The Role of Labor-Income Risk in Household Risk-Taking ⋆

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Abstract

In fifteen European countries, China, and the US, stocks and business equity as a share of total household assets are represented by an increasing and convex function of income/wealth. A parsimonious model fitted to the data shows why background labor-income risk can explain much of this risk-taking pattern. Uncontrollable labor-income risk stresses middle-income households more because labor income is a larger fraction of their total lifetime resources compared with the rich. In response, middle-income households reduce (controllable) financial risk. Richer households, having less pressure, can afford more risk-taking. The poor take low risk because they avoid jeopardizing their subsistence consumption.

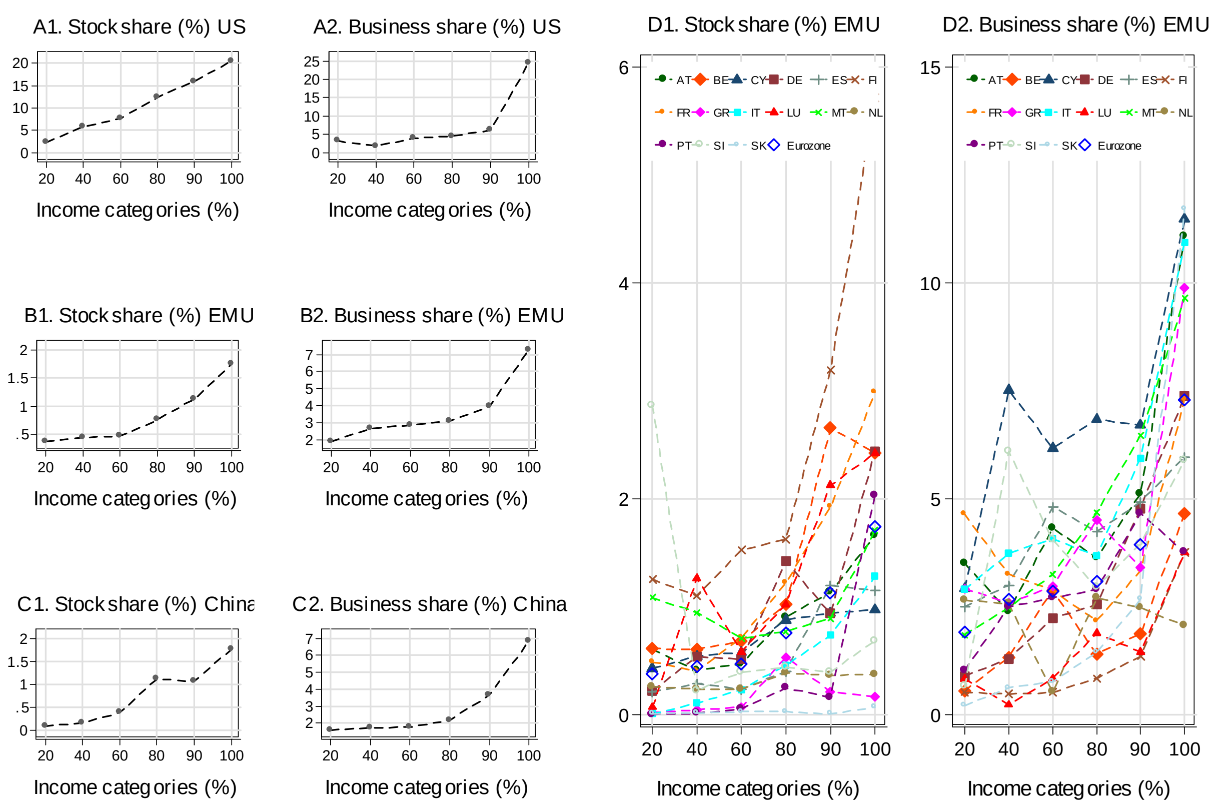
Keywords: background risk, household-portfolio shares, business equity, subsistence consumption, wealth inequality

JEL classification: G11, D91, D81, D14, D11, E21

1. Introduction

Figure 1 shows household-portfolio shares of stocks and business equity, plotted per household-income category.[1] In fifteen European-Union (EU) countries, China, and the US, risky asset shares are an increasing and convex function of household resources; as a household becomes richer, its shares of risky assets increase in an accelerating manner. The pattern demonstrated by Figure 1 is the same as patterns reported in recent studies using administrative data from Sweden and Norway.[2]

Figure 1: Portfolio share on risky assets by income categories (per equivalent adult and before tax)



[1] See Appendix A for an explanation of the data used in Figure 1.
[2] For the increasing and convex patterns of risk-taking in Sweden, see Bach, Calvet, and Sodini (2016, Figures 3 and 4), who present the mean returns of household risky assets and their standard deviations; for Norway, see Fagereng, Guiso, Malacrino, and Pistaferri (2016, Figure 1).

That risk-taking is an increasing and convex function of household resources in all these countries raises a natural question. *Is there a common reason why rich households undertake so much financial risk, while poor and middle-income households hesitate to do that?* In this paper, we seek to find an answer, examining common features that household background income risk may have in different economic areas.

The question of what explains the risk-taking pattern of Figure 1 is important for a recent literature of general interest. Given the recently documented evidence of wealth inequality (see Piketty, 2014), there has been a renewed interest in explaining the determinants of the top-1% wealth. In that literature, Benhabib, Bisin, and Luo (2017, p. 596) stress that "returns to wealth that are increasing in wealth" can explain the thick upper tail of the wealth distribution. In Figure 1, more risk-taking is accompanied by higher portfolio returns, lending support to this explanation of the heavy wealth tail, but this wealth-distribution literature assumes exogenous differential wealth returns. An explanation of these differential wealth returns as outcomes of conscious household choices is an important open question to enhance our understanding of the determinants of wealth dispersion.

The pioneering study that posed the question of why the rich take more risk than other households is Wachter and Yogo (2010). They distinguish between two categories of goods, basic goods and luxuries; their suggestion is that the rich invest more in risky assets because they are risking losses in mostly luxury consumption.[3] Similarly, Achury et al. (2012) introduce subsistence consumption to a simple Merton (1969, 1971) model using one broad consumer basket, suggesting that the poor do not invest in risky assets because they are strongly averse to losing their subsistence consumption. However, none of these studies explains why middle-class households are so reluctant to take financial risk. Building on

[3] This idea has been implicit in Browning and Crossley (2000). Importantly, Wachter and Yogo (2010) also employ the same utility function for all households, as we also do in this paper.

an enormous household-finance literature that demonstrates the importance of background labor-income risk for understanding portfolio choice, we explore whether labor-income risk is the reason holding back middle-class households from taking financial risk.[4]

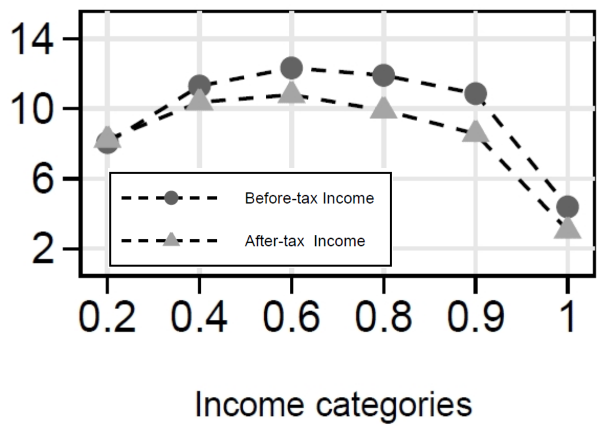


Figure 2 Income-asset ratio (US data)

To explain the pattern of Figure 1, we suggest a parsimonious household-portfolio model that uses the same utility function for all households. The fitting of this model to the risk-taking data reveals one common mechanism. Middle-income households reduce financial risk-taking because they try to cope with the high risk pressure caused by background income risk in their lifetime resources. Middle-income households have this higher background-risk pressure because income is a bigger fraction of their lifetime resources (e.g., see Figure 2 for the US). In contrast, rich households, in which income is a smaller fraction of their lifetime resources, can afford to take more financial risk. The poor avoid financial risk because they do not want to risk their subsistence consumption. This mechanism is closely supported by

[4] Examples from the literature that demonstrate the importance of background risk in household-portfolio analysis are Campbell and Viceira (1999, 2001), Davis and Willen (2000), Heaton and Lucas (2000a), Viceira (2001), Carroll (2002), Cocco, Gomes, and Maenhout (2005), Polkovnichenko (2007), and Fagereng, Guiso, and Pistaferri (2016), among others. Pure empirical studies on background risk include Guiso, Jappelli, Terlizzese (1996), Angerer and Lam (2009), and Bonaparte, Korniotis, and Kumar (2014).

the empirical findings of Fagereng, Guiso, and Pistaferri (2016) in Norway.[5]

A key reason we suggest that this mechanism is a global source of risk-taking reluctance for middle-income households is the ubiquity of another common empirical fact. In different countries, the income distribution is less dispersed and less skewed compared to the wealth distribution. The shape characteristics of income and wealth distributions drive the fact that labor income is a large fraction of middle-class lifetime resources, and a smaller fraction of lifetime resources of the rich, as Figure 2 suggests.

Our mechanism points to a direct synergy between the household-finance literature and the new literature on replicating the shape characteristics of income and wealth distributions (see e.g., Achdou, Han, Lasry, Lions, and Moll, 2015). On the one hand, commonly observed distribution-shape characteristics drive the replication of the risk-taking pattern of Figure 1. On the other hand, as pointed out by Benhabib, Bisin, and Luo (2017), replicating the common risk-taking pattern of Figure 1 in a general-equilibrium model can assist in replicating features of wealth distribution, and especially its thick upper tail. We believe this synergy opens a promising new agenda for future research.

A separate contribution of our paper is that we jointly explain household business shares and household stock shares. The importance of distinguishing stocks from business equity in household-portfolio analysis has been corroborated by results in Heaton and Lucas (2000b), Moskowitz and Vissing-Jorgensen (2002), Polkovnichenko (2003), and more recently in Palia, Qi, and Wu (2014), and Kartashova (2014). These studies note that (family) business equity constitutes a separate source of risk with different returns, exhibiting high concentration among the rich. These features of business equity make it a salient ingredient in explaining

[5] Specifically, they report an increasing convex pattern of the impact regarding background risk on households' risky portfolio share across the wealth distribution. This is most obvious for households with wealth above the 60th percentile, see Fagereng, Guiso, and Pistaferri (2016, Figure 3, and Table 8, Panel B).

the risk-taking behavior of the rich.[6]

2. Model

2.1 Observable budget-constraint characteristics

2.1.1 Income process

Time is continuous. At any instant $t \in [0, \infty)$ a household receives a labor income stream, $y(t)$, that evolves according to the geometric process

$$\frac{dy(t)}{y(t)} = \mu_y dt + \sigma_y dz_y(t) \ , \tag{1}$$

with $\sigma_y > 0$, $\mu_y \geq 0$, with $z_y(t)$ being a Brownian motion, and for a given $y(0) = y_0 > 0$.[7]

2.1.2 Asset returns

The household also possesses an initial stock of financial wealth, $a_0 \in \mathbb{R}$, and has the potential to invest this wealth in a risk-free asset with return r_f, and also in a set of $N \geq 1$ risky

[6] Recent studies, such as Christelis, Georgarakos, and Haliassos (2013), and Badarinza, Campbell, and Ramadorai (2016), analyze the importance and the cross-country differences of the role of private business equity among other household choices. Our study explores fundamental roles of business-equity risks in household risk-taking that may be common across countries.

[7] Notice the equivalence between the continuous-time representation in (1) and its discrete-time permanent-income hypothesis counterpart in Carroll (1992, 1997). In particular, Carroll (1992, p. 65) uses a discrete-time stochastic framework in which income, Y_t, following his notation, is governed by $\ln(Y_t) = \ln(P_t) + \ln(V_t)$, $\ln(V_t) \sim N\left(0, \sigma_V^2\right)$, i.i.d. over time, with P_t denoting the permanent-labor-income component which obeys $\ln(P_{t+1}) = \ln(G) + \ln(P_t) + \ln(N_{t+1})$, and in which $\ln(N_t) \sim N\left(0, \sigma_N^2\right)$, i.i.d. over time. Combining these two equations leads to,

$$\ln(Y_{t+1}) - \ln(Y_t) = \ln(G) + \ln(\varepsilon_{t+1}) \ , \tag{2}$$

in which $\ln(\varepsilon_{t+1}) = \ln(N_{t+1}) + \ln(V_{t+1}) - \ln(V_t)$. Given the assumption that $\ln(N_t)$ and $\ln(V_t)$ are independent, which is stated in Carroll (1992, p. 70), it follows that $\ln(\varepsilon_{t+1}) \sim N\left(0, \sigma_N^2 + 2\sigma_V^2\right)$, i.i.d. over time. After applying Itô's Lemma on (1) and stochastically integrating over a time interval $[t, t + \Delta t]$ for all $t \geq 0$ and any $\Delta t \geq 0$, we obtain,

$$\ln[y(t + \Delta t)] - \ln[y(t)] = \left(\mu_y - \frac{\sigma_y^2}{2}\right)\Delta t + \sigma_y\left[z_y(t + \Delta t) - z_y(t)\right] \ . \tag{3}$$

Setting $\Delta t = 1$, $\mu_y - \sigma_y^2/2 = \ln(G)$, and $\sigma_y^2 = \sigma_N^2 + 2\sigma_V^2$, makes equations (3) and (2) coincide.

assets. The price of risky asset $i \in \{1, ..., N\}$, denoted by $p_i(t)$, is governed by the process

$$\frac{dp_i(t)}{p_i(t)} = R_i dt + \mathbf{e}_i \boldsymbol{\sigma} d\mathbf{z}^T(t) \ , \tag{4}$$

in which $\mathbf{z}(t) \equiv [z_1(t) \quad z_2(t) \quad \cdots \quad z_N(t)]$ is a row vector of Brownian motions with $z_i(t)$ being associated with asset $i \in \{1, ..., N\}$. The $N \times N$ matrix $\boldsymbol{\sigma}$ is derived from the decomposition of the covariance matrix, $\boldsymbol{\Sigma}$, which refers to risks of the N risky assets only. In particular, $\boldsymbol{\Sigma} = \boldsymbol{\sigma}\boldsymbol{\sigma}^T$. Finally, $\mathbf{e}_i$ is a $1 \times N$ vector in which the value 1 is in position $i \in \{1, ..., N\}$, while all other elements are zero.

2.1.3 Correlation between labor-income growth and asset returns

Labor income is correlated with risky asset $i \in \{1, ..., N\}$ through the correlation coefficient $\rho_{y,i}$. Specifically,

$$z_y(t) = \sqrt{1 - \rho_{y,1}^2 - ... - \rho_{y.N}^2} z_0(t) + \rho_{y,1} z_1(t) + ... + \rho_{y,N} z_N(t) \ , \tag{5}$$

in which $z_0(t)$ is also a Brownian motion. If $\rho_{y,1}^2 + ... + \rho_{y,N}^2 \neq 1$, then labor-income risk is uninsurable. If, instead, $\rho_{y,1}^2 + ... + \rho_{y,N}^2 = 1$, then labor risk can be eliminated by trading financial assets. We analyze the data using the general and empirically plausible case of uninsurable labor-income risk ($\rho_{y,1}^2 + ... + \rho_{y,N}^2 \neq 1$). Nevertheless, one of the contributions of this paper is the derivation of a closed-form solution for the special case with insurable labor-income risk ($\rho_{y,1}^2 + ... + \rho_{y,N}^2 = 1$). That particular closed-form solution helps us in dealing with the technical problems of calibrating household-portfolio models with infinitely-lived agents, explained in Appendix B.[8] Below we provide more details on our solution approach.

