Observing Each Other's Observations in a Bayesian Coordination Game [1]

Dominik Grafenhofer

Max Planck Institute for Research on Collective Goods, Kurt-Schumacher-Str. 10, 53113 Bonn, Germany. Email: grafenhofer@coll.mpg.de.

Wolfgang Kuhle

Max Planck Institute for Research on Collective Goods, Kurt-Schumacher-Str. 10, 53113 Bonn, Germany. Email: kuhle@coll.mpg.de.

Abstract: *We study a Bayesian coordination game where agents receive private information on the game's payoff structure. In addition, agents receive private signals that inform them of each other's private information. We show that once agents possess these different types of information, there exists a coordination game in the evaluation of this information. Even though the precisions of both signal types is exogenous, the precision with which agents forecast each other's actions in equilibrium turns out to be endogenous. As a consequence, there exist multiple equilibria which differ with regard to the way that agents weight their private information to forecast each other's actions. Finally, even though all players' signals are of identical quality, it turns out that efficient equilibria are asymmetric.*

Keywords: Coordination Games, Equilibrium Selection, Primary Signals, Secondary Signals

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1 Introduction

Games with strategic complementarities give players a strong incentive to choose mutually consistent strategies. In reality, such choices are often complicated by the fact that players know neither the game's exact payoffs nor the other player's actions. In such environments

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player's have to rely on different "pieces" of private information to predict the other player's actions and thus their own payoffs from playing a particular strategy. That is, players try to sense whether and to which extend, the other player may be "leaning" towards a particular action or that the other player might "misunderstand" the game, or, respectively, may be under the "wrong impression" as to the situation. The current paper studies how players use diverse pieces of private information in coordination games. Players receive two types of private information (i) a "primary signal" that informs them of the game's payoff structure and (ii) a new "secondary signal" that informs players of each other's private believes over the games payoff structure. Where the "secondary" signal informs one player of the believes, biases, optimism and "wrong impressions" that the other player may have concerning the game. The key prediction of our model is that such heterogeneous pieces of information induce a coordination game in the evaluation of this information. As a consequence, there emerge multiple equilibria which differ with regard to the way that agents weight their private information to forecast each other's actions. And, even though all players' signals are of identical quality, it turns out that efficient equilibria are asymmetric.

Compared to the literature, we find that the new signal type, which partially informs players of each other's believes regarding the games payoffs, changes the predictions as to which equilibria are played. In particular Rubinstein (1989) and Carlsson and van Damme (1993), show that players play unique symmetric threshold equilibria in coordination games where agents rely only on "primary information" which informs them of the game's payoff structure. In the current game, where players receive both "primary" signals on the game's unknown coefficients and "secondary" signals which allow agents to partially observe each other's information, we show that there emerges a coordination game regarding the evaluation of this information. In particular, there emerges a class of asymmetric equilibria in which agents weight their private information asymmetrically. The significance of this new equilibrium class lies in the fact that asymmetric equilibria Pareto-dominate the symmetric ones with which they coexist.

More precisely, the players of Rubinstein (1989) and Carlsson and van Damme (1993) reason only indirectly about each other. They receive a signal regarding the game's payoffs, which they use to update their beliefs regarding the game's coefficients. Second, knowing that the other agent's signal is *correlated* with their own, they use their signal to infer the other player's posterior beliefs. In the present paper, we argue that players usually know more than that. Often players can observe *directly* parts of the other player's observation. That is, in the context of the coordinated attack interpretation of

the Rubinstein (1989) "electronic mail game", the general may observe that his messenger is off to a "good start". Thus, the chances that he eventually arrives at the other camp are better than usual. Put differently, the sender of the "primary message" knows that the other player most likely received a message indicating that a particular game was chosen. Or, alternatively, one general can observe from a distance that someone entered the other general's camp. While he cannot be sure whether it was his messenger or someone else, this observation induces him to revise upward the probability with which the other general received the news. On the other hand, if the sender sees that the messenger is off to a bad start, then he knows that it is less likely that the message will reach its receiver. If one player conditions his actions on the *"primary signal"*, which informs him of the particular game chosen by nature, the other player will have an incentive to condition on his *"secondary signal"*, which informs him of the *"primary signal"* that the other player received, and vice versa. Such asymmetric evaluation of signals maximizes the precision with which players can anticipate each other's actions. That is, even though the precision of both signal types is exogenous, the precision with which agents anticipate each other's actions in equilibrium turns out to be endogenous. Consequently, there exist multiple *asymmetric* equilibria that differ regarding the way that agents weight the different types of private information that they receive, to reason about each other's actions.

