the limit, the seller's reservation price (which we denote by P_{DL}) is equal to

$$P_{DL}(q) = \begin{cases} \frac{((\alpha_2+\alpha_1)\underline{v})^2}{2c} & \text{if } q < \bar{q}_{DL} \\ 2c & \text{if } q > \bar{q}_{DL}. \end{cases}$$

In the limit, the buyer purchases (without delay) the two units from the types smaller than $\bar{q}_{DL}$ at the price $\frac{((\alpha_2+\alpha_1)\underline{v})^2}{2c}$. Then, there is an impasse of real time $1 - \left(\frac{((\alpha_2+\alpha_1)\underline{v})}{2c}\right)^2$.

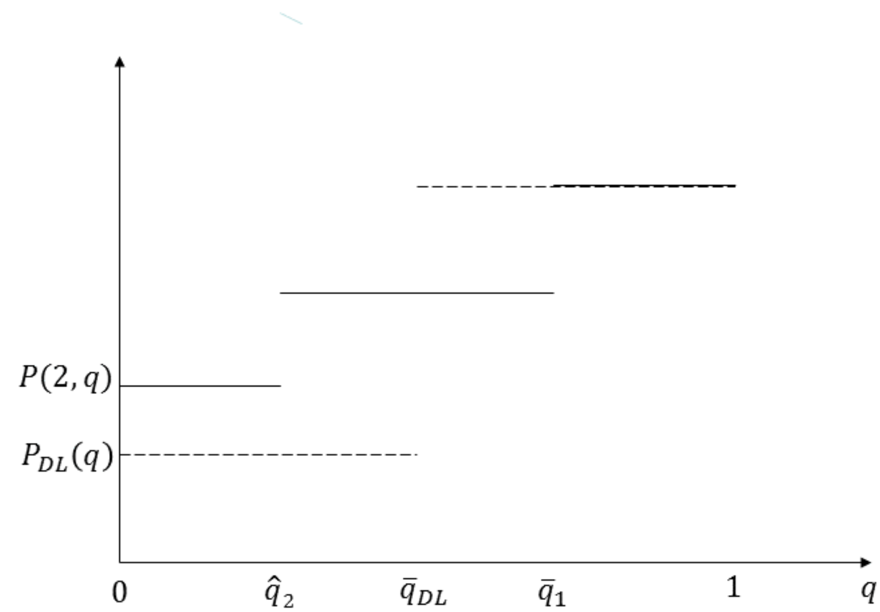

Figure 1: Reservation price for two units (benchmark model: solid line; DL: dashed line)

The bargaining process ends as soon as the impasse is resolved (the buyer purchases the two units from all the remaining types at the price $2c$). Thus, the buyer's limiting equilibrium payoff is equal to

$$W_{DL}(0) = \int_0^{\bar{q}_{DL}} ((\alpha_2 + \alpha_1) v(s) - P_{DL}(s)) \, ds = \bar{q}_{DL} (\alpha_2 + \alpha_1) \underline{v} \left(1 - \frac{((\alpha_2 + \alpha_1) \underline{v})}{2c}\right).$$

It is immediate to see that the buyer is better off when the good is indivisible. In fact, $W_{DL}(0) > W(2,0)$ since $\bar{q}_{DL} > \bar{q}_2 > \hat{q}_2$ and $\frac{((\alpha_2+\alpha_1)\underline{v})}{2c} < \frac{((\alpha_2+\alpha_1)\underline{v})}{c+P\left(1,\bar{q}_1^-\right)}$.

When the good is indivisible, the buyer pays a large price (equal to $2c$) when the belief is above $\bar{q}_{DL}$. This, however, has a positive effect on the buyer's initial payoff. A large price after the impasse implies a long delay and a severely reduced price before the impasse (see Figure 1).

Suppose now that the good is divisible and the state is $(2,q)$ with $q \in (\bar{q}_{DL}, \bar{q}_1)$. The buyer's payoff is still positive if he pays $2c$ for the two units. However, a more profitable strategy is now available. Specifically, the buyer is strictly better off by making a universal offer (for the first unit) and then proceeding with cream-skimming offers for the remaining unit. It follows that in the interval $(\bar{q}_{DL}, \bar{q}_1)$ the seller's reservation price (for two units) is lower in the model with divisibility than in the model with indivisibility (again, see Figure

1). This, in turn, has two effects. First, in the initial phase of the bargaining process (i.e., before the first impasse is reached), the buyer trades with a smaller set of types when the good is divisible ($\hat{q}_2 < \bar{q}_{DL}$).[19] Second, since there is double delay at each impasse, the buyer pays a larger initial price in the model with divisibility ($P(2,0) > P_{DL}(0)$). Clearly, both effects have a negative impact on the buyer's payoff.

6.2 Three or More Units

We now turn to the case in which the good is divided into three or more units. Our first goal is to illustrate when the penultimate impasse of the bargaining process will occur.

We start by investigating whether the penultimate impasse is when there are two or more than two units left for trade. It turns out that we need to distinguish between two cases, depending on whether $\bar{q}_3$ is greater or smaller than $\hat{q}_2$ (recall that $\bar{q}_3$ and $\hat{q}_2$ are defined in equations (7) and (8), respectively). Here and in what follows, we restrict attention to a generic set of parameters and assume that $\hat{q}_2 \neq \bar{q}_3$. More generally, we assume that $\bar{q}_{k+1} \neq \hat{q}_k$ for $k \in \{2, \ldots, m-1\}$ (this case is generic).

Case $\hat{q}_2 > \bar{q}_3$: there is an impasse at $(2, \hat{q}_2)$.

First, assume that $\hat{q}_2 > \bar{q}_3$. We claim that, in the limit, the penultimate impasse occurs at $(2, \hat{q}_2)$. Suppose, by contradiction, that the penultimate impasse is at (k, q) for some $k > 2$. The fact that $\hat{q}_2 > \bar{q}_3 > \bar{q}_4 > \ldots > \bar{q}_m$ implies that for $q \in (\hat{q}_2, \bar{q}_1)$ the buyer's payoff from purchasing $(k-1)$ units from all the types at the price $(k-1)c$ and the last unit from the types in $(q, \bar{q}_1)$ at the price $P\left(1, \bar{q}_1^-\right)$ is positive and bounded away from zero. Therefore, if there is an impasse at (k, q), it must be the case that $q < \hat{q}_2$.[20]

Consider now the state (k, q') for some $q' \in (q, \hat{q}_2)$. In the limit, the buyer gets the same payoff as if he buys $(k-1)$ units from the types above q' at the price $(k-1)c$, and the last unit at the price $P\left(1, \bar{q}_1^-\right)$ (respectively c) from the types smaller (respectively larger) than $\bar{q}_1$. This course of action is represented by the dashed line in Figure 2.

A different course of action (represented by the solid line in Figure 2) is to purchase $(k-2)$ units at the price $(k-2)c$ and the last two units from the types in $(q', \hat{q}_2)$ at the price $P\left(2, \hat{q}_2^-\right)$, reaching an impasse. After the impasse at $(2, \hat{q}_2)$ is resolved, the buyer purchases the penultimate unit at the price c and the last unit at the price $P\left(1, \bar{q}_1^-\right)$ (respectively c) from the types smaller (respectively larger) than $\bar{q}_1$.[21]

[19] Notice that the strategy of making a universal offer yields a strictly positive payoff when the buyer's belief lies in the interval $(\hat{q}_2, \bar{q}_{DL})$.

[20] Clearly, the impasse cannot occur at (k, q) with $q \geqslant \bar{q}_1$. Recall that for any $q \geqslant \bar{q}_1$, the buyer's payoff from buying $k' > 1$ units at the price $k'c$ from the types $[q, 1]$ is strictly positive.

[21] Notice that, in the limit, the offers made after the resolution of the impasse yield a payoff equal to zero (in fact, we have $W(2, \hat{q}_2) = 0$). Nevertheless, we specify these offers to facilitate the comparison with the

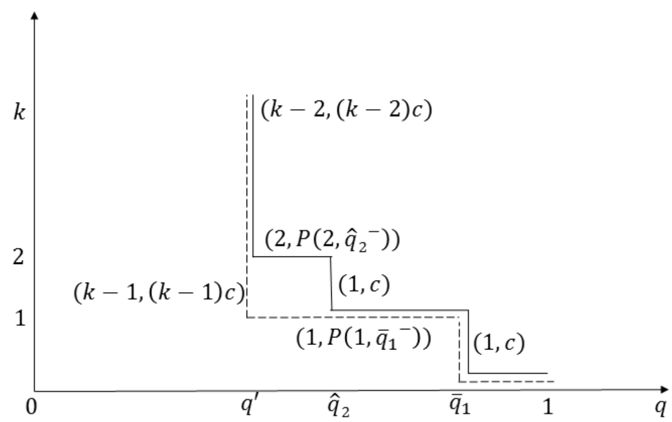

Figure 2: Penultimate Impasse with Two Units Remaining

As one can see in Figure 2, the difference between these two courses of action consists in the total price that the buyer pays to the types in $(q', \hat{q}_2)$ for the last two units of the good. The total price is $c + P\left(1, \bar{q}_1^-\right)$ under the first course, and $P\left(2, \hat{q}_2^-\right)$ under the second one. Recall that $P\left(2, \hat{q}_2^-\right)$ is bounded away from $P\left(2, \hat{q}_2^+\right) = c + P\left(1, \bar{q}_1^-\right)$. Thus, in the limit, the second course of action yields a larger payoff.

We conclude that for δ_n sufficiently close to one, the buyer would have a profitable deviation. In a similar way, we can rule out the case in which there is only one impasse at $(1, \bar{q}_1)$.

Case $\hat{q}_2 < \bar{q}_3$: there are no impasses when two units are left for trade.

Consider now the case $\hat{q}_2 < \bar{q}_3$, and notice that this implies $\hat{q}_2 < \hat{q}_3$.[22] The buyer's continuation payoff $W(3, q)$ is strictly positive if $q > \hat{q}_3$. Thus, in the limit, the transition from the state $(3, q)$, with $q \in (\hat{q}_3, \bar{q}_1)$ is instantaneous. This, in turn, implies that for $q \in (\hat{q}_3, \bar{q}_1)$ the limit reservation price for three units is equal to

$$P(3, q) = 2c + P\left(1, \bar{q}_1^-\right).$$

Consider now the state $(3, q)$ with $q \in (\hat{q}_2, \hat{q}_3)$. If the buyer makes a universal offer for one or two units his (limit) payoff is negative. In fact, in either case, the buyer ends

first course of action.

[22] The payoff $\int_q^1 ((\alpha_3 + \alpha_2) v(s) - 2c)\, ds + \int_q^{\bar{q}_1} \left(\alpha_1 \underline{v} - P\left(1, \bar{q}_1^-\right)\right) ds$ can be decomposed as the sum of the following two components: $\int_q^1 (\alpha_3 v(s) - c)\, ds$ and $\int_q^1 (\alpha_2 v(s) - c)\, ds + \int_q^{\bar{q}_1} \left(\alpha_1 \underline{v} - P\left(1, \bar{q}_1^-\right)\right) ds$. Both components are positive if $q \geqslant \bar{q}_3 > \hat{q}_2$ and negative if $q \leqslant \hat{q}_2 < \bar{q}_3$. It follows that $\hat{q}_3$ must belong to the interval $(\hat{q}_2, \bar{q}_3)$.

up paying $2c + P\left(1, \bar{q}_1^-\right)$ to the types below $\bar{q}_1$, and $3c$ to the types above $\bar{q}_1$ (this yields a negative payoff since $q < \hat{q}_3$). We conclude that the equilibrium offers must be cream-skimming. Also, for $q \in (\hat{q}_2, \hat{q}_3)$, the reservation price $P(3, q)$ must be bounded away from $2c + P\left(1, \bar{q}_1^-\right)$ otherwise the buyer's payoff would be negative.

Using an argument similar to the one developed in Section 6.1, we show that if the state $(3, q)$ is reached (either on or off path) and $q \in (\hat{q}_2, \hat{q}_3)$, then there is an impasse at $(3, \hat{q}_3)$ of real time $1 - \left(\frac{(\alpha_3+\alpha_2+\alpha_1)\underline{v}}{2c+P\left(1,\bar{q}_1^-\right)}\right)^2$, and for $q' < \hat{q}_3$ the reservation price $P(3, q')$ is equal to

$$P\left(3, \hat{q}_3^-\right) = \frac{\left((\alpha_3 + \alpha_2 + \alpha_1)\,\underline{v}\right)^2}{2c + P\left(1, \bar{q}_1^-\right)}.$$

It is not difficult to check that

$$P\left(3, \hat{q}_3^-\right) < c + P\left(2, \hat{q}_2^-\right) < c + P\left(2, \hat{q}_2^+\right) = 2c + P\left(1, \bar{q}_1^-\right). \tag{9}$$

The first inequality shows that there is a significant drop in price from $P\left(3, \hat{q}_3^+\right)$ to $P\left(3, \hat{q}_3^-\right)$ at the state $(3, \hat{q}_3)$. In particular, this drop is larger than the drop from $P\left(2, \hat{q}_2^+\right)$ to $P\left(2, \hat{q}_2^-\right)$ at the state $(2, \hat{q}_2)$.[23] The reason involves the magnitude by which the reservation price exceeds the buyer's valuation. The difference between $P\left(3, \hat{q}_3^+\right)$ and the buyer's valuation for three units is larger than the difference between $P\left(2, \hat{q}_2^+\right)$ and the valuation for two units. This, together with the fact than an impasse doubles the delay necessary to bring the reservation price to the buyer's valuation, accounts for the large discontinuity in price at $(3, \hat{q}_3)$.

Finally, we use the fact that cream-skimming offers for three units are relatively attractive to show that, on the limiting equilibrium path, there cannot be an impasse at $(2, \hat{q}_2)$. If this were the case, we could find a state (k, q) with $k > 2$ and $q < \hat{q}_2$ such that the buyer purchases $(k - 2)$ units from all the remaining types and then follows the equilibrium behavior with two units.[24] This course of action is represented by the dashed line in Figure 3.

Consider now a different course of action (represented by the solid line in Figure 3). First, the buyer purchases $(k - 3)$ units at the price $(k - 3)\,c$ (from all the remaining types). Then he purchases the last three units from the types in $(q, \hat{q}_3)$ at the price $P\left(3, \hat{q}_3^-\right)$, and the third to last unit at the price c. After this, he follows the equilibrium behavior with two units.

[23] In fact, notice that $P\left(3, \hat{q}_3^+\right) = c + P\left(2, \hat{q}_2^+\right)$.

[24] This means that the buyer purchases the last two units at the price $P\left(2, \hat{q}_2^-\right)$ from the types in $(q, \hat{q}_2)$, and the penultimate unit at the price c from all the types above $\hat{q}_2$. Finally, he buys the last unit at the price $P\left(1, \bar{q}_1^-\right)$ (c) from the types smaller (larger) than $\bar{q}_1$.

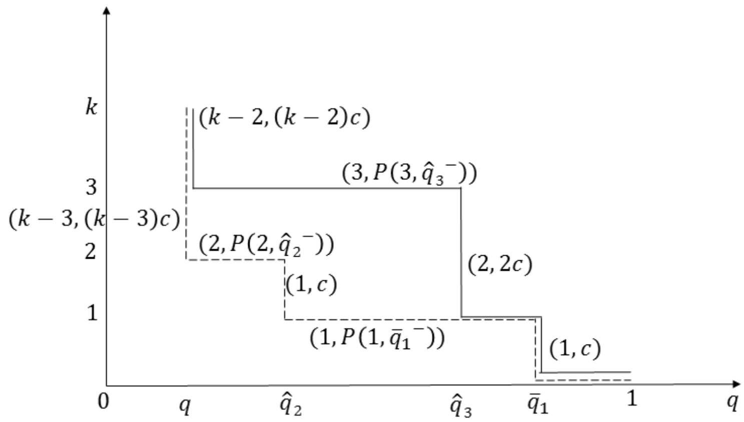

Figure 3: Penultimate Impasse with More than Two Units Remaining

As is evident from Figure 3, these two courses of action differ with the respect to the price that the buyer will pay to the types in $(q, \hat{q}_3)$ for the last three units. Under the second course, the buyer pays $P\left(3, \hat{q}_3^-\right)$. In contrast, under the first course, the buyer pays $c + P\left(2, \hat{q}_2^-\right)$ to the types in $(q, \hat{q}_2)$, and $2c + P\left(1, \bar{q}_1^-\right)$ to the types in $(\hat{q}_2, \hat{q}_3)$. In both cases, the buyer benefits from the significant drop in price for three units around the state $(3, \hat{q}_3)$ (see inequality 9) and strictly prefers the second course of action to the first one. We conclude that without an impasse at $(2, \hat{q}_2)$, the buyer would have a profitable deviation for δ_n sufficiently close to one.

To sum up, we have shown that the penultimate impasse is at $(2, \hat{q}_2)$ if and only if $\hat{q}_2 > \bar{q}_3$. If, instead, $\hat{q}_2 < \bar{q}_3$, then we compare $\hat{q}_3$ and $\bar{q}_4$. Similar arguments to those developed above allow us to conclude that the penultimate impasse is at $(3, \hat{q}_3)$ if and only if $\hat{q}_3 > \bar{q}_4$. In general, we show that the penultimate impasse is at $\left(\hat{k}, \hat{q}_{\hat{k}}\right)$, where we set $\bar{q}_{m+1} = 0$ and let $\hat{k}$ denote the smallest $k = 2, \ldots, m$ for which $\hat{q}_k > \bar{q}_{k+1}$.

Once the penultimate impasse at $\left(\hat{k}, \hat{q}_{\hat{k}}\right)$ has been determined, we treat the last $\hat{k}$ units as a single unit and reapply our algorithm. This will give us the third to last impasse. In general, our algorithm takes an impasse as given and applies the procedure above to identify the impasse that comes right before. We provide a formal description of the algorithm in Appendix D.

Below we state and discuss some general properties of the m-limiting equilibrium outcome for arbitrary values of m (these properties are derived in Appendix D).

Proposition 5 *The m-limiting equilibrium outcome involves a set of impasses*

$$\{(k_1, \tilde{q}_{k_1}), \ldots, (k_J, \tilde{q}_{k_J})\}$$

where $J \in \{1, \ldots, m\}$, $m \geqslant k_1 > \ldots > k_J = 1$, and $\tilde{q}_{k_1} < \ldots < \tilde{q}_{k_J} = \bar{q}_1$.

The impasses satisfy the following properties:

i) $\tilde{q}_{k_j} \in \left(\bar{q}_{k_j+1}, \bar{q}_{k_j}\right)$ for every $j \in \{1, \ldots, J-1\}$;

ii) $W\left(k_j, \tilde{q}_{k_j}\right) = 0$ for every $j \in \{1, \ldots, J\}$;

iii) For every $j \in \{1, \ldots, J-1\}$, we have

$$P\left(k_j, \tilde{q}^{+}_{k_j}\right) = \left(k_j - k_{j+1}\right) c + P\left(k_{j+1}, \tilde{q}^{-}_{k_{j+1}}\right) \tag{10}$$

and

$$P\left(k_j, \tilde{q}^{-}_{k_j}\right) = \frac{\left(\left(\alpha_{k_j} + \ldots + \alpha_1\right) \underline{v}\right)^2}{P\left(k_j, \tilde{q}^{+}_{k_j}\right)} = \frac{\left(\left(\alpha_{k_j} + \ldots + \alpha_1\right) \underline{v}\right)^2}{\left(k_j - k_{j+1}\right) c + P\left(k_{j+1}, \tilde{q}^{-}_{k_{j+1}}\right)}. \tag{11}$$

Similarly to the case with two units, the m-limiting equilibrium outcome is characterized by long periods of inactivity (impasses) followed by bursts of trade. As illustrated in Section 6, the last impasse occurs when there is one unit left for trade (and the buyer's belief is $\bar{q}_1$). The impasses differ both with respect the number of remaining units and the belief. The burst of trade between two consecutive impasses, say $\left(k_j, \tilde{q}_{k_j}\right)$ and $\left(k_{j+1}, \tilde{q}_{k_{j+1}}\right)$, consists of the universal offer for $\left(k_j - k_{j+1}\right)$ units, followed by a cream-skimming offer that is accepted by the low types in $\left(\tilde{q}_{k_j}, \tilde{q}_{k_{j+1}}\right)$. The rejection of this offer increases the buyer's belief and leads to another long period of inactivity.

We saw above that the penultimate impasse occurs at the state at $(2, \hat{q}_2)$ when $\hat{q}_2 \in (\bar{q}_3, \bar{q}_2)$. Proposition 5 i) shows that this property extends to all the other impasses (except the last one which occurs at $(1, \bar{q}_1)$).[25] The reason is similar to the case discussed above. If $\tilde{q}_{k_{j+1}}$ is larger than $\bar{q}_{k_{j+1}}$, then there cannot be an impasse at $\left(k_{j+1}, \tilde{q}_{k_{j+1}}\right)$ since the buyer's payoff is strictly positive. If, on the other hand, $\tilde{q}_{k_{j+1}}$ is smaller than $\bar{q}_{k_{j+1}+1}$, then the reservation price of the types close to $\tilde{q}_{k_j}$ for $k_{j+1}+1$ units is particularly attractive to the buyer, who then prefers to take advantage of this and avoid making a universal offer for $\left(k_j - k_{j+1}\right)$ units after the impasse $\left(k_j, \tilde{q}_{k_j}\right)$ is resolved. Again, there cannot be an impasse at $\left(k_{j+1}, \tilde{q}_{k_{j+1}}\right)$.

Clearly, at every impasse, the limit continuation payoff of the buyer must be equal to zero otherwise he would have an incentive to speed up trade.

Finally, recall that in equilibrium the low types are indifferent between accepting and rejecting cream-skimming offers (that are rejected by the high types), and their reservation

[25] Recall that we set $\bar{q}_{m+1} = 0$.

price is equal to their discounted continuation payoff after the rejection of the offer. In the limit, the transition from the state $\left(k_j, \tilde{q}_{k_j}\right)$ to the state $\left(k_{j+1}, \tilde{q}_{k_{j+1}}\right)$ is without delay and, thus, equation (10) must hold. Equation (11) follows from the fact that there is double delay at each impasse.

Decreasing Gains from Trade and New Profitable Trade Opportunities

As we have seen with detail in this section, the equilibrium dynamics with decreasing gains from trade are very different from the dynamics of the benchmark case of constant gains. To explain this point, recall that divisibility does not play any role when the gains from trade are constant. The equilibrium involves a long delay after which the buyer breaks even purchasing all the m units left at the price mc. Importantly, the buyer could not profitably deviate by making a universal offer for $k < m$ units at the price kc as he would also break even with this offer. This is no longer true when the gains from trade are decreasing. To see why, consider a putative limiting outcome with only one delay which occurs when there are m units left for trade. In contrast to the benchmark case, a universal offer for $k < m$ units brings two advantages: a direct payoff advantage and a strategic advantage. The direct payoff advantage is due to the fact that the average value of the first k units is higher than the average value of all the m units. The strategic advantage is more subtle. Consider the continuation game after the purchase of $k < m$ units. The buyer's valuation for each of the units left is small, which necessarily forces another delay. This delay, in turn, decreases the continuation value of low-type sellers, making them less reluctant to sell their goods in the current period at lower prices. As a consequence, the buyer profits from purchasing the good from a subset of low-type sellers at a low price. The combination of these two effects makes universal partial offers relatively more attractive when gains from trade are decreasing. These effects bring new profitable trade opportunities and ultimately lead to the gradual sale of high-quality goods. In the next section, we further explore the impact of these new trading opportunities allowing the good to be arbitrarily more divisible.

7 Limiting Equilibrium Outcome

Our analysis above shows that when the gains from trade are decreasing divisibility has a significant impact on the pattern of trade. To gain additional insights about trading dynamics, we consider the case in which the good becomes more and more divisible (i.e., is divided into smaller and smaller units). In addition to being a reasonable assumption in a number of environments (for example, the buyer of a firm can propose to purchase any fraction of it), this approach helps us to understand the role of divisibility, bringing new

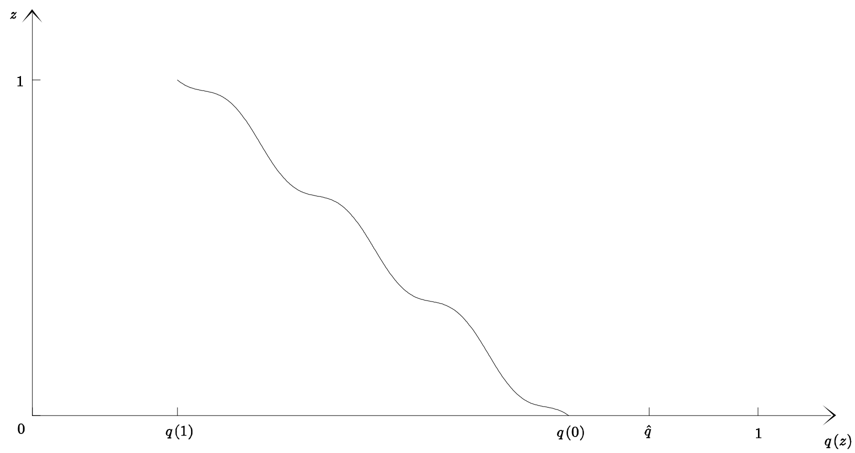

Figure 4: The function $q(z)$

insights into bargaining.