The evolution of assets is governed by the budget constraint,

$$da(t) = \left\{ \left\{ \boldsymbol{\phi}(t) \mathbf{R}^T + \left[1 - \boldsymbol{\phi}(t) \mathbf{1}^T\right] r_f \right\} a(t) + y(t) - c(t) \right\} dt + a(t) \boldsymbol{\phi}(t) \boldsymbol{\sigma} d\mathbf{z}^T(t) \ , \tag{6}$$

[8] Two early studies numerically solving infinitely-lived-agent household portfolio models and explaining calibration difficulties are Haliassos and Michaelides (2002, 2003).

in which $\mathbf{R} = [R_1 \quad \cdots \quad R_N]$ is a row vector containing all mean asset returns and $\boldsymbol{\phi}(t) = [\phi_1(t) \quad \cdots \quad \phi_N(t)]$ is a row vector containing the chosen fraction of financial wealth invested in risky asset i, for all $i \in \{1, ..., N\}$ at any time $t \geq 0$ ($\mathbf{A}^T$ denotes the transpose of any matrix $\mathbf{A}$). In addition, there is a borrowing constraint, $a \geq b$. We do not impose any short-selling restrictions on $\boldsymbol{\phi}(t)$.

2.2 Preferences

The problem faced by a household is to maximize its lifetime expected utility subject to constraints (6) and (1). Our utility specification involves a small, yet influential step away from the continuous-time formulation and parameterization of recursive "Epstein-Zin-Weil" preferences, suggested by Duffie and Epstein (1992a,b).[9] In particular, we use a subsistence-consumption level χ, defining expected utility as,

$$J(t) = E_t \left[\int_t^{\infty} f(c(\tau) \ , \ J(\tau)) \, d\tau \right] \ , \tag{7}$$

in which $f(c, J)$ is a normalized aggregator of continuation utility, J, and current consumption, c, with

$$f(c, J) \equiv \rho(1-\gamma) \cdot J \cdot \frac{\left\{ \frac{c-\chi}{[(1-\gamma)J]^{\frac{1}{1-\gamma}}} \right\}^{1-\frac{1}{\eta}} - 1}{1 - \frac{1}{\eta}} \ , \tag{8}$$

and in which $\chi \geq 0$ and $\rho, \eta, \gamma > 0$.[10]

[9] For the discrete-time version of Epstein-Zin-Weil utility function without subsistence consumption see Epstein and Zin (1989) and Weil (1989).

[10] If $\gamma = 1/\eta$, then expected utility converges to the case of time-separable preferences with hyperbolic-absolute-risk-aversion (HARA) momentary utility. If $\chi = 0$ (standard formulation), then η denotes the household's elasticity of intertemporal substitution and γ is the coefficient of relative risk aversion. Notice that Koo (1998) has provided theoretical analysis to a model that is similar to ours but he has restricted his attention to the constant-relative-risk aversion utility function using time-separable preferences without subsistence consumption. Other notable analyses with time-separable preferences are Duffie et al. (1997) and Henderson (2005).

2.3 Solution Approach

In equilibrium, continuation utility, $J^*(t)$, is a value function depending on the household's assets and labor income, therefore, $J^*(t) = V(a(t), y(t))$ for all $t \geq 0$. With infinitely-lived households and constraints with time-invariant state-space representation, the optimization problem of the households falls in the category of stationary discounted dynamic programming. So, the time index is dropped from the Hamilton-Jacobi-Bellman equation (HJB) which is given by,

$$0 = \max_{c \geq \chi, \boldsymbol{\phi}} \Bigg\{ f(c, V(a, y)) + \left\{ \left[\boldsymbol{\phi} \mathbf{R}^T + \left(1 - \boldsymbol{\phi} \mathbf{1}^T\right) r_f \right] a + y - c \right\} \cdot V_a(a, y)$$

$$+ \frac{1}{2} a^2 \boldsymbol{\phi} \boldsymbol{\sigma} \boldsymbol{\sigma}^T \boldsymbol{\phi}^T \cdot V_{aa}(a, y) + \mu_y y \cdot V_y(a, y)$$

$$+ \frac{1}{2} (\sigma_y y)^2 \cdot V_{yy}(a, y) + \sigma_y a y \boldsymbol{\phi} \boldsymbol{\sigma} \boldsymbol{\rho}_y^T \cdot V_{ay}(a, y) \Bigg\}, \quad (9)$$

subject to $a \geq b$, in which V_x denotes the first partial derivative with respect to variable $x \in \{a, y\}$, V_{xx} is the second partial derivative with respect to x, the notation for the cross-derivative is obvious, and $\boldsymbol{\rho}_y = \begin{bmatrix} \rho_{y,1} & \cdots & \rho_{y,N} \end{bmatrix}$ is a row vector containing all correlation coefficients between each of asset returns and the income process. Finally, r_f denotes the return of investment in the risk-free asset.

We use a recursive numerical method (Chebyshev-polynomial projections) in order to solve HJB equation (9). There are two crucial technical concerns. First, calibrating the model is not straightforward in order to guarantee that the maximization problem is well-defined.[11] Second, even for a well-calibrated model, a good initial guess on the value function

[11]Calibrating the variance-covariance matrix of asset returns is especially challenging. In Appendix B we provide a simple example that demostrates why value functions are fragile in household-portfolio models and also sensitive to the choice of the variance-covariance matrix of asset returns.

V is needed.

To address these concerns we provide a closed-form solution for the special case of insurable labor-income risk. Based on that closed-form solution we can use minimum-distance fitting to calibrate that particular special case to the data and to derive a first guess for the value function V. Then, proceeding in small steps, i.e., changing parameter values gradually (for example, the homotopy approach in Eaves and Schmedders, 1999), we solve the problem given by equation (9), re-optimizing the parameters of the uninsurable labor-income-risk case by minimum-distance fitting.

2.3.1 A special case: insurable labor-income risk ($\rho_{y,s}^2 + \rho_{y,b}^2 = 1$)

The first-order conditions of the problem expressed by (9) are,[12]

$$f_c\left(c, V\left(a, y\right)\right) = V_a\left(a, y\right) \ , \tag{10}$$

$$\boldsymbol{\phi}^T = \left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(\mathbf{R}^T - r_f \mathbf{1}^T\right)\frac{V_a\left(a, y\right)}{-a \cdot V_{aa}\left(a, y\right)} - \sigma_y \frac{y}{a}\left(\boldsymbol{\rho}_y \boldsymbol{\sigma}^{-1}\right)^T \frac{V_{ay}\left(a, y\right)}{V_{aa}\left(a, y\right)} \ . \tag{11}$$

Subject to two loose parametric assumptions and assumptions on the initial conditions, the optimal vector of portfolio shares is given by (see Appendix C),

$$\begin{aligned}\boldsymbol{\phi}^* = \boldsymbol{\Phi}\left(a, y\right) = {} & \frac{1}{\gamma}\left(\mathbf{R} - r_f \mathbf{1}\right)\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(1 - \frac{\frac{\chi}{r_f}}{a}\right) \\ & + \left[\frac{1}{\gamma}\left(\mathbf{R} - r_f \mathbf{1}\right)\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1} - \sigma_y \boldsymbol{\rho}_y \boldsymbol{\sigma}^{-1}\right]\frac{\frac{y}{r_y}}{a} \ ,\end{aligned} \tag{12}$$

[12]In this special case we focus on interior solutions. Section 1.3 in our Online Appendix discusses how we ensure that consumption is above subsistence and how we treat borrowing constraints in our applications.

with $r_y \equiv r_f - \mu_y + \sigma_y \left(\mathbf{R} - r_f \mathbf{1}\right) \left(\boldsymbol{\rho}_y \boldsymbol{\sigma}^{-1}\right)^T$.[13] The term $y(t)/r_y$ is the present value of expected lifetime labor earnings at time $t \geq 0$.[14] So, the sum $(a + y/r_y)$ equals the present value of total expected lifetime resources. The term χ/r_f is the present value of lifetime subsistence needs which uses the risk-free rate as its discount factor.[15] In light of these observations, the term $(a + y/r_y - \chi/r_f)$ equals the discretionary expected lifetime resources.

The most complicated analytical aspect of determining the dependence of portfolio shares, $\boldsymbol{\phi}$, on total asset holdings, a, and income, y, is the role played by the variance-covariance matrix of risky assets. In the case of two risky assets ($N = 2$), the covariance matrix is,

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_s^2 & \rho_{s,b}\sigma_s\sigma_b \\ \rho_{s,b}\sigma_s\sigma_b & \sigma_b^2 \end{bmatrix},$$

in which σ_i is the standard deviation of asset $i \in \{s, b\}$, with subscript "s" denoting "stocks" and subscript "b" denoting "business equity", while $\rho_{i,j}$ denotes the correlation coefficient between two risky assets $i, j \in \{s, b\}$. In the special case of insurable labor-income risk, the stochastic structure of the problem with $N = 2$ involves three volatility parameters, σ_s, σ_b,

[13] Although closed-form solutions are rare, there are remarkably many studies reporting closed-form solutions on portfolio choice and techniques for discovering such solutions. Examples include Merton (1973), Bodie et al. (2009) and Detemple and Rindisbacher (2005, 2010).

[14] Since labor income is insurable, the effective discount factor, r_y, which is used to calculate the present value of expected lifetime labor earnings, involves three opportunity-cost ingredients. These ingredients are the risk-free rate, r_f, the trend of income, μ_y, and a term involving the excess returns and risks of other assets, $\left(\mathbf{R} - r_f\mathbf{1}\right)\left(\boldsymbol{\sigma}^{-1}\right)^T$. In addition, $r_y = r_f - \mu_y + \sigma_y \left(\mathbf{R} - r_f\mathbf{1}\right)\left(\boldsymbol{\sigma}^{-1}\right)^T \boldsymbol{\rho}_y^T$, takes into account the correlations of income with the risky assets, $\boldsymbol{\rho}_y$, and income volatility, σ_y. In particular, notice that $y(t) = y_0 \cdot e^{\mu_y t + \sigma_y z_y(t)}$ (see equation (1)) while equation (5) combined with the condition $\boldsymbol{\rho}_y \boldsymbol{\rho}_y^T = 1$ gives $z_y(t) = \boldsymbol{\rho}_y \cdot \mathbf{z}^T(t)$.

[15] To see why the discount factor of lifetime subsistence needs is the risk-free rate alone, consider the special case of a household with minimum assets, $\underline{a}$, such that $\underline{a} + y/r_y = \chi/r_f$, i.e. total expected lifetime resources equal subsistence needs (in slight violation of Assumption 1). In this special case, equation (12) implies that the household holds a portfolio of risky assets, $\boldsymbol{\phi}^* \cdot \underline{a} = -\sigma_y y/r_y \boldsymbol{\rho}_y \boldsymbol{\sigma}^{-1}$ which enables it to perfectly insure against labor-income risk. In this way, the equilibrium consumption profile of such a household is $c^*(t) = \chi$ for all $t \geq 0$. So, the ability to insure against labor-income risk enables the household to avoid consumption fluctuations and to meet the condition $c(t) \geq \chi$ with equality at all times. Since this special household does not have any opportunity left for fluctuations in total income through its savings behavior (its total income is equal to χ for all $t \geq 0$), its intertemporal opportunity cost is determined solely by the risk-free rate r_f.

and σ_y, and two correlation coefficients, $\rho_{s,b}$ and $\rho_{y,s}$, since correlation $\rho_{y,b}$ can be deduced from the labor-risk-insurability constraint $\rho_{y,s}^2 + \rho_{y,b}^2 = 1$. As demonstrated in Appendix B, optimal portfolio choices, and even the existence of a solution to the portfolio-choice problem, are sensitive to the variance-covariance parameters used. In addition, there is no perfect agreement in the literature regarding business-equity returns, its riskiness and correlations with other risky variables.[16] For these reasons, the careful calibration procedure of our model is likely to suggest values for R_b, σ_b, $\rho_{s,b}$, and $\rho_{b,y}$, that work in practice and that also open new questions for the empirical literature about business-equity returns.

2.3.2 Replicating the convexity pattern of risk-taking under insurable labor-income risk ($\rho_{y,s}^2 + \rho_{y,b}^2 = 1$)

In Appendix D, we show that for $N = 2$, the exact solution described by (12) can be summarized by,

$$\phi_i^* = \kappa_{0,i} - \kappa_{1,i}\frac{1}{a} - \kappa_{2,i}\frac{y}{a} \ , \quad i \in \{s, b\} \ . \tag{13}$$

If we switch off the role of income in equation (13) by setting $\kappa_{2,i} = 0$, then with $\kappa_{1,i} > 0$ risk-taking is an increasing function of wealth.[17] However, with $\kappa_{2,i} = 0$ and $\kappa_{1,i} > 0$, risk-taking takes the form of an increasing and *concave* function in wealth, failing to reconcile the empirical pattern of Figure 1.

Considering $\kappa_{2,i} \neq 0$ in equation (13), stylized empirical features concerning income and wealth distributions play a particular role. Specifically, as the income distribution is less dispersed and less skewed compared to the wealth distribution, the income-to-asset ratio, y/a decreases in total household resources on average. This feature is conveyed by the pattern of Figure 2, which also shows that y/a is quite high for middle-class households. Considering a value with $\kappa_{2,i} < 0$ would add this high amount of y/a that characterizes middle-income

[16]See, for example, the difference between Moskowitz and Visssing-Jorgensen (2002) and Kartashova (2014).
[17]We provide the exact formulas of $\kappa_{0,i}$, $\kappa_{1,i}$, and $\kappa_{2,i}$ of equation (13) in Appendix D.

households to ϕ_i, making middle-income households take more risk. Therefore, $\kappa_{2,i} < 0$ would contribute to strengthening the concave pattern that is already implied by the term $\kappa_{1,i}/a$. Instead, $\kappa_{2,i} > 0$ is consistent with a reduced value of ϕ_i for middle-income households.

Reducing the value of ϕ_i for middle-income households is necessary for replicating the observed pattern of risk-taking: that ϕ_i is increasing and convex in lifetime household resources, $a + y/r_y$. *So, we call* $\kappa_{2,i}$ *"the convexity factor"*, emphasizing that $\kappa_{2,i} > 0$, is the only way to match the risk-taking pattern observed in the data.