We formalize this intuition in the context of the Rubinstein (1989) electronic mail game. First, we introduce the basic information structure, where agents purely rely on the correlation of their private observations to infer the other agent's beliefs. That is, in equilibrium, agents rely on the fact that they did not receive a confirmation of their last message, which may mean either that their last message did not reach the receiver, or that the receiver's reply was lost. Within this setting, we recall the main insight, namely that agents play a unique risk-dominant equilibrium. In the following, we refer to the basic signals from the electronic mail game as *primary signals*. In the main part, we introduce a *secondary signal* which allows agents to make additional inference on the other agent's observations. That is, we introduce a noisy signal that allows players to reason *directly* about the probability with which their (primary) signal reached the receiver. For the modified setting, players can coordinate on multiple equilibria if players observe each other's signal with great precision.

To interpret our findings, we compare two classes of equilibria: (i) symmetric equilibria and (ii) asymmetric equilibria. The distinguishing feature of a symmetric equilibrium will be that agents weight their two signal types equally. In an asymmetric equilibrium one agent leans heavily on the *secondary signal*, while the other agent has an incentive to lean

heavily on his *primary signal* and vice versa. The key feature is therefore that the two signals are "cross complements". That is, if player one relies heavily on his *secondary signal*, then the other player has an incentive to rely on his *primary signal*. Such an asymmetric weighting of private signals enables agents to maximize the precision with which they can forecast each other's actions. Finally, to emphasize the importance of the class of asymmetric equilibria, we show that asymmetric equilibria dominate symmetric equilibria on efficiency grounds.

Related Literature: The main contribution of the present paper is the introduction of a new class of private signals. Namely, signals about the other player's signals. For the generalized signal structure we show that multiple equilibria emerge once private signals are sufficiently precise. Moreover, we show that some of these equilibria are *asymmetric*. Compared to the literature, we note that Rubinstein (1989), Carlsson and van Damme (1993), and Frankel et al. (2003) have studied two-action coordination games, where agents receive what we call *primary signals* that allow them to make inference on the game's unknown coefficients.[2] Moreover, through the correlation of private information, agents can reason about each other's posteriors and actions. Regarding equilibrium selection, these studies predict that unique symmetric equilibria are ensured once private signals are sufficiently precise. The present example shows that the existence of secondary *private* signals can invert this finding: multiple equilibria, symmetric and asymmetric, are ensured once the private signals are sufficiently precise.

Regarding different types of signals, Morris and Shin (2004), Hellwig (2002), Metz (2002), and Angeletos and Werning (2006) emphasize the role of public signals/common priors in the global games framework, showing that such signals restore equilibrium multiplicity if public signals are sufficiently precise compared to the private signal; we give an example where multiplicity arises in pure private signal environments. A further class of signals was introduced by Minelli and Polemarchakis (2003), Angeletos and Werning (2006), and Dasgupta (2007), who study environments where agents observe each other's actions. Such signals tend to induce unique equilibria in the two-player games of Minelli and Polemarchakis (2003), where signals over each other's actions are perfectly revealing. Angeletos and Werning (2006), and Dasgupta (2007) study public signals that partially reveal the other player's actions. They show that multiplicity may emerge if the public

[2]See Frankel et al. (2003) for a broad literature overview on equilibrium selection through what we call *primary signals*. Carlsson and van Damme (1993), pp. 1008-1010, and Morris and Shin (2007) for a detailed comparison of their "global games", which rely on continuous distributions, with the "electronic mail game" and its discrete information structure.

signal is of high quality. Kuhle (2015) gives an example where the public signal's quality reduces the number of equilibria. Finally, we note that the aforementioned models feature symmetric equilibria, while the current study emphasizes the existence of asymmetric equilibria. To close, Rubinstein (1989) points out that equilibrium multiplicity may reemerge once there is a technical upper bound for the number of exchanged messages. Similarly, multiplicity also obtains in the model of Binmore and Samuelson (2001), where agents can *decide* whether or not to send electronic messages which are costly.