More precisely, we proceed as follows. We assume that the good has measure one. Suppose that the measure of the good on the table is $z > 0$ and that the buyer purchases a measure $z' \leqslant z$ from type q. The buyer's valuation is given by:

$$v(q)\left(\int_{z-z'}^{z} \alpha(u)\, du\right),$$

where $\alpha : [0,1] \to \mathbb{R}_{++}$ is a smooth and strictly increasing function, and $\alpha(0) = 1$. Type q's valuation of any measure z of the good is equal to zero if $q \leqslant \hat{q}$, and equal to $z\bar{c}$ if $q > \hat{q}$. Thus, $\bar{c}$ represents the cost of the whole good when the quality is high. We assume that $\bar{v} > \bar{c} > \underline{v} > 0$.

For every $z \in [0,1]$, we define $q(z) \in [0, \hat{q})$ implicitly by

$$\int_{q(z)}^{1} [\alpha(z)\, v(s) - \bar{c}]\, ds = 0. \tag{12}$$

Suppose that the measure z of the good is still available for trade and the buyer purchases one infinitesimal unit at the price $\bar{c}$. If the buyer's belief is equal to $q(z)$, then he breaks even.

The function $q(\cdot)$ (depicted in Figure 4) is smooth and strictly decreasing since $\alpha(\cdot)$ is smooth and strictly increasing. For future reference, we let $\psi(\cdot)$ denote the inverse of $q(\cdot)$.

We take the parameters $(\underline{v}, \bar{v}, \bar{c}, \hat{q}, r)$ and the function α as given and divide the good into $m = 1, 2, \ldots$ units of measure $1/m$ each. If the seller's type is q, the buyer's valuation of the $(m-k+1)$-th unit, $k=1,\ldots,m$, is equal to $\alpha_k v(q)$, where

$$\alpha_k := \int_{\frac{k-1}{m}}^{\frac{k}{m}} \alpha(u)\, du.$$

The cost of each unit for type q is equal to zero if $q \leqslant \hat{q}$, and equal to $\frac{\bar{c}}{m}$ if $q > \hat{q}$. Notice that since the function α is increasing, the gains from trade decrease as the parties engage in more and more transactions.

For every $m = 1, 2, \ldots,$ we compute the m-limiting equilibrium outcome. Recall that this is the limit, as the bargaining frictions vanish, of the equilibrium outcome of the game in which the good is divided into m parts. We know from Section 6.2 that the m-limiting equilibrium outcome is characterized by a sequence of impasses $\left\{\left(z_1^m, q_1^m\right), \ldots, \left(z_{N_m}^m, q_{N_m}^m\right)\right\}$, for some $N_m \leqslant m$. Thus, the j-th impasse occurs at the state $\left(z_j^m, q_j^m\right)$, where z_j^m denotes the *fraction* of the good left for trade and q_j^m denotes the buyer's belief. It is more convenient to work with the fraction of the good z_j^m left for trade as we vary the number of units m. Obviously, at the state $\left(z_j^m, q_j^m\right)$ the number of remaining units is equal to $m z_j^m$.

Of course, the impasses $\left\{\left(z_1^m, q_1^m\right), \ldots, \left(z_{N_m}^m, q_{N_m}^m\right)\right\}$ satisfy all the properties in Proposition 5. In particular, we have $\left(z_{N_m}^m, q_{N_m}^m\right) = \left(\frac{1}{m}, q\left(\frac{1}{m}\right)\right)$.

For every m and every $k = 1, \ldots, m$, we let $P_m(k, \cdot)$ denote the limit of the reservation price for k units when the bargaining frictions vanish (recall from Lemma 3 that the limit is well defined).

Finally, fix m and consider the m-limiting equilibrium outcome. We let $z_m(t)$ and $q_m(t)$ denote the fraction of the good left for trade and the buyer's belief, respectively, at time $t \in \mathbb{R}_+$ if all the cream-skimming offers proposed before time t are rejected (this is the case, for example, if the quality of the good is high). The functions $z_m(\cdot)$ and $q_m(\cdot)$ are well defined since there is a unique m-limiting equilibrium outcome.

Our goal is to characterize the limit, as m grows large, of the m-limiting equilibrium outcome. We refer to this as the limiting equilibrium outcome.[26]

To ease notation, in what follows we focus on the extreme case of adverse selection and assume that the buyer's expected valuation of the first infinitesimal unit is smaller than the cost of the seller's high types:

$$\int_0^1 \left[\alpha(1)\, v(s) - \bar{c}\right] ds < 0. \tag{13}$$

[26] This order of limits (first with respect to the length of the period, then with respect to the size of each unit) allows us to use the algorithm described in Section 6.2. It is an open question whether the same limiting outcome is obtained if the order of limits is exchanged.

It is easy to show that if $\int_0^1 [\alpha(0) v(s) - \bar{c}]\, ds \geqslant 0$, then for any m the m-limiting equilibrium outcome coincides with the first best and the Coase conjectures obtains.[27] Similarly, if $\int_0^1 [\alpha(z) v(s) - \bar{c}]\, ds = 0$ for some $z \in (0, 1]$, and the good is infinitely divisible, then the Coase conjecture applies, but only for a measure $1 - z$ of the good.[28]

We are now ready to state the main results (Theorem 1 and Corollary 1 below). Then we present a sketch of the proof, and provide the intuition behind our findings. The full proof of Theorem 1 is in Appendix C.

Theorem 1 *The limiting equilibrium outcome satisfies the following properties:*

i)

$$\lim_{m\to\infty} \max_{j\in\{2,\dots,N_m\}} q_j^m - q_{j-1}^m = 0$$

ii)

$$\lim_{m\to\infty} \max_{j\in\{2,\dots,N_m\}} z_{j-1}^m - z_j^m = 0$$

iii)

$$\lim_{m\to\infty} z_1^m = 1$$

iv)

$$\lim_{m\to\infty} \max_{j\in\{1,\dots,N_m\}} \left| q_j^m - q\left(z_j^m\right) \right| = 0$$

v)

$$\lim_{m\to\infty} \max_{j\in\{1,\dots,N_m\}} \left| P_m\left(m z_j^m, \left(q_j^m\right)^-\right) - \underline{v} \int_0^{z_j^m} \alpha(s)\, ds \right| =$$
$$\lim_{m\to\infty} \max_{j\in\{1,\dots,N_m\}} \left| P_m\left(m z_j^m, \left(q_j^m\right)^+\right) - \underline{v} \int_0^{z_j^m} \alpha(s)\, ds \right| = 0.$$

As the good becomes more and more divisible, the number of impasses goes to infinite while their size shrinks to zero. Between two consecutive impasses, the trading dynamics are as follows. First, the buyer makes a universal offer to purchase an arbitrarily small fraction of the good (thus, in the limit, the high-quality good is traded smoothly over time). Then the buyer makes a cream-skimming offer and purchase the remaining fraction of the good from an arbitrarily small set of low types. In the limit, the price of this offer coincides the buyer's valuation of the remaining fraction of the (low-quality) good.

Recall the definition of the functions $z_m(\cdot)$ and $q_m(\cdot)$ above. The monotonicity of these functions guarantees that the sequence $\{z_m(\cdot), q_m(\cdot)\}_{m=1,\dots}$ has a convergent subsequence. By Theorem 1, every subsequence converges to the same limit, which we denote by $(z^*(t), q^*(t))$.

[27] This means that in the limit, as the length of the period between consecutive offers vanishes, all the m units (of either quality) are traded without delay.

[28] In this case, the results derived in this section apply to the remaining measure z of the good.

Theorem 1 also allows us to characterize the functions $z^*(t)$ and $q^*(t)$. Property iv) immediately implies that for every $t \geqslant 0$

$$z^*(t) = \psi(q^*(t)),$$

where, recall, $\psi(\cdot)$ is the inverse of $q(\cdot)$ (defined in equation (12)). Thus, $q^*(0) = q(1)$ since $z^*(0) = 1$.

Finally, notice that in equilibrium the low types must be indifferent between accepting the cream-skimming offer at time t and accepting the cream-skimming offer at time $t + \Delta t$ (while mimicking the high types between t and $t+\Delta t$). Property v) of Theorem 1 shows that when m grows large the price of a cream-skimming offer approaches the buyer's valuation (for the low-quality good). Putting these observations together, we have:

$$\begin{aligned} \underline{v}\int_0^{\psi(q^*(t))}\alpha(u)du &= -\int_t^{t+\Delta t} e^{-r(\tau-t)}\psi'(q^*(\tau))\, q^{*\prime}(\tau)\bar{c}d\tau + e^{-r\Delta t}\underline{v}\int_0^{\psi(q^*(t+\Delta t))}\alpha(u)du = \\ -\psi'(q^*(t))\, q^{*\prime}(t)\bar{c}\Delta t &+ (1-r\Delta t)\underline{v}\int_0^{\psi(q^*(t))}\alpha(u)du + \underline{v}\alpha(\psi(q^*(t)))\psi'(q^*(t))\, q^{*\prime}(t)\,\Delta t + o(\Delta t). \end{aligned} \tag{14}$$

Finally, using equation (14) and taking the limit as Δt goes to zero, we obtain $q^{*\prime}(t)$. The following corollary summarizes our findings.

Corollary 1 *The functions $z^*(\cdot)$ and $q^*(\cdot)$ satisfy, for every $t \geqslant 0$, the following conditions:*

$$\begin{aligned} q^{*\prime}(t) &= \frac{r\underline{v}\int_0^{\psi(q^*(t))}\alpha(u)du}{\psi'(q^*(t))(\underline{v}\alpha(\psi(q^*(t)))-\bar{c})} \\ z^{*\prime}(t) = \psi'(q^*(t))\, q^{*\prime}(t) &= \frac{r\underline{v}\int_0^{\psi(q^*(t))}\alpha(u)du}{(\underline{v}\alpha(\psi(q^*(t)))-\bar{c})}. \end{aligned} \tag{15}$$

Furthermore, $q^*(0) = q(1)$, $z^*(0) = 1$, and $z^*(t) = \psi(q^*(t))$ for every t.

The limiting equilibrium outcome can be interpreted as follows. At time zero, the buyer makes a cream-skimming offer for the entire good (measure one) at a price $\underline{v}\left(\int_0^1 \alpha(u)du\right)$. This offer is accepted by the low types in the interval $[0, q(1)]$. Thus, if the offer is rejected, the buyer's belief jumps to $q(1)$. At each time $t > 0$, the buyer engages in two actions simultaneously: he makes a universal offer for an infinitesimal quantity (which, of course, is accepted by all the remaining types), and a cream-skimming offer to purchase the remaining fraction $\psi(q^*(t))$ of the good at the price $\underline{v}\left(\int_0^{\psi(q^*(t))}\alpha(u)du\right)$. The low types accept the cream-skimming offer smoothly over time in such a way that the belief evolves according to $q^*(\cdot)$.

We remark that $z^*(t) > 0$ for every $t \in \mathbb{R}_+$ and that $\lim_{t\to\infty} z^*(t) = 0$. That is, as the good becomes infinitely divisible, the game takes an arbitrarily long time to finish. Nonetheless, the fraction of the good available for trade vanishes over time.

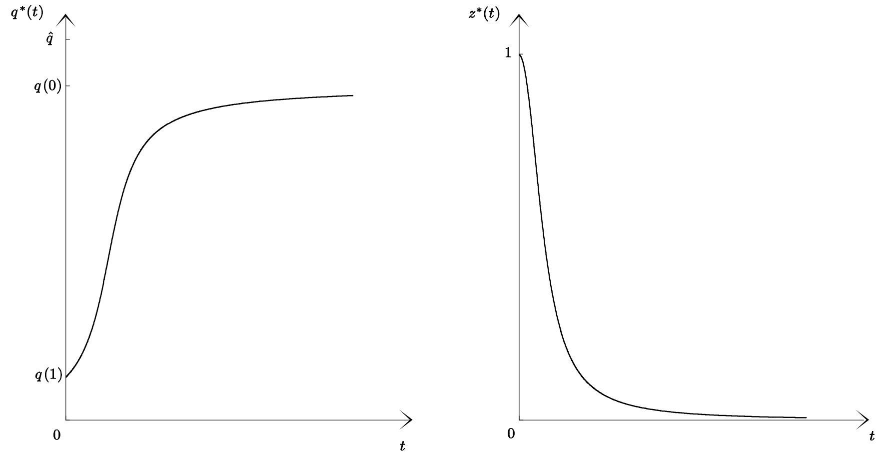

Figure 5: The Limiting Equilibrium Outcome

Figure 5 illustrates a typical trade path in the limiting equilibrium outcome.

The proof of Theorem 1 is based on the analysis (carried out in Appendix C) of a dynamic system linking one impasse to the next. In particular, we show that if the last impasse occurs when the fraction of the good left for trade is small (this is necessarily the case when m is large), then each of the prior impasses is also small. Formally, our analysis demonstrates that for every $\varepsilon > 0$, we can find $\eta > 0$ such that if $z^m_{N_m} < \eta$, then $\max_{j \in \{2,\ldots,N_m\}} z^m_{j-1} - z^m_j < \varepsilon$. Although the formal proof is somewhat involved and relegated to the appendix, the following fact and its proof convey most of the intuition with ease and expediency.

Fact 1 *Consider the sequence of impasses* $\left\{\left(z^m_1, q^m_1\right), \ldots, \left(z^m_{N_m}, q^m_{N_m}\right)\right\}_{m=1,\ldots}$. *Suppose that there exists a sequence* $\left\{z^m_{j_m}, q^m_{j_m}\right\}_{m=1,\ldots}$, *with* $j_m > 1$ *for every* m, *such that*

$$\lim_{m\to\infty} P_m\left(mz^m_{j_m}, \left(q^m_{j_m}\right)^+\right) - P_m\left(mz^m_{j_m}, \left(q^m_{j_m}\right)^-\right) = 0.$$

Then we have

$$\lim\nolimits_{m\to\infty} z^m_{j_m-1} - z^m_{j_m} = 0$$

$$\lim\nolimits_{m\to\infty} P_m\left(mz^m_{j_m-1}, \left(q^m_{j_m-1}\right)^+\right) - P_m\left(mz^m_{j_{m z^m_{j_m-1}}}, \left(q^m_{j_m-1}\right)^-\right) = 0.$$

The driving forces behind Fact 1 are closely related to our finding that the high-quality good is sold gradually over time. Consider any (non initial) impasse occurring at the state

$\left(z^m_{j_m}, q^m_{j_m}\right)$ in the m-limiting equilibrium outcome. Assume that the difference between the reservation prices before and after this impasse, $P_m\left(mz^m_{j_m}, \left(q^m_{j_m}\right)^+\right) - P_m\left(mz^m_{j_m}, \left(q^m_{j_m}\right)^-\right)$, is small. We start showing that this implies that the size of the previous delay, measured by $z^m_{j_m-1} - z^m_{j_m}$, will also be small when the good is very divisible (m is large).

First, observe that according to Proposition 5 ii), the buyer's limit payoff at the previous impasse $\left(z^m_{j_m-1}, q^m_{j_m-1}\right)$ is equal to zero. This payoff can be decomposed into two components: *universal-offer component* and *cream-skimming offer component*, which we illustrate below.

Universal-Offer component: This part represents the payoff from purchasing a measure $\left(z^m_{j_m-1} - z^m_{j_m}\right)$ of the good at the price $\left(z^m_{j_m-1} - z^m_{j_m}\right)\bar{c}$ when the belief is equal to $q^m_{j_m-1}$. Formally, the universal-offer component is equal to

$$\int_{z^m_{j_m}}^{z^m_{j_m-1}} \int_{q^m_{j_m-1}}^{1} \left(\alpha\left(u\right)v\left(s\right) - \bar{c}\right) ds du.$$

Recall from Proposition 5 ii) that $q^m_{j_m-1}$ is smaller than the belief at which the buyer would break even if he had to make a universal offer for the first of the $mz^m_{j_m-1}$ remaining units. Also, recall that the gains from trade are decreasing. Therefore, the universal-offer component is negative and decreasing in the fraction of the good that is traded.

Cream-skimming offer component: This part represents the payoff from purchasing the remaining fraction of the good $z^m_{j_m}$ at the price $P_m\left(mz^m_{j_m}, \left(q^m_{j_m}\right)^-\right)$ from all the low types in the set $\left(q^m_{j_m-1}, q^m_{j_m}\right)$. The cream-skimming component is positive (the price $P_m\left(mz^m_{j_m}, \left(q^m_{j_m}\right)^-\right)$ is smaller than the buyer's valuation of the fraction $z^m_{j_m}$ of the low-quality good) and equal to

$$\int_{q^m_{j_m-1}}^{q^m_{j_m}} \left[\underline{v}\int_0^{z^m_{j_m}} \alpha\left(u\right)du - P_m\left(mz^m_{j_m}, \left(q^m_{j_m}\right)^-\right)\right] dq$$

From Proposition 5 iii) we have

$$P_m\left(mz^m_{j_m}, \left(q^m_{j_m}\right)^-\right) = \frac{\left(\underline{v}\int_0^{z^m_{j_m}} \alpha\left(u\right)du\right)^2}{P_m\left(mz^m_{j_m}, \left(q^m_{j_m}\right)^+\right)}.$$

It is easy to check that this, together with $P_m\left(mz^m_{j_m}, \left(q^m_{j_m}\right)^-\right) < P_m\left(mz^m_{j_m}, \left(q^m_{j_m}\right)^+\right)$, implies

$$P_m\left(mz^m_{j_m}, \left(q^m_{j_m}\right)^-\right) < \underline{v}\int_0^{z^m_{j_m}} \alpha\left(u\right)du < P_m\left(mz^m_{j_m}, \left(q^m_{j_m}\right)^+\right).$$

Thus, as the prices $P_m\left(mz_{j_m}^m,\left(q_{j_m}^m\right)^-\right)$ and $P_m\left(mz_{j_m}^m,\left(q_{j_m}^m\right)^+\right)$ get arbitrarily close to each other, the cream-skimming offer component shrinks to zero.

This, in turn, implies that the universal-offer component must also converge to zero (recall that the sum of the two components is equal to zero). This is possible only if $z_{j_m-1}^m$ converges to $z_{j_m}^m$ as m grows large, which proves the first result in Fact 1.

We now turn to the second result. Again, from Proposition 5 iii) we have

$$P_m\left(mz_{j_m-1}^m,\left(q_{j_m-1}^m\right)^-\right)=\left(\frac{\underline{v}\int_0^{z_{j_m-1}^m}\alpha\left(u\right)du}{P_m\left(mz_{j_m}^m,\left(q_{j_m}^m\right)^-\right)+\left(z_{j_m-1}^m-z_{j_m}^m\right)\bar{c}}\right)^2 P_m\left(mz_{j_m-1}^m,\left(q_{j_m-1}^m\right)^+\right).$$

We have shown that as m goes to infinity, $P_m\left(mz_{j_m}^m,\left(q_{j_m}^m\right)^-\right)$ converges to $\underline{v}\int_0^{z_{j_m}^m}\alpha\left(u\right)du$, and the difference $\left(z_{j_m-1}^m-z_{j_m}^m\right)$ shrinks to zero. We conclude that the first term of the right-hand side (i.e., the term in parenthesis) in the equation above must converge to one. This establishes the second result in Fact 1.

In summary, we conclude that when the good is very divisible, a small discontinuity in the reservation price at a given impasse is preceded by both a small transaction of the high-quality good and a small discontinuity in the reservation price at the impasse that comes right before. This is the key element leading to Theorem 1. In fact, from Proposition 5 we know that the last impasse occurs when there is one unit left for trade. Furthermore, when this impasse is resolved, the reservation price (for the last unit of the good) is equal to $\bar{c}/m$. As m grows large, the discontinuity in the reservation price at the last impasse converges to zero. The logic above immediately implies that the penultimate impasse must also be small, entailing a small third-to-last impasse, and so on.

We remark that since the number of impasses increases without bounds as the good becomes arbitrarily divisible, the proof of Theorem 1 has to show that there are no cumulative effects and all the impasses remain uniformly small when the last impasse is sufficiently small. This requires a somewhat more involved argument and is relegated to the appendix.

As we saw in the last section, direct payoff and strategic advantages give rise to new profitable partial offers when the gains from trade are decreasing. The number of such profitable trade opportunities increase without bound as the good becomes more divisible and as a result high-quality goods are smoothly sold over time in the limit.

7.1 Gradual Sale of High-quality Goods

A common result in the literature on dynamic markets with adverse selection and indivisibility is the sale of low-quality goods at the beginning of the relationship, followed by periods of market freezing, after which high-quality goods are sold. Theorem 1 shows that adverse selection unravels in a very different way when the good is divisible and the gains from trade are decreasing. In such situations, we predict that the gradual sale of high-quality goods arises as the main signaling device. Our model delivers novel and testable implications for situations in which two parties dynamically trade for portfolio diversification such as negotiations over securitized assets in over-the-counter markets. In times of uncertainty about fundamentals, we expect that sellers with valuable assets make small transaction over time, while sellers who hold lemons also engage in sales of large quantities.

7.2 Coasean Outcome in the Market for Lemons

In his influential work, Coase (1972) conjectured that a monopolist who lacks commitment power ends up selling the good almost immediately at a price equal to the lowest valuation among all consumers. In our environment, this conjecture would signify the buyer's purchase of the entire good immediately at price $\bar{c}$. However, this cannot be an equilibrium outcome of our model, as the buyer would pay a price larger than the expected valuation of the good. Thus, we conclude that, in equilibrium, there must necessarily be delay. If we wish to conjecture that the competition between the buyer's present and future selves brings his profits to the lowest possible level, we must consider an adaptation of the Coase conjecture to an environment with interdependent values.

We say that there is a *Coasean outcome in the market for lemons* (Coasean outcome for brevity) if at any time $t \geqslant 0$ the buyer breaks even both if he makes a universal offer for the first of the infinitesimal units still available and if he makes the cream-skimming offer. Theorem 1 shows that the limiting outcome of the model with decreasing gains from trade is Coasean.

We conclude that the combination of adverse selection and divisibility has dramatic effects on payoffs. When the good is infinitely divisible, both the buyer and the high type sellers get a payoff equal to zero. On the other hand, the low types' payoff coincides with the buyer's valuation of the low-quality good. This result is in stark contrast to the results obtained by DL in the one-unit model, where the buyer's payoff is strictly positive and the low types get less than the buyer's valuation.[29]

When the good is indivisible, negotiations necessarily involve long impasses. In fact,

[29] It is easy to show that our limiting equilibrium outcome is more efficient than the limiting outcome in DL. More formally, there exist $\bar{m}$ such that for any $m > \bar{m}$ the expected social welfare (i.e., the discounted sum of the realized trading surplus) of the m-limiting equilibrium outcome is higher than the social welfare in the limiting outcome of the model with indivisibility (1-limiting equilibrium outcome). However, the limiting equilibrium outcome is not constrained efficient. There exists a direct incentive compatible mechanism which achieves a larger welfare than the limiting equilibrium outcome. Finally, the social welfare of the m-limiting equilibrium outcome is not necessarily monotonic in m.

the buyer is willing to pay a large price only if he is sufficiently optimistic about the quality of the good. On the other hand, the owner of the low-quality good is willing to wait for a long period until he can sell the good at a price much larger than his cost. Because of the long impasses, the price at which the buyer is able to purchase the good at the beginning of the negotiations is severely reduced. Therefore, he obtains a strictly positive payoff.

When the good is divisible, purchasing all the remaining units at a large price is only one of many options available to the buyer. When this option yields a low payoff, the buyer prefers to purchase only a fraction of the remaining good. Compared to DL's model, this lowers the price that the owner of the low-quality good is able to charge in the middle of the negotiations. However, lower payments in the middle of the negotiations correspond to shorter impasses and higher prices paid in the early stages of the negotiations. Consequently, the buyer's initial payoff is drastically reduced and converges to zero in the limit.

Our analysis shows that the possibility to make (arbitrary) partial offers is detrimental to the buyer's payoff when the gains from trade are decreasing. In the standard model with an indivisible object, the buyer would be willing to pay (ex-ante) a positive price to lose the ability to adjust the offer as he becomes more optimistic about the quality of the good. Similarly, in our model, the buyer would be willing to pay a positive price to lose the ability to make partial offers.

7.3 Comparative Statics

In this section, we investigate how the limiting equilibrium outcome depends on the primitives of the model. This will allow us to determine the effects of the economic fundamentals on the speed of trade.

For some small $\varepsilon > 0$, consider a different valuation function $\tilde{\alpha}$ such that $\tilde{\alpha}(u) \in (\alpha(u), \alpha(u) + \varepsilon)$ for every $u \in [0,1]$. Define the function $\tilde{\psi}(\cdot)$ accordingly. Let $(\tilde{z}^*(\cdot), \tilde{q}^*(\cdot))$ be the functions describing the new Coasean outcome. We claim that $\tilde{z}^*(t) < z^*(t)$ for every $t > 0$. Assume towards a contradiction that this is not the case. First, notice that

$$\tilde{z}^{*\prime}(0) = \tilde{\psi}'(\tilde{q}^*(0))\, \tilde{q}^{*\prime}(0) =$$
$$\frac{r\underline{v}\int_0^1 \tilde{\alpha}(u)du}{\underline{v}\tilde{\alpha}(1) - \bar{c}} < \frac{r\underline{v}\int_0^1 \alpha(u)du}{\underline{v}\alpha(1) - \bar{c}} = z^{*\prime}(0) < 0.$$

Let $\underline{t}$ denote the smallest $t > 0$ for which $\tilde{z}^*(t) = z^*(t)$ (our contradiction hypothesis guarantees that $\underline{t}$ exists). It follows that

$$\tilde{z}^{*\prime}(\underline{t}) = \frac{r\underline{v}\int_0^{\tilde{z}^*(\underline{t})} \tilde{\alpha}(u)du}{\underline{v}\tilde{\alpha}(\tilde{z}^*(\underline{t})) - \bar{c}} < \frac{r\underline{v}\int_0^{z^*(\underline{t})} \alpha(u)du}{\underline{v}\alpha(z^*(\underline{t})) - \bar{c}} = z^{*\prime}(\underline{t}) < 0.$$

However, this and the fact that $\tilde{z}^*(t) < z^*(t)$ for every $t < \underline{t}$ imply that $\tilde{z}^*(\underline{t}) < z^*(\underline{t})$, which contradicts the definition of $\underline{t}$.