The intuition behind a positive value of $\kappa_{2,i}$ is based on labor-income-risk diversification incentives. Middle-income households whose income, y, is a big fraction of their lifetime resources, $a + y/r_y$, choose to take less controllable financial risk in order to cope with the high uncontrollable labor-income risk carried by y. On the contrary, for richer households, y is a smaller fraction of their lifetime resources, so they can afford to add financial risk to their total lifetime resources.[18]

In this special case of $\rho_{y,s}^2 + \rho_{y,b}^2 = 1$ with (13) being the closed-form solution to the model, background labor-income risk is fully insurable. Nevertheless, the motive of choosing

[18] That labor-income risk diversification is a crucial mechanism driving the positivity of the "convexity factor" $\kappa_{2,i}$ is obvious from a concise version of its formula. Specifically, a concise way to express $\kappa_{2,i}$, $i \in \{s, b\}$ (see Appendix D) is given by,

$$\kappa_{2,s} = \left[\frac{1}{\gamma} \cdot \frac{\sigma_s}{1 - \rho_{s,b}^2} \cdot \left(\frac{R_s - r_f}{\sigma_s} - \rho_{s,b} \frac{R_b - r_f}{\sigma_b} \right) + \frac{\rho_{s,b}\sigma_s}{\sigma_b \sqrt{1 - \rho_{s,b}^2}} Cov\left(y, b\right) - Cov\left(y, s\right) \right] \frac{1}{r_y \sigma_s^2} , \qquad (14)$$

and

$$\kappa_{2,b} = \left[\frac{1}{\gamma} \cdot \frac{\sigma_b}{\sqrt{1 - \rho_{s,b}^2}} \left(\frac{R_b - r_f}{\sigma_b} - \rho_{s,b} \frac{R_s - r_f}{\sigma_s} \right) - Cov\left(y, b\right) \right] \frac{1}{r_y \sigma_b^2 \sqrt{1 - \rho_{s,b}^2}} . \qquad (15)$$

Both equations (14) and (15) convey that, apart from comparisons between the Sharpe ratios of the two risky assets, $\kappa_{2,i}$ can become negative if the covariances of labor-income risk and risky-asset returns, $Cov\left(y, b\right)$ and $Cov\left(y, s\right)$, are high. In other words, with sufficiently low covariance of labor-income risk and risky-asset returns, diversification of background labor-income risk is possible. With $\kappa_{2,i} > 0$, $i \in \{s, b\}$, the cross-sectional effect of labor-income-risk diversification is that middle-income households (with higher y/a) will reduce their portfolio risk-taking more than rich households.

lower financial risk in order to cope with background risk is present, no matter if labor-income risk is ultimately fully insurable or not. In the next section we demonstrate that even if background labor-income risk is uninsurable ($\rho_{y,s}^2 + \rho_{y,b}^2 \neq 1$), the exact formulas of $\kappa_{0,i}$, $\kappa_{1,i}$, and $\kappa_{2,i}$ of equation (13) are a good approximation of the true solution of the model, and that $\kappa_{2,i} > 0$ holds in all calibration exercises leading to a risk-taking pattern that is increasing and convex in income/wealth. So, *even under uninsurable labor-income risk, the motive of reducing background labor-income risk is the key reason behind the strong reluctance of middle-income households to undertake financial risk; this is the predominant mechanism behind replicating the observed pattern, that risk-taking is increasing and convex in household resources.*

Our model's ability to detect the underlying mechanism through simple inspection of whether $\kappa_{2,i} > 0$ or not, relies on the analytical sophistication of our decision rule for portfolio shares. Specifically, our decision rule in (13) is of the form $\phi_i^* = \Phi^i(a, y)$, $i \in \{s, b\}$, i.e., it distinguishes household resources between assets and income. This feature differentiates our study from previous literature seeking to understand mechanisms behind household portfolios through calibrated models, in which portfolio shares are a function of cash on hand (see, for example, Gomes and Michaelides, 2005).

3. Fitting the Model to the Data

3.1 Calibration

Our calibration strategy follows two steps. In the first step we calibrate the model using the empirically implausible special case of insurable labor-income risk ($\rho_{y,s}^2 + \rho_{y,b}^2 = 1$). In this special case the closed-form solution given by (13) allows us to use minimum-distance techniques in order to find the calibrating parameters that best fit the data.[19] Using these initial calibrating parameters, we take the second step, which is to gradually change parameter values while employing recursive numerical methods in order to solve equation (9) for the general case of uninsurable-labor-income risk ($\rho_{y,s}^2 + \rho_{y,b}^2 \neq 1$).[20] For every recursive solution to the uninsurable labor-income-risk version based on equation (9), we re-calibrate model parameters through minimum-distance fitting as well.

Table 1: Calibrated Parameters across Model Specifications

Parameters	*US*		*EU*		*CN*	
$\rho_{ys}^2 + \rho_{yb}^2$	1	0.75	1	0.75	1	0.75
	(%)		*(%)*		*(%)*	
σ_y	12.0	12.0	6.0	6.2	13.0	13.5
μ_y	1.7	1.7	0.6	0.8	4.5	4.5
r_f	3.7	3.7	3.7	3.1	2.7	2.7
R_s	7.0	6.7	7.0	7.0	12.9	13.3
R_b	11.1	11.8	24.9	22.4	64.9	65.0
σ_s	20.9	18.0	22.9	21.8	56.2	59.0
σ_b	30.0	29.5	42.3	36.5	90.0	90.1
ρ_{sy}	48.5	41.3	29.3	30.0	25.5	22.3
ρ_{sb}	-7.5	-7.6	9.9	9.6	11.1	11.6
ρ_{by}	87.4	76.1	95.6	81.3	96.7	83.7
γ	3.316	4.439	10.677	12.000	8.121	8.826
η	0.160	0.163	0.160	0.165	0.160	0.165
ρ	0.03	0.03	0.03	0.03	0.03	0.03
	(in USD)		*(in Euros)*		*(in RMB)*	
χ	1437	1603	2070	2140	3000	3000
PPP adjusted 2007 USD	1437	1603	1785	1845	1491	1491

Table 1 provides all calibrating parameters in two cases: the case of insurable labor-

[19]In Appendix B we explain why portfolio-choice models with infinitely-lived agents have fragile value functions. This fragility motivates using special cases in which the closed-form solution serves as a guide in order to calibrate the model through minimum-distance fitting to data.

[20]The practice of starting from calibrated parameters of a well-behaved solution in order to change parameter values in a gradual, step-by-step fashion, is the homotopy approach, explained in Garcia and Zangwill (1981), and also in Eaves and Schmedders (1999). All recursive numerical methods, including a full explanation of how borrowing constraints are treated numerically, appear in our Online Computational Appendix.

income risk ($\rho^2_{y,s} + \rho^2_{y,b} = 1$), and a plausibly calibrated case of uninsurable labor-income risk in which $\rho^2_{y,s} + \rho^2_{y,b} = 0.75$. Figure 3 shows the goodness of fit of our calibrated model for $\rho^2_{y,s} + \rho^2_{y,b} = 1$, while Figure 4 shows the same for $\rho^2_{y,s} + \rho^2_{y,b} = 0.75$. Although we ultimately rely on minimum-distance techniques, we need to anchor some paramenter values in order to start implementing a minimum-distance approach to find fitting values for different parameters.

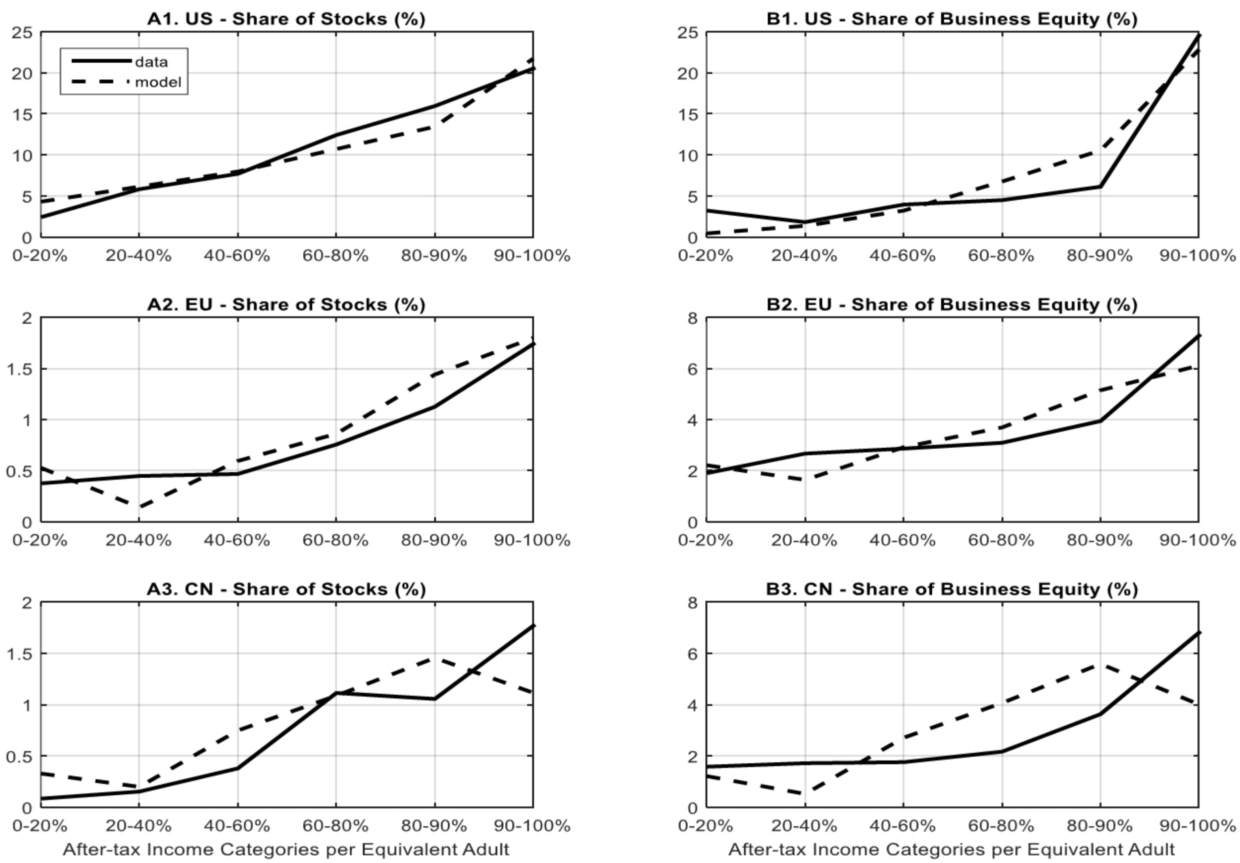


Figure 3 – Benchmark calibration of the special case of the model with insurable labor-income risk ($\rho^2_{y,s} + \rho^2_{y,b} = 1$), using the closed-form solution. "CN" denotes China.

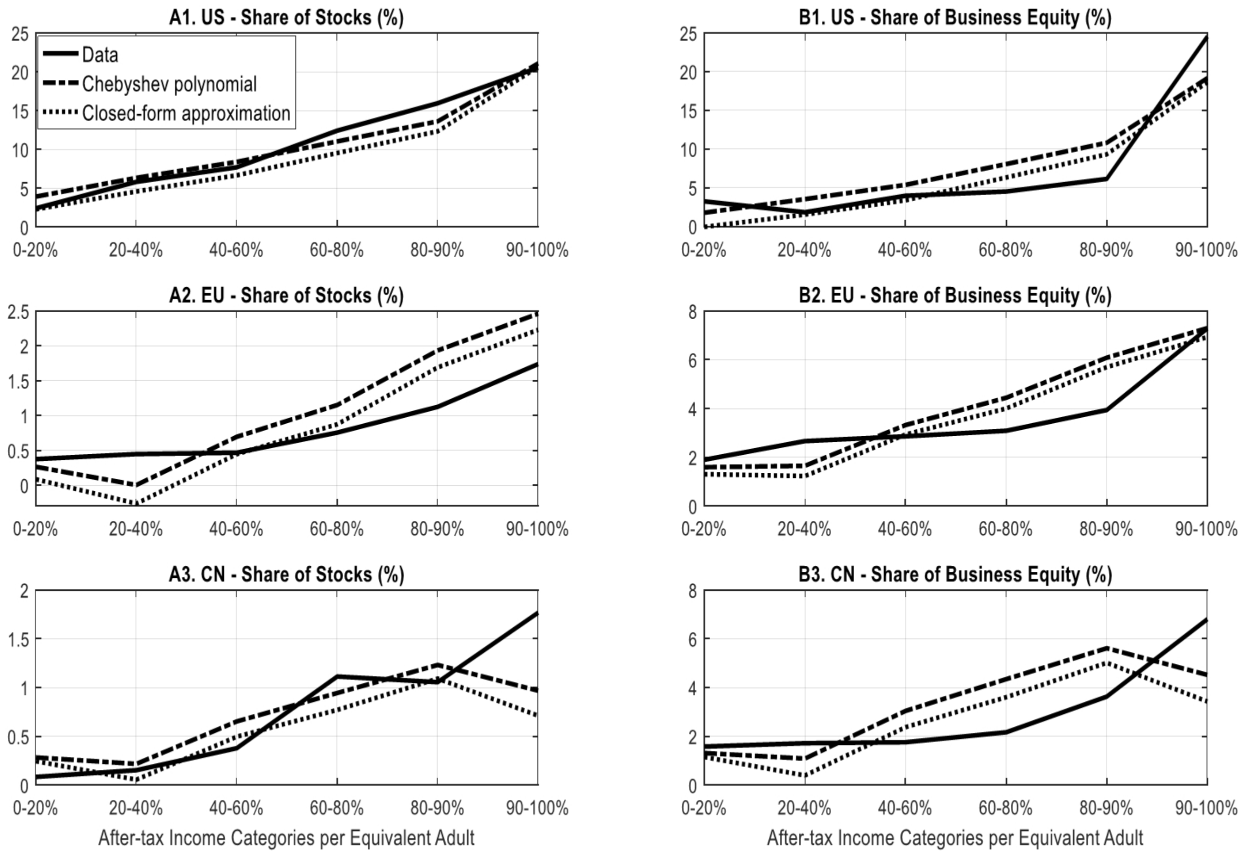


Figure 4 – Calibration of the general version of the model with uninsurable labor-income risk ($\rho_{y,s}^2 + \rho_{y,b}^2 = 0.75$). "CN" denotes China.

Setting labor-income risk, σ_y, equal to 12.0%, is within the ballpark of a standard parametrization motivated by micro data (see, for example, Gomes and Michaelides (2003 p. 736) for details).[21] In Europe, labor protection regulation reduces labor-income fluctuations; therefore, we pick more moderate values, while Chinese labor-income fluctuations resemble those in the US. In the US, we set the mean labor-income growth to 1.7%, and we use different values for Europe and China, reflecting the real-economy growth experiences of the two economic regions and population-aging trends.

[21]For the definition of after-tax incomes, and income-tax calculations see the Online Data Appendix for details on tax rates and also Grant, Koulovatianos, Michaelides, and Padula (2010, Table 2, p. 968). For the value of σ_y, see also Storesletten, Telmer, and Yaron (2004).

In the US, our stock returns, R_s, and their volatility, σ_s, are close to their long-term values of 8% and 19% (see Guvenen, 2009, Table II, p. 1725) and close to the values 6% and 18% used by Gomes and Michaelides (2003). Our calibration exercise worked better by giving the risk-free rate, r_f, a rather generous 3.70%, compared to the standard value close to 2% (see, for example, Gomes and Michaelides (2003) and Guvenen (2009)). While our implied equity premium is rather low (3.3%), it is not uncommon in the household-finance literature to consider such values. For example, an equity premium of 2.5% is within the range of values examined by Gomes and Michaelides (2003). European stock markets have similar features, so our calibrating values for R_s, σ_s, and r_f in the EU area are similar to those in the US. In China, however, which is an emerging market, the stock returns from the Shangai A-Share Index (using data from 1992-2013), R_s and σ_s are 11% and 76% in the data, so we picked numbers around 13% and 59% in our calibration.[22]

A crucial preference parameter that we can calibrate based on survey data, is subsistence consumption, χ. The monthly amount of USD 120 that we use for all economic regions (slightly higher for the EU) is within the range of survey evidence of monthly subsistence consumption reported by Koulovatianos et al. (2007, 2014), ranging between USD 111 and 302.[23]

After anchoring the values of all parameters above, R_s, σ_s, r_f, μ_y, σ_y, and χ, we perform minimum-distance fitting in order to match portfolio shares, ϕ_s and ϕ_b, observed in the data, using our closed-form formulas from Appendix D. Our minimum-distance exercise

[22] Details on the Chinese stock-market data can be provided by the authors upon request.

[23] Econometric studies such as those of Atkeson and Ogaki (1996), Ogaki and Zhang (2001), and Donaldson and Pendakur (2006) do not reject the existence of subsistence consumption levels. Yet, issues of econometric model specification affect the robustness of subsistence estimates. Here, we rely on estimates from surveys regarding living standard comparisons across households; an adult needs an annual amount of approximately 3,000 US dollars in order to just survive. Our calibration in this paper refers to US dollars in year 2007. For the survey evidence see Koulovatianos et al. (2007, 2014) who use data from six countries derived by using the survey method first suggested by Koulovatianos et al. (2005), finding annual subsistence costs per person between 1,300-3,600 US dollars.

then implies a number of parameters for business equity that best match the data.