The paper is organized as follows. Section 2 outlines our electronic mail game. In Section 2.1, we recall the uniqueness result for the modified game without secondary signals. Section 3 contains the main result. Sections 5 concludes.

2 A symmetric electronic mail game

There are two players 1 and 2. Each has two actions A and B to choose from. And there is uncertainty about which game G_a or G_b the two players are going to play. Games a and b differ regarding their payoffs. Nature selects game a with probability $1-p$ and game b with probability $p < \frac{1}{2}$. The game's payoffs are:

Game G_a

	A	B
A	M, M	$0, -L$
B	$-L, 0$	$0, 0$

Game G_b

	A	B
A	$0, 0$	$0, -L$
B	$-L, 0$	M, M

Moreover, we assume $L > M > 0$. Hence, players face a coordination problem in both states of the world: if players coordinate on actions A (B) in state a (b), they receive M each, while coordination on B (A) yields 0 to both players. However, if players fail to coordinate, i.e., choose different actions, then the player who plays B receives $-L$, and the payoff for playing A is 0. Players receive private information on the game's fundamental before they choose an action. The probability p, the payoff structure, and the forthcoming *communication protocol* are common knowledge among players.

Before players choose action A or B simultaneously, they receive information T_1 and T_2 respectively: In state a, both players get information $T_1 = T_2 = 0$. In state b, *one player is randomly selected* with probability $\frac{1}{2}$, and informed of the true state b. The selected player i then sends a message to player j. The message, however, is lost with probability ε. Upon receiving a message, player j sends a confirmation back to player i which is also lost with probability ε. These messages are exchanged until finally one

message is lost and communication ends. Players 1 and 2 now choose their actions based on the number of messages T_1 and T_2 that they received.

The present game therefore differs from the Rubinstein (1989) game in that it is random which player spots the actual game selected, and starts to inform the other player. Moreover, we assume that both players do not know who was selected to send the first message.[3] This symmetric structure accommodates a more natural interpretation of the *asymmetric* equilibria that players play once we introduce *secondary signals* that inform players about each other's *primary signals* T_1 and T_2. That is, unlike Rubinstein (1989), we endow players with signals of identical quality and assume that the probability with which they observe the true state of nature, is the same (1/2) across players. We make this assumption to interpret and emphasize the emergence of asymmetric equilibria where players differ with regard to the use of their private information. The propositions in this paper regarding the existence of multiple asymmetric equilibria continue to hold once we set the probability with which player 1 observes the true state of nature to one as in Rubinstein (1989). Finally, in the context of the coordinated attack interpretation of the electronic mail game, it seems natural that players do not necessarily know that "player 1" always observes the true state of nature first.

2.1 Rubinstein's Equilibrium

Before turning to our main findings, we restate the uniqueness result of Rubinstein (1989) for our symmetric mail game.

Proposition 1. *There exists only one equilibrium in which player 1 plays A in the state of nature a. In this equilibrium, both players play A, irrespective of the number of received messages T_1 and T_2.*

Proof. See Appendix A for the proof via induction. □

Proposition 1 recalls the inductive equilibrium selection mechanism that operates through higher-order beliefs: If player 1 plays A for $T_1 = 0$, then player 2 also plays A, and this induces both players to always play A when $T_i > 0, i = 1, 2$. That is, even though both players $T_i > 0, i = 1, 2$ know that game b was selected, players still play (A, A), despite the fact that (B, B) would be payoff-dominant. However, as in Rubinstein (1989), there exists a second equilibrium, where both players play (B, B) all the time,

[3] In Appendix D, we derive our main result for the original Rubinstein (1989) game, where player 1 is informed of the state of nature with probability 1.

receiving a zero payoff. This equilibrium does not exist if in game G_a the payoff from playing (B, B) is negative, rather than 0, for both players[4]. In this case, there exists only one unique equilibrium, where both players play A. Such a modification of payoffs, which may be introduced throughout the paper, would bring us closer to the formulation of Carlsson and van Damme (1993), where there exist unique strict equilibria for certain signal values.[5]