A similar argument shows that a decrease in the cost $\bar{c}$, an increase in the value $\underline{v}$, and an increase in the discount rate r all decrease $z^*(t)$ for every $t > 0$. It is also easy to check that a small change in the value $\bar{v}$ or in the probability $\hat{q}$ does not affect the speed of trade. The analysis above allows us to state the following fact.

Fact 2 *In the Coasean outcome, trade of the high-quality good happens faster when adverse selection is less severe (i.e., when α or $\underline{v}$ increases or when $\bar{c}$ decreases) or when the parties are less patient.*

Let us provide some intuition for Fact 2. First, consider an increase in α or in $\underline{v}$. In this case, the low types' payoffs from accepting cream-skimming offers increase. On the other hand, the payoff from accepting a universal offer (for an infinitesimal quantity) remains constant. Consequently, a smaller delay is necessary to induce the low types to reveal their private information. Similarly, a decrease in $\bar{c}$ makes the universal offers less attractive for the low types (in other words, the low types are less tempted to pool with the high types), and hence trade happens faster.

Fact 2 yields sharp predictions about the effect of a change of the primitives on the timing at which the high-quality good is sold. On the other hand, when the seller's type is low, the quantity of the low-quality good that is still available for trade at time $t > 0$ is a random variable. In particular, if the seller's type is smaller than $q^*(t)$, then the remaining quantity is zero. Otherwise, it is equal to $z^*(t)$. Thus, if we let $g^*(t)$ denote the expected quantity of the low-quality good that is available at time $t > 0$, we have

$$g^*(t) = \left(\frac{\hat{q} - q^*(t)}{\hat{q}}\right) z^*(t).$$

When adverse selection becomes less severe (i.e., α or $\underline{v}$ increases or $\bar{c}$ decreases) there are two opposing effects on the timing at which the low-quality good is sold. On the one hand, a decrease in adverse selection decreases $q(1)$.[30] Hence, the quantity of the low-quality good that is available right after $t = 0$, $g^*(0^+) = \left(\frac{\hat{q}-q^*(1)}{\hat{q}}\right)$, increases. On the other hand, Fact 2 establishes that a decrease in adverse selection increases the speed of trade of the high-quality good. For a given $q(1)$, this effect increases the speed of trade of the low-quality good and hence decreases the expected quantity that is available at t. It is easy to construct examples showing that the quantity of the low-quality good still available for trade at time t is non-monotonic in α, $\underline{v}$ or $\bar{c}$.

[30] Remember that $q(1)$ satisfies $\int_{q(1)}^{1} [\alpha(1)v(s) - \bar{c}]\, ds = 0$ (see equation (12)).

Finally, the effect of a change in r, $\bar{v}$, or $\hat{q}$ is unambiguous. In fact, it is simple to check that the trade of the low-quality good happens faster ($g^*(t)$ decreases for every $t > 0$) when r or $\hat{q}$ increase, or when $\bar{v}$ decreases. Clearly, there is less delay when the parties are less patient. Also, recall that a change in $\hat{q}$ or $\bar{v}$ does not affect the speed of trade of the high-quality good (the value of $z^*(t)$ remains unchanged for every t) and that the buyer must break even when he makes a universal offer. This observation, together with an increase in $\hat{q}$ (or a decrease in $\bar{v}$), immediately implies that the fraction $\frac{\hat{q}-q^*(t)}{\hat{q}}$ of the low types that the buyer still faces at time t must decrease. Therefore, the speed of trade of the low-quality good increases since the low types are quicker to accept cream-skimming offers.

8 Increasing Gains

In this section, we consider the case of increasing gains from trade and assume that $\alpha_m < \ldots < \alpha_1$. The assumption of increasing gains is natural when the different units are complementary. For example, when different shareholders dispute the control of a firm, the marginal value of an additional share is often increasing in the number of shares already owned, as bigger shareholders can exert more influence on the firm's decisions.

The case of increasing gains is similar to the case of constant gains from trade. In particular, in equilibrium the buyer uses only cream-skimming offers (this is true for any discount factor δ).

To see why universal offers are not optimal for the buyer, consider the model in which the parties bargain over the last k units and assume that these units are *indivisible* (as in DL). When the gains from trade are increasing, the buyer's valuation for an additional unit is decreasing in the number of units already traded. This implies that the unit price is decreasing in k (the total number of units). In equilibrium, the buyer takes advantage of the low prices associated to large transactions and purchases all the units at once.

Proposition 6 *Suppose that the gains from trade are increasing: $\alpha_m < \ldots < \alpha_1$. Then all equilibrium offers are cream-skimming.*

The proof of Proposition 6 is slightly more involved than the proof of Proposition 3.[31] We provide it in Appendix F.

Clearly, Proposition 6 implies that in the limit, as the bargaining frictions vanish, the equilibrium outcome is as in DL. The buyer purchases all the units from a subset of low

[31] At the cost of a slightly lengthier proof, one can simultaneously accommodate the cases of constant and increasing gains from trade.

types without delay. Then there is an impasse. Once the impasse is resolved, the buyer trades all the units with all the remaining types (without further delay).

In the limit, as δ goes to one, the equilibrium outcome with increasing and constant gains is very different from the equilibrium outcome with decreasing gains. However, this is not a failure of upper-hemicontinuity of the equilibrium. Fix the total number of units m. It is possible to show that for every δ there exists $\varepsilon_\delta > 0$ such that if $|\alpha_m - \alpha_1| < \varepsilon_\delta$, then all equilibrium offers are cream-skimming. However, as δ approaches one, ε_δ shrinks to zero, which is consistent with our finding that when the gains are decreasing and the bargaining frictions vanish universal offers occur on the equilibrium path.

9 Menus of Offers

In this section, we examine an extension of the model involving the types of trading instruments available to the buyer. In the benchmark model, the buyer can choose in every period the number of the remaining units that he wants to purchase. Theorem 1 suggests that when the gains from trade are decreasing having many trading opportunities is not beneficial to the buyer. According to this line of reasoning, one would expect that the buyer would not benefit from a larger set of trading opportunities. To assess this conjecture, we extend the contract space by allowing the buyer to offer menus of offers. In what follows, we restrict attention to the case of decreasing gains from trade.[32]

As in the benchmark model, an offer (k, p) specifies the number of unit k that the parties trade and the transfer $p \geqslant 0$ that the seller is entitled to receive. A menu $\mathcal{M}$ is a set of offers. In particular, when the number of units on the table is $k = 1, \ldots, m$, the set of available menus is the set of all compact subsets of $\{1, ..., k\} \times \mathbb{R}_+$. Upon being offered a menu $\mathcal{M}$, the seller can either accept one offer in $\mathcal{M}$ or reject all the offers.

We continue to restrict attention to stationary (perfect Bayesian) equilibria. In our context, this means that the decision of any seller's type depends on the number of units left for trade and the menu $\mathcal{M}$ offered by the buyer.[33] Furthermore, we impose the additional requirement that all the seller's high types agree on their decisions at any point in time. Formally, we consider stationary equilibria in which the seller's strategy satisfies the following property. For every t and for every history $((\mathcal{M}_0, a_0), \ldots, (\mathcal{M}_{t-1}, a_{t-1}), \mathcal{M}_t)$ of proposed menus and seller's decisions, there exists an action that is chosen by all the types

[32] It is easy to check that when the gains from trade are constant or increasing our results extend to the case in which the buyer offers menus.

[33] As in the benchmark model, the low types' decision also depends on whether or not the buyer is convinced that the quality of the good is low (there are no stationary equilibria if we do not allow for this possibility).

larger than $\hat{q}$.

Under this refinement, the buyer's beliefs (on the equilibrium path) take a very simple form. Similarly to the benchmark model, after every on-path history, the buyer believes that the seller's type is uniformly distributed over the set $[\tilde{q}, 1]$ for some $\tilde{q} \in [0, \hat{q}]$.[34] Our refinement implies that the high types accept an offer if and only if it yields a non-negative current payoff.[35] This is in line with the high types' behavior in the equilibrium of the benchmark model.

In Appendix G, we construct a stationary equilibrium of the game in which the buyer can propose menus with at most two offers. We show that, as the bargaining frictions vanish, the outcome of the equilibrium described in Appendix G converges to the m-limiting outcome of the benchmark model (where m denotes the total number of units available for trade). Furthermore, the assumption of allowing the buyer to make at most two offers is not restrictive. Consider the game in which the buyer can propose arbitrary menus (arbitrary compact subsets of the set of feasible offers). Under our refinement (and for generic values of the parameters), all stationary equilibria of this game are outcome equivalent to the equilibrium in Appendix G. This shows that the main result of this paper (Theorem 1) is robust to the introduction of menus. In particular, the analysis in this section corroborates our finding that the buyer does not benefit from additional trading instruments.

10 Concluding Remarks

This paper studies bargaining with interdependent values and divisible goods. We show that when the gains from trade are decreasing a new pattern of trade, gradual trading, arises in these markets and that the possibility of purchasing fractions of the good is detrimental to the uninformed buyer. In the limit, when offers are frequent and the good is arbitrarily divisible, the high-quality good is sold smoothly over time and the buyer's payoff converges to the lowest possible level. When adverse selection is particularly severe, the buyer's limiting equilibrium payoff is equal to zero.

Throughout the paper we have made a few simplifying assumptions which make the analysis tractable. In particular, we have assumed that the quality of the good can take only two values (i.e., the seller has only two types). This assumption guarantees the

[34] In contrast, if we allow the high types to make different choices, then the beliefs are not "unidimensional" (in the sense that one has to keep track of the fractions of *both* the low *and* the high types still present in the game), and the analysis is not tractable. Moreover, it is an open question whether there exist equilibria that do not satisfy our refinement.

[35] Under our refinement, it is without loss of generality (in terms of equilibrium outcomes) to restrict attention to equilibria in which the menus proposed by the buyer contains at most one offer that yields a non-negative payoff to the high types.

existence of stationary equilibria. We are unable to show that stationary equilibria exist in a model with more than two types. However, under some conditions on the primitives of the model, the main result of the paper extends to the case of finitely many types.

Consider the model described in Section 7 but assume that $c(q)$ and $v(q)$ are two (non-decreasing) step functions with finitely many discontinuity points. Without loss, we assume that $c(0) = 0$ and let $\hat{q} = \sup_{\{q:c(q)=0\}} q$ denote the probability that the good is of the worst quality. Also, suppose that for $z = 0, 1$, there exists $q(z) \in (0, \hat{q})$

$$\int_{q(z)}^{1} [\alpha(z) v(s) - c(1)] \, ds = 0.$$

In words, the belief at which the buyer breaks even when he purchases any unit of the good at the price $c(1)$ corresponds to a type with the lowest quality of the good. It is not difficult to show that under these assumptions Theorem 1 continues to hold.

An important question in the literature on bargaining concerns the role of the gap between the buyer and the seller's valuations of the good. Similarly to DL, we have assumed that the gains from trade are strictly positive (for every type and every unit). How do our results change if we assume that the gains are weakly positive? In the case of decreasing (increasing) gains from trade, the gap-assumption affects only the last (first) unit and our main results continue to hold. In particular, our characterization of the limiting equilibrium outcome remains valid if we assume that $\alpha : [0, 1] \to \mathbb{R}_{++}$ is a smooth and strictly increasing (or strictly decreasing) function and $a(z) v(q) - c(q) \geqslant 0$ for every z and every q. [36]

In our model, the buyer learns about the quality of the good only through the seller's behavior. This assumption is reasonable in a number of important applications and our model constitutes a useful theoretical benchmark to study bargaining over divisible objects. However, it would be interesting to extend the model to allow for additional forms of learning: endogenous, for example, in the form of learning via the consumption of parts of the good, or exogenous (as in Daley and Green (2012) and (2016). We leave the study of bargaining with learning for future work, but conjecture that the driving forces behind our results will emerge even with learning and will lead to the gradual sale of the high-quality good.

[36] It is easy to see that when the gains from trade are constant, the violation of the gap-assumption leads to an equilibrium outcome in which the high-quality good is not traded at all and only (some of) the low types reach an immediate agreement (for all the units) with the buyer. We also refer the reader to DL for a discussion of the gap-assumption in the case of constant gains from trade.

Appendix A: Existence of Stationary Equilibria

Proof of Lemma 1.

Given a perfect Bayesian equilibrium $(\sigma_B, \sigma_S, \mu)$, we let $H_k(\sigma_B, \sigma_S, \mu)$, $k = 1, \ldots, m$, denote the set of histories h^t after which there are k units left for trade and such that $\mu([0, \hat{q}] | h^t) = 1$.

First, we show that the claim is true if $h^t \in H_1(\sigma_B, \sigma_S, \mu)$, that is, there is only one unit left for trade. Define $\bar{u}_L$ as the supremum, over all the perfect Bayesian equilibria, of the continuation payoffs of the low types $q \leqslant \hat{q}$ at histories $h^t \in H_1(\sigma_B, \sigma_S, \mu)$. Assume towards a contradiction that $\bar{u}_L > 0$. Take $\varepsilon = \left(\frac{1-\delta}{2}\right) \bar{u}_L$ and notice that there exists a perfect Bayesian equilibrium $(\sigma_B, \sigma_S, \mu)$ and a history $\bar{h}^t \in H_1(\sigma_B, \sigma_S, \mu)$ at which the buyer makes the proposal $\varphi_t = (1, p)$ for some $p \in [\bar{u}_L - \varepsilon, \bar{u}_L]$. We claim that the conditional probability, given the beliefs $\mu\left(\bar{h}^t\right)$, that the proposal φ_t is accepted must be one. To see this, notice that if φ_t is accepted with a probability of less than one (given $\mu\left(\bar{h}^t\right)$), then $\left(\bar{h}^t, (\varphi_t, R)\right) \in H_1(\sigma_B, \sigma_S, \mu)$ (i.e., $\mu\left([0, \hat{q}] | \bar{h}^t, (\varphi_t, R)\right) = 1$) and the continuation payoff of the low types is at most $\bar{u}_L$. But then it is not optimal for a low type to reject φ_t since $\bar{u}_L - \varepsilon > \delta \bar{u}_L$. However, using a similar argument it is easy to show that the conditional probability, given the belief $\mu\left(\bar{h}^t\right)$, that the proposal $\varphi'_t = \left(1, \bar{u}_L - \left(\frac{3}{2}\right)\varepsilon\right)$ is accepted is also one (since $\bar{u}_L - \left(\frac{3}{2}\right)\varepsilon > \delta \bar{u}_L$). Thus, the buyer has a profitable deviation at $\bar{h}^t$ since he strictly prefers the proposal φ'_t to φ_t.

Next, assume that the claim is true for any $h^t \in H_1(\sigma_B, \sigma_S, \mu) \cup \ldots \cup H_{k-1}(\sigma_B, \sigma_S, \mu)$, $k = 2, \ldots, m$. We show that the claim is also true for any $h^t \in H_k(\sigma_B, \sigma_S, \mu)$. Again, towards a contradiction, let $\bar{u}_L > 0$ be the supremum, over all the perfect Bayesian equilibria $(\sigma_B, \sigma_S, \mu)$, of the continuation payoffs of the low types $q \leqslant \hat{q}$ at histories $h^t \in H_k(\sigma_B, \sigma_S, \mu)$. Take $\varepsilon = \left(\frac{1-\delta}{2}\right) \bar{u}_L$ and notice that there exist a perfect Bayesian equilibrium $(\sigma_B, \sigma_S, \mu)$ and a history $\bar{h}^t \in H_k(\sigma_B, \sigma_S, \mu)$ at which the buyer makes the proposal $\varphi_t = (k', p)$ for some $k' = 1, \ldots, k$ and $p \in [\bar{u}_L - \varepsilon, \bar{u}_L]$. Using the induction hypothesis and an argument similar to the one presented in the previous paragraph, we conclude that the conditional probability, given $\mu\left(\bar{h}^t\right)$, that the proposal $\varphi_t = (k', p)$ is accepted is equal to one. However, the same is true for the proposal $\varphi'_t = \left(k', \bar{u}_L - \left(\frac{3}{2}\right)\varepsilon\right)$ which is, therefore, strictly preferred to φ_t. Again, this shows that the buyer has a profitable deviation at $\bar{h}^t$ and concludes our proof. ■

Proof of Lemma 2.

The first step of the proof is to show that, at any history, the equilibrium continuation payoff of the high types is zero. From the characterization in DL, we know that this is true at every history h^t at which there is only one unit left for trade. For $k = 1, \ldots, m-1$, we assume that the continuation payoff (in any stationary equilibrium) of the high types $q > \hat{q}$ is zero when k or fewer units are left for trade. We show that the continuation payoff of the high types is also zero when the number of remaining units is $k+1$.

Define $\bar{u}_H$ as the supremum, over all stationary equilibria, of the continuation payoffs of the high types $q > \hat{q}$ at histories in which the remaining units are $k+1$. Trivially, we have $\bar{u}_H \in [0, (\alpha_1 + \ldots + \alpha_{k+1}) \bar{v}]$. We claim that $\bar{u}_H = 0$. By contradiction, assume that $\bar{u}_H > 0$. Take $\varepsilon = \left(\frac{1-\delta}{2}\right) \bar{u}_H$ and notice that there exist a stationary equilibrium $(\sigma_B, \sigma_S, \mu)$ and a history $\bar{h}^t$ at which the high types $q > \hat{q}$ obtain the continuation payoff $u_H \in [\bar{u}_H - \varepsilon, \bar{u}_H]$ by accepting a certain offer $\varphi_t = (k', k'c + u_H)$ for some $k' \in \{1, \ldots, k+1\}$. We claim that the conditional probability, given the beliefs $\mu\left(\bar{h}^t\right)$, that the proposal φ_t is accepted is one. First, every type $q > \hat{q}$ must accept φ_t at $\bar{h}^t$. This is because every high type gets at least $\bar{u}_H - \varepsilon$ by accepting φ_t, whereas he gets at most $\delta \bar{u}_H < \bar{u}_H - \varepsilon$ by rejecting φ_t. Thus, our claim is true if $\mu\left([0, \hat{q}] \,| \bar{h}^t\right) = 0$. If, on the other hand, $\mu\left([0, \hat{q}] \,| \bar{h}^t\right) > 0$ and the conditional probability that φ_t is accepted is not one, then $\mu\left([0, \hat{q}] \,| \bar{h}^t, (\varphi_t, R)\right) = 1$ and the continuation payoff of the low types $q \leqslant \hat{q}$ will be zero (recall Lemma 1 above). But then it is not optimal for the low types to reject φ_t.

Consider now the proposal $\varphi'_t = \left(k', k'c + u_H - \frac{\varepsilon}{2}\right)$. Using an argument similar to the one above, it is easy to see that the conditional probability, given $\mu\left(\bar{h}^t\right)$, that the proposal φ'_t is accepted is also one. Since in a stationary equilibrium the continuation payoff of the buyer depends only on the number of units left and his beliefs about the seller's type, φ'_t is a profitable deviation at $\bar{h}^t$. Thus, we conclude that $\bar{u}_H$ is equal to zero.

Fix a stationary equilibrium $(\sigma_B, \sigma_S, \mu)$ and let h^t denote an arbitrary history. Suppose that the buyer makes the offer $\varphi_t = (k, p)$ at h^t and consider type $q > \hat{q}$. The fact that q's continuation payoff is zero immediately implies that he must accept the offer φ_t if $p > kc$ and must reject it if $p < kc$. Recall that the definition of stationarity requires that q accepts the offer if and only if $p \geqslant P(k, k', q)$. It thus follows that, in any stationary equilibrium, $P(k, k', q) = k'c$ for every k, k' and $q > \hat{q}$. ■

Proof of Proposition 1.

Recall that $W(k, q)$ denotes the buyer's expected payoff when the state is (k, p) and satisfies equation (1). Let $Y(k, q)$ denote the arg max correspondence in (1). As mentioned in Section 3.1, the solution can either be of the form (k, q'), for some $q' \in [q, \hat{q}] \cup \{1\}$, or of the form (k', q) for some $k' = 1, \ldots, k-1$. In the first case, the buyer makes the cream-skimming offer $(k, P(k, q'))$ and purchases all the remaining k units from the types in the interval $[q, q']$.[37] In the second case, the buyer makes a universal offer and purchases $(k - k')$ units from the types in $[q, 1]$.

[37] Below, we show that if (k, q') is a solution of (1), then $P(k, q'') > P(k, q')$ for every $q'' > q'$.

Of course, for $q > \hat{q}$ equation (1) implies

$$W(k,q) = (1-q)\left[(\alpha_k + \ldots + \alpha_1)\,\bar{v} - kc\right].$$

Although we have a continuum of low types, it is obvious that, at every history, all of them must get the same continuation payoff (otherwise, some of these types would have an incentive to deviate and mimic the behavior of other low types). We refer to this payoff as the payoff of the low type. We let $Z_L(k,q)$ denote the low type's payoff when the state is (k,q). When the set $Y(k,q)$ is a singleton, the buyer's optimal behavior is uniquely determined. When $Y(k,q)$ contains more than one element, we select the solution in $Y(k,q)$ that yields the lowest payoff to the low types. Therefore, for any $k = 1,\ldots,m$ and any $q \in [0,\hat{q}]$, $Z_L(k,q)$ satisfies:

$$Z_L(k,q) = \min\left\{\left\{\min_{(k,q')\in Y(k,q)} P(k,q')\right\}, \left\{\min_{(k',q)\in Y(k,q)} \left[(k-k')\,c + \delta Z_L(k',q)\right]\right\}\right\}. \quad (16)$$

We let $t(k,q)$ denote the solution to (16). (If there are multiple solutions, then there exists at least one solution of the form (k',q) and we pick the one with the lowest k'.)

Finally, for every $k = 1,\ldots,m$, we define $P(k,\cdot) : [0,1] \to \mathbb{R}_+$ to be the largest increasing function that is (weakly) below the function $\delta Z_L(k,\cdot)$. Formally, $P(k,\cdot)$ satisfies the following condition.

Condition 2 *For every q, $P(k,q)$ satisfies:*
(i) $P(k,q) \leqslant \delta Z_L(k,q)$;
(ii) $P(k,q) \leqslant P(k,q')$ for every $q' > q$;
(iii) For every $\eta \in (P(k,q), \delta Z_L(k,q))$ there exists $q' > q$ such that $\delta Z_L(k,q') < \eta$.

Next, we show that it is never optimal for the buyer to choose a point in the interior of a flat segment of $P(k,\cdot)$. That is, we show that if $t(k,q) = (k,q')$, then $P(k,q'') > P(k,q')$ for every $q'' > q'$.

By contradiction, suppose $t(k,q) = (k,q')$ and $P(k,q'') = P(k,q')$ for $q'' > q'$. Suppose also that $t(k,q') = (k',q')$ and $t(k',q') = (k',q''')$ for some $q''' > q''$. Notice that we have

$$P(k,q'') = P(k,q') \leqslant \delta(k-k')\,c + \delta^2 P(k',q''').$$

Given the state (k,q), consider the following strategy. The buyer buys k parts from the types in $[q,q'']$ at the price $P(k,q'') = P(k,q')$. Then if the seller rejects the offer, the buyer purchases $(k-k')$ units from all the types. Finally the buyer purchases k' parts from the types in $(q'', q''']$. The difference between the buyer's payoff from following this strategy and $W(k,q)$ is equal to:

$$\int_{q'}^{q''}\left[(\alpha_k+\ldots+\alpha_1)v(s)-P(k,q'')\right]ds-\delta\int_{q'}^{q''}\left[(\alpha_k+\ldots+\alpha_{k'+1})v(s)-(k-k')c\right]ds-$$
$$\delta^2\int_{q'}^{q''}\left[(\alpha_{k'}+\ldots+\alpha_1)v(s)-P(k',q''')\right]ds\geqslant\int_{q'}^{q''}\left[(\alpha_k+\ldots+\alpha_1)v(s)\right]ds-$$
$$\delta\int_{q'}^{q''}\left[(\alpha_k+\ldots+\alpha_{k'+1})v(s)\right]ds-\delta^2\int_{q'}^{q''}\left[(\alpha_{k'}+\ldots+\alpha_1)v(s)\right]ds>0.$$

Here we considered a specific continuation strategy following the state (k, q'). In a similar way, it is easy to show that if the buyer does not choose the endpoint of a flat segment, then there is a strictly profitable deviation for any possible continuation strategy. We omit the details.

A quadruplet of functions $(W(\cdot), P(\cdot), Z_L(\cdot), t(\cdot))$ satisfying equations (1) and (16), and Condition 2 defines the following stationary equilibrium. First, at any history of the game, type $q > \hat{q}$ accepts an offer to sell k units if and only if the transfer is at least kc. Consider now type $q < \hat{q}$. Suppose that there are k units left for trade and type q has never accepted a transfer smaller than $k'c$ to sell $k' \leqslant m - k$ units. In this case, type q accepts the proposal (k'', p), $k'' \leqslant k$, if and only if $p \geqslant \min\{P(k,q), k''c\}$.[38] On the other hand, if type q has already accepted a transfer smaller than $k'c$ to sell $k' \leqslant m - k$ units, then he accepts any offer.