Interesting and robust are the implications that the mean and standard deviation of business-equity returns, R_b and σ_b, are 11.1% and 30% in the US. The value $R_b = 11.1\%$ is not far from the average estimates in Moskowitz and Vissing-Jorgensen (2002, Table 4, p. 756), and Kartashova (2014, Table 5, p. 3308). Regarding our model's implication that $\sigma_b = 30\%$, Moskowitz and Vissing-Jorgensen (2002, p. 765) mention: "*[...] the annual standard deviation of the smallest decile of public firm returns is 41.1 percent. A portfolio of even smaller private firms is likely to be as volatile.*" It can be difficult to estimate idiosyncratic risks borne by a household. Unobservable limitations in outside options, such as frictions in relocating a business if other family incomes could increase by relocating, imperfect insurance from theft, etc., may justify that a value for σ_b in the order of 30% may still be low. In China, Bai, Hsieh and Qian (2006, Figure 12, p. 85) indicate returns to capital by province that, in most cases, range between $20-60\%$. In an emerging economy such as China, private business-equity returns should range well above these return figures, therefore we pick values around 65% for R_b in China. Similarly, as indicated in Bai, Hsieh and Qian (2006, Figure 13, p. 86), returns to capital across provinces are also very volatile. Picking $\sigma_b = 90\%$ for China is a natural choice, keeping the Sharpe ratio of private business equity about twice as much as the Sharpe ratio of stock returns in the US which is 30% in the data (see Guvenen, 2009, Table II, p. 1725). Given the inter-country differences in the EU area, its lack of deep capital-market integration compared with the US, and the emerging-market features of private business in peripheral EU countries, the values of R_b and σ_b of 23% and 37% are reasonable (and comparable with the ranges estimated by Kartashova, 2014, Table 5, p. 3308, for the US).

Regarding the correlation between labor income shocks and stock returns, ρ_{ys}, Gomes and Michaelides (2003 p. 736) suggest an educated value of 30% for the US, but also try higher values. Our implied value for $\rho_{y,s}$ is 41.3% in the case of uninsurable labor-income risk.[24] The associated value for $\rho_{y,b}$ satisfying the relationship $\rho_{y,s}^2 + \rho_{y,b}^2 = 0.75$ is 76.1%. This high correlation between business equity and family income may be plausible as a large fraction of households have family businesses and tend to employ family members. In the EU and in China, the correlation of $\rho_{y,b}$ is similarly high.

Finally, in all our calibration exercises, the correlation between stock returns and business-equity returns, $\rho_{s,b}$, is always close to zero, in all economic regions. This robust implication of the model agrees with evidence in Palia, Qi, and Wu (2014, Table 2, p. 1702), in which the mean of $\rho_{s,b}$, is close to zero and the median is zero for the US. Apparently, the multitude of regional and idiosyncratic risks added to private business equity contribute to generating this trivial correlation between stock returns and business-equity returns.

3.2 Detecting the underlying mechanism under uninsurable labor-income risk

Our discussion in Section 2.3.2 above made a specific point. In the context of insurable background labor-income risk, the exact solution given by equation (13) implies that the only way to achieve the convexity feature in the risk-taking pattern of Figure 1 is to have $\kappa_{2,i} > 0$, $i \in \{s, b\}$. Therefore, we called $\kappa_{2,i}$ "the convexity factor".

Although in the case of uninsurable labor-income risk ($\rho_{y,s}^2 + \rho_{y,b}^2 \neq 1$) equation (13) is invalid, Figure 4 uses (13) and demonstrates a feature that is important for distinguishing the underlying risk-taking incentives and mechanism. The dotted lines of Figure 4 use the parameter values corresponding to the case of $\rho_{y,s}^2 + \rho_{y,b}^2 = 0.75$ in Table 1, after imputing these parameter values into the formula of equation (13), and specifically in equations (14)

[24] In our Online Appendix we find that $\rho_{y,s}$ is 51% for college graduates and 32% for the total population using annual frequency.

and (15). Figure 4 shows that this wrongly applied closed-form solution, represented by the dotted lines, is a good approximation of the true solution, which is shown by the dashed lines of Figure 4. Given this close approximation, reporting the value of $\kappa_{2,i}$ even in the case of $\rho^2_{y,s}+\rho^2_{y,b} = 0.75$ is a good way to reveal the underlying mechanism of risk-taking. Specifically, having $\kappa_{2,i} > 0$ is sufficient to show that middle-class households take less financial risk because they try to relieve themselves from the high level of background-income risk that they have to bear.

Table 2: The calibrated "convexity factor" $\kappa_{2,i}$

$\rho^2_{ys} + \rho^2_{yb}$	1		0.75	
	$\kappa_{2,s}$	$\kappa_{2,b}$	$\kappa_{2,s}$	$\kappa_{2,b}$
US	1.069	1.586	1.203	1.406
EU	0.208	0.445	0.255	0.404
CN	0.148	0.653	0.150	0.637

Note: $\kappa_{2,i}$ in eqn. (13).

Table 2 presents the values of the "convexity factor", $\kappa_{2,i}$, $i \in \{s, b\}$, for all model simulations in Figures 3 and 4. In all three countries/economic regions and in all cases, $\kappa_{2,i}$ is strictly positive, reconfirming the role of background-income risk in household risk-taking suggested in this paper.[25]

There is recent evidence supporting our model's mechanism regarding the role of background-income risk in household risk-taking. Fagereng, Guiso, and Pistaferri (2016) use administrative data from Norway to identify background risk. In a regression of household risk-taking against background risk and wealth, they find a negative coefficient against background risk and a positive coefficient between risk-taking and the interaction term of background risk and lagged wealth (see Fagereng, Guiso, and Pistaferri, 2016, Table 8, Panel B therein).

[25] The goodness of fit of our model even in the case of $\rho^2_{y,s} + \rho^2_{y,b} = 1$ is consistent with the findings of Benzoni, Collin-Dufresne, and Goldstein (2007) for lifecycle portfolio-choice models.

These findings support the interpretation of $\kappa_{2,i} > 0$ from equations (14) and (15): diversification of background income risk is the motivation behind decreasing portfolio shares, ϕ_i. In addition, middle-class and poor households bear high background risk (see Fagereng, Guiso, and Pistaferri, 2016, Figure 3). This evidence supports our model's interpretation of a key feature of middle-income households: that the main reason these households bear higher background-labor income risk is their higher income-to-asset ratios, y/a.

Importantly, our model can also replicate saving rates of the rich and the poor within the ranges suggested by Dynan et al. (2004). These saving rates are also increasing in household resources.[26]

3.3 Sensitivity analysis and discrete-time applications

In Appendix E we perform a sensitivity analysis by further relaxing the relationship linking $\rho_{y,s}$ with $\rho_{y,b}$. Specifically, we consider $\rho_{y,s}^2 + \rho_{y,b}^2 = 0.5$. The main message is that our mechanism prevails, although the goodness of fit worsens in some cases. Our benchmark case with $\rho_{y,s}^2 + \rho_{y,b}^2 = 0.75$ seems to be a good representation of the data at least at the level of the model's abstraction and parsimony.

In Appendix F, we perform another important extension. We translate our model to its discrete-time counterpart. This is an essential exercise, because decisions under uncertainty in continuous-time models take into account only two moments of Brownian-motion shocks. On the contrary, decisions under uncertainty in discrete-time models take into account higher moments of random shocks. Due to this difference between the discrete-time and the continuous-time settings, continuous-time-model parameters that fit data targets well are not appropriate for the discrete-time model. Yet, parameters from the closed-form solution of the continuous-time setting that best fit the data serve as an ideal starting point

[26] These saving-rate results can be provided by the authors upon request.

in a homotopy approach for calibrating and solving the discrete-time model (see Garcia and Zangwill, 1981, and in Eaves and Schmedders, 1999). The optimal continuous-time parameters put the discrete-time model in the area of well-behaved solutions, with a well-defined value function of the portfolio-choice model. Using the continuous-time parameters with diversifiable labor-income risk as a starting guess, we recursively solve the discrete-time model with diversifiable labor-income risk. This solution of the discrete-time model with diversifiable labor-income risk is the initial value of the homotopy process toward the final goal which is the undiversifiable labor-income risk version with data-compatible correlation parameters.

3.3.1 Borrowing constraints and risk of failing to meet subsistence needs

In all simulations, we have imposed the constraints $c \geq \chi$, and $a \geq 0$, i.e., we have placed a borrowing limit. The borrowing limit is less relevant to the target group of stockholders that we are calibrating, since these households already have some financial assets, and the probability of a binding borrowing constraint is negligible in the calibration. The constraint $c \geq \chi$ is more likely to bind. The calibrating parameters indicated by the closed-form solution in the case of $\rho_{y,s}^2 + \rho_{y,b}^2 = 1$, keep this likelihood low, with negligible impact on our simulations, even in the case of $\rho_{y,s}^2 + \rho_{y,b}^2 = 0.5$, in which labor-uninsurability concerns are substantial. An intuitive explanation for the state described by $c < \chi$ is homelessness or failure to meet daily calorie-intake needs. Perhaps our target group of risky-asset-holders, conditional on their risky-asset-holding status, has initial conditions, a and y, that already give these households the opportunity to choose savings and risk-taking strategies that keep them well within an interior solution. Including non-stockholders with lower initial wealth is beyond the scope of our analysis, but an interesting future extension to pursue.

4. Implications of the risk-taking mechanism

4.1 Implications for explaining the wealth distribution

Recent studies have brought to surface more detailed empirical facts about the wealth distribution, (see, for example, Piketty, 2014, Figure 10.6, p. 349).[27] These empirical observations have triggered a renewed interest in modeling the wealth distribution as an equilibrium outcome of market activity. A puzzling feature is the thickness of the upper tail of the wealth distribution, that standard heterogeneous-agent models, e.g., Huggett (1993), Aiyagari (1994), and Krusell and Smith (1998) cannot replicate. Recently, Benhabib, Bisin, and Zhu (2011, 2015) demonstrated analytically that the main reason such older heterogeneous-agent models cannot replicate the heavy wealth tail is the homogeneity of investment returns across the rich and the poor. Benhabib, Bisin, and Luo (2015) suggest, among other ideas, that introducing differential capital-income risk and returns is one of the ways to replicate such interesting features of the wealth distribution. However, their calibration exercise uses exogenous differences in risk and returns between the rich and the poor. To reach the ultimate goal of this inequality research agenda, which is to evaluate how inequality responds to policy, one needs to dig deeper in order to endogenize both the sources of portfolio choice and the wealth distribution.

Our findings have direct implications for endogenizing both portfolio choice and the wealth distribution in future extensions. The mechanism suggested by our model requires that the wealth distribution has a heavy upper tail in order to replicate the increasing and convex pattern of risk-taking in household resources. Correspondingly, for replicating the heavy wealth-distribution tail, Benhabib, Bisin, and Luo (2015) suggest differential idiosyncratic returns across the rich and the poor that are in line with the observed risk-taking

[27] Interesting findings also pertain to the link between entrepreneurship and inequality. See, for example, Cagetti and De Nardi (2006), De Nardi, Doctor, and Krane (2007), and also De Nardi and Giulio (2017).

patterns. This potential synergy between mechanisms is promising to jointly explain the stylized empirical observations regarding risk-taking patterns and shapes of wealth and income distributions.

4.2 Implications for redistributive income taxes: a potential paradox?

The mechanism supported by our model also has implications for the effects of taxation on household risk-taking. Making progressive income taxation tighter is likely to provide incentives to richer households to undertake even more financial risk. First, income taxes, and especially progressive income taxes, reduce the income-to-asset ratio of the rich even more. Second, as is empirically supported by Grant, Koulovatianos, Michaelides, and Padula (2010), higher taxes suppress effective labor-income risk, leading to an insurance effect. Both effects of taxation make the rich less urged to reduce labor-income risk, and more willing to increase riskiness in their asset portfolios, by pursuing more investments with higher mean returns.

So, paradoxically, more redistributive income taxation can potentially lead to thicker upper tails of the wealth distribution, by providing incentives to the rich to switch to investments with higher returns. Such general-equilibrium effects can be analyzed through models that jointly endogenize household portfolios and the wealth sistribution. Early research on this interplay includes Gomes and Michaelides (2008) and Guvenen (2009). These studies focus more on asset-pricing questions and less on the features of the resulting wealth distribution. Continuous-time models such as Achdou, Han, Lasry, Lions, and Moll (2015, Section 4), may share analytical features of our study more directly, if combined with insights from recent research by Benhabib, Bisin, and Luo (2017) that focuses on replicating the tails of the wealth distribution. Studying taxation questions in such frameworks that take into account the portfolio-choice channel, is a promising agenda for future research.

5. Conclusion

In two influential articles that survey the household-finance literature, Campbell (2006) and Guiso and Sodini (2013) emphasize the empirical importance of general and idiosyncratic background risks faced by households in explaining household risk-taking. Both survey articles emphasize the need for more "positive household finance" which examines what investment choices household actually make (see Campbell, 2006, p. 1554, and Guiso and Sodini, 2013, p. 1399). A rapidly growing literature that uses administrative data helps us to understand numerous details of household risk-taking choices with confidence.[28] This literature promotes the development of new "normative" or "equilibrium household-finance theories", i.e., theories that explain what households should do, based on some theoretical reasoning that may involve bounded rationality (see Campbell, 2006, pp. 1554-1555). While we do not object to the fruitful development of new theories that include concepts from behavioral economics, here we have reverted to using a simple variant of the Merton (1969, 1971) "normative" model in order to investigate a general, big-picture mechanism that may explain an ubiquitous pattern of risk-taking: that household stock and business-equity risk-taking is increasing in income/wealth at an accelerating pace. Specifically, we have employed a parsimonious model that uses the same utility function for all households in order to answer the question "*why do rich households undertake so much financial risk, while poor and middle-income households hesitate to do that?*".

Our model has demonstrated a specific role that labor-income risk plays in household risk-taking. Labor-income risk makes middle-income households reluctant to take financial risk.

[28]Early examples of this literature include Calvet, Campbell, and Sodini (2007, 2009a, 2009b) and Calvet and Sodini (2014).

This reluctance is capable of explaining the substantial gap between risk-taking behaviors of rich and middle-income households that is observed in the data. The wide rich/middle-class risk-taking gap is observed across fifteen European countries, China, and the US. Specifically, the shares of stocks and business equity in total household assets are an increasing and convex function of total household resources. This ubiquitous pattern has been documented by research using administrative data, that are free from sampling error, by Bach, Calvet, and Sodini (2016) in Sweden, and by Fagereng, Guiso, Malacrino, and Pistaferri (2016) in Norway.

The mechanism behind the role of labor-income risk is intuitive. For middle-income households, labor income is a big fraction of their total lifetime resources. For this reason, as confirmed by administrative data in Norway (see Fagereng, Guiso, and Pistaferri 2016, Figure 3), the relative pressure of the uncontrollable background labor-income risk that must be borne by middle-income households is high. To cope with this uncontrollable risk, middle-income households respond by reducing their controllable financial risk-taking. Our model demonstrated that this explanation is common across the three major economic regions we examined, the US, the EU, and China (see Table 2, and equation (13), demostrating that income-to-assets ratios are multiplied by negative coefficients in risk-taking equations in our calibrated exercises). This mechanism agrees with the indicative evidence from administrative Norwegian data, which posits that background risk correlates negatively with risk-taking; moreover, background risk multiplied by wealth has a positive correlation with risk-taking (see Fagereng, Guiso, and Pistaferri, 2016, Table 8, Panel B).

The mechanism we have proposed relies on another stylized fact across countries. The income distribution is less dispersed and less skewed compared to wealth distribution. These distributional features are the main reason why labor income is a large fraction of middle-class lifetime resources, and a smaller fraction of lifetime resources of the rich (decreasing income-to-assets ratios in income/wealth).