3 Observing each other's observations

Let us now add a secondary signal Z_1 and Z_2 as another source of private information: player i not only gets information T_i but also observes

$$Z_i := \begin{cases} T_j & \text{with probability } 1-\psi \\ T_j + 1 & \text{with probability } \psi\,. \end{cases}$$

The secondary signal Z_1 informs player 1 of the primary signal T_2 that player 2 received. As such, the secondary signal carries two types of information. First, it allows player 1 to reason about the true fundamental of the game. That is, through its dependence on T_2, Z_1 is correlated with nature's choice of a fundamental. Second, and more importantly, Z_1 allows player 1 to look *directly* at T_2. In turn, this direct look at T_2 informs him about the probability with which player 2 plays A or B. In the following main propositions 3, 5 and 6, we show that this "direct look" at the other player's signal will induce asymmetric equilibria, in which players weight their signals Z and T differently. That is, if player 1 conditions his actions mainly on his primary signal T_1, then player 2 will have an incentive to weight signal Z_2 heavily and vice versa. Put differently, the signals T_i, Z_j deliver complementary information for the purpose of coordination, while the signals T_i, T_j do only so to a lesser degree.

To underscore the significance of these asymmetric equilibria, we proceed in three steps. First, we show that they exist. Second, we describe the symmetric equilibria, where agents weight their signals symmetrically. Third, we show that the asymmetric equilibria welfare-dominate symmetric equilibria. Before we study the asymmetric equilibria, we note that the Rubinstein (1989) equilibrium carries over to the environment where agents receive primary and secondary signals.

[4]This would correspond to the existence of a strict dominance region as in global games.

[5]See Carlsson and van Damme (1993), pp. 1008-1010, and Morris and Shin (2007) for a detailed discussion of the relation between global games and mail games.

Proposition 2. *When information T_1, Z_1 and T_2, Z_2 are available to players, there exists an equilibrium in which both players play A irrespective of the information received.*

Proof. Suppose player 1 thinks that player 2 plays A for sure. Irrespective of (T_1, Z_1), the following holds: Choosing B will yield a payoff $-L$, while taking action A will secure him a payoff of M. The same argument can be made for player 2, and thus we have established that the strategy profile (A, A) is an equilibrium. □

In this equilibrium, both players receive a zero payoff, even in those situations where they know that playing (B, B) would yield a higher payoff. However, players can use their private signals to coordinate on an alternative class of equilibria:

Proposition 3. *If the secondary signals' precision is sufficiently high (ψ sufficiently small), there exist two asymmetric threshold equilibria for every $n \in \{1, 2, 3, \dots\}$: In one equilibrium, player 1 plays B if and only if $T_1 \geq n + 1$ (which implies $Z_1 \geq n$), and player 2 plays B if and only if $Z_2 \geq n + 1$ and $T_2 \geq n$. Reversing the roles of players 1 and 2 yields a second equilibrium.*

Proof. Let us consider the first equilibrium with cutoff n.

1. Take the behavior of player 2 as given. There are three cases to consider:
   (a) $T_1 < n$: Player 1 is sure that $Z_2 \leq n$ and hence plays A.
   (b) $T_1 = n$: With probability $1-\psi$ (ψ) player 2's information is $Z_2 = n$ ($Z_2 = n+1$). Playing A secures a payoff of zero for sure; playing B yields an expected payoff larger than $(1 - \psi)(-L) + \psi M$, which is the first player's payoff from B, when player 2 always plays B given $Z_2 > n$. Thus, for $\psi \leq \frac{L}{L+M} =: \psi_1$ playing A is optimal.
   (c) $T_1 \geq n+1$: Player 1 is sure that $Z_2 \geq n+1$ and $T_2 \geq n$, hence finds it optimal to play B.
2. Equivalently, now take the behavior of player 1 as given.
   (a) $Z_2 \leq n$: Player 2 knows that $T_1 \leq n$, and thus plays A.
   (b) $Z_2 > n + 1$: Player 2 knows that $T_1 \geq n + 1$, and thus plays B.
   (c) $Z_2 = n + 1$: Here we have to take care of four sub-cases:
      i. $T_2 = n - 1$: Hence $T_1 = n$ for sure and player 2 thus chooses A.