We now turn to the buyer. In the first period, the buyer makes an offer consistent with $t(m, 0)$.[39] Suppose that the seller has never accepted an offer less than $k'c$ for k' units and that the state is (k, q). Then the buyer makes an offer consistent with $t(k, q)$.[40] Finally, suppose that in the past the seller has accepted a transfer less than $k'c$ for k' units. Then the buyer offers zero to purchase all the remaining units.

As far as the buyer's beliefs are concerned, they are pinned down by Bayes' rule except for the following two cases. First, suppose that the state is (k, q), the buyer makes the offer (k', p), with $k' < k$ and $p < P(k, k', q)$, and the offer is accepted. In this case, we assume that the buyer assigns probability one to the fact that the good is of low quality. Second, assume that the state is (k, q), the buyer makes the offer (k', p), with $k' \leqslant k$ and

[38] This defines the functions $P(k, k', \cdot)$ for $k > k'$ and $q \in [0, \hat{q}]$.

[39] This means that if $t(m, 0) = (m, q)$ for some $q > 0$, then the buyer proposes to purchase all the m units at the price $P(m, q)$. If instead $t(m, 0) = (k, 0)$ for some $k < m$, then the buyer purchases $(m - k)$ units at the price $(m - k)c$.

[40] If the previous offer p for k units was not in the range of $P(k, \cdot)$, then the buyer randomizes among the offers consistent with the elements of $Y(k, q)$ so as to rationalize the low types' acceptance decision of the offer p.

$p \geqslant k'c$, and the offer is rejected. In this case, the buyer does not update his belief and the state remains (k, q).

Given how we construct the quadruplet $(W(\cdot), P(\cdot), Z_L(\cdot), t(\cdot))$ and how we define the buyer's strategy, it is immediate to see that the seller cannot profitably deviate. It is also straightforward to check that the buyer's behavior is optimal if we restrict the buyer to choosing between cream-skimming and universal offers. To conclude that we indeed have an equilibrium, it remains to show that it is not optimal for the buyer to purchase a fraction of the remaining units only from the low types. Suppose the state is (k, q), for some $q \leqslant \hat{q}$, and consider the offer (k', p), with $k' < k$ and $P(k, k', q) \leqslant p < k'c$.[41] Let $[q, q']$ denote the set of types who accept this offer. Given the seller's strategy, we have $p \geqslant P(k, q'')$ for $q'' \leqslant q'$ and $p < P(k, q'')$ for $q'' > q'$. Consider now the offer (k, p). This offer is also accepted by the types in $[q, q']$. Clearly, the buyer strictly prefers the offer (k, p) to the offer (k', p) (while the two offers specify the same payment, the discounted consumption is larger under the former offer than under the latter).

To conclude the proof of Proposition 1 it remains to show that there exists a quadruplet $(W(\cdot), P(\cdot), Z_L(\cdot), t(\cdot))$ satisfying equations (1) and (16), and Condition 2.

Claim 1 *For every $k = 1, \ldots, m$, there exists $\bar{q} < \hat{q}$ such that for $q \in [\bar{q}, \hat{q}]$*

$$W(k, q) = \int_q^1 [(\alpha_k + \ldots + \alpha_1) v(s) - kc] \, ds > 0$$
$$Z_L(k, q) = kc$$
$$P(k, q) = \delta kc$$
$$t(k, q) = (k, 1).$$

Proof. If the buyer proposes to buy all the k units at a price smaller than kc, his expected payoff is bounded above by

$$(\hat{q} - q)(\alpha_k + \ldots + \alpha_1) \underline{v} + (1 - \hat{q}) \delta [(\alpha_k + \ldots + \alpha_1) \bar{v} - kc],$$

which is smaller than

$$\int_q^1 [(\alpha_k + \ldots + \alpha_1) v(s) - kc] \, ds$$

[41] Suppose that the state is (k, q) and consider the offer (k', p), with $k' < k$ and $p < P(k, k', q)$. Below we show that $W(k'', q'') > 0$ for all k'' and $q'' < 1$. This immediately implies that the offer (k', p) is not optimal. This is because the offer is rejected with probability one and the buyer will get at most $\delta W(k, q) < W(k, q)$.

for q sufficiently close to $\hat{q}$. Using an inductive argument, it is also easy to check that buying $k' < k$ units from all the types (i.e., at the price $k'c$) is not optimal when q is close to $\hat{q}$. ■

For $k = 1$, the quadruplet $(W(1,\cdot), P(1,\cdot), Z_L(1,\cdot), t(1,\cdot))$ is as in DL. For $k = 2, \ldots, m$, we assume that $(W(\cdot), P(\cdot), Z_L(\cdot), t(\cdot))$ are defined for $k' < k$ and extend the construction to k. Claim 1 also allows us to assume that $(W(k,\cdot), P(k,\cdot), Z_L(k,\cdot), t(k,\cdot))$ are defined over the interval $[q_n, 1]$ for some $q_n < \hat{q}$ and $W(k,q) > 0$ for every $q \in [q_n, 1]$. We now extend the quadruplet $(W(k,\cdot), P(k,\cdot), Z_L(k,\cdot), t(k,\cdot))$ to the interval $[q_{n+1}, 1]$ for some $q_{n+1} < q_n$.

For $q \in [0, q_n]$, define $\tilde{W}(k,q)$ and $\tilde{X}(k,q)$ as follows:

$$\tilde{W}(k,q) = \max_{q' \geqslant q_n} \int_q^{q'} [(\alpha_k + \ldots + \alpha_1) v(s) - P(k,q')] \, ds + \delta W(k,q'), \tag{17}$$

$$\tilde{X}(k,q) = \arg\max_{q' \geqslant q_n} \int_q^{q'} [(\alpha_k + \ldots + \alpha_1) v(s) - P(k,q')] \, ds + \delta W(k,q').$$

It is easy to check that if $q' \in \tilde{X}(k,q)$, then $P(k,q'') > P(k,q')$ for every $q'' > q'$ (i.e., the maximum is never achieved on the interior of a flat segment of P). The objective function in (17) has strictly increasing differences in q at all maximizers q'. Thus, $\tilde{X}$ is a nondecreasing correspondence: if $q > q'$, then $\tilde{q} \geqslant \tilde{q}'$ for any pair $\tilde{q} \in \tilde{X}(k,q)$, $\tilde{q}' \in \tilde{X}(k,q')$. Also, from the theorem of the maximum, $\tilde{X}$ is upper hemicontinuous.

We let $\tilde{x}(k,q) \in [q_n, 1]$ denote the smallest element in $\tilde{X}(k,q)$. We also define for $q \in [0, q_n]$ the acceptance function $\tilde{P}(k,q) = \delta P(k, \tilde{x}(k,q))$.

Next, for $q \in [0, q_n]$, define $\hat{W}(k,q)$ as follows:

$$\hat{W}(k,q) = \max_{q' \in [q, q_n]} \int_q^{q'} \left[(\alpha_k + \ldots + \alpha_1) v(s) - \tilde{P}(k,q') \right] ds + \delta \tilde{W}(k,q').$$

We let $\hat{x}(k,q) \in [q, q_n]$ denote the smallest element of the arg max correspondence in the above expression. Finally, define $q_{n+1} = \max\left\{ q \in [0, q_n] : \tilde{W}(k,q) \leqslant \hat{W}(k,q) \right\}$ whenever the set is nonempty and $q_{n+1} = 0$ otherwise. Notice that $\tilde{W}(k,q_n) > 0$ ($\tilde{W}(k,q_n)$ is equal to $\delta W(k,q_n) > 0$ if we choose $q' = q_n$ in (17)), and $\hat{W}(k,q_n) = \delta \tilde{W}(k,q_n) < \tilde{W}(k,q_n)$. By the theorem of the maximum, $\tilde{W}$ and $\hat{W}$ are continuous and, therefore, $q_{n+1} < q_n$. By definition of q_{n+1}, we have $\tilde{W}(k,q) > \hat{W}(k,q)$ for $q > q_{n+1}$. It is easy to show that $\tilde{W}(k,q) \leqslant \hat{W}(k,q)$ for $q \leqslant q_{n+1}$ (the function $\tilde{W}(k,q) - \hat{W}(k,q)$ is increasing in $[0, q_{n+1}]$ and is equal to zero at q_{n+1}).

Next, we claim that $\tilde{W}(k,q) > 0$ for all $q \in [q_{n+1}, q_n]$. For q_n we have $\tilde{W}(k,q_n) \geqslant \delta W(k,q_n) > 0$. For $q \in [q_{n+1}, q_n)$, let $x(k,q) \in [q_n, 1]$ be the largest element in $\tilde{X}(k,q)$

and let ε be such that $q+\varepsilon \in (q, q_n)$. Since $x(k,q)$ is feasible at $q+\varepsilon$, we have

$$\tilde{W}(k, q+\varepsilon) \geqslant \int_{q+\varepsilon}^{x(k,q)} [(\alpha_k + \ldots + \alpha_1) v(s) - P(k, x(k,q))] \, ds + \delta W(k, x(k,q)).$$

It then follows from

$$\tilde{W}(k,q) = \int_{q}^{x(k,q)} [(\alpha_k + \ldots + \alpha_1) v(s) - P(k, x(k,q))] \, ds + \delta W(k, x(k,q))$$

that

$$\tilde{W}(k, q+\varepsilon) \geqslant \tilde{W}(k,q) - \int_{q}^{q+\varepsilon} [(\alpha_k + \ldots + \alpha_1) v(s) - P(k, x(k,q))] \, ds. \tag{18}$$

Since $q+\varepsilon < q_n$ and $\tilde{W}(k,q) > \hat{W}(k,q)$, we have

$$\tilde{W}(k,q) > \int_{q}^{q+\varepsilon} \left[(\alpha_k + \ldots + \alpha_1) v(s) - \tilde{P}(k, q+\varepsilon)\right] ds + \delta \tilde{W}(k, q+\varepsilon). \tag{19}$$

Substituting (18) into (19) yields

$$(1-\delta)\tilde{W}(k,q) >$$
$$\int_{q}^{q+\varepsilon} \{(1-\delta)(\alpha_k + \ldots + \alpha_1) v(s) - \delta [(P(k, \tilde{x}(k, q+\varepsilon)) - P(k, x(k,q)))]\} \, ds.$$

The proof is complete if we show that

$$\lim_{\varepsilon \downarrow 0} [(P(k, \tilde{x}(k, q+\varepsilon)) - P(k, x(k,q)))] = 0. \tag{20}$$

Because $\tilde{X}(k,q)$ is a nondecreasing upper hemicontinuous correspondence, we have $\lim_{\varepsilon \downarrow 0} \tilde{x}(k, q+\varepsilon) = x(k,q)$. Therefore, equality (20) can fail only if $P(k,\cdot)$ has a discontinuity at $x(k,q)$. But then we must have $\tilde{x}(k, q+\varepsilon) = x(k,q)$ for a sufficiently small ε.

Next, we define the quadruplet $(W^1(k,\cdot), P^1(k,\cdot), Z_L^1(k,\cdot), t^1(k,\cdot))$. For $q > q_n$, we let $(W^1(k,\cdot), P^1(k,\cdot), Z_L^1(k,\cdot), t^1(k,\cdot))$ be equal to $(W(k,\cdot), P(k,\cdot), Z_L(k,\cdot), t(k,\cdot))$. For $q \in [q_{n+1}, q_n]$ define

$$W^1(k,q) = \max \left\{ \tilde{W}(k,q), \left\{ \int_q^1 [(\alpha_k + \ldots + \alpha_{k'+1}) v(s) - (k-k') c] \, ds + \delta W(k', q) \right\}_{k'=1,\ldots,k-1} \right\},$$

and let $t^1(k,q)$ be the solution with the lowest continuation payoff to the low type. (If there are multiple solutions with the same continuation payoff, then there exists at least one solution of the form (k', q), and we pick the one with the lowest k'.)

If $t^1(k,q) = (k, \tilde{x}(k,q))$, then let $Z_L^1(k,q) = \delta Z_L(k, \tilde{x}(k,q))$. If $t^1(k,q) = (k',q)$, then let $Z_L^1(k,q) = (k-k')c + \delta Z_L(k',q)$. Finally, for $q \in [q_{n+1}, q_n]$, we define $P^1(k,q)$ to be the largest increasing function that is (weakly) below the function $\delta Z_L^1(k,\cdot)$.

We now inductively define a sequence of quadruplets $\left\{\left(W^\ell(k,\cdot), P^\ell(k,\cdot), Z_L^\ell(k,\cdot), t^\ell(k,\cdot)\right)\right\}_{\ell=1,2,\ldots,}$. Given $\left(W^\ell(k,\cdot), P^\ell(k,\cdot), Z_L^\ell(k,\cdot), t^\ell(k,\cdot)\right)$, we define the next element of the sequence as follows:

$$W^{\ell+1}(k,q) = \max\left\{\left(\max_{q'\in[q,1]} \int_q^{q'} \left[(\alpha_k + \ldots + \alpha_1) v(s) - P^\ell(k,q')\right] ds + \delta W^\ell(k,q')\right),\right.$$
$$\left.\left\{\int_q^1 \left[(\alpha_k + \ldots + \alpha_{k'+1}) v(s) - (k-k')c\right] ds + \delta W(k',q)\right\}_{k'=1,\ldots,k-1}\right\}.$$

We let $t^{\ell+1}(k,q)$ be the solution to the above problem with the lowest continuation payoff to the low type which we denote by $Z^{\ell+1}(k,q)$. Finally, $P^{\ell+1}(k,\cdot)$ is the largest increasing function below $\delta Z^{\ell+1}(k,\cdot)$.

Claim 2 *There exists ℓ^* such that*

$$\left(W^{\ell^*}(k,\cdot), P^{\ell^*}(k,\cdot), Z_L^{\ell^*}(k,\cdot), t^{\ell^*}(k,\cdot)\right) = \left(W^{\ell^*+1}(k,\cdot), P^{\ell^*+1}(k,\cdot), Z_L^{\ell^*+1}(k,\cdot), t^{\ell^*+1}(k,\cdot)\right).$$

Proof. Since $W^1(k,\cdot)$ is strictly positive on $[q_{n+1},1]$, there exists $\Delta > 0$, such that $W^1(k,q) > \Delta$ for every $q \in [q_{n+1},1]$. For each ℓ, $W^\ell(k,\cdot)$ is uniformly continuous and $W^\ell(k,q) \geqslant W^1(k,q)$. This implies that there exists $\varepsilon_\ell > 0$ such that for every $q \in [q_{n+1},1]$, if $t^\ell(k,q) = (k,q')$, then $q' > q + \varepsilon_\ell$.

Recall that Δ is a lower bound to $W^\ell(k,\cdot)$ for each ℓ. Let T be an integer such that

$$\Delta > \left(\frac{1}{T} + \delta^T\right)(\alpha_k + \ldots + \alpha_1)\bar{v}.$$

If the claim fails, then we can find ℓ, $q \in [q_{n+1}, q_n)$, and a sequence $q = q^0 < q^1 < \ldots < q^T < q + \frac{1}{T}$ such that for every $\tau = 1,\ldots,T$, $t^\ell(k, q^{\tau-1}) = (k, q^\tau)$. But then we have the following contradiction:

$$\Delta < W^\ell(k,q) < \left(\frac{1}{T} + \delta^T\right)(\alpha_k + \ldots + \alpha_1)\bar{v} < \Delta.$$

■

At this point we are ready to extend $(W(k,\cdot), P(k,\cdot), Z_L(k,\cdot), t(k,\cdot))$ to the interval $[q_{n+1}, q_n]$ by setting them equal to $\left(W^{\ell^*}(k,\cdot), P^{\ell^*}(k,\cdot), Z_L^{\ell^*}(k,\cdot), t^{\ell^*}(k,\cdot)\right)$.

Finally, we show that it takes finitely many steps to extend the function $W(k,\cdot)$ to the whole unit interval. By contradiction, suppose that $\lim_{n\to\infty} q_n = q^* > 0$. We distinguish between the following two cases.

Case 1. First, consider the case in which there exists a sequence of points $\{q_j\}_{j=1}^{\infty}$ converging (from above) to q^* such that for each q_j

$$W(k, q_j) = \int_{q_j}^{q'_j} \left[(\alpha_k + \ldots + \alpha_1) v(s) - P\left(k, q'_j\right)\right] ds + \delta W\left(k, q'_j\right) \tag{21}$$

for some $q'_j > q_j$.

From this it follows that $\lim_{q \to q^*} P(k, q) = 0$ and, therefore, $\inf_{(q^*,1]} W(k, q) > 0$. We now choose $\varepsilon > 0$ to satisfy

$$[(\alpha_k + \ldots + \alpha_1) \bar{v} + \delta] \varepsilon < (1 - \delta) \inf_{(q^*,1]} W(k, q). \tag{22}$$

Because of the uniform continuity of $W(k, \cdot)$, given ε we can find $\eta \in (0, \varepsilon)$, such that for every $(q, q') \in (q^*, 1]^2$, if $|q - q'| < \eta$, then $|W(k, q) - W(k, q')| < \varepsilon$.

The fact that the sequence $\{q_j\}_{j=1}^{\infty}$ converges to q^* and satisfies (21) implies that there exists $\hat{\jmath} = 1, 2, \ldots,$ such that $|q_{\hat{\jmath}} - q'_{\hat{\jmath}}| < \eta < \varepsilon$. If we let $\underline{w}$ denote the minimum between $W(k, q_{\hat{\jmath}})$ and $W\left(k, q'_{\hat{\jmath}}\right)$ we have the following contradiction:

$$\underline{w} \leqslant W(k, q_{\hat{\jmath}}) \leqslant (\alpha_k + \ldots + \alpha_1) \bar{v}\varepsilon + \delta W\left(k, q'_{\hat{\jmath}}\right) \leqslant$$
$$(\alpha_k + \ldots + \alpha_1) \bar{v}\varepsilon + \delta(\underline{w} + \varepsilon) < \underline{w}$$

where the last inequality follows from (22) and $\underline{w} \geqslant \inf_{(q^*,1]} W(k, q)$.

Case 2. Consider now the case in which there exists an interval $(q^*, \breve{q})$ and $k' < k$ such that for each $q \in (q^*, \breve{q})$, $t(k, q) = (k', q)$.

In this case, we can find $\hat{n}$ such that for every $n \geqslant \hat{n}$, $t(k, q_n) = (k', q_n)$ and $t(k', q_n) = (k', \tilde{q})$ for some $\tilde{q} > q_{\hat{n}}$. Furthermore, the sequence $\{q_n\}_{n=\hat{n}}^{\infty}$ converges (from above) to q^*. First, we show that $\lim_{n\to\infty} W(k, q_n) = 0$. To see this, recall our definition of $\tilde{W}$, $\tilde{P}$ and $\hat{W}$. The function $\tilde{W}$ is uniformly continuous. Fix $\varepsilon > 0$, and let $\eta \in (0, \varepsilon)$ be such $\left|\tilde{W}(k, q) - \tilde{W}(k, q')\right| < \varepsilon$ for every (q, q') with $|q - q'| < \eta$. Furthermore, there exists n' such that $q_n - q_{n+1} < \eta$ for every $n > n'$. Therefore, if $n > n'$ we have

$$\tilde{W}(k, q_{n+1}) = \hat{W}(k, q_{n+1}) = \max_{q' \in [q_{n+1}, q_n]} \int_q^{q'} \left[(\alpha_k + \ldots + \alpha_1) v(s) - \tilde{P}(k, q')\right] ds + \delta \tilde{W}(k, q') \leqslant$$
$$\varepsilon (\alpha_k + \ldots + \alpha_1) \bar{v} + \delta \sup_{q' \in [q_{n+1}, q_n]} \tilde{W}(k, q') \leqslant \varepsilon [(\alpha_k + \ldots + \alpha_1) \bar{v} + \delta] + \delta \tilde{W}(k, q_{n+1})$$

and, therefore,

$$\tilde{W}(k, q_{n+1}) \leqslant \frac{\varepsilon [(\alpha_k + \ldots + \alpha_1) \bar{v} + \delta]}{(1 - \delta)}.$$

This implies $\lim_{n\to\infty}\tilde{W}(k,q_n)=0$. On the other hand, for $n>\hat{n}$, $\tilde{W}(k,q_{n+1})\geqslant -\varepsilon kc+\delta W(k,q_n)$. Thus, we have $\lim_{n\to\infty}W(k,q_n)=0$.

Next, we claim that

$$\delta\left[(k-k')\,c+\delta P(k',\tilde{q})\right]\geqslant(\alpha_k+\ldots+\alpha_1)\,\underline{v}.$$

The left hand side is an upper bound to the reservation price $P(k,q_n)$ for k units of any type q_n with $n\geqslant\hat{n}$. If the inequality is violated, then $W(k,q_n)$ is bounded below by

$$(q_{\hat{n}}-q_n)\left[(\alpha_k+\ldots+\alpha_1)\,\underline{v}-\delta\left[(k-k')\,c+\delta P(k',\tilde{q})\right]\right]>0$$

contradicting the fact that $\lim_{n\to\infty}W(k,q_n)=0$.

For $q<q_{\hat{n}}$, suppose the buyer adopts the following strategy. First, he purchases k units at the price $P(k,q_{\hat{n}})\leqslant\delta\left[(k-k')\,c+\delta P(k',\tilde{q})\right]$ from the types in $[q,q_{\hat{n}}]$. Then he purchases $(k-k')$ units from all the types in $[q_{\hat{n}},1]$. Finally, he follows the optimal strategy given $(k',q_{\hat{n}})$. The buyer's payoff from adopting such a strategy is weakly larger than $R(q)$ defined by

$$R(q)=\int_q^{q_{\hat{n}}}\left[(\alpha_k+\ldots+\alpha_1)\,\underline{v}-\delta\left[(k-k')\,c+\delta P(k',\tilde{q})\right]\right]ds+$$
$$\delta\int_{q_{\hat{n}}}^1\left[(\alpha_k+\ldots+\alpha_{k'+1})\,v(s)-(k-k')\,c\right]ds+\delta^2\int_{q_{\hat{n}}}^{\tilde{q}}\left[(\alpha_{k'}+\ldots+\alpha_1)\,\underline{v}-P(k',\tilde{q})\right]ds+\delta^3W(k',\tilde{q}).$$

Notice that $R(q)$ is increasing in q.

For $q<q_{\hat{n}}$, let $V(q)$ be equal to

$$V(q)=\int_q^{q_{\hat{n}}}\left[(\alpha_k+\ldots+\alpha_{k'+1})\,\underline{v}-(k-k')\,c\right]ds+\int_{q_{\hat{n}}}^1\left[(\alpha_k+\ldots+\alpha_{k'+1})\,v(s)-(k-k')\,c\right]ds+$$

$$\delta\int_q^{q_{\hat{n}}}\left[(\alpha_{k'}+\ldots+\alpha_1)\,\underline{v}-P(k',\tilde{q})\right]ds+\delta\int_{q_{\hat{n}}}^{\tilde{q}}\left[(\alpha_{k'}+\ldots+\alpha_1)\,\underline{v}-P(k',\tilde{q})\right]ds+\delta^2W(k',\tilde{q})$$

and notice that $W(k,q)=V(q)$ for $q\in(q^*,q_{\hat{n}}]$. Let $\bar{q}$ be such that $R(\bar{q})=V(\bar{q})$. Such $\bar{q}$ is well defined since it is the solution to the following equation:

$$\left[\int_{q_{\hat{n}}}^1\left[(\alpha_k+\ldots+\alpha_{k'+1})\,v(s)-(k-k')\,c\right]ds+\delta\int_{q_{\hat{n}}}^{\tilde{q}}\left[(\alpha_{k'}+\ldots+\alpha_1)\,\underline{v}-P(k',\tilde{q})\right]ds+\delta^2W(k',\tilde{q})\right]=$$
$$\int_{\bar{q}}^{q_{\hat{n}}}\left[(\alpha_{k'}+\ldots+\alpha_1)\,\underline{v}+(k-k')\,c+\delta P(k',\tilde{q})\right]ds. \tag{23}$$

For $q\in(q^*,q_{\hat{n}}]$, the left hand side is larger than the right hand side in (23) since $V(q)$ is the value of the optimal strategy. Of course, the inequality is reversed as $\bar{q}$ goes to $-\infty$. Thus $\bar{q}$ exists and is weakly smaller than q^*.

Using (23), we obtain

$$R(\bar{q}) = V(\bar{q}) \geqslant (q_{\hat{n}} - \bar{q})(\alpha_k + \ldots + \alpha_1)\underline{v} > 0.$$

Recall that $\bar{q} \leqslant q^*$ and $R(\cdot)$ is increasing in q. For every $n \geqslant \hat{n}$, we have

$$W(k, q_n) = V(q_n) \geqslant R(q_n) \geqslant R(\bar{q}) > 0$$

which contradicts the fact that $\lim_{n\to\infty} W(k, q_n) = 0$. ■

Appendix B: Uniqueness

Proof of Proposition 2.

First, we prove the result under the assumption that $P(k, \cdot)$ is weakly increasing and, without loss of generality, left-continuous for all k. Second, we give a general argument.

Step 1: *For generic parameters all stationary equilibria with increasing reservation price functions are outcome equivalent.*

DL establish that when the parties trade a single unit there is (for a generic set of parameters) a unique left-continuous reservation price function $P(1, \cdot)$, which is equal to the one constructed in Appendix A. First, we consider equilibria satisfying Condition 1 below.