This alliance between stylized facts has implications for research that pursues explanations of the observed fat upper tails of wealth distributions through general-equilibrium models. In two survey articles, Benhabib and Bisin (2017), and Benhabib, Bisin, and Luo (2017) stress that it is necessary to have higher stochastic idiosyncratic returns in order to explain the thickness of the upper tail of the wealth distribution. Heterogeneous-agent models, such as Krusell and Smith (1998) and Achdou, Han, Lasry, Lions, and Moll (2015), that pursue fitting the features of both income and wealth distributions simultaneously, will benefit from introducing portfolio choice in a general-equilibrium framework. Based on suggestions by Benhabib, Bisin and Luo (2017) about adding new features to general-equilibrium models, the one stylized fact, which is the increasing and convex pattern of household risk-taking in wealth, will assist the generation of fat upper tails of wealth distribution in such models. In turn, fat upper tails of the wealth distribution will assist in the replication of the increasing and convex pattern of household risk-taking in wealth. We believe our suggested methods of calibration can guide continuous-time heterogeneous-agent approaches, as in Achdou, Han, Lasry, Lions, and Moll (2015, Section 4, on their portfolio-choice application), thereby opening a promising new agenda for understanding interlinkages between heterogeneity in financial choices of households and their role in explaining the income/wealth distribution. In addition, such extensions will shed new light on questions of effectiveness of redistributive income taxation. Our analysis suggests that more redistributive income taxation can lead to adverse effects that could even thicken the upper tails of wealth distribution. Achdou, Han, Lasry, Lions, and Moll (2015, Section 4.4, examining capital income taxation) offer tools for initiating such a research agenda. Exploring such policy-response mechanisms is of interest to both policymakers and the public.

6. Appendix A – Explaining the data used in Figure 1

The data sources for the US are the Survey of Consumer Finances (SCF), referring to year 2007 (before the crisis), for the EU it is the Eurosystem Household Finance and Consumption Survey (HFCS) in year 2013, and for China it is the China Household Finance Survey (CHFS) in year 2013 as well. The household-income categorization in Figure 1 follows the convention proposed by the SCF Chartbook in the US and in the EU (see, Bucks et al. 2009, and European Central Bank, 2013): income categories include both labor income and capital income.

In Bach, Calvet, and Sodini (2016, see Figures 3 and 4) in which they present the mean returns of household risky assets and their standard deviations in Sweden, and also in Fagereng, Guiso, Malacrino, and Pistaferri (2016, Figure 1 therein), the wealth percentiles they employ include young and old households. We do the same mixing of young and old households in the income categories we use in Figure 1. This descriptive strategy abstracts from lifecycle effects in household risk-taking. Table A.1 shows that the average and median ages corresponding to each income category do not exhibit a strong age bias, varying only slightly across income categories (the EU survey excludes pensioners and, unlike the US and the EU survey, the Chinese survey has not oversampled from the richest population group). The indication that ages are almost symmetrically distributed around the means of our income categories provides confidence that we can search for fundamental common reasons on why the rich/poor distinction leads to such a common risk-taking pattern depicted by in Figure 1.

For the US we use data before the subprime crisis of 2008, whereas for EU countries and China we use the databases from 2013, that also enable comparisons across EU countries. US household-portfolio data include direct stockholding, mutual funds, and retirement accounts.

EU data include only direct stockholding and mutual funds, which is one reason that EU portfolio shares are lower compared to the US. Chinese data include direct stockholding, mutual funds and some other indirect stockholding data, such as Exchange Traded Funds (ETFs). In Online Data Appendices we provide details about our data sources, and database structure and quality.

Table A.1: Average and median age by income category

	US		**EU**		**China**	
Income Category	Mean	Median	Mean	Median	Mean	Median
$0-20\%$	50	47	46	46	53	53
$20-40\%$	51	49	46	45	50	50
$40-60\%$	50	48	45	44	49	48
$60-80\%$	48	47	46	46	49	49
$80-90\%$	50	50	46	47	48	48
$90-100\%$	53	53	48	49	42	40

7. Appendix B – Why value functions of household-portfolio models are fragile

Value-funciton fragility arises even in the simplest partial-equilibrium C-CAPM models with infinitely-lived agents. In order to see the problem, consider the simplest Merton (1969, 1971) model in discrete time. Consumption and portfolio shares of n risky assets, $\{\phi_{i,t}\}_{i=1}^{n}$, are chosen throughout an infinite horizon $t = 0, 1, ...$, in order to maximize $E_0 \left(\sum_{t=0}^{\infty} \beta^t c_t^{1-\gamma} / (1-\gamma) \right)$, with $\gamma > 0$, $\gamma \neq 1$, $\beta \in (0,1)$, subject to $a_{t+1} = R_{p,t} a_t - c_t$, in which a is the value of total assets, c is consumption, and $R_{p,t} \equiv \sum_{i=1}^{n} \phi_{i,t-1} R_{i,t} + \left(1 - \sum_{i=1}^{n} \phi_{i,t-1}\right) R_f$, in which R_f is the gross risk-free rate and $R_{i,t}$ is the gross return of risky asset i, with $R_{i,t}$ being i.i.d. over time for all $i \in \{1, ..., n\}$. This model has a simple analytical solution, in which optimal portfolio shares are constant over time, $\boldsymbol{\phi}^* = \{\phi_{i,t}^* = \phi_i^*\}_{i=1}^{n}$ for all t, solving the $n \times n$ system of equations given by,

$$E_t \left[\left(R_{p,t+1}|_{\boldsymbol{\phi}^*}\right)^{-\gamma} \left(R_{j,t+1} - R_f\right) \right] = 0 \ , \quad j = 1, ..., n \ , \tag{16}$$

in which

$$R_{p,t+1}|_{\boldsymbol{\phi}^*} \equiv \sum\nolimits_{i=1}^{n} \phi_i^* R_{i,t+1} + \left(1 - \sum\nolimits_{i=1}^{n} \phi_i^*\right) R_f \ .$$

The value function of this problem is

$$V(a) = \left\{ 1 - \left\{ \beta E_t \left[\left(R_{p,t+1}|_{\boldsymbol{\phi}^*}\right)^{1-\gamma} \right] \right\}^{1/\gamma} \right\}^{-\gamma} \frac{a^{1-\gamma}}{1-\gamma} \ . \tag{17}$$

Equation (17) implies that the value function $V(a)$ is well-defined only if

$$E_t \left[\left(R_{p,t+1}|_{\boldsymbol{\phi}^*}\right)^{1-\gamma} \right] < \frac{1}{\beta} \ . \tag{18}$$

Yet, as (16) indicates, the nonviolation of (18) is sensitive to asset-return covariance-matrix parameters. Slight changes in this variance-covariance matrix can influence optimal portfolio choices, $\boldsymbol{\phi}^*$ in ways that can cause a failure of (18) and non-existence of $V(a)$.

To summarize, once the ex-post portfolio choice is imposed on the effective household-specific return, the value function may no longer be well-defined if some parameters are changed (violation of the transversality condition). One source of parameter sensitivity is the covariance matrix among different risky assets. Even for solving this simple model, educated calibration guesses on parameters are required. Simpler cases with analytical solutions can serve as a guide for discovering calibrating parameters that overcome this non-existence problem. These specific calibrated parameters can indicate the "ballpark" within which more complicated models can be solved numerically.

8. Appendix C – Derivation of the closed-form solution for insurable labor-income risk

We make two technical assumptions that enable us to secure interiority of solutions and analytical tractability. The rationale behind these assumptions becomes more obvious after the statement of Proposition 1, so we provide intuition at a later point.

Assumption 1 *Initial conditions are restricted so that,*

$$a_0 + \frac{y_0}{r_y} > \frac{\chi}{r_f} .$$

Assumption 2 *The parameter restriction,*

$$\frac{1}{\eta} > 1 - \frac{\rho}{r_f + \frac{\nu}{2\gamma}} \ , \ \text{ with } \nu \equiv (\mathbf{R} - r_f \mathbf{1}) \left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1} \left(\mathbf{R}^T - r_f \mathbf{1}^T\right) \ ,$$

holds.

Proposition 1 provides the formulas of the analytical solution to the model.

Proposition 1

If $\rho_{y,1}^2+...+\rho_{y,N}^2=1$*, short selling is allowed, and Assumptions 1 and 2 hold, the solution to the problem expressed by the HJB equation given by (9) is a decision rule for portfolio choice given by (12), and a decision rule for consumption,*

$$c^* = C(a,y) = \xi\left(a+\frac{y}{r_y}-\frac{\chi}{r_f}\right)+\chi\ , \tag{19}$$

in which

$$\xi = \rho\eta + (1-\eta)\,r_f - \frac{(\eta-1)}{2\gamma}\nu\ , \tag{20}$$

while the value function is given by,

$$V(a,y) = \rho^{-\eta\frac{1-\gamma}{1-\eta}}\xi^{\frac{1-\gamma}{1-\eta}}\frac{\left(a+\frac{y}{r_y}-\frac{\chi}{r_f}\right)^{1-\gamma}}{1-\gamma}\ .$$

Proof of Proposition 1

We make a guess on the functional form of the value function, namely,

$$V(a,y) = b\frac{(a+\psi y-\omega)^{1-\gamma}}{1-\gamma}\ , \tag{21}$$

which implies,

$$V_a(a,y) = b\,(a+\psi y-\omega)^{-\gamma}\ , \tag{22}$$

and also

$$f_c(c,V(a,y)) = \rho b^{1-\frac{1-\frac{1}{\eta}}{1-\gamma}}\,(a+\psi y-\omega)^{\frac{1}{\eta}-\gamma}\,(c-\chi)^{-\frac{1}{\eta}}\ . \tag{23}$$

From (22), (23) and (10) it is,

$$c = \rho^{\eta} b^{-\eta\frac{1-\frac{1}{\eta}}{1-\gamma}}\,(a+\psi y-\omega)+\chi\ . \tag{24}$$

Similarly, calculating the appropriate partial derivatives and substituting them in (11), gives,

$$\boldsymbol{\phi}^T = \frac{1}{\gamma}\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(\mathbf{R}^T - r_f\mathbf{1}^T\right)\left(1+\psi\frac{y}{a}-\frac{\omega}{a}\right) - \sigma_y\psi\frac{y}{a}\left(\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\right)^T \ . \tag{25}$$

Substituting (24), (21), (8), (25), and all derivatives stemming from (21) into the HJB given by (9) results in,

$$\begin{aligned}
\rho b\frac{(a+\psi y-\omega)^{1-\gamma}}{1-\frac{1}{\eta}} &= \frac{\rho^\eta b^{1-\eta\frac{1-\frac{1}{\eta}}{1-\gamma}}}{1-\frac{1}{\eta}}(a+\psi y-\omega)^{1-\gamma}+\\
&+b(a+\psi y-\omega)^{-\gamma}\left\{\frac{1}{\gamma}(\mathbf{R}-r_f\mathbf{1})\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(\mathbf{R}^T-r_f\mathbf{1}^T\right)(a+\psi y-\omega)-\right.\\
&\left.-\sigma_y\psi y\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\left(\mathbf{R}^T-r_f\mathbf{1}^T\right)+r_fa+y-\chi-\rho^\eta b^{-\eta\frac{1-\frac{1}{\eta}}{1-\gamma}}(a+\psi y-\omega)\right\}-\\
&-\frac{\gamma}{2}a^2b(a+\psi y-\omega)^{-\gamma-1}\left\{\frac{1}{\gamma}(\mathbf{R}-r_f\mathbf{1})\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(1+\psi\frac{y}{a}-\frac{\omega}{a}\right)-\sigma_y\frac{\psi y}{a}\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\right\}\times\\
&\times\boldsymbol{\sigma}\boldsymbol{\sigma}^T\left\{\frac{1}{\gamma}\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(\mathbf{R}^T-r_f\mathbf{1}^T\right)\left(1+\psi\frac{y}{a}-\frac{\omega}{a}\right)-\sigma_y\frac{\psi y}{a}\left(\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\right)^T\right\}+\\
&+\psi b(a+\psi y-\omega)^{-\gamma}\mu_y y-\frac{\gamma}{2}b\psi^2(\sigma_y y)^2(a+\psi y-\omega)^{-\gamma-1}-\gamma\sigma_y ayb\psi(a+\psi y-\omega)^{-\gamma-1}\times\\
&\times\left\{\frac{1}{\gamma}(\mathbf{R}-r_f\mathbf{1})\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(1+\psi\frac{y}{a}-\frac{\omega}{a}\right)-\sigma_y\frac{\psi y}{a}\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\right\}\boldsymbol{\sigma}\boldsymbol{\rho}_y^T \ .
\end{aligned} \tag{26}$$

After some algebra, (26) leads to,

$$\begin{aligned}
\frac{\rho-\frac{1}{\eta}\rho^\eta b^{-\eta\frac{1-\frac{1}{\eta}}{1-\gamma}}}{1-\frac{1}{\eta}}-\frac{1}{2\gamma}(\mathbf{R}-r_f\mathbf{1})\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(\mathbf{R}^T-r_f\mathbf{1}^T\right) &= r_f\frac{a+\frac{1+\psi\left[\mu_y-\sigma_y(\mathbf{R}-r_f\mathbf{1})\left(\boldsymbol{\rho}\boldsymbol{\sigma}^{-1}\right)^T\right]}{r_f}y-\frac{\chi}{r_f}}{a+\psi y-\omega}\\
&+\frac{1}{2}\gamma\left(\frac{\sigma_y\psi y}{a+\psi y-\omega}\right)^2\left(\boldsymbol{\rho}_y\boldsymbol{\rho}_y^T-1\right) \ .
\end{aligned} \tag{27}$$

Since we have assumed that $\rho_{y,1}^2+...+\rho_{y,N}^2=1$, $\boldsymbol{\rho}_y\boldsymbol{\rho}_y^T=1$, and the last term of the right-hand side in (27) vanishes. Moreover, we set

$$\omega = \chi/r_f \tag{28}$$

and

$$\psi = \frac{1 + \psi \left[\mu_y - \sigma_y \left(\mathbf{R} - r_f \mathbf{1}\right) \left(\boldsymbol{\rho}_y \boldsymbol{\sigma}^{-1}\right)^T\right]}{r_f} \ , \tag{29}$$

which gives

$$\psi = 1/r_y \ . \tag{30}$$

After substituting (29) into (27) we obtain

$$\frac{\rho - \frac{1}{\eta}\rho^{\eta} b^{-\eta \frac{1-\frac{1}{\eta}}{1-\gamma}}}{1 - \frac{1}{\eta}} - \frac{1}{2\gamma} \left(\mathbf{R} - r_f \mathbf{1}\right) \left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1} \left(\mathbf{R}^T - r_f \mathbf{1}^T\right) = r_f \ . \tag{31}$$

Solving (31) for $\rho^{\eta} b^{-\eta(1-1/\eta)/(1-\gamma)}$ gives,

$$\rho^{\eta} b^{-\eta \frac{1-\frac{1}{\eta}}{1-\gamma}} = \xi \ , \tag{32}$$

in which ξ is given by equation (20). Moreover, substituting (30) and (28) into (25) leads to (12). Substituting formulas (30) and (28) in (21) reveals that Assumption 1 is both necessary and sufficient in order that $V(a, y)$ be well-defined. From (20) the requirement that $\xi > 0$ is equivalent to the condition given by Assumption 2 in order to guarantee that, under Assumption 1 and equation (19), $c \geq \chi$ for all (a, y), completing the proof. $\square$

9. Appendix D – Characterization of portfolio shares in the case of two risky assets

Proof of equations (14) and (15)