ii. $T_2 = n$: Defining $\lambda_\psi := P(T_1 \leq n | T_2 = n \wedge Z_2 = n+1) = \frac{\psi}{\psi + \frac{1-\varepsilon}{2}(1-\psi)}$, the payoff for playing B can be written as $\lambda_\psi(-L) + (1-\lambda_\psi)M$. From this we obtain a boundary $\psi_2 := \frac{(1-\varepsilon)M}{2L-(1-\varepsilon)M} > 0$, which ensures that for all $\psi \leq \psi_2$ playing B is optimal for player 2. That is, for $\psi \leq \psi_2$ the expected payoff of playing B is non-negative.

iii. $T_2 = n+1$: We can repeat the same argument using $\mu_\psi := P(T_1 \leq n | T_2 = n+1 \wedge Z_2 = n+1) = \frac{\psi}{\psi + (1-\varepsilon)(1-\psi)}$. It holds that $\mu_\psi < \lambda_\psi$, such that for all $\psi \leq \psi_2$ playing B is optimal for player 2.

iv. $T_2 = n+2$: Hence $T_1 = n+1$ for sure, and player 2 chooses B.

Again, we can choose ψ sufficiently small, i.e. $\psi \leq \min\{\psi_1, \psi_2\}$, such that the strategy profile from the proposition is indeed an equilibrium. □

To interpret the equilibria in Proposition 3 we note that players weight primary and secondary signals asymmetrically. That is, if player 1 switches from playing A to playing B for signal pairs $T_1 \geq n+1, Z_1 \geq n$, then player 2 switches from A to B for signal values $T_2 \geq n, Z_2 \geq n+1$. And, as the proof shows, signals where the trigger strategy requires values greater or equal $n+1$ carry the main information regarding the other player's signals and actions. On the contrary, signals where the trigger strategy requires values greater or equal n carry little information on other player's signals. More precisely, player 1 relies in his inference about the other player's action on the fact that $T_1 \geq n+1$ informs him of the fact that $T_2 \geq n, Z_2 \geq n+1$. Hence, player 1 relies on his primary signal to infer the action of player 2. The main reason for player 1's reliance on his primary signal T_1, is that player 2 conditions his actions on $T_2 \geq n, Z_2 \geq n+1$. That is, as steps $2.(\text{c})i - iv$ in the proof show, player 2 relies on his secondary signal to infer the action of player 1. In turn, player 1's reliance on the secondary signal Z_1 justifies player 2's reliance on the primary signal... This complementarity between player 1's primary and player 2's secondary signal ensures that asymmetric weighting of signals is an equilibrium. Put differently, players face a coordination game in the weighting of their private signals; players can *choose their cutoff values T_i, Z_i in a way that makes it easy for their counterpart to assess whether their requirement for playing B is met or not.* In the present case, this means leaning on the primary signal once the opponent leans on the secondary signal and vice versa.

4 Welfare

The main purpose of the following propositions 4-6 is to emphasize the role of asymmetric equilibria further. First, we show that there also exist symmetric equilibria, where agents weight their signals equally. Moreover, we show that not every configuration of cutoffs is an equilibrium. Second, proposition 5 underscores that multiple equilibria emerge once private signals are of high quality. Finally, proposition 6 documents that asymmetric equilibria, in which agents exploit the complementarity between primary and secondary signals, welfare dominate the symmetric equilibria of proposition 4.

Proposition 4. *If the secondary signals' precision is sufficiently high, there exist symmetric monotone equilibria for every* $n \in \{1, 2, 3, \dots\}$*, where players weight their signals equally such that both players play B if and only if* $T_i \geq n+1$ *and* $Z_i \geq n+1$*. There exist no symmetric monotone equilibria, where both players play B if and only if* $T_i \geq n+1$ *and* $Z_i \geq n+2$ *(or* $T_i \geq n+2$ *and* $Z_i \geq n+1$*).*

Proof. See Appendix B. □

Propositions 2-4 close our discussion on equilibrium multiplicity. As remarked earlier, we follow Rubinstein (1989) and focus on equilibria where players always play A if nature selects game a. This rules out equilibria, where players play B, B all the time. Regarding the current propositions, it also rules out additional equilibria involving $n = 0$.