Condition 1: *At any history in which there are k units remaining, the buyer either makes a cream-skimming offer or makes a universal offer for $k' < k$ units.*

Assume that for some $k = 2, \ldots, m$, there is a generic set of parameters for which for all $k' < k$, $P(k', \cdot)$ constructed in Appendix A is the unique reservation price function in all stationary equilibria. It is straightforward to show that there is $\bar{q} < 1$ such that if the state is (k, q') with $q' \geqslant \bar{q}$, then the buyer makes the offer (k, kc) and this offer is accepted with probability one in all stationary equilibria. Define

$$q^* := \inf \left\{ \begin{array}{c} q' : \text{there exists a generic set of parameters such that for all } q > q' \\ P(k,q) \text{ constructed in Appendix A is the unique} \\ \text{reservation price in all stationary equilibria} \end{array} \right\}.$$

We show that there is $\varepsilon > 0$ and a generic set of parameters such that for all $q > q^* - \varepsilon$, our $P(k, q)$ is the unique reservation price in all stationary equilibria.

Consider a state (k, q) with $q < q^*$. First, impose that the buyer is obliged to make a cream-skimming offer $(k, P(k, \tilde{q}))$ for some $\tilde{q} \geqslant q$. Using the same argument as in Appendix A we conclude that there exists $\varepsilon > 0$ such that if $q > q^* - \varepsilon$ then in any stationary equilibria, the buyer's optimal offer is $(k, P(k, q'))$ with $q' > q^* + \varepsilon$. Let us now restrict our attention to the states (k, q) with $q \in (q^* - \varepsilon, q^*]$.

Consider the choice between a cream-skimming offer and a universal offer for $k' < k$ units. Observe that, by assumption, for every (k', q) with $k' < k$ and $q \in [0, 1]$ or $k' = k$ and $q > q^*$, $P(k', q)$ is unique in all stationary equilibria. Hence, the low type's continuation payoff at the state (k', q) is the same in all stationary equilibria. Thus, for a generic set of parameters, the buyer has a unique best response at (k, q). This implies that the unique (left-continuous) extension of P to (k, q) with $q > q^* - \varepsilon_1$ is the reservation price function that we constructed in Appendix A. This, in turn, shows the uniqueness of the reservation price functions under the assumption that the functions are weakly increasing and the buyer makes only cream-skimming or universal offers.

Next, we show that Condition 1 holds generically (this guarantees the uniqueness of stationary equilibrium outcomes when the reservation price functions are increasing). It follows from DL that Condition 1 holds for $k = 1$. Assume that for some $k = 2, \ldots, m$, Condition 1 holds for all $k' < k$. Notice that there exists $\tilde{q} \in (0, 1)$ such that whenever the state is (k, q) and $q > \tilde{q}$ the buyer makes the offer (k, kc). Define

$$q^* := \inf \left\{ \begin{array}{c} q' : \text{there exists a generic set of parameters such that for all } q > q' \\ \text{Condition 1 holds at the state } (k, q) \end{array} \right\}.$$

We now show that there exists $\varepsilon > 0$ such that generically Condition 1 holds at every state (k, q) with $q > q^* - \varepsilon$. By contradiction, assume that this is not true. It is straightforward to show that there exist $\varepsilon > 0$, a stationary equilibrium and a state (k, q), with $q > q^* - \varepsilon$, such that the following two conditions are satisfied. First, the buyer makes the offer (k', p) with $k' < k$ and $p < k'c$. Second, if the offer is rejected, then the state becomes (k, q') with $q' > q^* + \varepsilon$.

From our analysis above, we may assume that for all $q > q^*$ the reservation prices coincide with those constructed in Appendix A. This and the fact that $q' > q^* + \varepsilon$ imply that the low types' continuation payoff from rejecting the offer (k', p) is at least $\delta Z_L (k, q')$ (recall that in Appendix A the reservation price functions are constructed to give the lowest continuation payoff to the seller's low types). Now it follows from Lemma 1 that the continuation payoff of the (low) types who accept the offer (k', p) is equal to zero. Thus, $p \geqslant \delta Z_L (k, q')$ since some low types accept the offer (k', p). But then the buyer has a profitable deviation at the state (k, q) by making the cream-skimming offer $(k, \delta Z_L (k, q'))$ (notice that this offer is weakly more likely to be accepted than the equilibrium offer (k', p)). This concludes the proof of Step 1.

Step 2: *General result*

Next, we claim that there is a generic set of parameters for which, for every stationary equilibrium, there is an outcome equivalent stationary equilibrium in which all the reservation price functions are increasing. This, together with Step 1, proves Proposition 2.

From the results in DL we know that the claim above is true when $m = 1$. Consider the game in which there are $m > 1$ parts and take a stationary equilibrium. Let $P(1, \cdot), \ldots, P(m, \cdot)$ denote the corresponding reservation price functions. We consider the following relabeling process, which does not change the outcome of the game.

i) *Relabeling the types for the first unit*

Since $P(1, \cdot) : [0, 1] \to \mathbb{R}$ is measurable, there exists a one-to-one (measurable) function $g : [0, 1] \to [0, 1]$ such that $\hat{P}(1, \cdot)$ defined by

$$\hat{P}(1, q) = P(1, g(q))$$

is increasing (almost everywhere). Without loss we assume that $\hat{P}(1, \cdot)$ is increasing everywhere and left-continuous.[42] For every $k = 2, ..., m$ define $\hat{P}(k, \cdot) : [0, 1] \to \mathbb{R}$ by:

$$\hat{P}(k, q) = P(k, g(q)).$$

From DL we know that $\hat{P}(1, \cdot)$ is a step function. We claim that for every $k \in \{2, ..., m\}$ the regions in which $\hat{P}(k, \cdot)$ fails to be monotonic lie in flat regions of $\hat{P}(1, \cdot)$. Using $m(\cdot)$ for the Lebesgue measure, the formal statement is as follows.

Claim 3 *Let Ω be the generic set of parameters under which Step 1 holds. Let $q_1 < q_2 < q_3 < q_4$ be such that $\hat{P}(1, \cdot)$ is constant both in the interval $[q_1, q_2]$ and in $[q_3, q_4]$, and $\hat{P}(1, q_1) < \hat{P}(1, q_3)$. Take $k \in \{2, ..., m\}$, $p \in \mathbb{R}$ and assume that $\hat{P}(k, q) \geqslant p$ for some subset of $[q_1, q_2]$ with positive measure. Finally, let $A := \{q \in [q_3, q_4] : \hat{P}(k, q) < p\}$. Then $m(A) = 0$.*

Proof. The proof is by contradiction. Define $B := \{q \in [q_1, q_2] : \hat{P}(k, q) \geqslant p\}$ and assume that $m(A) > 0$ and $m(B) > 0$.

Consider now the following history. At the beginning of the game the buyer purchases $m - k$ units from all the types at the price $(m - k)c$. Then he makes an offer for the k remaining units which is accepted by all the types in A and is rejected by all the types in B. Following this rejection the buyer purchases $k - 1$ units from all the remaining types at the price $(k - 1)c$. Let $C \subset [0, 1]$ denote the set of types who still have one unit to trade.

[42] This is possible because the set of discontinuities of $\hat{P}(1, \cdot)$ is countable.

Notice that C contains "gaps". We would like to relabel C in such a way that it contains no "gaps". To do so, define a mapping $h : C \to [0,1]$ implicitly by:

$$m(q' : q' \in C \cap [q,1]) = 1 - h(q).$$

Let $\underline{q} = \inf\{h(q) : q \in C\}$, and notice that $m(C) = 1 - \underline{q}$ and for any $\tilde{q} \in (\underline{q}, 1)$ we have

$$m(q' : q' \in C, h(q') \geqslant \tilde{q}) = \int_{q \in C \cap [\inf\{q' : h(q') \geqslant \tilde{q}\},1]} dm(q) = 1 - \tilde{q}.$$

Next, we define a function $\tilde{P}(1, \cdot) : [1 - \underline{q}, 1] \to \mathbb{R}$ by:

$$\tilde{P}(1, q) := \sup\{\hat{P}(1, h(\tilde{q})) : \ h(\tilde{q}) < q\}.$$

The function $\tilde{P}(1, \cdot)$ is increasing since both $\hat{P}(1, \cdot)$ and $h(\cdot)$ are increasing. By changing $\tilde{P}(1, \cdot)$ in a set of measure zero if necessary, we may assume that $\tilde{P}$ is left-continuous. Therefore, it follows from the analysis in Step 1 that the functions $\tilde{P}(1, \cdot)$ and $\hat{P}(1, \cdot)$ must coincide on the interval $[1 - \underline{q}, 1]$. However, we now show that they differ. Indeed, define q^* and q^{**} by

$$q^* := \sup\{q : \hat{P}(1, q) \leqslant \hat{P}(1, q_3)\}, \quad q^{**} := \sup\{q : \tilde{P}(1, q) \leqslant \hat{P}(1, q_3)\}.$$

The fact that $m(A) > 0$ and $m(B) > 0$ implies $q^* < q^{**}$ and, therefore, $m(\{q \geqslant 1 - \underline{q} : \hat{P}(1, q) > \tilde{P}(1, q)\}) > 0$. Thus, we conclude that the parameters of the model do not belong to Ω.■

ii) *Inductively relabeling the types for the next unit*

Recall that $P(1, \cdot), \ldots, P(m, \cdot)$ denote the equilibrium reservation price functions. Assume, as an induction hypothesis, that for some $\tilde{k} = 2, \ldots, m-1$ there exists a one-to-one (measurable) function $g : [0,1] \to [0,1]$ such that $\hat{P}(k, \cdot)$, $k = 1, \ldots, \tilde{k}$, defined by

$$\hat{P}(k, q) = P(k, g(q))$$

is an increasing and left-continuous step function. For every $k' = \tilde{k} + 1, \ldots, m$ define $\hat{P}(k', \cdot) : [0,1] \to \mathbb{R}$ by:

$$\hat{P}(k', q) = P(k', g(q)).$$

The proof is complete if we show that for every $k' > \tilde{k}$ the regions in which $\hat{P}(k', \cdot)$ fails to be monotonic lie in flat regions of $\hat{P}(1, \cdot), \ldots, \hat{P}\left(\tilde{k}, \cdot\right)$.

Claim 4 *Let Ω be the generic set of parameters under which Step 1 holds. Fix $k = 1, \ldots, \tilde{k}$ and let $q_1 < q_2 < q_3 < q_4$ be such that $\hat{P}(k, \cdot)$ is constant both in the interval $[q_1, q_2]$ and in $[q_3, q_4]$, and $\hat{P}(k, q_1) < \hat{P}(k, q_3)$. Take $k' \in \tilde{k}+1, \ldots, m$, $p \in \mathbb{R}$ and assume that $\hat{P}(k', q) \geqslant p$ for some subset of $[q_1, q_2]$ with positive measure. Finally, let $A := \{q \in [q_3, q_4] : \hat{P}(k', q) < p\}$. Then $m(A) = 0$.*

Proof Define $B := \{q \in [q_1, q_2] : \hat{P}(k', q) \geqslant p\}$ and assume towards a contradiction that $m(A) > 0$ and $m(B) > 0$. Consider now the following history. At the beginning of the game the buyer purchases $m - k'$ units from all the types at the price $(m - k')\, c$. Then he makes an offer for the k' remaining units, which is accepted by all the types in A and is rejected by all the types in B. Following this rejection, the buyer purchases $k' - k$ units from all the remaining types at the price $(k' - k)\, c$. Proceeding analogously to the proof of Claim 3 we conclude that in the model with k units the outcome of stationary equilibria with increasing reservation price functions is not unique. Then, it follows from Step 1 that the parameters of the model do not belong to Ω and. This concludes the proof of Proposition 2. ■

Appendix C: Proof of Theorem 1

In this appendix, we provide the proof of Theorem 1. For each $m = 1, 2, \ldots,$ we let

$$A_m := \left\{ \left(z_1^m, q_1^m\right), \ldots, \left(z_{N_m}^m, q_{N_m}^m\right) \right\}$$

denote the set of impasses of the m-limiting equilibrium outcome. It is convenient to reverse the order adopted in Section 7 and assume that $\left(z_{N_m}^m, q_{N_m}^m\right)$ is the *first* impasse of the m-limiting equilibrium outcome, and $\left(z_1^m, q_1^m\right) = \left(\frac{1}{m}, q\left(\frac{1}{m}\right)\right)$ is the *last* impasse. Therefore, $q_1^m > q_2^m > \ldots > q_{N_m}^m$.

In what follows, we prove the first two results in Theorem 1.

Claim 5 *The limiting equilibrium outcome satisfies the following properties:*

i)

$$\lim_{m \to \infty} \max_{j \in \{2, \ldots, N_m\}} \left| q_j^m - q_{j-1}^m \right| = 0$$

ii)

$$\lim_{m \to \infty} \max_{j \in \{2, \ldots, N_m\}} \left| z_{j-1}^m - z_j^m \right| = 0.$$

The first equation in Claim 5 establishes that the largest probability that a cream-skimming offer is accepted converges to zero. The second equation shows that the size of the largest universal offer also shrinks to zero.

Once we have established Claim 5, it is immediate to derive the remaining results in Theorem 1 and show that trade of an infinitely divisible high-quality good will occur gradually over time. This result leads to the differential equation (15) which determines the rate at which the information is revealed and the good is sold.

To prove Claim 5, we derive a dynamic system that links the beliefs at (the limits of) three consecutive impasses (see equation (32) below). The system follows from the following three conditions. First, the buyer's payoff from trading between two impasses is equal to zero. Second, there is double delay at every impasse. Finally, the seller's low types are indifferent between accepting and rejecting cream-skimming offers. Below is the formal derivation of equation (32). We then analyze the dynamic system and show that, in the limit, the distance between the beliefs at two consecutive impasses must shrink to zero.

Assume towards a contradiction that

$$\limsup_{m} \max_{A_m} \left| q_u^m - q_{u-1}^m \right| > \varepsilon,$$

for some $\varepsilon > 0$. By taking a subsequence if necessary, we may assume that there exists a sequence $\left\{ \left(\left(z_{r_m+1}^m, q_{r_m+1}^m \right), \left(z_{r_m}^m, q_{r_m}^m \right) \right) \right\}_{m=1}^{\infty}$, with $\left(\left(z_{r_m+1}^m, q_{r_m+1}^m \right), \left(z_{r_m}^m, q_{r_m}^m \right) \right) \in A_m \times A_m$ for every m, converging to $\left(\left(z_1, q_1 \right), \left(z_0, q_0 \right) \right)$ such that

$$q_0 - q_1 = \nu > 0. \tag{24}$$

First, we claim that $z_j = \psi\left(q_j\right)$, for $j = 0, 1$. Here we provide the proof for $j = 1$ (the proof for $j = 0$ is similar and therefore omitted). It follows from Proposition 5-i) that $q_{r_m+1}^m$ lies between $\bar{q}_{j_m+1}$ and $\bar{q}_{j_m}$, for some $j_m = 1, \ldots, m-1$, where $\bar{q}_{j_m+1}$ and $\bar{q}_{j_m}$ satisfy

$$\int_{\bar{q}_{j_m+1}}^{1} \left[\left(\int_{\frac{j_m}{m}}^{\frac{j_m+1}{m}} \alpha\left(u\right) du \right) v\left(s\right) - \frac{\bar{c}}{m} \right] ds = 0,$$

$$\int_{\bar{q}_{j_m}}^{1} \left[\left(\int_{\frac{j_m-1}{m}}^{\frac{j_m}{m}} \alpha\left(u\right) du \right) v\left(s\right) - \frac{\bar{c}}{m} \right] ds = 0.$$

Since α' is continuous, we conclude that $\bar{q}_{j_m+1}$ and $\bar{q}_{j_m}$ become uniformly close as m goes to infinity, which leads to the desired conclusion.

Second, Proposition 5-ii) implies that the buyer's payoff from transiting from the state $\left(z_{r_m+1}^m, q_{r_m+1}^m \right)$ to the state $\left(z_{r_m}^m, q_{r_m}^m \right)$ must be equal to zero. There are two different ways to express the payoff from this transition. We can either assume that the buyer first makes the universal offer to purchase $m\left(z_{r_m+1}^m - z_{r_m}^m \right)$ units from all the types above $q_{r_m+1}^m$. Then he proposes to purchase $m z_{r_m}^m$ units at the price $P_m\left(m z_{r_m}^m, \left(q_{r_m}^m \right)^- \right)$. This offer is accepted by all the (low) types in the interval $\left[q_{r_m+1}^m, q_{r_m}^m \right]$. Alternatively, we can assume that the buyer first makes the offer to purchase $mz^m_{r_m+1}$ units at the price $P_m\left(mz^m_{r_m+1}, \left(q^m_{r_m+1}\right)^+\right)$. Again, this cream-skimming offer is accepted by all the types in $\left[q^m_{r_m+1}, q^m_{r_m}\right]$. If the offer is rejected, then the buyer makes the universal offer to purchase $m\left(z^m_{r_m+1} - z^m_{r_m}\right)$ from all the types above $q^m_{r_m}$. It turns out to be more convenient to work with the second expression. Thus, we have:

$$\begin{aligned}\left(q^m_{r_m} - q^m_{r_m+1}\right)\left[\int_0^{z^m_{r_m+1}} \alpha(u)\underline{v}du - P_m\left(mz^m_{r_m+1}, \left(q^m_{r_m+1}\right)^+\right)\right] + \\ \int_{z^m_{r_m}}^{z^m_{r_m+1}}\left(\int_{q^m_{r_m}}^1 [\alpha(u)v(s) - \bar{c}]\,ds\right)du = 0.\end{aligned} \tag{25}$$

The expression in (25) is continuous in $\left(z^m_{r_m+1}, q^m_{r_m+1}, z^m_{r_m}, q^m_{r_m}\right)$ and $P_m\left(mz^m_{r_m+1}, \left(q^m_{r_m+1}\right)^+\right)$. Hence, since the sequence $\left\{\left(\left(z^m_{r_m+1}, q^m_{r_m+1}\right), \left(z^m_{r_m}, q^m_{r_m}\right)\right)\right\}_{m=1}^\infty$ converges to $((z_1, q_1), (z_0, q_0))$, it follows that the sequence $\left\{P_m\left(mz^m_{r_m+1}, \left(q^m_{r_m+1}\right)^+\right)\right\}$ is convergent and the limit, which we denote by $\bar{P}_1$, must satisfy

$$(q_0 - q_1)\left[\int_0^{z_1} \alpha(u)\underline{v}du - \bar{P}_1\right] + \int_{z_0}^{z_1}\left(\int_{q_0}^1 [\alpha(u)v(s) - \bar{c}]\,ds\right)du = 0. \tag{26}$$

The two components of the buyer's payoff from the transition between (the limits of) two consecutive impasses can be easily visualized in Figure 6.

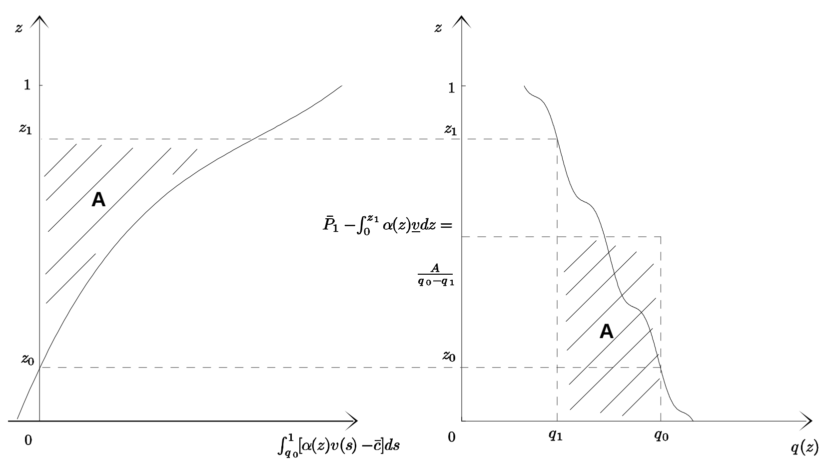

Figure 6: The buyer's payoff from transiting between two consecutive impasses

Recall that $z_j = \psi(q_j)$, for $j = 0, 1$, and $\psi'(q) < 0$ for every q. Thus, we may change variables and write equation (26) as

$$(q_0 - q_1)\left[\int_0^{\psi(q_1)} \alpha(u)\,\underline{v}du - \bar{P}_1\right] + \int_{q_1}^{q_0}\left[-\psi'(u)\left(\int_{q_0}^{1}\left[\alpha(\psi(u))\,v(s) - \bar{c}\right]ds\right)\right]du = 0. \tag{27}$$

Next, we show that $q_0 < q(0)$ (and, therefore, $z_0 > 0$). The proof is by contradiction. It follows easily from Proposition 5-iii) that if $z_0 = 0$, then $\bar{P}_1 = z_1\bar{c}$. But then as m goes to infinity, the buyer's payoff from transiting from the state $\left(z^m_{r_m+1}, q^m_{r_m+1}\right)$ to the state $\left(z^m_{r_m}, q^m_{r_m}\right)$ converges to

$$(q(0) - q_1)\left[\int_0^{z_1} \alpha(u)\,\underline{v}du - z_1\bar{c}\right] + \int_0^{z_1}\left(\int_{q(0)}^1 \left[\alpha(u)\,v(s) - \bar{c}\right]ds\right)du =$$
$$\int_0^{z_1}\left(\int_{q_1}^1 \left[\alpha(u)\,v(s) - \bar{c}\right]ds\right)du < 0,$$

which contradicts equation (26).

Therefore, for m large there is an impasse $\left(z^m_{r_m-1}, q^m_{r_m-1}\right)$ that comes right after $\left(z^m_{r_m}, q^m_{r_m}\right)$. Again, by taking a subsequence if necessary, assume that the sequence $\left\{\left(z^m_{r_m-1}, q^m_{r_m-1}\right)\right\}$ converges and let (z_{-1}, q_{-1}) denote its limit. We claim that $q_{-1} \in (q_0, q(0))$. The proof that $q_{-1} < q(0)$ is identical to the proof above that $q_0 < q(0)$. We now show that $q_{-1} > q_0$. Recall that $q^m_{r_m-1} > q^m_{r_m}$ for every m, and assume, by contradiction, that $q_{-1} = q_0$. Also, recall that $\left(z^m_{r_m+1}, q^m_{r_m+1}\right)$ and $\left(z^m_{r_m}, q^m_{r_m}\right)$ are two consecutive impasses. Thus, it follows from Proposition 5 that if $q_{-1} = q_0$, then

$$\bar{P}_1 = (z_1 - z_0)\,\bar{c} + \int_0^{z_0} \alpha(u)\,\underline{v}du.$$

This, in turn, implies that, as m goes to infinity, the buyer's payoff from transiting from $\left(z^m_{r_m+1}, q^m_{r_m+1}\right)$ to $\left(z^m_{r_m}, q^m_{r_m}\right)$ converges to

$$(q_0 - q_1)\left[\int_0^{z_1} \alpha(u)\,\underline{v}du - (z_1 - z_0)\,\bar{c} - \int_0^{z_0} \alpha(u)\,\underline{v}du\right] + \int_{z_0}^{z_1}\left(\int_{q_0}^1 \left[\alpha(u)\,v(s) - \bar{c}\right]ds\right)du =$$
$$\int_{z_0}^{z_1}\left(\int_{q_1}^1 \left[\alpha(u)\,v(s) - \bar{c}\right]ds\right)du < 0,$$

which, again, contradicts equation (26) (see Figure 7).