The decomposition of matrix $\mathbf{\Sigma}$ is

$$\mathbf{\Sigma} = \boldsymbol{\sigma}\boldsymbol{\sigma}^T = \begin{bmatrix} \sigma_s & 0 \\ \rho_{s,b}\sigma_b & \sigma_b\sqrt{1-\rho_{s,b}^2} \end{bmatrix} \cdot \begin{bmatrix} \sigma_s & \rho_{s,b}\sigma_b \\ 0 & \sigma_b\sqrt{1-\rho_{s,b}^2} \end{bmatrix}, \tag{33}$$

with

$$\boldsymbol{\sigma}^{-1} = \frac{1}{\sigma_s\sigma_b\sqrt{1-\rho_{s,b}^2}} \begin{bmatrix} \sigma_b\sqrt{1-\rho_{s,b}^2} & 0 \\ -\rho_{s,b}\sigma_b & \sigma_s \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma_s} & 0 \\ \frac{-\rho_{s,b}}{\sigma_s\sqrt{1-\rho_{s,b}^2}} & \frac{1}{\sigma_b\sqrt{1-\rho_{s,b}^2}} \end{bmatrix}, \tag{34}$$

so,

$$\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1} = \begin{bmatrix} \rho_{y,s} & \rho_{y,b} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sigma_s} & 0 \\ \frac{-\rho_{s,b}}{\sigma_s\sqrt{1-\rho_{s,b}^2}} & \frac{1}{\sigma_b\sqrt{1-\rho_{s,b}^2}} \end{bmatrix}$$

or,

$$\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1} = \begin{bmatrix} \frac{\rho_{y,s}}{\sigma_s} - \frac{\rho_{y,b}\rho_{s,b}}{\sigma_s\sqrt{1-\rho_{s,b}^2}} & \frac{\rho_{y,b}}{\sigma_b\sqrt{1-\rho_{s,b}^2}} \end{bmatrix}. \tag{35}$$

Notice that since,

$$\mathbf{\Sigma}^{-1} = \left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1} = \frac{1}{\sigma_s^2\sigma_b^2\left(1-\rho_{s,b}^2\right)} \begin{bmatrix} \sigma_b^2 & -\rho_{s,b}\sigma_s\sigma_b \\ -\rho_{s,b}\sigma_s\sigma_b & \sigma_s^2 \end{bmatrix}, \tag{36}$$

after some algebra, the term $(1/\gamma)\left(\mathbf{R} - r_f\mathbf{1}\right)\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}$ in (12) is,

$$\frac{1}{\gamma}\left(\mathbf{R} - r_f\mathbf{1}\right)\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1} = \frac{1}{\gamma} \cdot \frac{1}{1-\rho_{s,b}^2} \begin{bmatrix} \frac{\frac{R_s-r_f}{\sigma_s} - \rho_{s,b}\frac{R_b-r_f}{\sigma_b}}{\sigma_s} & \frac{\frac{R_b-r_f}{\sigma_b} - \rho_{s,b}\frac{R_s-r_f}{\sigma_s}}{\sigma_b} \end{bmatrix}. \tag{37}$$

After combining (37) and (35) with (12), and after imposing the constraint $\rho_{y,s}^2 + \rho_{y,b}^2 = 1$, we obtain,

$$
\begin{aligned}
\kappa_{0,s} &= \frac{1}{\gamma}\cdot\frac{1}{1-\rho_{s,b}^2}\cdot\frac{\frac{R_s-r_f}{\sigma_s}-\rho_{s,b}\frac{R_b-r_f}{\sigma_b}}{\sigma_s} \\
\kappa_{1,s} &= \frac{1}{\gamma}\cdot\frac{1}{1-\rho_{s,b}^2}\cdot\frac{\frac{R_s-r_f}{\sigma_s}-\rho_{s,b}\frac{R_b-r_f}{\sigma_b}}{\sigma_s}\cdot\frac{\chi}{r_f} \\
\kappa_{2,s} &= \left[\frac{1}{\gamma}\cdot\frac{1}{1-\rho_{s,b}^2}\cdot\frac{\frac{R_s-r_f}{\sigma_s}-\rho_{s,b}\frac{R_b-r_f}{\sigma_b}}{\sigma_s}-\sigma_y\left(\frac{\rho_{y,s}}{\sigma_s}-\frac{\sqrt{1-\rho_{y,s}^2}\cdot\rho_{s,b}}{\sigma_s\sqrt{1-\rho_{s,b}^2}}\right)\right]\frac{1}{r_y}\ , \qquad (38)
\end{aligned}
$$

and,

$$
\begin{aligned}
\kappa_{0,b} &= \frac{1}{\gamma}\cdot\frac{1}{1-\rho_{s,b}^2}\cdot\frac{\frac{R_b-r_f}{\sigma_b}-\rho_{s,b}\frac{R_s-r_f}{\sigma_s}}{\sigma_b} \\
\kappa_{1,b} &= \frac{1}{\gamma}\cdot\frac{1}{1-\rho_{s,b}^2}\cdot\frac{\frac{R_b-r_f}{\sigma_b}-\rho_{s,b}\frac{R_s-r_f}{\sigma_s}}{\sigma_b}\cdot\frac{\chi}{r_f} \\
\kappa_{2,b} &= \left[\frac{1}{\gamma}\cdot\frac{1}{1-\rho_{s,b}^2}\cdot\frac{\frac{R_b-r_f}{\sigma_b}-\rho_{s,b}\frac{R_s-r_f}{\sigma_s}}{\sigma_b}-\sigma_y\cdot\frac{\sqrt{1-\rho_{y,s}^2}}{\sigma_b\sqrt{1-\rho_{s,b}^2}}\right]\frac{1}{r_y}\ . \qquad (39)
\end{aligned}
$$

Equations (38) and (39) prove their concise versions given by (14) and (15). Finally, since $r_y = r_f - \mu_y + \sigma_y\left(\mathbf{R} - r_f\mathbf{1}\right)\left(\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\right)^T$, after combining equation (35) with the constraint $\rho_{y,s}^2 + \rho_{y,b}^2 = 1$, we obtain,

$$
r_y = r_f - \mu_y + \sigma_y\left[\frac{R_s-r_f}{\sigma_s}\cdot\left(\rho_{y,s}-\frac{\sqrt{1-\rho_{y,s}^2}\cdot\rho_{s,b}}{\sqrt{1-\rho_{s,b}^2}}\right)+\frac{R_b-r_f}{\sigma_b}\cdot\frac{\rho_{y,b}}{\sqrt{1-\rho_{s,b}^2}}\right]\ . \qquad (40)
$$

Equation (40) reveals that apart from r_f, μ_y, and σ_y, a linear relationship between the Sharpe ratios weighted by expressions involving the correlation coefficients $\rho_{y,s}$ and $\rho_{s,b}$ plays a key role in determining the magnitude of r_y which critically affects the level of lifetime labor income y/r_y. equation (40). $\square$

10. Appendix E – Sensitivity analysis

We repeat our analysis here for $\rho_{ys}^2 + \rho_{yb}^2 = 0.50$ as a robustness check. The corresponding calibration parameters across different economic regions are provided in Table E.1, while Figure E.1 demonstrates the minimum-distance fit of the model to the data. The dotted line shows the closed form solution using equation (13). Unsurprisingly, with $\rho_{ys}^2 + \rho_{yb}^2 = 0.50$, we find wider distances between the Chebyshev approximation and the closed form solution than Figure 4 in the paper. Nevertheless, the parameters achieved under the closed-form solution still behave well used as the starting initial guess for calibrating the "true" model when labor income risk is uninsurable.

Table E.1: Calibrated Parameters across Model Specifications

Parameters	*US*	*EU*	*CN*
$\rho_{ys}^2 + \rho_{yb}^2$	0.50	0.50	0.50
	(%)	*(%)*	*(%)*
σ_y	12.1	7.1	13.5
μ_y	1.7	0.08	4.5
r_f	3.7	2.9	2.9
R_s	6.7	7.0	13.0
R_b	14.3	21.5	58.1
σ_s	15.9	20.5	56.3
σ_b	30.2	37.6	87.6
ρ_{sy}	30.2	30.0	22.5
ρ_{sb}	-7.5	10.0	10.4
ρ_{by}	63.9	64.1	67.0
γ	6.576	11.890	9.567
η	0.160	0.165	0.167
ρ	0.03	0.03	0.03
	(in 2007 USD)	*(in 2013 Euro)*	*(in 2013 RMB)*
χ	1800	2140	3000
In 2007 USD (ppp adjusted)	1800	1845	1491

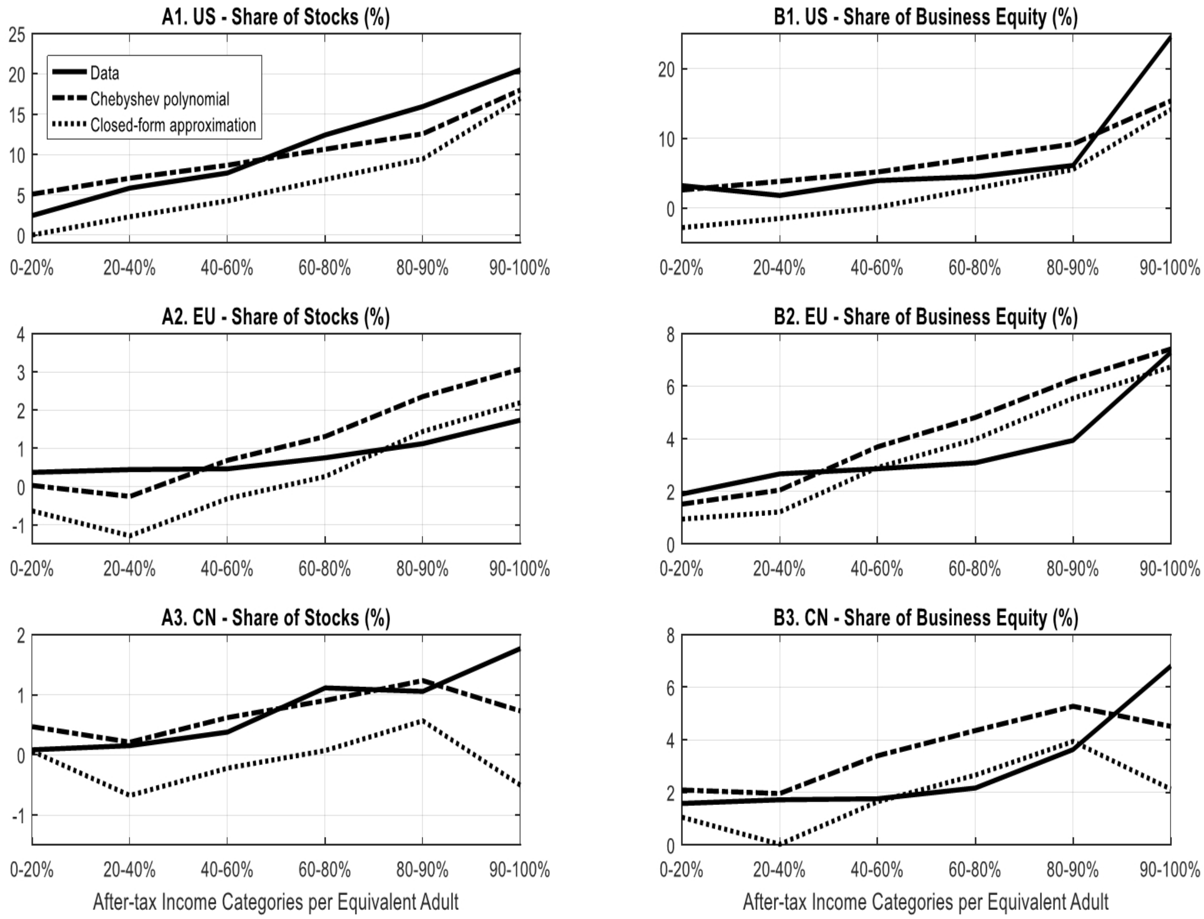


Figure E.1 – Calibration of the general version of the model with uninsurable labor-income risk ($\rho^2_{y,s} + \rho^2_{y,b} = 0.50$). "CN" denotes China.

In our Online Computational Appendix A.3, based on Panel-Study-of-Income-Dynamics (PSID) income data and Standard and Poors (S&P) stock-index data, we present evidence that empirical estimates of ρ_{ys} range between 32-51%.

In brief, the closed-form solution has served as a useful calibration guide even for $\rho^2_{ys} + \rho^2_{yb} = 0.50$, allowing us to understand which parameter combinations can achieve satisfactory data fitting. Trying the closed-form solution formula leads to an informative approximation of the true solution (Chebyshev polunomial) depicted by Figure E.1. Although for $\rho^2_{ys} + \rho^2_{yb}$

far away from 1 the closed-form solution is mathematically incorrect, it seems that the closed-form formula is directly useful for understanding the underlying role of background labor-income risk in risk-taking.[29] Specifically, Table E.2 presents the values of the "convexity factor" again, in comparison with the $\rho_{ys}^2 + \rho_{yb}^2 = 1$ case (see Table 2). Again, in all three countries/economic regions, $\kappa_{2,i}$ is strictly positive, demonstrating that the positive value of $\kappa_{2,i}$ is necessary for the model, espcially for fitting the increasing and convex risk-taking pattern in household resources.

Table E.2: The calibrated "Convexity factor" $\kappa_{2,i}$

$\rho_{ys}^2 + \rho_{yb}^2$	1		0.50	
	$\kappa_{2,s}$	$\kappa_{2,b}$	$\kappa_{2,s}$	$\kappa_{2,b}$
US	1.069	1.586	1.080	1.304
EU	0.208	0.445	0.397	0.305
CN	0.148	0.653	0.350	0.670

Note: $\kappa_{2,i}$ in eqn. (13).

[29] This numerical proximity between risky asset shares derived by the closed-form solution and the numerical solution in the case of uninsurable labor-income risk, may be sensitive to changes in parameter values. Yet, such an investigation is beyond the scope of the present study.

11. Appendix F – Extension to a Discrete-time analysis

For the discrete-time analogue to the continuous-time version of the model with two risky assets, using Epstein-Zin-Weil (EZW) preferences, the Bellman equation is,[30]

$$V\left(a_t, y_t\right) = \max_{\left(c_t, \phi_t\right)} \frac{\left\{ (1-\beta)\left(c_t-\chi\right)^{1-\frac{1}{\eta}} + \beta\left\{(1-\gamma)\, E_t\left[V\left(R_{p,t+1}a_t + y_t - c_t\ ,\ y_{t+1}\right)\right]\right\}^{\frac{1-\frac{1}{\eta}}{1-\gamma}} \right\}^{\frac{1-\gamma}{1-\frac{1}{\eta}}}}{1-\gamma}\ , \tag{41}$$

in which,

$$R_{p,t+1} \equiv \left(R^s_{t+1} - r_f\right)\phi^s_t + \left(R^b_{t+1} - r_f\right)\phi^b_t + r_f\ , \tag{42}$$

and with,

$$\ln\left(y_{t+1}\right) - \ln\left(y_t\right) = \mu_y + \varepsilon_{y,t+1}\ ,\ \ \varepsilon_{y,t+1} \sim N\left(0, \sigma_y^2\right)\ , \tag{43}$$

$$\ln\left(P_{s,t+1}\right) - \ln\left(P_{s,t}\right) = R_s + \varepsilon_{s,t+1}\ ,\ \ \varepsilon_{s,t+1} \sim N\left(0, \sigma_s^2\right)\ , \tag{44}$$

$$\ln\left(P_{b,t+1}\right) - \ln\left(P_{b,t}\right) = R_b + \varepsilon_{b,t+1}\ ,\ \ \varepsilon_{b,t+1} \sim N\left(0, \sigma_b^2\right)\ , \tag{45}$$

where $P_{s,t}$ and $P_{b,t}$ denote the stock price and the business equity price in period t, while,

$$\frac{Cov\left(\varepsilon_{s,t+1}, \varepsilon_{y,t+1}\right)}{\sigma_s\sigma_y} = \rho_{ys}\ , \tag{46}$$

$$\frac{Cov\left(\varepsilon_{b,t+1}, \varepsilon_{y,t+1}\right)}{\sigma_b\sigma_y} = \rho_{yb}\ , \tag{47}$$

$$\frac{Cov\left(\varepsilon_{s,t+1}, \varepsilon_{b,t+1}\right)}{\sigma_s\sigma_b} = \rho_{sb}\ , \tag{48}$$

[30]See Epstein and Zin (1989) and Weil (1989).