One might suspect[6] that multiplicity depends on the relative precisions of primary and secondary signals, i.e., a high ε/ψ ratio may be required. This, however, is not the case:

Proposition 5. *There exist upper bounds* $\bar{\varepsilon} > 0$ *and* $\bar{\psi} > 0$*, such that the equilibria described in propositions 3 and 4 exist for all combinations of* $\varepsilon \leq \bar{\varepsilon}$ *and* $\psi \leq \bar{\psi}$*.*

Proof. For propositions 3 and 4 to hold, we need a sufficiently small error probability for the secondary signal, i.e., $\psi \leq \min[\psi_1, \psi_2, \psi_3]$, where $\psi_1 = \frac{L}{L+M}$, $\psi_2 = \frac{(1-\varepsilon)M}{2L-(1-\varepsilon)M}$, and $\psi_3 = \frac{(1-\varepsilon)M}{L+(1-\varepsilon)M}$. It therefore suffices for show that the limits of ψ_1, ψ_2, ψ_3 for $\varepsilon \to 0$ are positive: First, observe that ψ_1 is positive and does not depend on ε. Second, $\lim_{\varepsilon \to 0} \psi_2 = \frac{M}{2L-M} > 0$. Finally, $\lim_{\varepsilon \to 0} \psi_3 = \frac{M}{L+M} > 0$. □

[6] In the public and private information frameworks of Hellwig (2002), Morris and Shin (2004), and Angeletos and Werning (2006), multiple symmetric equilibria emerge once public signals or priors are sufficiently precise *relative* to private signals.

Our results therefore differ fundamentally from those obtained by Carlsson and van Damme (1993), Frankel et al. (2003), and Morris and Shin (2007), where equilibrium selection works best once private information is very precise. The equilibria in Proposition 3 rely on a coordination game in the evaluation of this information. And this incentive to coordinate is strongest once private signals are very informative.

Finally, we argue that asymmetric equilibria deserve special scrutiny since they are welfare-dominant:

Proposition 6. *If the secondary private signals are very precise, asymmetric equilibria of Proposition 3 welfare-dominate the symmetric ones of Proposition 4 for every given cutoff n. Furthermore, the asymmetric equilibria described in Proposition 3 for $n = 1$ welfare-dominate those where $n > 1$.*

Proof. See Appendix C. □

That is, once agents exploit the complementarity in weighting the primary and secondary signal, which gives rise to the asymmetric equilibria of proposition 3, they can anticipate each other's actions with great precision. This increases expected utility since it reduces the probability that, e.g., nature selects game b, but players play (A, A), or, worse, (A, B).

5 Discussion

In coordination games, such as the coordinated attack problem, we often have a sense as to the believes that the other player may hold on the game that he is involved in. That is, one player may sense that the other player is an "optimist" with regard to a certain action or, similarly, that the other player likely holds a particular "mistaken view" of the situation. The current model incorporates such aspects by giving players noisy information over each other's information. The analysis of such an information structure shows that players who posses different pieces of information face a coordination game as to how they use their private signals to forecast each other's actions. This coordination game in the evaluation of information has multiple symmetric and asymmetric equilibria. Comparison of these equilibria shows that asymmetric equilibria, where players exploit the complementarity between primary and secondary signals, welfare dominate symmetric ones.

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A Proof of Proposition 1

The proof is parallel to the one in Rubinstein (1989). First, we establish that player i plays A when $T_i = 0$. Player i considers two possible scenarios:

1. With probability $(1-p)$, game G_a is played.
2. With probability $\frac{1}{2}p\varepsilon$, Player j was selected, game G_b is played, and the message from player j to player i was lost.

Hence, we find a lower bound $\tilde{A}$ for i's payoff from playing A and an upper bound $\tilde{B}$ for i's payoff from playing B:

$$\pi(A) \geq \frac{(1-p)M + \frac{1}{2}p\varepsilon 0}{(1-p) + \frac{1}{2}p\varepsilon} =: \tilde{A} \qquad \pi(B) \leq \frac{-(1-p)L + \frac{1}{2}p\varepsilon M}{(1-p) + \frac{1}{2}p\varepsilon} =: \tilde{B}$$