Using the same argument described above, we conclude that the sequence $\left\{P_m\left(mz^m_{r_m}, \left(q^m_{r_m}\right)^+\right)\right\}$ has a limit, which we denote by $\bar{P}_0$, and that

$$(q_{-1} - q_0)\left[\int_0^{\psi(q_0)} \alpha(u)\,\underline{v}du - \bar{P}_0\right] + \int_{q_0}^{q_{-1}}\left[-\psi'(u)\left(\int_{q_{-1}}^{1}\left[\alpha(\psi(u))\,v(s) - \bar{c}\right]ds\right)\right]du = 0. \tag{28}$$

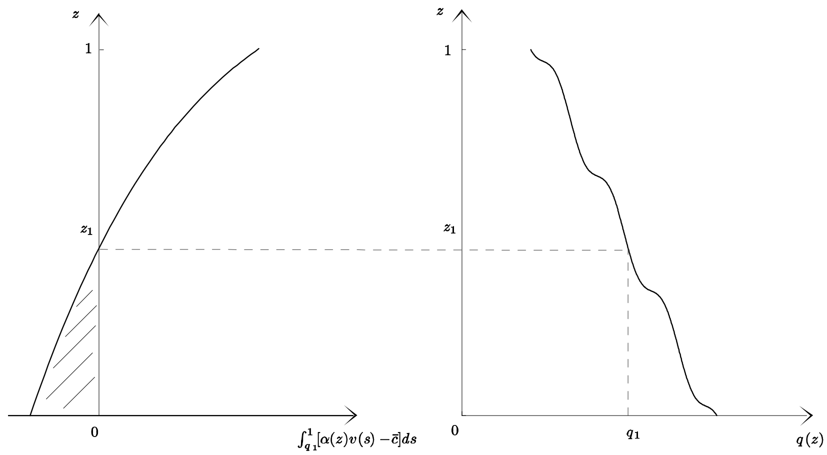

Figure 7: Profitable deviation between two impasses

The next step is to show that we can express $\bar{P}_1$ in terms of (q_1, q_0, q_{-1}). We then substitute this expression into (27) and obtain an equation in (q_1, q_0, q_{-1}). From equation (28), we have

$$\bar{P}_0 = \int_0^{\psi(q_0)} \alpha(u)\,\underline{v}du + \left(\frac{\int_{q_0}^{q_{-1}} \left[-\psi'(u) \left(\int_{q_{-1}}^{1} \left[\alpha(\psi(u))\, v(s) - \bar{c} \right] ds \right) \right] du}{q_{-1} - q_0} \right). \tag{29}$$

It easy to show that the sequence of prices $\left\{ P_m \left(m z_{r_m}^m, \left(q_{r_m}^m \right)^- \right) \right\}$ admits a limit which we denote by $\underline{P}_0$. Using the fact that there is double delay at every impasse and equation (29), we obtain

$$\underline{P}_0 = \left(\frac{\int_0^{\psi(q_0)} \alpha(u)\underline{v}du}{\bar{P}_0} \right) \int_0^{\psi(q_0)} \alpha(u)\,\underline{v}du =$$

$$\left(\frac{\int_0^{\psi(q_0)} \alpha(u)\underline{v}du}{\int_0^{\psi(q_0)} \alpha(u)\underline{v}du + \left(\frac{\int_{q_0}^{q_{-1}} \left[-\psi'(u) \left(\int_{q_{-1}}^{1} \left[\alpha(\psi(u))v(s) - \bar{c} \right] ds \right) \right] du}{q_{-1} - q_0} \right)} \right) \int_0^{\psi(q_0)} \alpha(u)\,\underline{v}du.$$

Recall that, for every m, the seller's low types are indifferent between selling the remaining $m z_{r_m+1}^m$ units at the price $P_m \left(m z_{r_m+1}^m, \left(q_{r_m+1}^m \right)^+ \right)$ and selling the first $m \left(z_{r_m+1}^m - z_{r_m}^m \right)$ at the price $\left(z_{r_m+1}^m - z_{r_m}^m \right) \bar{c}$ and the remaining $m z_{r_m}^m$ at the price $P_m \left(m z_{r_m}^m, \left(q_{r_m}^m \right)^- \right)$. Thus, in the limit, as m goes to infinity, we have (recall that $z_1 - z_0 = \psi(q_1) - \psi(q_0)$)

$$\bar{P}_1 = (\psi(q_1) - \psi(q_0))\,\bar{c} + \underline{P}_0 =$$

$$(\psi(q_1) - \psi(q_0))\,\bar{c} + \left(\frac{\int_0^{\psi(q_0)} \alpha(u)\underline{v}du}{\int_0^{\psi(q_0)} \alpha(u)\underline{v}du + \left(\frac{\int_{q_0}^{q_{-1}} \left[-\psi'(u) \left(\int_{q_{-1}}^{1} [\alpha(\psi(u))v(s) - \bar{c}] ds \right) \right] du}{q_{-1} - q_0} \right)} \right) \int_0^{\psi(q_0)} \alpha(u)\,\underline{v}du. \tag{30}$$

Finally, we substitute equation (30) into equation (27) and obtain an equation linking the beliefs at (the limits of) three consecutive impasses:

$$\Phi(q_1, q_0, q_{-1}) := \int_0^{\psi(q_1)} \alpha(u)\,\underline{v}du + \left(\frac{\int_{q_1}^{q_0} \left[-\psi'(u) \left(\int_{q_0}^{1} [\alpha(\psi(u))v(s) - \bar{c}] ds \right) \right] du}{q_0 - q_1} \right) -$$

$$(\psi(q_1) - \psi(q_0))\,\bar{c} - \left(\frac{\int_0^{\psi(q_0)} \alpha(u)\underline{v}du}{\int_0^{\psi(q_0)} \alpha(u)\underline{v}du + \left(\frac{\int_{q_0}^{q_{-1}} \left[-\psi'(u) \left(\int_{q_{-1}}^{1} [\alpha(\psi(u))v(s) - \bar{c}] ds \right) \right] du}{q_{-1} - q_0} \right)} \right) \int_0^{\psi(q_0)} \alpha(u)\,\underline{v}du = 0. \tag{31}$$

Since $q_{-1} < q(0)$, for m large there is an impasse $\left(z^m_{r_m-2}, q^m_{r_m-2}\right)$ that comes right after $\left(z^m_{r_m-1}, q^m_{r_m-1}\right)$. As usual, by taking a subsequence if necessary, assume that the sequence $\left\{\left(z^m_{r_m-2}, q^m_{r_m-2}\right)\right\}$ converges and let (z_{-2}, q_{-2}) denote its limit. It is also easy to show that $\Phi(q_0, q_{-1}, q_{-2}) = 0$, where the function Φ is defined in (31).

We proceed inductively and obtain a sequence of limits of impasses $\{(z_i, q_i)\}_{i=1,0,-1,\ldots}$. For every $i = 1, 0, -1, \ldots$, we have $z_i = \psi(q_i)$ and

$$\Phi(q_i, q_{i-1}, q_{i-2}) = \int_0^{\psi(q_i)} \alpha(u)\,\underline{v}du + \left(\frac{\int_{q_i}^{q_{i-1}} \left[-\psi'(u) \left(\int_{q_{i-1}}^{1} [\alpha(\psi(u))v(s) - \bar{c}] ds \right) \right] du}{q_{i-1} - q_i} \right) -$$

$$(\psi(q_i) - \psi(q_{i-1}))\,\bar{c} - \left(\frac{\int_0^{\psi(q_{i-1})} \alpha(u)\underline{v}du}{\int_0^{\psi(q_{i-1})} \alpha(u)\underline{v}du + \left(\frac{\int_{q_{i-1}}^{q_{i-2}} \left[-\psi'(u) \left(\int_{q_{i-2}}^{1} [\alpha(\psi(u))v(s) - \bar{c}] ds \right) \right] du}{q_{i-2} - q_{i-1}} \right)} \right) \int_0^{\psi(q_{i-1})} \alpha(u)\,\underline{v}du = 0. \tag{32}$$

Also, for every $i = 1, 0, -1, \ldots$, we let $\bar{P}_i$ denote the limit, as m goes to infinity, of $P_m\left(mz^m_{r_m+i}, \left(q^m_{r_m+i}\right)^+\right)$.

By construction, the sequence $\{q_i\}$ is increasing and bounded above by $q(0)$. Therefore, we have

$$\lim_{i \to -\infty} (q_{i-1} - q_i) = 0. \tag{33}$$

In what follows, we analyze the dynamic system in (32) and show that if an impasse is small, then all the impasses that precede it are also small. Formally, we demonstrate that there exists $\varepsilon > 0$ such that if $(q_{i-1} - q_i) < \varepsilon$, for some i, then $(q_{i'-1} - q_{i'}) < \frac{\nu}{2}$ for every $i' = 1, 0, \ldots, i+1$. This contradicts equation (24), showing that $\limsup_m \max_{A_m} \left|q_u^m - q_{u-1}^m\right| = 0$ and completing the proof of Claim 5.

Consider the sequence $\left\{\left(q_i, \bar{P}_i\right)\right\}_{i=1,0,-1,\ldots}$ constructed above and recall that for every $i = 1, 0, -1, \ldots$ we have $\Phi\left(q_i, q_{i-1}, q_{i-2}\right) = 0$ (see equation (32)) and

$$\bar{P}_i = \int_0^{\psi(q_i)} \alpha\left(u\right) \underline{v} du + \left(\frac{\int_{q_i}^{q_{i-1}} \left[-\psi'(u) \left(\int_{q_{i-1}}^{1} \left[\alpha\left(\psi\left(u\right)\right) v(s) - \bar{c} \right] ds \right) \right] du}{q_{i-1} - q_i} \right) \tag{34}$$

Also recall that the sequence $\{q_1, q_0, q_{-1}, \ldots\}$ is increasing and bounded above by $q\left(0\right)$. Therefore, the sequence is convergent, and we denote its limit by $q_{-\infty}$.

In Appendix E, we establish the following two facts.

Fact 3 *There exists $\eta^* > 0$ such that for every $i = 0, -1, \ldots,$ if $q_{i-1} - q_i < \eta^*$, then $q_i - q_{i+1} < \frac{4}{3}\left(q_{i-1} - q_i\right)$.*

Fact 4 *There exist two constants $b_1 > 0$ and $b_2 > 0$ such that for every $i = 1, 0, -1, \ldots,$ we have*

i)

$$\frac{\left(\bar{P}_i - \int_0^{\psi(q_i)} \alpha\left(u\right) \underline{v} du\right) - \left(\bar{P}_{i-1} - \int_0^{\psi(q_{i-1})} \alpha\left(u\right) \underline{v} du\right)}{q_{i-1} - q_i} \leqslant b_1 (q_{i-1} - q_i), \tag{35}$$

and ii)

$$q_{i-1} - q_i \leqslant b_2 \left(\bar{P}_i - \int_0^{\psi(q_i)} \alpha\left(u\right) \underline{v} du \right). \tag{36}$$

Claim 5 follows from Lemmas 4-6 below.

Lemma 4 *Let j and j' be two integers satisfying $1 \geqslant j' > j$, and consider the beliefs $q_{j'} < q_{j'-1} < \ldots < q_j$. Let ε and M be two positive numbers such that $q_{i-1} - q_i < \varepsilon$ for every $i = j+1, \ldots, j'$, and $q_j - q_{j'} < M^{-1}$. Then for every $i = j+1, \ldots, j'$, we have*

$$\bar{P}_i - \int_0^{\psi(q_i)} \alpha\left(u\right) \underline{v} du < \bar{P}_j - \int_0^{\psi(q_j)} \alpha\left(u\right) \underline{v} du + \varepsilon b_1 M^{-1}.$$

Proof. Notice that $(q_{i-i} - q_i) < \varepsilon$ implies

$$\bar{P}_i - \int_0^{\psi(q_i)} \alpha(u)\, \underline{v} du =$$

$$\bar{P}_{i-1} - \int_0^{\psi(q_{i-1})} \alpha(u)\, \underline{v} du + \left(\frac{\bar{P}_i - \int_0^{\psi(q_i)} \alpha(u) \underline{v} du - \left(\bar{P}_{i-1} - \int_0^{\psi(q_{i-1})} \alpha(u) \underline{v} du \right)}{q_{i-i} - q_i} \right) (q_{i-i} - q_i) <$$

$$\bar{P}_{i-1} - \int_0^{\psi(q_{i-1})} \alpha(u)\, \underline{v} du + \varepsilon b_1 (q_{i-i} - q_i),$$

where the inequality follows from equation (35). Therefore for every $i = j+1, \ldots, j'$, we have

$$\bar{P}_i - \int_0^{\psi(q_i)} \alpha(u)\, \underline{v} du < \bar{P}_j - \int_0^{\psi(q_j)} \alpha(u)\, \underline{v} du + \varepsilon b_1 \sum_{i'=j+1}^{j'} (q_{i'-i} - q_{i'}) <$$

$$\bar{P}_j - \int_0^{\psi(q_j)} \alpha(u)\, \underline{v} du + \varepsilon b_1 M^{-1}.$$

■

Lemma 5 *Let j and j' be two integers satisfying $0 \geqslant j' > j$, and let ε and M be two positive numbers such that $\varepsilon < \eta^*$, $q_{i-1} - q_i < \varepsilon$ for every $i = j+1, \ldots, j'$, $\bar{P}_j - \int_0^{\psi(q_j)} \alpha(u)\, \underline{v} du < \left(\frac{\varepsilon}{3b_2}\right)$, and $M > 3b_1 b_2$. If $q_j - q_{j'} < M^{-1}$, then $q_{j'} - q_{j'+1} < \varepsilon$.*

Proof. We have

$$q_{j'-1} - q_{j'} \leqslant b_2 \left(\bar{P}_{j'} - \int_0^{\psi(q_{j'})} \alpha(u)\, \underline{v} du \right) <$$

$$b_2 \left(\bar{P}_j - \int_0^{\psi(q_j)} \alpha(u)\, \underline{v} du + \varepsilon b_1 M^{-1} \right) <$$

$$b_2 \left(\left(\frac{\varepsilon}{3b_2} \right) + \varepsilon b_1 M^{-1} \right) < \frac{2}{3} \varepsilon,$$

where the first inequality follows from Fact 4 (equation (36)), the second follows from Lemma 4, and the last two inequalities are an immediate consequence of the assumptions in Lemma 5. Finally, the inequality above together with Fact 3 and the assumption $\varepsilon < \eta^*$ implies

$$q_{j'} - q_{j'+1} < \frac{4}{3} (q_{j'-1} - q_{j'}) = \frac{4}{3} \frac{2}{3} \varepsilon < \varepsilon.$$

■

Lemma 6 *For every $\varepsilon > 0$ there exists $\kappa > 0$ such that for every $i = 0, -1, \ldots,$ the following is true. If max $\left\{ q_{i-1} - q_i, \bar{P}_i - \int_0^{\psi(q_i)} \alpha(u)\, \underline{v} du \right\} < \kappa$, then $q_{j-1} - q_j < \varepsilon$ for every $j = 1, 0, \ldots, i+1$.*

Proof. Take $\bar{M} > 3b_1b_2$ and an integer N such that $\left(\frac{q_{-\infty}-q_1}{N}\right) < \left(\frac{1}{2\bar{M}}\right)$, and $q_0 > q_1 + \left(\frac{q_{-\infty}-q_1}{N}\right)$. Consider the partition of $[q_1, q_{-\infty}]$ into N intervals. Let the integer m^* be such

$$q_1 + m^* \left(\frac{q_{-\infty} - q_1}{N}\right) \leqslant q_0 < q_1 + (m^* + 1) \left(\frac{q_{-\infty} - q_1}{N}\right),$$

and for $m = m^* + 1, \ldots, N - 1$ define

$$j_m := \max \left\{ j : q_j \geqslant q_1 + m \left(\frac{q_{-\infty} - q_1}{N}\right) \right\}.$$

Also define $j_N = -\infty$. Consider $m = m^*+1, \ldots, N$. It follows from Lemma 4 and Lemma 5 that for every $\varepsilon_m > 0$, there exists $\kappa_m > 0$ such that $\max \left\{ q_{i-1} - q_i, \bar{P}_i - \int_0^{\psi(q_i)} \alpha(u)\, \underline{v} du \right\} < \kappa_m$, for some $i = j_{m-1} - 1, j_{m-1} - 2, \ldots, j_m$, implies $\max \left\{ q_{i'-1} - q_{i'}, \bar{P}_{i'} - \int_0^{\psi(q_{i'})} \alpha(u)\, \underline{v} du \right\}$ $< \varepsilon_m$ for every $i' = j_{m-1}, \ldots, i + 1$.

This, together with Fact 3, immediately implies Lemma 6. ■

Recall that $\lim_{i \to -\infty} q_i - q_{i-1} = 0$ (see equation (33)). Using this fact and equation (34), it is easy to check that

$$\lim_{i \to -\infty} \bar{P}_i - \int_0^{\psi(q_i)} \alpha(u)\, \underline{v} du = 0. \tag{37}$$

Combining Lemma 6 with equation (33) and equation (37) we obtain a contradiction to equation (24). This concludes the proof of Claim 5. ■

References

[1] Ashcraft and Schuermann (2008): "Understanding the Securitization of Subprime Mortgage Credit." Federal Reserve of New York Staff Reports.

[2] Ausubel, L. M., and R. J. Deneckere (1992): "Bargaining and the Right to Remain Silent," *Econometrica*, 60, 597-625.

[3] Camargo, B. and B. Lester (2014): "Trading Dynamics in Decentralized Markets with Adverse Selection," *Journal of Economic Theory*, 153, 534-568.

[4] Coase, R. H. (1972): "Durability and Monopoly," *Journal of Law and Economics*, 15, 143-149.

[5] Daley, D., and B. Green (2012): "Waiting for News in the Market for Lemons," *Econometrica*, 80, 1433-1504.

[6] Daley, D., and B. Green (2016): "Bargaining and News," mimeo.

[7] Downing, C., Jaffee, D. Wallace, N. (2009). "Is the Market for Mortgage-Backed Securities a Market for Lemons?". *Review of Financial Studies* 22, 2457-2494.

[8] Deneckere, R. J., and M.-Y. Liang (2006): "Bargaining with Interdependent Values," *Econometrica*, 74, 1309-1364.

[9] Evans, R. (1989): "Sequential Bargaining with Correlated Values," *Review of Economic Studies*, 56, 499-510.

[10] Faltin-Traeger and Mayer (2012): "Lemons and CDOs," mimeo.

[11] Fuchs, W., A. Öry and A. Skrzypacz (2016): "Transparency and Distressed Sales under Asymmetric Information," *Theoretical Economics,* 11, 1103–1144.

[12] Fuchs, W., and A. Skrzypacz (2013a): "Bridging the Gap: Bargaining with Interdependent Values," *Journal of Economic Theory*, 148, 1226-1236.

[13] Fuchs, W., and A. Skrzypacz (2013b): "Costs and Benefits of Dynamic Trading in a Lemons Market," mimeo.

[14] Fudenberg, D., D. K. Levine, and J. Tirole (1985): "Infinite-Horizon Models of Bargaining with One-Sided Incomplete Information," in *Game Theoretic Models of Bargaining*, ed. by A. Roth. Cambridge University Press.

[15] Fudenberg, D., and J. Tirole (1991): *Game Theory*, The MIT Press.

[16] Gerardi, D., J. Hörner and L. Maestri (2014): "The Role of Commitment in Bilateral Trade," *Journal of Economic Theory*, 154, 578-603.

[17] Gorton, G., and A. Metrick (2012): "Securitization," in Constantinides, G., M. Harris, and R. Stulz (eds), *The Handbook of Economics and Finance,* 2A, North Holland.

[18] Gul, F., and H. Sonnenschein (1988): "On Delay in Bargaining with One-Sided Uncertainty," *Econometrica*, 56, 601-611.

[19] Gul, F., H. Sonnenschein, and R. Wilson (1986): "Foundations of Dynamic Monopoly and the Coase Conjecture," *Journal of Economic Theory*, 39, 155-190.

[20] Hörner, J. and N. Vieille (2009): "Public vs. Private Offers in the Market for Lemons," *Econometrica*, 77, 29-70.

[21] Inderst, R. (2005): "Matching Markets with Adverse Selection," *Journal of Economic Theory*, 121, 145-166.

[22] Kannan, R. and C. K. Krueger (1996): *Advanced Analysis on the Real Line*, Springer.

[23] Kaya, A. and K. Kim (2015): "Trading Dynamics with Private Buyer Signals in the Market for Lemons," mimeo.

[24] Kim, K. (2017): "Information about Sellers' Past Behavior in the Market for Lemons," *Journal of Economic Theory*, forthcoming.

[25] Lewis, M. (2010): "The Big Short: Inside The Doomsday Machine' W. W. Norton & Company.

[26] McLean B. and J. Nocera (2011): "All the Devils Are Here: The Hidden History of the Financial Crisis, " Portfolio/Penguin Press.

[27] Moreno, D. and M. Wooders (2010): "Decentralized Trade Mitigates the Lemons Problem," *International Economic Review,* 121, 51, 383-399.

[28] Moreno, D. and M. Wooders (2016): "Dynamic Markets for Lemons: Performance, Liquidity, and Policy Intervention," *Theoretical Economics*, 11, 601-639.

[29] Ryan, S. Tucker, J. and Y. Zhou (2016): "Securitization and Insider Trading" *Accounting Review*, 91, 649-675.

[30] Samuelson, W. (1984): "Bargaining under Asymmetric Information," *Econometrica*, 52, 995-1005.

[31] Vincent, D. R. (1989): "Bargaining with Common Values," *Journal of Economic Theory*, 48, 47-62.

Online Appendix: Omitted Proofs (Not for Publication)

Appendix D: Algorithm

Proof of Proposition 4.

First, suppose, by contradiction, that the last impasse occurs at (k, q') for some $k > 1$. We claim that $q' < \bar{q}_1$. To see this, notice that there exists $\gamma > 0$ such that for every $q \geqslant \bar{q}_1$

$$W(k, q) \geqslant \int_q^1 ((\alpha_k + \ldots + \alpha_1) v(s) - kc) \, ds > \gamma.$$

The fact that the function $W(k, \cdot)$ is strictly bounded below from zero implies that, as $\delta_n \to 1$, the time it takes for the buyer to buy the remaining k units from the types above $\bar{q}_1$ converges to zero. This result follows from well known Coasean forces.

The fact that the last impasse is at (k, q') implies that for $q \in (q', \bar{q}_1)$ we have

$$W(k, q) = \int_q^1 ((\alpha_k + \ldots + \alpha_1) v(s) - kc) \, ds <$$
$$\int_q^1 ((\alpha_k + \ldots + \alpha_2) v(s) - (k-1) c) \, ds + \int_q^{\bar{q}_1} \left(\alpha_1 v(s) - P\left(1, \bar{q}_1^-\right)\right) ds,$$

which is the payoff from buying the first $(k-1)$ units from all the remaining types at the price $(k-1) c$ and the last unit from the types in $[q, \bar{q}_1]$ at the price $P\left(1, \bar{q}_1^-\right)$. Thus, for δ_n sufficiently close to one, the buyer has a profitable deviation at (k, q).

Suppose now that no impasse occurs on the equilibrium path. Then for $q < \bar{q}_1$, we have

$$W(m, q) = \int_q^1 ((\alpha_m + \ldots + \alpha_1) v(s) - mc) \, ds <$$
$$\int_q^1 ((\alpha_m + \ldots + \alpha_2) v(s) - (m-1) c) \, ds + \int_q^{\bar{q}_1} \left(\alpha_1 v(s) - P\left(1, \bar{q}_1^-\right)\right) ds.$$

and, again, for δ_n sufficiently close to one, the buyer has a profitable deviation at (m, q). This shows that the last impasse is at $(1, \bar{q}_1)$. The fact that this impasse is of real time $1 - \left(\frac{\alpha_1 \underline{v}}{c}\right)^2$ follows from DL. ■

We now develop the necessary notation to provide a formal description of the algorithm which describes the m-limiting equilibrium outcome. We let D denote the set of the following triples:

$$D := \{(k, q, p) : k = 1, \ldots, m-1, \ q > \bar{q}_{k+1} \text{ and } p < (\alpha_k + \ldots + \alpha_1) \underline{v}\}.$$

Fix a triple $(k, q, p) \in D$. For each state (k', q'), with $k' = k+1, \ldots, m$ and $q' < q$, consider the following course of action. The buyer makes the universal offer for $(k' - k)$

units (which is accepted by all the types in $[q',1]$) and then purchases the last k units from the types in $[q',q]$ at the price p. We let $\chi(k',k,q,p)$ denote the type q' at which the buyer breaks even. Formally, for $k'=k+1,\ldots,m$, we let $\chi(k',k,q,p)$ be implicitly defined by

$$\int_{\chi(k',k,q,p)}^{1}\left(\left(\alpha_{k'}+\ldots+\alpha_{k+1}\right)v(s)-(k'-k)c\right)ds+\int_{\chi(k',k,q,p)}^{q}\left(\left(\alpha_{k}+\ldots+\alpha_{1}\right)\underline{v}-p\right)ds=0,$$

provided that the solution to the above equation exists and is positive. In the other cases, we set $\chi(k',k,q,p)$ equal to zero.

For each $k'=k+1,\ldots,m$, we compare $\chi(k',k,q,p)$ with zero and $\bar{q}_{k'+1}$ (recall that this is the type at which the buyer breaks even if he trades the $(m-k)$-th unit at the price c). We let $\phi_1(k,q,p)$ denote the smallest integer k' for which $\chi(k',k,q,p)$ is strictly larger than the other two quantities (see below for the case in which such an integer does not exist). Formally, suppose that $\chi(k',k,q,p)>\max\{0,\bar{q}_{k'+1}\}$ for some $k'=k+1,\ldots,m-1$. Then we let $\phi_1(k,q,p)$ be equal to

$$\phi_1(k,q,p):=\arg\min\left\{k'=k+1,\ldots,m-1\text{ s.t. }\chi(k',k,q,p)>\max\{0,\bar{q}_{k'+1}\}\right\}.$$

If instead $\chi(k',k,q,p)\leqslant\max\{0,\bar{q}_{k'+1}\}$ for every $k'=k+1,\ldots,m-1$, then we let $\phi_1(k,q,p)$ be equal to m.

Finally, we let

$$\phi_2(k,q,p):=\chi(\phi_1(k,q,p),k,q,p)$$

denote the critical type q' at which the buyer breaks even if he purchases $(\phi_1(k,q,p)-k)$ units at the price $(\phi_1(k,q,p)-k)c$ from the types in $[q',1]$ and the last k units at the price p from the types in $[q',q]$.[43]

We are now ready to provide a formal description of our algorithm.

Proposition 7 *Suppose that, in the limit, the $(j-1)$-th to last impasse is at (k_{j-1},q_{j-1}) with $k_{j-1}<m$.*

i) If $\phi_2\left(k_{j-1},q_{j-1},P\left(k_{j-1},q_{j-1}^{-}\right)\right)=0$, then there are no other impasses.

ii) If $\phi_2\left(k_{j-1},q_{j-1},P\left(k_{j-1},q_{j-1}^{-}\right)\right)>0$, then the j-th to last impasse is at

$$\left(\phi_1\left(k_{j-1},q_{j-1},P\left(k_{j-1},q_{j-1}^{-}\right)\right),\phi_2\left(k_{j-1},q_{j-1},P\left(k_{j-1},q_{j-1}^{-}\right)\right)\right).$$

This impasse is also the first one if $\phi_1\left(k_{j-1},q_{j-1},P\left(k_{j-1},q_{j-1}^{-}\right)\right)=m$. Furthermore, the impasse is of real time $1-\left(\frac{\left(\left(\alpha_{k_j}+\ldots+\alpha_1\right)\underline{v}\right)}{(k_j-k_{j-1})c+P\left(k_{j-1},q_{j-1}^{-}\right)}\right)^2$, and

$$P\left(k_j,q_j^{-}\right)=\frac{\left(\left(\alpha_{k_j}+\ldots+\alpha_1\right)\underline{v}\right)^2}{(k_j-k_{j-1})c+P\left(k_{j-1},q_{j-1}^{-}\right)}<\left(\alpha_{k_j}+\ldots+\alpha_1\right)\underline{v}$$

$$P\left(k_j,q_j^{+}\right)=(k_j-k_{j-1})c+P\left(k_{j-1},q_{j-1}^{-}\right).$$

[43] Recall that we set $\chi(\phi_1(k,q,p),k,q,p)$ equal to zero if the critical type q' is negative or does not exist.