R_t^s in equation (42) is given by,

$$R_t^s = e^{R_s + \varepsilon_{s,t}} \ , \tag{49}$$

R_t^b in equation (42) is given by,

$$R_t^b = e^{R_b + \varepsilon_{b,t}} \ , \tag{50}$$

y_t is given by,

$$y_t = e^{\mu_y + \varepsilon_{y,t}} \ , \tag{51}$$

and (a_0, y_0, ϕ_0) are given.

The computational algorithm is fully explained in our Online Computational Appendix. In the interest of brevity, we only use the US case as an illustrating example here. In Panel A1/A2 of Figure F.1 we can see that trying the best-fitting parameters of the continuous-time model (directly from the closed form solution, see Table 1), does not lead to a good match of the discrete-time model to the data. We report the calibrating values in Table F.1 that match the discrete-time model to the risky-asset-holding data (Panel A1/A2 of Figure F.1) as $\rho_{ys}^2 + \rho_{yb}^2 = 1$.[31] In Panel B1/B2 of Figure F.1 we show that the parameter values used in the case of $\rho_{ys}^2 + \rho_{yb}^2 = 1$ cannot match the data for $\rho_{ys}^2 + \rho_{yb}^2 \in \{0.75, 0.50\}$, indicating that further re-parameterization is needed. We re-calibrate the model and show the goodness of fit of the model against the data in Panels C1/C2 and D1/D2 of Figure F.1, with the corresponding calibrated parameters presented in Table F.1. An important message here is that the guidance we have had from the continuous-time model allows us to get in the ballpark of best-matching parameters to data, achieving our calibration goals through a homotopy approach.

[31]We have achieved this goodness of fit through a trial-and-error approach but using the continuous-time calibration parameters that correspond to the closed form solution as the starting guess.

Table F.1: Calibrated Parameters in Discrete Time

Parameters	*US*			
$\rho_{ys}^2 + \rho_{yb}^2$	1 (Closed-form)	1 *(discrete time)*	0.75	0.50
	(%)			
σ_y	0.120	0.138	0.138	0.129
μ_y	0.017	0.010	0.010	0.010
r_f	0.037	0.045	0.036	0.033
R_s	0.070	0.075	0.073	0.070
R_b	0.111	0.120	0.122	0.109
σ_s	0.209	0.188	0.221	0.174
σ_b	0.300	0.287	0.287	0.216
ρ_{sy}	0.485	0.477	0.440	0.338
ρ_{sb}	-0.075	-0.034	-0.039	0.028
γ	3.316	5.750	5.494	8.946
η	0.160	0.175	0.179	0.119
β	0.971	0.980	0.980	0.967
	(in 2007 USD)			
χ	1437	1800	1801	1810

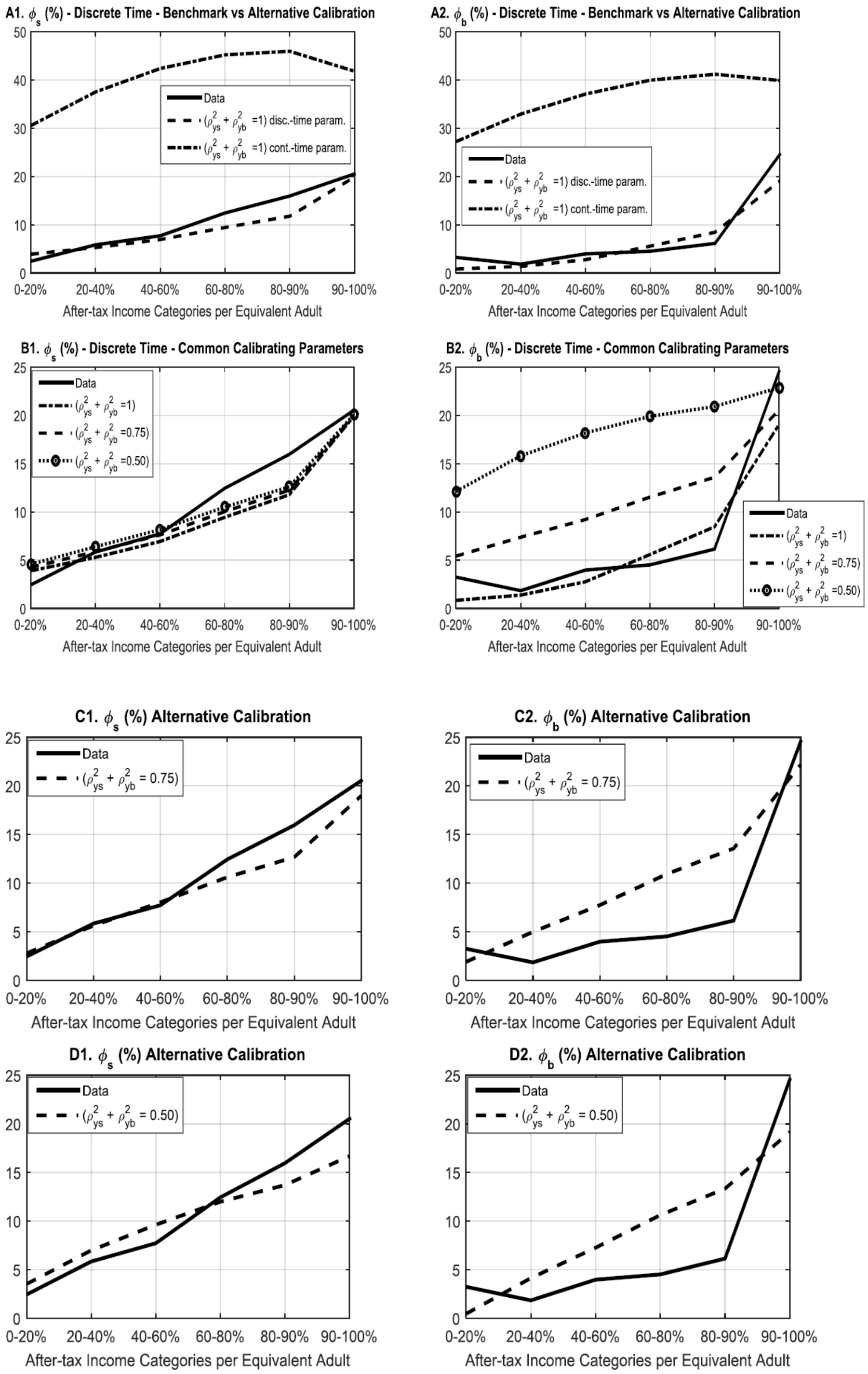


Figure F.1 US - Discrete-time numerical simulations, using exponential projection on the model with two risky assets.

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Online Computational Appendices

for

The Role of Labor-Income Risk in Household Risktaking

1. Appendix A – Simulating the continuous-time model

1.1 Algebraic manipulations

The first-order conditions of the problem expressed by equation (10) in the main body of the paper are,

$$f_c\left(c, V\left(a, y\right)\right) = V_a\left(a, y\right) \ , \tag{1}$$

$$\boldsymbol{\phi}^T = \left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(\mathbf{R}^T - r_f\mathbf{1}^T\right)\frac{V_a\left(a, y\right)}{-a \cdot V_{aa}\left(a, y\right)} - \sigma_y\frac{y}{a}\left(\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\right)^T\frac{V_{ay}\left(a, y\right)}{V_{aa}\left(a, y\right)} \ . \tag{2}$$

Based on equation (8) in the paper,

$$f_c\left(c, V\right) = \rho\left[\left(1-\gamma\right)V\right]^{1-\frac{1-\frac{1}{\eta}}{1-\gamma}}\left(c - \chi\right)^{-\frac{1}{\eta}} \ . \tag{3}$$

In order to make notation somewhat easier to follow, set,

$$\theta \equiv \frac{1-\frac{1}{\eta}}{1-\gamma} \ . \tag{4}$$

Combining (4) with (3) we obtain,

$$f_c\left(c, V\right) = \rho\left[\left(1-\gamma\right)V\right]^{1-\theta}\left(c - \chi\right)^{\theta(1-\gamma)-1} \ . \tag{5}$$

Combining (5) with (1) gives,

$$c^* = C\left(a, y\right) = \chi + \left\{\rho^{-1}V_a^* \cdot \left[\left(1-\gamma\right)V^*\right]^{\theta-1}\right\}^{\frac{1}{\theta(1-\gamma)-1}} \ , \tag{6}$$

in which V^* is the fixed point of the Hamilton-Jacobi-Bellman (HJB) equation given by equation (10) in the paper. From equation (8) in the paper,

$$f\left(c^*, V^*\right) = \frac{\rho}{\theta\left(1-\gamma\right)}\left(c^* - \chi\right)^{\theta(1-\gamma)}\left[\left(1-\gamma\right)V^*\right]^{1-\theta} - \frac{\rho}{\theta}V^* \ . \tag{7}$$

Finally, (2) implies,

$$\left(\boldsymbol{\phi}^*\right)^T = \left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(\mathbf{R}^T - r_f\mathbf{1}^T\right)\frac{V_a^*}{-a\cdot V_{aa}^*} - \sigma_y\frac{y}{a}\left(\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\right)^T\frac{V_{ay}^*}{V_{aa}^*}\ . \tag{8}$$

The max operator on the right-hand side of the HJB equation which is given by (10) in the paper, can be discarded at the fixed point, V^*. Using equations (6), (7), and (8), we can incorporate the optimizers $\boldsymbol{\phi}^*$and c^* in the HJB equation, in order to obtain,

$$\begin{aligned}1 = \Bigg\{ & \frac{\rho}{\theta(1-\gamma)}\left(c^* - \chi\right)^{\theta(1-\gamma)}\left[(1-\gamma)V^*\right]^{1-\theta} + \left\{\left[\boldsymbol{\phi}^*\mathbf{R}^T + \left(1 - \boldsymbol{\phi}^*\mathbf{1}^T\right)r_f\right]a + y - c^*\right\}\cdot V_a^* \\ & + \frac{1}{2}a^2\boldsymbol{\phi}^*\boldsymbol{\sigma}\boldsymbol{\sigma}^T\left(\boldsymbol{\phi}^*\right)^T\cdot V_{aa}^* + \mu_y y\cdot V_y^* \\ & + \frac{1}{2}\left(\sigma_y y\right)^2\cdot V_{yy}^* + \sigma_y a y\boldsymbol{\phi}\boldsymbol{\sigma}\boldsymbol{\rho}_y^T\cdot V_{ay}^*\Bigg\} \Big/ \left(\frac{\rho}{\theta}V^*\right)\ .\end{aligned} \tag{9}$$

Equation (9) is a second-order (bivariate) partial differential equation, which we solve numerically. Yet, the numerical solution of partial differential equations can be challenging in terms of numerical accuracy, rounding problems, or error-accumulation problems. For example, in a slightly alternative version of equation (9), the term $(\rho/\theta)\,V^*$ would be on the left-hand side of (9); in the present version of (9) we have divided both sides of that alternative version by the term $(\rho/\theta)\,V^*$; we have done so, because, for successful calibrating parameter values of the model, the numerical values of V^* are often small numbers in the order of 10^{-15}; such small-valued functions V^* usually neglect convergence criteria, and a resolution of this problem is to normalize the HJB equation, as we did in (9).

1.2 Chebyshev-polynomial approximation

The Chebyshev-approximated function we use has the form,

$$V(a,y) \simeq \sum_{i=0}^{\nu_1-1} \sum_{j=0}^{\nu_1-1} \theta_{ij} T_i(X(a)) T_j(X(y)) \ , \tag{10}$$

in which $T_j(x)$ is the Chebyshev polynomial of degree $j \in \{0, 1, ...\}$, given by,

$$T_j(x) = \cos(j \cdot \arccos(x)) \ , \tag{11}$$

with,

$$T_j'(x) = \frac{\partial \cos(j \cdot \arccos(x))}{\partial x} = j \frac{\sin(j \arccos(x))}{\sqrt{1-x^2}} \ , \tag{12}$$

$$T_j''(x) = \frac{\partial^2 \cos(j \cdot \arccos(x))}{\partial x^2} = \frac{1}{1-x^2} \left[x \cdot \frac{j \cdot \sin(j \arccos(x))}{\sqrt{1-x^2}} - j^2 \cdot \cos(j \cdot \arccos(x)) \right] \ ,$$

and based on formulas (11) and (12) we have the concise formula for the second derivative,

$$T_j''(x) = \frac{1}{1-x^2} \left[x \cdot T_j'(x) - j^2 \cdot T_j(x) \right] \ . \tag{13}$$

Regarding functions $X(a)$ and $X(y)$ in (10), notice that the domain of $T_j(x)$ is $[-1, 1]$. Thanks to linearity properties of vector spaces it is straightforward to implement the Chebyshev projection method to values $a \in [\underline{a}, \bar{a}]$ and $y \in [\underline{y}, \bar{y}]$ through the linear transformation,

$$X(z) = \frac{2}{\bar{z} - \underline{z}} \cdot z - \frac{\bar{z} + \underline{z}}{\bar{z} - \underline{z}} \ , \quad z \in \{a, y\} \ , \tag{14}$$

in which $\underline{a}$ and $\underline{y}$ are the smallest values of the grids for a and y, while $\bar{a}$ and $\bar{y}$ are the largest values of a and y.

1.2.1 Forming the endogenous Chebyshev grids

Chebyshev polynomials can avoid accumulating rounding errors as the polynomial degree of the approximating function increases. While using state-space grids, this ability stems from the "discrete-orthogonality properties" of Chebyshev polynomials. These properties hold at specific gridpoints on the interval $[-1,1]$, at values $\bar{x}_k$, such that $T_n(x_k)=0$, $k\in\{1,...,n\}$, with,[1]

$$\bar{x}_k=\cos\left(\frac{2k-1}{2n}\pi\right),\quad k\in\{1,...,n\}\ . \tag{15}$$

Using m gridpoints for each dimension, a and y, the $m\times 1$ vector which is computed by (15) is denoted by $\bar{\mathbf{x}}$, and it is called the "Chebyshev nodes". In order to project the gridpoints given by $\bar{\mathbf{x}}$ back onto variables a and y, use the inverse transformation of (14), in order to create the corresponding $m\times 1$ vectors, $\mathbf{a}_{grid}=\bar{\mathbf{a}}$, and $\mathbf{y}_{grid}=\bar{\mathbf{y}}$, namely,

$$\bar{\mathbf{a}}=A(\bar{\mathbf{x}})=\frac{(\bar{\mathbf{x}}+1)(\bar{a}-\underline{a})}{2}+\underline{a}\ , \tag{16}$$

and

$$\bar{\mathbf{y}}=Y(\bar{\mathbf{x}})=\frac{(\bar{\mathbf{x}}+1)\left(\bar{y}-\underline{y}\right)}{2}+\underline{y}\ . \tag{17}$$

1.2.2 Best-fitting the two-dimensional Chebyshev polynomial to a known function

Let's assume that we have an $m_a\times 1$ grid for a, $\bar{\mathbf{a}}$, calculated using (16), and an $m_y\times 1$ grid for y, $\bar{\mathbf{y}}$, calculated using (17). Any known function, $V(a,y)$, can map the grid of Chebyshev nodes (discretized domain) to an $m_a\times m_y$ matrix, $\bar{\mathbf{V}}$, defined as,

$$\bar{\mathbf{V}}=[\bar{v}_{k,\ell}]=[V(\bar{a}_k,\bar{y}_\ell)]=[V(A(\bar{x}_{a,k}),Y(\bar{x}_{y,\ell}))]\ ,\quad k\in\{1,...,m_a\},\ell\in\{1,...,m_y\}, \tag{18}$$

in which $A(\cdot)$ and $Y(\cdot)$ are given by (16) and (17). Let's also assume that the Chebyshev polynomial degree for dimension a is ν_a, and ν_y for dimension y. In order to achieve a best Chebyshev polynomial fitting of the functional form given by (10) on the elements of

[1] This error-minimizing property of gridpoints $\{x_k\}_{k=1}^{n}$ with $T_n(x_k)=0$ can be proved formally. See, for example, Judd (1992) and further references therein.

matrix $\bar{\mathbf{V}}$, we minimize least-squares residuals. The formulas for the optimal Chebyshev-approximation estimator $\hat{\theta}_{i,j}$ are given by (see, for example, Heer and Maußner, 2005, Ch. 8, p. 441),

$$\begin{aligned}
\hat{\theta}_{0,0} &= \frac{1}{m_a m_y}\sum_{k=1}^{m_a}\sum_{\ell=1}^{m_y}\bar{v}_{k,\ell} && (19)\\
\hat{\theta}_{i,0} &= \frac{2}{m_a m_y}\sum_{k=1}^{m_a}\sum_{\ell=1}^{m_y}\bar{v}_{k,\ell}T_i\left(\bar{x}_{a,k}\right) && (20)\\
\hat{\theta}_{0,j} &= \frac{2}{m_a m_y}\sum_{k=1}^{m_a}\sum_{\ell=1}^{m_y}\bar{v}_{k,\ell}T_j\left(\bar{x}_{y,\ell}\right) && (21)\\
\hat{\theta}_{i,j} &= \frac{4}{m_a m_y}\sum_{k=1}^{m_a}\sum_{\ell=1}^{m_y}\bar{v}_{k,\ell}T_i\left(\bar{x}_{a,k}\right)T_j\left(\bar{x}_{y,\ell}\right)\ , && (22)
\end{aligned}$$

for $i \in \{1, ..., \nu_a - 1\}$ and $j \in \{1, ..., \nu_y - 1\}$. For convenience, we can summarize the optimal-fitting conditions given by equations (19) through (22) using some particular matrix arrays.