It holds that $\tilde{A} > \tilde{B}$, and thus player i plays A. The induction step from $t-1$ to t is identical to the original Rubinstein one: assume that both players play A when they receive a $T_i < t$. Consider that player i gets information $T_i = t$. For the following argument we denote the probability that player i was informed first that game G_b is played by $\kappa_t \in [0,1]$. The posterior probability of player j having received information $T_j = t-1$ is given by

$$z_t := \frac{\kappa_t\varepsilon + 1 - \kappa_t}{\kappa_t(\varepsilon + (1-\varepsilon)\varepsilon) + 1 - \kappa_t} > \frac{1}{2}\,.$$

In other words, the posterior probability of player j playing A is larger than $\frac{1}{2}$, and thus playing A is optimal for player i as well: Playing A yields 0, while playing B has expected payoff $z_t(-L) + (1-z_t)M < 0$.

B Proof of Proposition 4

We start by proving the first statement. Without loss of generality, we have to check only if player i's best response to player j's equilibrium strategy is consistent with player i's equilibrium strategy. We have to check the following cases of information that player i might receive:

1. $Z_i \leq n$: Player i knows that $T_j \leq Z_i \leq n$ and that j player A, hence plays A.
2. $Z_i = n+1$:
   (a) $T_i = n$: player i knows that $Z_j = n$ and that player j plays A, which leads j to play A as well.

(b) $T_i = n+1$: Clearly $Z_j \geq n+1$ and $T_j \in \{n, n+1\}$. To determine the payoff of playing B, the conditional distribution of T_j has to be taken into account:

$$\frac{(1-\varepsilon)^{2n}\varepsilon\psi}{(1-\varepsilon)^{2n}\varepsilon\psi + (1-\varepsilon)^{2n+1}\varepsilon(1-\psi)}(-L) + \frac{(1-\varepsilon)^{2n+1}\varepsilon(1-\psi)}{(1-\varepsilon)^{2n}\varepsilon\psi + (1-\varepsilon)^{2n+1}\varepsilon(1-\psi)}M\,.$$

Hence playing B is optimal if $\psi < \psi_3 := \frac{(1-\varepsilon)M}{L+(1-\varepsilon)M}$.

(c) $T_i > n+1$: thus $Z_j \geq n+2$ and $T_j \geq n+2$. Player j plays B and the optimal response of i is B.

3. $Z_i = n+2$: Player i knows that $T_j \geq n+1$.

(a) $Z_i = n+2$ implies that $T_i < n$ is not feasible.

(b) $T_i = n$: $Z_j \geq n+1$ with probability ψ, hence the payoff of playing B is $(1-\psi)(-L) + \psi M$, which is negative for $\psi < \psi_1$, the case when playing A is optimal for player i.

(c) $T_i > n$: Therefore $Z_j \geq n+1$ for sure, and thus both players play B.

4. $Z_i > n+2$: Player i knows that $T_j \geq Z_i - 1 > n+1$. It also holds that $T_i \geq n+1$ and thus $Z_j \geq n+1$. Therefore player j plays B and player i's best response is to play B as well.

Hence we have established the first part of the proposition for $\psi \leq \min\{\psi_1, \psi_3\}$.

To prove the second part of the proposition we provide a counter example: Suppose $Z_1 = n+1$ and $T_1 = n+2$: player 1 now plays A. This is not a best response since player 1 knows that $T_2 \geq n+1$ and $Z_2 \geq n+2$, and thus that 2 plays B with certainty. Hence, equilibria where players play B iff $T_i \geq n+1$ and $Z_i \geq n+2$ cannot exist.

C Proof of Proposition 6

We prove the second part of the statement first. That is, we compute the total welfare loss in the asymmetric equilibria of Proposition 3 given n (sum of expected surplus losses of player 1 and 2) compared to hypothetical perfect coordination between both players. Note that in state a neither miscoordination nor coordination on the wrong action can occur. In state b

1. coordination on the wrong action (A, A) happens with probability

$$p\left[1 - (1-\varepsilon)^{2(n-1)} + (1-\varepsilon)^{2(n-1)}\varepsilon(1 + (1-\psi)(1-\varepsilon)) + \frac{1}{2}(1-\varepsilon)^{2n}(1-\psi)\right]$$

2. miscoordination (B, A) happens when $T_1 = n$, $T_2 \geq n$, and $Z_2 = n + 1$. The associated probability is $p(1-\varepsilon)^{2n-1}\varepsilon\psi$.