Proof. The proof of Proposition 7 is split into a series of lemmata. To simplify the notation, we consider the case $j = 2$. In other words, we take as given the fact that the last impasse is at $(1, \bar{q}_1)$ and characterize the penultimate impasse. The proof for arbitrary values of j is analogous to the case analyzed here. Also, for notational convenience, we use $\hat{q}_k$, $k = 2, \ldots, m$, to denote $\chi\left(k, 1, \bar{q}_1, P\left(1, \bar{q}_1^-\right)\right)$. We remind the reader that $\hat{q}_k$ is implicitly defined by

$$\int_{\hat{q}_k}^{\bar{q}_1} \left((\alpha_k + \ldots + \alpha_1)\underline{v} - (k-1)c - P\left(1, \bar{q}_1^-\right)\right) ds + \int_{\bar{q}_1}^{1} \left((\alpha_k + \ldots + \alpha_2) v(s) - (k-1)c\right) ds = 0.$$

provided that the solution to the above equation exists and is positive. Otherwise we set $\hat{q}_k$ equal to zero.

Of course, $\hat{q}_k < \bar{q}_1$ for every $k > 1$. We also point out that for generic values of the parameters, $\hat{q}_k \neq \bar{q}_{k+1}$. In the following, we restrict our attention to generic cases.

It is easy to see that if $\hat{q}_k > 0$, then

$$(\alpha_k + \ldots + \alpha_1)\underline{v} < (k-1)c + P\left(1, \bar{q}_1^-\right).$$

Lemma 7 *Consider $k = 2, \ldots, m-1$ and suppose that $0 < \hat{q}_k < \bar{q}_{k+1}$. Then $\hat{q}_{k+1} > \hat{q}_k$.*

Proof. Notice that

$$\begin{gathered}
\int_{\hat{q}_k}^{\bar{q}_1} \left((\alpha_{k+1} + \ldots + \alpha_1)\underline{v} - kc - P\left(1, \bar{q}_1^-\right)\right) ds + \int_{\bar{q}_1}^{1} \left((\alpha_{k+1} + \ldots + \alpha_2) v(s) - kc\right) ds = \\
\int_{\hat{q}_k}^{\bar{q}_1} \left((\alpha_k + \ldots + \alpha_1)\underline{v} - (k-1)c - P\left(1, \bar{q}_1^-\right)\right) ds + \int_{\bar{q}_1}^{1} \left((\alpha_k + \ldots + \alpha_2) v(s) - (k-1)c\right) ds + \\
\int_{\hat{q}_k}^{\bar{q}_{k+1}} \left(\alpha_{k+1}\underline{v} - c\right) ds + \int_{\bar{q}_{k+1}}^{1} \left(\alpha_{k+1} v(s) - c\right) ds = \\
\int_{\hat{q}_k}^{\bar{q}_{k+1}} \left(\alpha_{k+1}\underline{v} - c\right) ds < 0.
\end{gathered}$$

Consider the function $G : \mathbb{R} \to \mathbb{R}$,

$$G(q) := \int_{q}^{\bar{q}_1} \left((\alpha_{k+1} + \ldots + \alpha_1)\underline{v} - kc - P\left(1, \bar{q}_1^-\right)\right) ds + \int_{\bar{q}_1}^{1} \left((\alpha_{k+1} + \ldots + \alpha_2) v(s) - kc\right) ds.$$

Notice that G is linear in q, $G(\bar{q}_1) > 0$ and $G(\hat{q}_k) < 0$. Therefore, the inequality above implies $\hat{q}_{k+1} \in (\hat{q}_k, \bar{q}_1)$. ■

Lemma 8 *Let $k \leqslant m$ and assume that for $j = 2, ..., k-1$ we have $\hat{q}_j \leqslant \max\{0, \bar{q}_{j+1}\}$. For $\delta_n \to 1$, consider a sequence of histories $h_n^{t_n}$ (on or off-path) associated to states (k, q_n) with $q_n \to q^* \in (\hat{q}_k, \bar{q}_1)$. Then, the state $(1, \bar{q}_1)$ is reached. Moreover, the real time required for this event converges to zero.*

Proof. By contradiction, let k be the minimal number such that the claim is violated. Without loss, assume that $q_n \in (\hat{q}_k, \bar{q}_1)$ for all n.

First, it is easy to see that the conclusion of Proposition 4 holds if the initial history is associated to a state (k, q) with $q < \bar{q}_1$. Therefore, we know that any continuation game must reach an impasse at $(1, \bar{q}_1)$. The proof will be concluded if we show that for any $\varepsilon > 0$ there exists $\gamma > 0$ such that for all $j \in \{2, ..., k\}$ and $q > \hat{q}_k + \varepsilon$ we have $W(j, q) > \gamma$.

Take $\varepsilon > 0$. For $j \in \{2, ..., k-1\}$ Lemma 7 implies $\hat{q}_k > \hat{q}_j$. Hence, for any $q \in (\hat{q}_k + \varepsilon, \bar{q}_1)$ and $\tilde{k} \in \{2, ..., k\}$ the payoff from buying $(\tilde{k} - 1)$ units in the first period and then following the equilibrium strategy for $(1, q)$ converges to

$$\int_q^{\bar{q}_1} \left((\alpha_{\tilde{k}} + \ldots + \alpha_1) \underline{v} - \left(\tilde{k} - 1\right) c - P\left(1, \bar{q}_1^-\right) \right) ds + \int_{\bar{q}_1}^1 \left((\alpha_{\tilde{k}} + \ldots + \alpha_2) v(s) - \left(\tilde{k} - 1\right) c \right) ds,$$

which is bounded away from zero. ■

Lemma 8 immediately implies the following corollary.

Corollary 2 *Let $k \leqslant m$ and assume that for $j = 2, ..., k-1$ we have $\hat{q}_j \leqslant \max\{0, \bar{q}_{j+1}\}$. Then for $q \in (\hat{q}_k, \bar{q}_1)$ we have*

$$P(k, q) = P(1, \bar{q}_1^-) + (k-1) c.$$

For $k > 1$ we define $\tilde{W}(k, \cdot)$ and $\tilde{P}(k, \cdot)$ by

$$\tilde{W}(k, q) := \int_q^{\bar{q}_1} (\alpha_k v(s) - c)\, ds + W(k-1, q)$$
$$\tilde{P}(k, q) := c + P(k-1, q).$$

Lemma 9 *Let $k \leqslant m$ and assume that for $j = 2, ..., k-1$ we have $\hat{q}_j \leqslant \max\{0, \bar{q}_{j+1}\}$ and $\hat{q}_k > 0$. For $\delta_n \to 1$, consider a sequence of histories $h_n^{t_n}$ (on or off-path) associated to states (k, q_n) with $q_n \to q^* \in (\hat{q}_{k-1}, \hat{q}_k)$. Then:*

i) The continuation game reaches the state $(k, \hat{q}_k)$;

ii) The real time required for this event converges to zero;

iii) There is an impasse at $(k, \hat{q}_k)$ of real time $1 - \left(\frac{\sum_{j \leqslant k} \alpha_j \underline{v}}{P\left(k, \hat{q}_k^+\right)}\right)^2$.

iv) For $q \in (\hat{q}_{k-1}, \hat{q}_k)$ we have:

$$P(k, q) = \left(\frac{\sum_{j \leqslant k} \alpha_j \underline{v}}{P\left(k, \hat{q}_k^+\right)}\right)^2 P\left(k, \hat{q}_k^+\right),$$

v) For $q \in (\hat{q}_{k-1}, \hat{q}_k)$ we have:

$$W(k, q) = \int_q^{\hat{q}_k} \left(\sum_{j \leqslant k} \alpha_j \underline{v} - P\left(k, \hat{q}_k^-\right) \right) ds.$$

Proof. We begin by arguing that there must be an impasse at $(k, \hat{q}_k)$.

From Lemma 8 we have $\tilde{W}(k, \hat{q}_k) = 0$ and $\tilde{W}(k, q) < 0$ for $q \in \left(\left(\frac{\hat{q}_{k-1}+\hat{q}_k}{2}\right), \hat{q}_k\right)$. First, taking a subsequence if necessary the continuation value of the low type at (k, q_n) converges to some v_L^*. We claim that $v_L^* < P(k, \hat{q}_k^+)$. Suppose this is not the case. Take a small $\varepsilon > 0$ and let $h_n^{t_1}$ the first history reached (with positive probability) by $h_n^{t_n}$ such that the state at $h_n^{t_n}$, (k_n, q_n^1) is such that either i) $k_n = k$ and $q_n^1 \geqslant \hat{q}_k + \varepsilon$ or ii) the buyer makes an offer for z units at a price zc. Using the fact that the continuation values are always positive and the reservation prices are weakly increasing, an upper bound to the buyer's payoff (for large n) is:

$$\int_{q_n}^{q_n^1} \left(\sum_{j \leqslant k} \alpha_j \underline{v} - P(k, \hat{q}_k^+) + \varepsilon\right) ds + \tilde{W}\left(k_n, q_n^1\right) + \varepsilon =$$

$$\tilde{W}(k, q_n) + \varepsilon + \varepsilon\left(q_n^1 - q_n\right)$$

which is negative for $\varepsilon < -\frac{\tilde{W}(k, q_n)}{2}$.

For some (small) $\eta > 0$, take the continuation game and define $h_n^{t_2(\eta)}$ the first history reached (with positive probability) by $h_n^{t_n}$ such that either the buyer buys z units ($z \in \{1, ..., k-1\}$) at a price cz or the continuation utility of the low type is at least $P(k, \hat{q}_k^+) - \eta$. Let (k, q_n^η) the state at $h_n^{t_2(\eta)}$. Taking a convergent subsequence, $q_n^\eta \to q^\eta$. Clearly $\lim_{\eta \to 0} q^\eta = \hat{q}_k$. The first argument of this Lemma establishes that $\liminf_{\eta \to 0} q^\eta \geqslant \hat{q}_k$, while Corollary 2 establishes $\limsup_{\eta \to 0} q^\eta \leqslant \hat{q}_k$. One can then take a small η such that: a) q_n^η is close to $\hat{q}_k$;b) For (k, q), such that $q \in (q_n, q_n^\eta)$ the buyer buys k units only from the low type; and c) $P_n(k, q_n^\eta)$ is close to $P(k, \hat{q}_{k+})$. After establishing a),b) and c), one can use an argument similar to that in DL to establish that there is an impasse at $(k, \hat{q}_k)$ corresponding to a real delay equal to $\left(\frac{\sum_{j \leqslant k} \alpha_j \underline{v}}{P\left(k, \hat{q}_k^+\right)}\right)^2$. This implies $P(k, \hat{q}_k^-) = \left(\frac{\sum_{j \leqslant k} \alpha_j \underline{v}}{P\left(k, \hat{q}_k^+\right)}\right)^2 P(k, \hat{q}_k^+)$. Therefore, iii) and iv) are established.

Now, consider a state (k, q), with $q \in (\hat{q}_{k-1}, \hat{q}_k)$. Since the reservation price of the low type converges to $P(k, \hat{q}_k^-) < \sum_{j \leqslant k} \alpha_j \underline{v}$ and the limit payoff from buying z units ($z \in \{1, ..., k-1\}$) at a price cz converges to $\tilde{W}(k, q_n) < 0$, i), ii) and v) follow immediately. ■

Lemma 10 *Let $k < m$ and assume that for $j = 2, ..., k$ we have $\hat{q}_j \leqslant \max\{0, \bar{q}_{j+1}\}$. Then, for any initial state (k', q'), $k' = k+1, \ldots, m$, $q' \geqslant 0$, and for any $q \geqslant q'$, there is no impasse at (k, q).*

Proof. First we show an important inequality. If $c \geqslant \alpha_{k+1}\underline{v}$, then we have:

$$c + \left(\frac{\left(\left(\alpha_k + \ldots + \alpha_1\right)\underline{v}\right)^2}{(k-1)c + P(1, \bar{q}_1^-)}\right) > \left(\frac{\left(\left(\alpha_k + \ldots + \alpha_1\right)\underline{v} + \alpha_{k+1}\underline{v}\right)^2}{(k-1)c + P(1, \bar{q}_1^-) + c}\right). \tag{38}$$

To simplify notation, write V for $(\alpha_k + \ldots + \alpha_1)\underline{v}$, P for $(k-1)c + P(1, \bar{q}_1^-)$ and v_{k+1} for $\alpha_{k+1}\underline{v}$. The inequality above is equivalent to:

$$c + \left(\frac{V^2}{P}\right) > \left(\frac{(V + v_{k+1})^2}{P + c}\right) \Leftrightarrow c\left(P^2 + Pc + V^2\right) > 2V v_{k+1} P + P v_{k+1}^2.$$

For the last inequality it suffices that

$$c\left(P^2 + Pc + V^2\right) > 2VPv_{k+1} + Pcv_{k+1} \Leftrightarrow c\left(P^2 + P(c - v_{k+1}) + V^2\right) > 2VPv_{k+1}.$$

For the last inequality, it suffices that

$$c\left(P^2 + V^2\right) > 2VPc \Leftrightarrow c(P - V)^2 > 0.$$

If $\hat{q}_k = 0$, there cannot be an impasse at any state (k, q) since $W(k, \cdot)$ is bounded away from zero.

Therefore, assume that $0 < \hat{q}_k < \hat{q}_{k+1}$. If the state is (k', q) and $q > \frac{\hat{q}_k + \hat{q}_{k+1}}{2}$, then there cannot be an impasse at (k, q) since $W(k, \cdot)$ is bounded away from zero.

It remains to consider the case $q \leqslant \frac{\hat{q}_k + \hat{q}_{k+1}}{2}$. Let (k', q''), $q'' \in [q', q]$, denote the state at which the buyer purchases $(k' - k)$ units from all the remaining types. We now show that there exists $\gamma > 0$ such that when the state is (k', q''), the limit payoff from buying $(k' - k)$ units from the types in $[q'', 1]$ is lower at least by γ than the payoff from buying $(k' - k - 1)$ units from the types in $[q'', 1]$ and then $(k + 1)$ units from the types in $(q'', \hat{q}_{k+1})$. If $q \in \left[\hat{q}_k, \left(\frac{\hat{q}_k + \hat{q}_{k+1}}{2}\right)\right]$ the result is trivial. Hence, assume $q < \hat{q}_k$.The payoff from the first strategy is:

$$\int_{q''}^{1} \left((\alpha_{k'} + \ldots + \alpha_{k+1}) v(s) - (k' - k) c\right) ds + \int_{q''}^{\hat{q}_k} \left((\alpha_k + \ldots + \alpha_1)\underline{v} - \left(\tfrac{((\alpha_k + \ldots + \alpha_1)\underline{v})^2}{(k-1)c + P(1, \bar{q}_1^-)}\right)\right) ds =$$
$$\int_{q''}^{\hat{q}_k} \left((\alpha_{k'} + \ldots + \alpha_1)\underline{v} - \left(\tfrac{((\alpha_k + \ldots + \alpha_1)\underline{v})^2}{(k-1)c + P(1, \bar{q}_1^-)}\right) - (k' - k) c\right) ds +$$
$$\int_{q''}^{1} \left((\alpha_{k\prime} + \ldots + \alpha_{k+1}) v(s) - (k' - k) c\right) ds <$$
$$\int_{q''}^{\hat{q}_k} \left((\alpha_{k'} + \ldots + \alpha_1)\underline{v} - \left(\tfrac{((\alpha_k + \ldots + \alpha_1)\underline{v})^2}{(k-1)c + P(1, \bar{q}_1^-)}\right) - c\right) ds +$$
$$\int_{q''}^{1} \left((\alpha_{k\prime} + \ldots + \alpha_{k+2}) v(s) - (k' - k - 1) c\right) ds$$

since $\int_{q''}^{1} (\alpha_{k+1} v(s) - c)\, ds < 0$.[44] Now, from inequality (38) the expression above is strictly smaller than

$$\int_{q''}^{\hat{q}_k} \left((\alpha_{\alpha_{k'}} + \ldots + \alpha_1)\underline{v} - \tfrac{((\alpha_{k+1} + \ldots + \alpha_1)\underline{v})^2}{(k-1)c + P(1, \bar{q}_1^-) + c}\right) ds +$$
$$\int_{q''}^{1} \left((\alpha_{k\prime} + \ldots + \alpha_{k+2}) v(s) - (k' - k - 1) c\right) ds$$

[44] We set $\alpha_{k\prime} + \ldots + \alpha_{k+2} = 0$ if $k' = k + 1$.

which, in turn, is strictly smaller than

$$\int_{q''}^{\hat{q}_{k+1}}\left(\left(\alpha_{k+1}+\ldots+\alpha_1\right)\underline{v}-\frac{\left(\left(\alpha_{k+1}+\ldots+\alpha_1\right)\underline{v}\right)^2}{(k-1)c+P(1,\bar{q}_1^-)+c}\right)ds+$$
$$\int_{q''}^{1}\left(\left(\alpha_{k\prime}+\ldots+\alpha_{k+2}\right)v(s)-\left(k'-k-1\right)c\right)ds$$

which is the payoff from the second strategy. ■

At this point it is convenient to define $\bar{q}_{m+1}=0$. We now need to distinguish between two cases:

(a) there exists $k=2,\ldots,m$, such that $\hat{q}_k>\max\{\bar{q}_{k+1},0\}$;

(b) for every $k=2,\ldots,m$, $\hat{q}_k\leqslant\max\{\bar{q}_{k+1},0\}$.

We start with case (a) and define $\hat{k}$ as

$$\hat{k}:=\inf\left\{k=2,\ldots,m-1:\ \hat{q}_k>\max\{\bar{q}_{k+1},0\}\right\}$$

The following Lemma follows directly from Lemma 9 and Lemma 10.

Lemma 11 *The limit functions* $P\left(\hat{k},\cdot\right)$ *and* $W\left(\hat{k},\cdot\right)$ *satisfy*

$$P\left(\hat{k},q\right):=\begin{cases}\left(\frac{\left(\alpha_{\hat{k}}+\ldots+\alpha_1\right)\underline{v}}{\left(\hat{k}-1\right)c+P\left(1,\bar{q}_1^-\right)}\right)^2\left(\left(\hat{k}-1\right)c+P\left(1,\bar{q}_1^-\right)\right) & \text{if } q<\hat{q}_{\hat{k}}\\ \left(\hat{k}-1\right)c+P\left(1,\bar{q}_1^-\right) & \text{if } q\in\left(\hat{q}_{\hat{k}},\bar{q}_1\right)\\ \hat{k}c & \text{if } q>\bar{q}_1\end{cases}$$

$$W\left(\hat{k},q\right):=\begin{cases}\int_q^{\hat{q}_{\hat{k}}}\left(\left(\alpha_{\hat{k}}+\ldots+\alpha_1\right)\underline{v}-P\left(\hat{k},q\right)\right)ds & \text{if } q\leqslant\hat{q}_{\hat{k}}\\ \begin{array}{l}\int_q^{\bar{q}_1}\left(\left(\alpha_{\hat{k}}+\ldots+\alpha_1\right)\underline{v}-P\left(\hat{k},q\right)\right)ds+\\ \int_{\bar{q}_1}^1\left(\left(\alpha_{\hat{k}}+\ldots+\alpha_2\right)v(s)-\left(\hat{k}-1\right)c\right)ds\end{array} & \text{if } q\in\left(\hat{q}_{\hat{k}},\bar{q}_1\right]\\ \int_q^1\left(\left(\alpha_{\hat{k}}+\ldots+\alpha_1\right)v(s)-\hat{k}c\right)ds & \text{if } q>\bar{q}_1\end{cases}$$

Lemma 12 *The penultimate impasse occurs at* $\left(\hat{k},\hat{q}_{\hat{k}}\right)$ *and is of real time* $1-\left(\frac{\left(\alpha_{\hat{k}}+\ldots+\alpha_1\right)\underline{v}}{\left(\hat{k}-1\right)c+P\left(1,\bar{q}_1^-\right)}\right)^2$. *Furthermore, if* $\hat{k}=m$, *then the impasse at* $\left(\hat{k},\hat{q}_{\hat{k}}\right)$ *is the first impasse.*

Proof. By contradiction, suppose the claim is false. There are two possibilities:

i) the penultimate impasse occurs at (k, q') for some $k = \hat{k}+1, \ldots, m$;

ii) there is only one impasse at $(1, \bar{q}_1)$.

We consider case i) and derive a contradiction. The proof for case ii) is similar to the proof for case i); thus, we omit the details.

The fact that $\hat{q}_{\hat{k}} > \bar{q}_{\hat{k}+1} > \bar{q}_{\hat{k}+2} > \ldots \bar{q}_m$ implies that for any $k = \hat{k}+1, \ldots, m$ and any $q' \in [\hat{q}_{\hat{k}}, \bar{q}_1]$

$$\int_{q'}^{\bar{q}_1} \left((\alpha_k + \ldots + \alpha_1) v(s) - (k-1) c - P\left(1, \bar{q}_1^-\right) \right) ds +$$

$$\int_{\bar{q}_1}^{1} \left((\alpha_k + \ldots + \alpha_2) v(s) - (k-1) c \right) ds > 0.$$

Thus, if the penultimate impasse occurs at (k, q'), then $q' < \hat{q}_{\hat{k}}$. For $q \in (q', \hat{q}_{\hat{k}})$, we have

$$W(k,q) = \int_{q}^{\bar{q}_1} \left((\alpha_k + \ldots + \alpha_1) v(s) - (k-1) c - P\left(1, \bar{q}_1^-\right) \right) ds +$$

$$\int_{\bar{q}_1}^{1} \left((\alpha_k + \ldots + \alpha_2) v(s) - (k-1) c \right) ds$$

$$= \int_{q}^{\hat{q}_{\hat{k}}} \left((\alpha_k + \ldots + \alpha_1) v(s) - (k-1) c - P\left(1, \bar{q}_1^-\right) \right) ds +$$

$$\int_{\hat{q}_{\hat{k}}}^{1} \left((\alpha_k + \ldots + \alpha_{\hat{k}+1}) v(s) - \left(k - \hat{k}\right) c \right) ds + W\left(\hat{k}, \hat{q}_{\hat{k}}\right),$$

where we used the expression for $W\left(\hat{k}, q\right)$ for $q \geqslant \hat{q}_{\hat{k}}$. Since $W\left(\hat{k}, \hat{q}_{\hat{k}}\right) = 0$ the expression above is:

$$\int_{q}^{\hat{q}_{\hat{k}}} \left((\alpha_k + \ldots + \alpha_{\hat{k}+1}) v(s) - \left(k - \hat{k}\right) c \right) ds +$$

$$\int_{q}^{\hat{q}_{\hat{k}}} \left((\alpha_{\hat{k}} + \ldots + \alpha_1) v(s) - \left(\hat{k} - 1\right) c - P\left(1, \bar{q}_1^-\right) \right) ds +$$

$$\int_{\hat{q}_{\hat{k}}}^{1} \left((\alpha_k + \ldots + \alpha_{\hat{k}+1}) v(s) - \left(k - \hat{k}\right) c \right) ds =$$

$$\int_{q}^{1} \left((\alpha_k + \ldots + \alpha_{\hat{k}+1}) v(s) - \left(k - \hat{k}\right) c \right) ds +$$

$$\int_{q}^{\hat{q}_{\hat{k}}} \left((\alpha_{\hat{k}} + \ldots + \alpha_1) v(s) - \left(\hat{k} - 1\right) c - P\left(1, \bar{q}_1^-\right) \right) ds <$$

$$\int_{q}^{1} \left((\alpha_k + \ldots + \alpha_{\hat{k}+1}) v(s) - \left(k - \hat{k}\right) c \right) ds +$$

$$\int_{q}^{\hat{q}_{\hat{k}}} \left((\alpha_{\hat{k}} + \ldots + \alpha_1) v(s) - P\left(\hat{k}, \hat{q}_{\hat{k}-}\right) \right) ds,$$

which is the (limit) payoff from buying $\left(k - \hat{k}\right)$ units at a price $\left(k - \hat{k}\right) c$.

Finally, it follows immediately from Lemma 9 and Lemma 11 that the impasse at $\left(\hat{k}, \hat{q}_{\hat{k}}\right)$ is of real time $1 - \left(\frac{\left(\alpha_{\hat{k}} + \ldots + \alpha_1\right) \underline{v}}{\left(\hat{k}-1\right) c + P\left(1, \bar{q}_1^-\right)} \right)^2$ and that no other impasses occur prior to it when $\hat{k} = m$. ■

We now turn to case (b).

Lemma 13 *Suppose that for every $k = 2, \ldots, m$, $\hat{q}_k \leqslant \max\{\bar{q}_{k+1}, 0\}$. Then in the limit there is only one impasse at $(1, \bar{q}_1)$.*

Proof. First, notice that for any $k = 2, \ldots, m$, if $\hat{q}_k = 0$, an impasse cannot occur at (k, q) since $W(k, \cdot)$ is bounded away from zero. Under the assumptions of the lemma, the fact an impasse cannot occur at $(k, \hat{q}_k)$ with $k = 2, \ldots, m-1$ and $\hat{q}_k \in (0, \bar{q}_{k+1})$ follows from Lemma 10. ■

11 Appendix E: Upper Bounds

In this appendix, we prove Fact 3 and Fact 4 stated in Appendix C.

For every $i = 0, -1, \ldots,$ we have $\Phi(q_{i+1}, q_i, q_{i-1}) = 0$ (recall that the function Φ is defined in equation (32)). Using straightforward algebra and defining the function $\hat{\alpha} = \alpha \circ \psi$ we obtain

$$(q_i - q_{i+1}) \frac{\int_{q_{i+1}}^{q_i} -\psi'(u)(\bar{c} - \hat{\alpha}(u)\underline{v})du}{q_i - q_{i+1}} = (q_i - q_{i+1}) \left(\frac{\int_{q_{i+1}}^{q_i} \left[-\psi'(u) \left(\int_{q_i}^{1} [\hat{\alpha}(u)v(s) - \bar{c}]ds \right) \right] du}{(q_i - q_{i+1})^2} \right) +$$
$$\left(\frac{\int_0^{\psi(q_i)} \alpha(u)\underline{v}du}{\int_0^{\psi(q_i)} \alpha(u)\underline{v}du + \left(\frac{\int_{q_i}^{q_{i-1}} \left[-\psi'(u) \left(\int_{q_{i-1}}^{1} [\hat{\alpha}(u)v(s) - \bar{c}]ds \right) \right] du}{q_{i-1} - q_i} \right)} \right) \left(\frac{\int_{q_i}^{q_{i-1}} \left[-\psi'(u) \left(\int_{q_{i-1}}^{1} [\hat{\alpha}(u)v(s) - \bar{c}]ds \right) \right] du}{(q_{i-1} - q_i)^2} \right) (q_{i-1} - q_i),$$

or equivalently

$$\frac{q_i - q_{i+1}}{q_{i-1} - q_i} = \left(\frac{\int_0^{\psi(q_i)} \alpha(u)\underline{v}du}{\int_0^{\psi(q_i)} \alpha(u)\underline{v}du + \left(\frac{\int_{q_i}^{q_{i-1}} \left[-\psi'(u) \left(\int_{q_{i-1}}^{1} [\hat{\alpha}(u)v(s) - \bar{c}]ds \right) \right] du}{q_{i-1} - q_i} \right)} \right) \cdot$$
$$\left(\frac{\frac{\int_{q_i}^{q_{i-1}} \left[-\psi'(u) \left(\int_{q_{i-1}}^{1} [\hat{\alpha}(u)v(s) - \bar{c}]ds \right) \right] du}{(q_{i-1} - q_i)^2}}{\frac{\int_{q_{i+1}}^{q_i} -\psi'(u)(\bar{c} - \hat{\alpha}(u)\underline{v})du}{q_i - q_{i+1}} - \frac{\int_{q_{i+1}}^{q_i} \left[-\psi'(u) \left(\int_{q_i}^{1} [\hat{\alpha}(u)v(s) - \bar{c}]ds \right) \right] du}{(q_i - q_{i+1})^2}} \right). \tag{39}$$

Consider the first term in the right-hand side of equation (39). It is bounded above by one. Hence, we obtain the following upper bound for $\frac{q_i - q_{i+1}}{q_{i-1} - q_i}$:

$$\frac{q_i - q_{i+1}}{q_{i-1} - q_i} \leqslant \frac{\frac{\int_{q_i}^{q_{i-1}} \left[-\psi'(u)\left(\int_{q_{i-1}}^{1}[\hat{\alpha}(u)v(s)-\bar{c}]ds\right)\right]du}{(q_{i-1}-q_i)^2}}{\frac{\int_{q_{i+1}}^{q_i} -\psi'(u)(\bar{c}-\hat{\alpha}(u)\underline{v})du}{q_i-q_{i+1}} - \frac{\int_{q_{i+1}}^{q_i}\left[-\psi'(u)\left(\int_{q_i}^{1}[\hat{\alpha}(u)v(s)-\bar{c}]ds\right)\right]du}{(q_i-q_{i+1})^2}}. \tag{40}$$

Using the fact that $\int_{q_i}^{1} [\hat{\alpha}(q_i)\, v(s) - \bar{c}]\, ds = 0$ and the mean value theorem, we obtain

$$\begin{gathered}\int_{q_{i+1}}^{q_i} \left[-\psi'(u)\left(\int_{q_i}^{1} [\hat{\alpha}(u)\, v(s) - \bar{c}]\, ds\right)\right] du = \\ \int_{q_i}^{1} v(s)\, ds \int_{q_{i+1}}^{q_i} \left(\psi'(u) \int_u^{q_i} \hat{\alpha}'(s)\, ds\right) du = \left(\frac{\psi'(q_i')\hat{\alpha}'\left(q_i''\right) \int_{q_i}^{1} v(s)ds}{2}\right)(q_i - q_{i+1})^2\end{gathered} \tag{41}$$

for some $(q_i', q_i'') \in [q_{i+1}, q_i]^2$.

In a similar way we obtain

$$\int_{q_i}^{q_{i-1}} \left[-\psi'(u)\left(\int_{q_{i-1}}^{1} [\hat{\alpha}(u)\, v(s) - \bar{c}]\, ds\right)\right] du = \left(\frac{\psi'(q_{i-1}')\hat{\alpha}'\left(q_{i-1}''\right) \int_{q_{i-1}}^{1} v(s)\, ds}{2}\right)(q_{i-1} - q_i)^2 \tag{42}$$

for some $\left(q_{i-1}', q_{i-1}''\right) \in [q_i, q_{i-1}]^2$.

Also, it follows from the mean value theorem that

$$\frac{\int_{q_{i+1}}^{q_i} -\psi'(u)\,(\bar{c} - \hat{\alpha}(u)\, \underline{v})\, du}{q_i - q_{i+1}} = -\psi'(q_i''')\,(\bar{c} - \hat{\alpha}(q_i''')\, \underline{v}), \tag{43}$$

for some $q_i''' \in [q_{i+1}, q_i]$.

Hence, from equations (40)-(43) we conclude that the function $\Upsilon(q_{i-1}, q_i, q_{i+1})$ defined by

$$\Upsilon(q_{i-1}, q_i, q_{i+1}) := \frac{\left(\frac{\psi'(q_{i-1}')\hat{\alpha}'\left(q_{i-1}''\right) \int_{q_{i-1}}^{1} v(s)ds}{2}\right)}{-\psi'(q_i''')\,(\bar{c} - \hat{\alpha}(q_i''')\, \underline{v}) - \left(\frac{\psi'(q_i')\hat{\alpha}'\left(q_i''\right) \int_{q_i}^{1} v(s)ds}{2}\right)},$$

is an upper bound to $\frac{q_i - q_{i+1}}{q_{i-1} - q_i}$.[45]

We let $h = q_{i-1} - q_i$ and express q_i as $q_{i-1} - h$. It easily follows from equation (32) that q_{i+1} is a (locally) well defined function of q_{i-1} and h and we write $q_{i+1}(q_{i-1}, h)$ for it.[46] Next, we define $\hat{\Upsilon}(q_{i-1}, h) := \Upsilon(q_{i-1}, q_{i-1} - h, q_{i+1}(q_{i-1}, h))$. We now compute the limit

[45] To simplify the notation, in the definition of the function Υ we suppress the dependence of q_i', q_i'' and q_i''' on q_{i+1} and q_i. Similarly, we suppress the dependence of q_{i-1}' and q_{i-1}'' on q_i and q_{i-1}.

[46] Notice that $\lim_{h \to 0} q_{i+1}(q_{i-1}, h) = q_{i-1}$.

of $\hat{\Upsilon}(q_{i-1},h)$ as h shrinks to zero. To do this, we first need to evaluate $\hat{\alpha}'(q)$. Recall that, by definition, $\int_q^1 (\hat{\alpha}(q)v(s)-\bar{c})\,ds = 0$. This immediately implies $\hat{\alpha}'(q) = \left(\frac{\hat{\alpha}(q)\underline{v}-\bar{c}}{\int_q^1 v(s)ds}\right)$ for every $q \leqslant \hat{q}$. Hence, for every q_{i-1} we have

$$\lim_{h\to 0}\hat{\Upsilon}(q_{i-1},h) = \frac{\frac{\left(\frac{\hat{\alpha}(q_{i-1})\underline{v}-\bar{c}}{\int_{q_{i-1}}^1 v(s)ds}\right)\int_{q_{i-1}}^1 v(s)ds}{2}}{(\hat{\alpha}(q_{i-1})\underline{v}-\bar{c}) - \left(\frac{\left(\frac{\hat{\alpha}(q_{i-1})\underline{v}-\bar{c}}{\int_{q_{i-1}}^1 v(s)ds}\right)\int_{q_{i-1}}^1 v(s)ds}{2}\right)} = 1. \tag{44}$$

Next, we state some simple mathematical facts (the proofs are standard and, therefore, omitted).

Claim 6 *Let f and $\tilde{f}$ be two continuous functions from $[0,1]\times[a,b]$ into $\mathbb{R}$. Assume that f and $\tilde{f}$ are four times continuously differentiable in the first argument. Assume that these derivatives are continuous in the second argument. There exists $M>0$ such that for every $y\in[a,b]$ and $x\in(0,1)$*

i) $\left|\frac{d}{dx}\left(\frac{\int_0^x f(s,y)ds}{x}\right)\right| < M$;

ii) $\left|\frac{d}{dx}\left(\frac{\int_0^x f(s,y)\int_0^s \tilde{f}(u,y)duds}{x^2}\right)\right| < M$.

Claim 6 is used to prove the next result.

Claim 7 *$\hat{\Upsilon}(q_{i-1},h)$ is differentiable in h. Furthermore, there exist $M>0$ and $h_1>0$ such that $h<h_1$ implies $\left|\frac{\partial\hat{\Upsilon}(q_{i-1},h)}{\partial h}\right| < M$ for all $q_{i-1}\in(0,q(0))$.*

We use the implicit function theorem to evaluate $\frac{\partial\hat{\Upsilon}(q_{i-1},h)}{\partial h}$. It is straightforward to bound this derivative for every q_{i-1} bounded away from $q(0)$. We use Claim 6 to bound the derivative for q_{i-1} close to $q(0)$. Straightforward but tedious algebra yields the Lipschitz constant.

Claim 7 and equation (44) imply the following result.

Claim 8 *For every $\varepsilon>0$ there exists $h_1>0$ such that $h<h_1$ implies $\Upsilon(q_{i-1},h)<1+\varepsilon$ for every $q_{i-1}\in(0,q(0))$.*

The next result is an immediate corollary of Claim 8 and allows us to prove Fact 3 (in Appendix C).

Corollary 3 *For every $\varepsilon > 0$ there exists $\eta > 0$ such that for every $i = 0, -1, \dots,$ if $q_{i-1} - q_i < \eta$, then $q_i - q_{i+1} < (1+\varepsilon)(q_{i-1} - q_i)$.*

We now turn to the proof of Fact 4. It follows from equation (34) that for every $i = 1, 0, -1, \dots,$ we have

$$\frac{\bar{P}_i - \int_0^{\psi(q_i)} \alpha(u)\underline{v}du - \left(\bar{P}_{i-1} - \int_0^{\psi(q_{i-1})} \alpha(u)\underline{v}du\right)}{q_{i-1}-q_i} = \frac{\int_{q_i}^{q_{i-1}} \left[-\psi'(u)\left(\int_{q_{i-1}}^1 [\hat{\alpha}(u)v(s)-\bar{c}]ds\right)\right]du}{(q_{i-1}-q_i)^2} - \frac{\int_{q_{i-1}}^{q_{i-2}} \left[-\psi'(u)\left(\int_{q_{i-2}}^1 [\hat{\alpha}(u)v(s)-\bar{c}]ds\right)\right]du}{(q_{i-2}-q_{i-1})^2} \left(\frac{q_{i-2}-q_{i-1}}{q_{i-1}-q_i}\right). \tag{45}$$

Similarly to what we did above, we let $h = q_{i-2} - q_{i-1}$ and express q_{i-1} as $q_{i-2} - h$ and compute q_i as function of q_{i-2} and h (and write $q_i(q_{i-2}, h)$ for it). It follows from equation (41) that as h goes to zero the right-hand side of (45) converges to

$$\left(\frac{\psi'(q_{i-2})\hat{\alpha}'(q_{i-2}) \int_{q_i}^1 v(s)\,ds}{2}\right)\left(1 - \frac{h}{q_{i-2} - h - q_i(q_{i-2}, h)}\right).$$

It follows from Corollary 3 that for every $\varepsilon > 0$ there exists $h_1 > 0$ such that $h < h_1$ implies $\frac{h}{q_{i-2}-h-q_i(q_{i-2},h)} > \frac{1}{1+\varepsilon}$ for every $q_{i-2} \in (0, q(0))$. This allows us to establish our next result.

Claim 9 *As h goes to zero the function* $\max\left\{\left(\frac{\psi'(q_{i-2})\hat{\alpha}'(q_{i-2})\int_{q_i}^1 v(s)ds}{2}\right)\left(1 - \frac{h}{q_{i-2}-h-q_i(q_{i-2},h)}\right), 0\right\}$ *converges uniformly (in q_{i-2}) to zero.*

Finally, using straightforward algebra one can show that there exists an upper bound for the function above, which implies the first lemma of this section.

Lemma 14 *There exists $b_1 > 0$ such that for every $i = 1, 0, -1, \dots,$*

$$\frac{\left(\bar{P}_i - \int_0^{\psi(q_i)} \alpha(u)\underline{v}du\right) - \left(\bar{P}_{i-1} - \int_0^{\psi(q_{i-1})} \alpha(u)\underline{v}du\right)}{q_{i-1} - q_i} \leqslant b_1(q_{i-1} - q_i).$$

Next, using equations (34) and (42) we obtain

$$\left(\bar{P}_i - \int_0^{\psi(q_i)} \alpha(u)\underline{v}du\right) = \left(\frac{\psi'(q'_{i-1})\hat{\alpha}'(q''_{i-1}) \int_{q_{i-1}}^1 v(s)\,ds}{2}\right)(q_{i-1} - q_i), \tag{46}$$

for some $\left(q'_{i-1}, q''_{i-1}\right) \in [q_i, q_{i-1}]^2$. Letting b_2 be a uniform bound for the reciprocal of the term in parenthesis in the right-hand side of (46), we obtain our second important lemma.

Lemma 15 *There exists $b_2 > 0$ such that for every $i = 1, 0, -1, \ldots,$*

$$q_{i-1} - q_i \leqslant b_2 \left(\bar{P}_i - \int_0^{\psi(q_i)} \alpha(u)\, \underline{v} du \right).$$

12 Appendix F: Increasing Gains

Proof of Proposition 6.

For every $k = 1, \ldots, m$, consider the DL's model in which the parties trade k *indivisible* units and let $W(k, \cdot)$ and $P(k, \cdot)$ denote the buyer's continuation payoff and the seller's reservation price, respectively. Also, for every $q \in [0, 1)$, let $t_k(q) > q$ be such that

$$W(k, q) = \int_q^{t_k(q)} [(\alpha_k + \ldots + \alpha_1) v(s) - P(k, t_k(q))] ds + \delta W(k, t_k(q)).$$

The proof of Proposition 6 is complete if we show that it is suboptimal for the buyer to make universal offers when the number of remaining units is $k = 1, \ldots, m$. By definition, our claim holds when there is only one unit left for trade. We now assume that the claim holds when the number of remaining units is at most $k-1$, $k = 2, \ldots, m$, and show that it also holds when there are k units left for trade.

By contradiction, suppose that there exist $k' < k$ and q such that

$$\begin{aligned} W(k, q) \leqslant \textstyle\int_q^1 [(\alpha_k + \ldots + \alpha_{k'+1}) v(s) - (k - k') c] ds + \delta W(k', q) < \\ \textstyle\int_q^1 [(\alpha_k + \ldots + \alpha_{k'+1}) v(s) - (k - k') c] ds + W(k', q), \end{aligned} \tag{47}$$

where the second inequality follows from the fact that $W(k', q)$ is strictly positive for every q (see DL). This, together with the fact that $t_k(q) \leqslant 1$, imply

$$\begin{aligned} \textstyle\int_q^1 [(\alpha_k + \ldots + \alpha_1) v(s) - kc] ds \leqslant W(k, q) < \\ \textstyle\int_q^1 [(\alpha_k + \ldots + \alpha_{k'+1}) v(s) - (k - k') c] ds + W(k', q), \end{aligned}$$

and, thus,

$$W(k', q) > \int_q^1 [(\alpha_{k'} + \ldots + \alpha_1) v(s) - k'c] ds. \tag{48}$$

Recall that $W(k',q)$ is the buyer's continuation payoff when he makes cream-skimming offers (which are, by assumption, optimal when there are k' units left for trade). Thus, there exists $T \in \mathbb{Z}_+$ such that

$$W(k',q) = \sum_{\tau=1}^{T} \delta^{t-1} \int_{t_{k'}^{\tau-1}(q)}^{t_{k'}^{\tau}(q)} \left[(\alpha_{k'} + \ldots + \alpha_1) v(s) - P(k', t_{k'}^{\tau}(q)) \right] ds,$$

where we define $t_{k'}^{0}(q) = q$, and for $\tau = 1, \ldots, T$, $t_{k'}^{\tau}(q) = t_{k'}\left(t_{k'}^{\tau-1}(q)\right)$ (of course, $t_{k'}^{T}(q) = 1$).

A lower bound to the continuation payoff $W(k,q)$ can be computed by assuming that the buyer purchases after $\tau - 1$ periods, $\tau = 1, \ldots, T$, the k units from the types between $t_{k'}^{\tau-1}(q)$ and $t_{k'}^{\tau}(q)$ at the price $P(k, t_{k'}^{\tau}(q))$. Thus, we have

$$W(k,q) \geqslant \sum_{\tau=1}^{T} \delta^{t-1} \int_{t_{k'}^{\tau-1}(q)}^{t_{k'}^{\tau}(q)} \left[(\alpha_k + \ldots + \alpha_1) v(s) - P(k, t_{k'}^{\tau}(q)) \right] ds. \tag{49}$$

Using equations (5)-(12) in DL (pages 1318-1319), it is easy to check that

$$\frac{P(k,q')}{k} \leqslant \frac{P(k',q')}{k'} \tag{50}$$

for every $q' \in [0,1]$.

Combining equations (47), (49), and (50), we obtain

$$\begin{gathered}
\sum_{\tau=1}^{T} \delta^{t-1} \int_{t_{k'}^{\tau-1}(q)}^{t_{k'}^{\tau}(q)} \left[(\alpha_k + \ldots + \alpha_1) v(s) - \tfrac{k}{k'} P(k', t_{k'}^{\tau}(q)) \right] ds \leqslant \\
\sum_{\tau=1}^{T} \delta^{t-1} \int_{t_{k'}^{\tau-1}(q)}^{t_{k'}^{\tau}(q)} \left[(\alpha_k + \ldots + \alpha_1) v(s) - P(k, t_{k'}^{\tau}(q)) \right] ds \leqslant W(k,q) < \\
\int_q^1 \left[(\alpha_k + \ldots + \alpha_{k'+1}) v(s) - (k-k') c \right] ds + W(k',q) = \\
\int_q^1 \left[(\alpha_k + \ldots + \alpha_{k'+1}) v(s) - (k-k') c \right] ds + \\
\sum_{\tau=1}^{T} \delta^{t-1} \int_{t_{k'}^{\tau-1}(q)}^{t_{k'}^{\tau}(q)} \left[(\alpha_{k'} + \ldots + \alpha_1) v(s) - P(k', t_{k'}^{\tau}(q)) \right] ds,
\end{gathered}$$

which implies (we compare the first and the last term after multiplying both of them by $\frac{k'}{k-k'}$)

$$\begin{gathered}
\sum_{\tau=1}^{T} \delta^{t-1} \int_{t_{k'}^{\tau-1}(q)}^{t_{k'}^{\tau}(q)} \left[\tfrac{k'}{k-k'} (\alpha_k + \ldots + \alpha_{k'+1}) v(s) - P(k', t_{k'}^{\tau}(q)) \right] ds < \\
\int_q^1 \left[\tfrac{k'}{k-k'} (\alpha_k + \ldots + \alpha_{k'+1}) v(s) - k'c \right] ds.
\end{gathered}$$

Recall that the gains from trade are increasing ($\alpha_m < \ldots < \alpha_1$). Thus, we have

$$\frac{k'}{k-k'}(\alpha_k + \ldots + \alpha_{k'+1}) < \alpha_{k'} + \ldots + \alpha_1.$$

It follows immediately from the last two inequalities that

$$W(k',q) = \sum_{\tau=1}^{T} \delta^{t-1} \int_{t_{k'}^{\tau-1}(q)}^{t_{k'}^{\tau}(q)} [(\alpha_{k'} + \ldots + \alpha_1) v(s) - P(k', t_{k'}^{\tau}(q))] \, ds <$$
$$\int_q^1 [(\alpha_k + \ldots + \alpha_{k'+1}) v(s) - k'c] \, ds,$$

which contradicts inequality (48) and concludes our proof.

13 Appendix G: Menus of Offers

As anticipated in Section 9, in this appendix we construct a stationary equilibrium of the game in which the buyer can propose menus with at most two offers. In particular, we focus on the equilibrium on-path behavior. The equilibrium off-path behavior and the buyer's beliefs are derived similarly to those in Appendix A (we omit the details).

In equilibrium, the buyer proposes two types of menus. Let $k = 1, \ldots, m$ denote the number of units on the table. A menu $\mathcal{M}$ of the first type takes the form $\mathcal{M} = \{(k,p), (k', k'c)\}$ for some $k' = 1, \ldots, k$ and $p < kc$. Thus, the first offer (k,p) is for all the remaining units and can be accepted only by the low types. The second offer $(k', k'c)$ is for a fraction of the remaining units and the price is such that the high types break even.

The second type of menus contains only one offer of the form (k,p) with $p \leqslant kc$ (i.e., the buyer proposes to purchase all the remaining units). Although a menu $\mathcal{M}$ of the second type contains only one offer, we find it convenient to denote it as $\mathcal{M} = \{(k,p), (0,0)\}$. In this case, we also say that the seller accepts the offer $(0,0)$ if he rejects the offer (k,p).

For every $k = 1, \ldots, m$ and every $k' = 0, \ldots, k-1$, we define the function $P_{\mathfrak{m}}(k,k',\cdot) : [0,1] \to \mathbb{R}_+$. The function $P_{\mathfrak{m}}(k,k',\cdot)$ is weakly increasing, left-continuous, and satisfies $P_{\mathfrak{m}}(k,k',q) = kc$ for every $q > \hat{q}$. Suppose that the buyer offers the menu $\mathcal{M} = \{(k,p), (k', k'c)\}$. In equilibrium, type q accepts the first offer if $p \geqslant P_{\mathfrak{m}}(k,k',q)$, and the second offer if $p < P_{\mathfrak{m}}(k,k',q)$.

As in the equilibrium of the benchmark model, the set of types who have not sold all their units is of the form $[q,1]$ for some $q \in [0,1)$. Thus, we continue to use (k,q) to denote an arbitrary state of the economy (where k denotes the number of units on the table). We also let $W_{\mathfrak{m}}(k,q)$ denote the buyer's expected payoff when the state is (k,q). Clearly, for every (k,q), $W_{\mathfrak{m}}(k,q)$ satisfies

$$W_{\mathfrak{m}}(k,q) = \max_{q' \geqslant q, k'=0,\ldots,k-1} \int_q^{q'} \left[(\alpha_k + \ldots + \alpha_1) v(s) - P(k,k',q')\right] ds + \\ \int_{q'}^1 \left[(\alpha_k + \ldots + \alpha_{k-k'+1}) v(s) - k'c\right] ds + \delta W(k-k', q'), \tag{51}$$

where we set $\alpha_k + \ldots + \alpha_{k-k'+1} = 0$ if $k' = 0$.

The existence of the functions $P_{\mathfrak{m}}(k,k',\cdot)$ and $W_{\mathfrak{m}}(k,\cdot)$ satisfying equation (51) and the other equilibrium conditions is established along the lines of Appendix A.

Also, we can repeat the analysis in Section 6 and Appendix D to develop an algorithm that pins down the limiting equilibrium outcome as the bargaining frictions vanish (and the discount factor converges to one). It follows from equation (51) that the algorithm for the model with menus shares several similarities with the algorithm of the benchmark model and delivers the same limiting equilibrium outcome (again, we omit the details).

Finally, we proceed as in Appendix B to show that generically all stationary equilibria (satisfying our refinement that all the high types always agree on their decisions) of the model with arbitrary menus are outcome equivalent to the equilibrium above (with two offers). In particular, we first establish outcome equivalence for equilibria in which the seller's reservation price functions are increasing. Clearly, there is a unique equilibrium outcome when the state is (k,q) and q is sufficiently close to $\hat{q}$. Then we assume that there is a unique outcome when the state is (k,q) and then establish the same result for some state (k,q') with $q' < q$ until we reach the state $(k,0)$. We then relax the monotonicity assumption. An argument similar to the final part of the proof of Proposition 2 (Appendix B) shows that it is possible to relabel the types in such a way that for any stationary equilibrium there exists an outcome equivalent equilibrium with increasing reservation price functions.