Using this, we can compute the welfare loss in equilibrium:

$$\begin{aligned} l_n &:= p \Bigg\{ (1-\varepsilon)^{2n-1}\varepsilon\psi(2M+L) \\ &+ \left[1 - (1-\varepsilon)^{2(n-1)} + (1-\varepsilon)^{2(n-1)}\varepsilon(1+(1-\psi)(1-\varepsilon)) + \frac{1}{2}(1-\varepsilon)^{2n}(1-\psi)\right] 2M \Bigg\} \\ &= p(1-\varepsilon)^{2n-1} \left\{ \varepsilon\psi(2M+L) + \left[-1 + \varepsilon(1+(1-\psi)(1-\varepsilon)) + \frac{1}{2}(1-\varepsilon)^2(1-\psi)\right] 2M \right\} + p2M\,. \end{aligned}$$

It is straightforward to see that the expression in curly brackets is negative for small ψ. Hence, l_n is increasing in n for small ψ.

The proof of the first part of the proposition requires computing the welfare loss in the symmetric equilibria of Proposition 3 given n. Again there are two types of losses:

1. coordination on the wrong action (A, A) happens with probability

$$p\left[1 - (1-\varepsilon)^{2n} + (1-\varepsilon)^{2n}\varepsilon(1-\psi)\right]$$

2. miscoordination (B, A) happens once $T_1 = n$, $T_2 \geq n$, and $Z_2 = n + 1$. Hence it happens with probability $p(1-\varepsilon)^{2(n-1)}\varepsilon\psi$.

Using these probabilities, we compute the welfare loss in equilibrium:

$$\tilde{l}_n = p(1-\varepsilon)^{2(n-1)} \left\{ \varepsilon\psi(2M+L) + (1-\varepsilon)^2 \left[-1 + \varepsilon(1-\psi)\right] 2M \right\} + p2M$$

Note that $l_n - \tilde{l}_n \xrightarrow[\psi \to 0]{} -p(1-\varepsilon)^{2n}\varepsilon M < 0$. Hence, for a small enough ψ it holds that $l_n < \tilde{l}_n$, i.e., welfare is higher in the asymmetric equilibria for every given n.

D Referee appendix

In this appendix we show that our main result, multiplicity of equilibria in the presence of primary and secondary signals, holds for the original asymmetric version of the electronic mail game of Rubinstein (1989). That is, we now assume that it is always player 1 who gets informed first, i.e., gets a message in case nature draws game b. Equivalently, we set the probability P, with which player 1 is informed first to $P = 1$ (rather than $1/2$, which is what we assumed in the main text). Other than that leave the signals Z, T unchanged. Our only deviation from Rubinstein (1989) is therefore the introduction of the secondary signal Z. Naturally, Proposition 2 holds without modification of the proof.

We now show for this simplified setting that multiple equilibria exist as in the main text. In particular, we prove that the asymmetric equilibria described in Proposition 3 still exist:

Proposition 7. *For small enough ψ there exists an asymmetric threshold equilibrium for every $n \in \{1, 2, 3, \dots\}$: player 1 plays B if and only if $T_1 \geq n+1$ (which implies $Z_1 \geq n$) and player 2 plays B if and only if $Z_2 \geq n+1$ and $T_2 \geq n$.*

Proof. The proof is mostly unchanged compared to the proof of Proposition 3. There are two exceptions:

2. Equivalently, now take the behavior of player 1 as given.

   (c) $Z_2 = n + 1$: Here we have to take care of four subcases:

   ii. $T_2 = n$: Note that $P(T_1 = n | T_2 = n \wedge Z_2 = n + 1) = \psi$, and thus, the payoff of playing B is given by $\psi(-L) + (1 - \psi)M$. From this we can determine $\bar{\psi}_2 := \frac{M}{L+M} > 0$ such that for all $\psi \leq \bar{\psi}_2$ playing B is optimal for player 2, i.e. where the expected payoff of playing B is non-negative.

   iii. $T_2 > n$: Hence $T_1 \geq n + 1$ for sure, player 2 chooses B.

Again, we can choose a small enough ψ, i.e., $\psi \leq \min\{\psi_1, \bar{\psi}_2\}$, such that the strategy profile from the proposition is indeed an equilibrium. □