Consider the matrices,

$$\mathbf{T}_a\left(X\left(\bar{\mathbf{a}}\right)\right) = \mathbf{T}_a\left(\bar{\mathbf{x}}_a\right) = \begin{bmatrix} T_0\left(\bar{x}_{a,1}\right) & T_1\left(\bar{x}_{a,1}\right) & \cdots & T_{\nu_a-1}\left(\bar{x}_{a,1}\right) \\ T_0\left(\bar{x}_{a,2}\right) & T_1\left(\bar{x}_{a,2}\right) & \cdots & T_{\nu_a-1}\left(\bar{x}_{a,2}\right) \\ \vdots & \vdots & \ddots & \vdots \\ T_0\left(\bar{x}_{a,m_a}\right) & T_1\left(\bar{x}_{a,m_a}\right) & \cdots & T_{\nu_a-1}\left(\bar{x}_{a,m_a}\right) \end{bmatrix},$$

and

$$\mathbf{T}_y\left(X\left(\bar{\mathbf{y}}\right)\right) = \mathbf{T}_y\left(\bar{\mathbf{x}}_y\right) = \begin{bmatrix} T_0\left(\bar{x}_{y,1}\right) & T_1\left(\bar{x}_{y,1}\right) & \cdots & T_{\nu_y-1}\left(\bar{x}_{y,1}\right) \\ T_0\left(\bar{x}_{y,2}\right) & T_1\left(\bar{x}_{y,2}\right) & \cdots & T_{\nu_y-1}\left(\bar{x}_{y,2}\right) \\ \vdots & \vdots & \ddots & \vdots \\ T_0\left(\bar{x}_{y,m_y}\right) & T_1\left(\bar{x}_{y,m_y}\right) & \cdots & T_{\nu_y-1}\left(\bar{x}_{y,m_y}\right) \end{bmatrix}.$$

Notice that $\mathbf{T}_a\left(\bar{\mathbf{x}}_a\right)$ is of size $m_a \times \nu_a$, while $\mathbf{T}_y\left(\bar{\mathbf{x}}_y\right)$ is an $m_y \times \nu_y$ matrix. Consider also the two matrices,

$$\mathbf{I}_{m_a} = \begin{bmatrix} \frac{1}{m_a} & 0 & \cdots & 0 \\ 0 & \frac{2}{m_a} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{2}{m_a} \end{bmatrix} ,$$

and

$$\mathbf{I}_{m_y} = \begin{bmatrix} \frac{1}{m_y} & 0 & \cdots & 0 \\ 0 & \frac{2}{m_y} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{2}{m_y} \end{bmatrix} ,$$

with $\mathbf{I}_{m_a}$ being of size $\nu_a \times \nu_a$, and with $\mathbf{I}_{m_y}$ being of size $\nu_y \times \nu_y$.

The $\nu_a \times \nu_y$ matrix $\hat{\mathbf{\Theta}}$ that contains all Chebyshev coefficients $\hat{\theta}_{i,j}$ for $i \in \{0, ..., \nu_a - 1\}$ and $j \in \{0, ..., \nu_y - 1\}$, as these are given by the optimal-fitting conditions (19) through (22), are summarized by,

$$\hat{\mathbf{\Theta}} \equiv \arg\min_{\mathbf{\Theta}} \sum_{k=1}^{m_a} \sum_{\ell=1}^{m_y} \left[\mathbf{T}_a (\bar{\mathbf{x}}_{a,k}) \cdot \Theta \cdot \mathbf{T}_y (\bar{\mathbf{x}}_{y,\ell})^\mathsf{T} - \bar{\mathbf{V}}_{k,\ell} \right]^2 = \mathbf{I}_{m_a} \cdot \mathbf{T}_a (\bar{\mathbf{x}}_a)^\mathsf{T} \cdot \bar{\mathbf{V}} \cdot \mathbf{T}_y (\bar{\mathbf{x}}_y) \cdot \mathbf{I}_{m_y} , \tag{23}$$

in which $\mathbf{T}_a (\bar{\mathbf{x}}_{a,k})$ and $\mathbf{T}_y (\bar{\mathbf{x}}_{y,l})$ are the k-th and ℓ-th row of matrices $\mathbf{T}_a (\bar{\mathbf{x}}_a)$ and $\mathbf{T}_y (\bar{\mathbf{x}}_y)$.

Finally, notice the matrix array,

$$\bar{\mathbf{V}} = \mathbf{T}_a (\bar{\mathbf{x}}_a) \cdot \hat{\mathbf{\Theta}} \cdot \mathbf{T}_y (\bar{\mathbf{x}}_y)^\mathsf{T} , \tag{24}$$

which is easy to verify from the expression given by (23) and the Chebyshev discrete-orthogonality conditions, which imply,

$$\mathbf{T}_a (\bar{\mathbf{x}}_a) \cdot \mathbf{I}_{m_a} \cdot \mathbf{T}_a (\bar{\mathbf{x}}_a)^\mathsf{T} = \mathbf{I}_{(m_a \times m_a)} \quad \text{and} \quad \mathbf{T}_y (\bar{\mathbf{x}}_y) \cdot \mathbf{I}_{m_y} \cdot \mathbf{T}_y (\bar{\mathbf{x}}_y)^\mathsf{T} = \mathbf{I}_{(m_y \times m_y)} ,$$

and in which $\mathbf{I}_{(m_a \times m_a)}$ and $\mathbf{I}_{(m_y \times m_y)}$ are identiy matrices of size $m_a \times m_a$ and $m_y \times m_y$.

1.2.3 Computing all partial derivatives efficiently, and dealing with the small values of the indirect utility function

Let

$$\mathbf{A} \equiv \mathbf{T}_a\left(X\left(\bar{\mathbf{a}}\right)\right) \text{ , and } \mathbf{Y} \equiv \mathbf{T}_y\left(X\left(\bar{\mathbf{y}}\right)\right) \text{ .} \tag{25}$$

Let also,

$$\mathbf{A}_1 \equiv \frac{\partial \mathbf{T}_a\left(X\left(\bar{\mathbf{a}}\right)\right)}{\partial a} = \frac{2}{\bar{a}-\underline{a}}\mathbf{T}_a'\left(X\left(\bar{\mathbf{a}}\right)\right) \text{ ,} \tag{26}$$

with,

$$\mathbf{T}_a'\left(X\left(\bar{\mathbf{a}}\right)\right) = \mathbf{T}_a'\left(\bar{\mathbf{x}}_a\right) = \begin{bmatrix} T_0'\left(\bar{x}_{a,1}\right) & T_1'\left(\bar{x}_{a,1}\right) & \cdots & T_{\nu_a-1}'\left(\bar{x}_{a,1}\right) \\ T_0'\left(\bar{x}_{a,2}\right) & T_1'\left(\bar{x}_{a,2}\right) & \cdots & T_{\nu_a-1}'\left(\bar{x}_{a,2}\right) \\ \vdots & \vdots & \ddots & \vdots \\ T_0'\left(\bar{x}_{a,m_a}\right) & T_1'\left(\bar{x}_{a,m_a}\right) & \cdots & T_{\nu_a-1}'\left(\bar{x}_{a,m_a}\right) \end{bmatrix} \text{ ,} \tag{27}$$

in which $T_j'(x)$ is computed using (12). Notice that the term $2/(\bar{a}-\underline{a})$ is the result of applying the chain rule of differentiation on $\mathbf{T}_a(X(a))$, in which $X(a)$ is given by (14). Similarly,

$$\mathbf{A}_2 \equiv \frac{\partial^2 \mathbf{T}_a\left(X\left(\bar{\mathbf{a}}\right)\right)}{\partial a^2} = \left(\frac{2}{\bar{a}-\underline{a}}\right)^2 \mathbf{T}_a''\left(X\left(\bar{\mathbf{a}}\right)\right) \text{ ,} \tag{28}$$

with,

$$\mathbf{T}_a''\left(X\left(\bar{\mathbf{a}}\right)\right) = \mathbf{T}_a''\left(\bar{\mathbf{x}}_a\right) = \begin{bmatrix} T_0''\left(\bar{x}_{a,1}\right) & T_1''\left(\bar{x}_{a,1}\right) & \cdots & T_{\nu_a-1}''\left(\bar{x}_{a,1}\right) \\ T_0''\left(\bar{x}_{a,2}\right) & T_1''\left(\bar{x}_{a,2}\right) & \cdots & T_{\nu_a-1}''\left(\bar{x}_{a,2}\right) \\ \vdots & \vdots & \ddots & \vdots \\ T_0''\left(\bar{x}_{a,m_a}\right) & T_1''\left(\bar{x}_{a,m_a}\right) & \cdots & T_{\nu_a-1}''\left(\bar{x}_{a,m_a}\right) \end{bmatrix} \text{ ,} \tag{29}$$

in which $T_j''(x)$ is computed using (13). We also produce matrices $\mathbf{Y}_1$ and $\mathbf{Y}_2$, in accordance with formulas (26), (27), (28), and (29).

For reasonable calibrating parameters, the numerical values of $V(a,y)$ are often small numbers in the order of 10^{-15}. The problem is that such small-valued functions circumvent loops with tight convergence criteria. In order to deal with this problem, we normalize $V(a,y)$, through the transformation,

$$V(a,y) \equiv \frac{\left[\tilde{V}(a,y)\right]^{1-\gamma}}{1-\gamma} . \tag{30}$$

Using (24), for any estimator $\hat{\mathbf{\Theta}}^{(n)}$, during the n-th iteration of a recursive process, we approximate the value of $\tilde{V}(a,y)$ by,

$$\tilde{V}^{(n)}(a,y) \simeq \mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^T . \tag{31}$$

According to (30) and (31),

$$V^{(n)}(a,y) \simeq \frac{\left[\mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^T\right]^{1-\gamma}}{1-\gamma} . \tag{32}$$

The transformation given by (32) allows us to achieve Chebyshev-polynomial coefficients (contained in in matrix $\hat{\mathbf{\Theta}}^{(n)}$) with values large enough for implementing a recursive numerical method that searches for a fixed point for matrix $\hat{\mathbf{\Theta}}^{(n)}$.

Using (32), the partial derivatives $V_a^{(n)}$ and $V_y^{(n)}$ are given by,

$$V_a^{(n)}(a,y) \simeq \left(\mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^T\right)^{-\gamma} \mathbf{A}_1\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^T , \tag{33}$$

and

$$V_y^{(n)}(a,y) \simeq \left(\mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^T\right)^{-\gamma} \mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}_1^T . \tag{34}$$

Using (33) and (34), we obtain,

$$V_{aa}^{(n)}(a,y) \simeq \left(\mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^T\right)^{-\gamma} \left[-\gamma\left(\mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^T\right)^{-1}\left(\mathbf{A}_1\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^T\right)^2 + \mathbf{A}_2\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^T\right] , \tag{35}$$

$$V_{yy}^{(n)}(a,y) \simeq \left(\mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^T\right)^{-\gamma}\left[-\gamma\left(\mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^T\right)^{-1}\left(\mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}_1^T\right)^2 + \mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}_2^T\right], \tag{36}$$

and

$$V_{ay}^{(n)}(a,y) \simeq \left(\mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^T\right)^{-\gamma}\left[-\gamma\left(\mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^T\right)^{-1}\left(\mathbf{A}_1\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^T\right)\left(\mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}_1^T\right) + \mathbf{A}_1\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}_1^T\right]. \tag{37}$$

1.2.4 Matrix array for computing all functions of the HJB equation using nonlinear regression techniques

Matrices described by equations (30) through (37) use the matrix array,

$$V(a,y) \simeq \underbrace{V_{matrix}}_{m_a \times m_y} \equiv \begin{bmatrix} V(a_1,y_1) & V(a_1,y_2) & \cdots & V\left(a_1,y_{m_y}\right) \\ V(a_2,y_1) & V(a_2,y_2) & \cdots & V\left(a_2,y_{m_y}\right) \\ \vdots & \vdots & \ddots & \vdots \\ V(a_{m_a},y_1) & V(a_{m_a},y_2) & \cdots & V\left(a_{m_a},y_{m_y}\right) \end{bmatrix}. \tag{38}$$

For V_{matrix} in (38) we use the $(m_a \cdot m_y) \times 1$ vector array,

$$V_{vector\_array} = (m_a \cdot m_y) \times 1 \left\{ \begin{bmatrix} V(a_1, y_1) \\ V(a_2, y_1) \\ \vdots \\ V(a_{m_a}, y_1) \\ ---- \\ V(a_1, y_2) \\ V(a_2, y_2) \\ \vdots \\ V(a_{m_a}, y_2) \\ ---- \\ \vdots \\ ---- \\ V\left(a_1, y_{m_y}\right) \\ V\left(a_2, y_{m_y}\right) \\ \vdots \\ V\left(a_{m_a}, y_{m_y}\right) \end{bmatrix} \right. \tag{39}$$

The array in (39) can be achieved by matching two $(m_a \cdot m_y) \times 1$ vectors,

$$\bar{\mathbf{a}}_{grid\_long} = \mathbf{1}_{(m_y \times 1)} \otimes \bar{\mathbf{a}} \ , \tag{40}$$

which corresponds to m_y stacked vectors $\bar{\mathbf{a}}$, and

$$\bar{\mathbf{y}}_{grid\_long} = \bar{\mathbf{y}} \otimes \mathbf{1}_{(m_a \times 1)} \ , \tag{41}$$

which is m_y stacked vectors of size $m_a \times 1$, with each $m_a \times 1$ vector having m_a identical elements, m_a times each element of $\bar{\mathbf{y}}$, stacked in the order of elements of $\bar{\mathbf{y}}$.

Using the vector array in (39), we express all matrices described by equations (30) through (37), using the Matlab command "`reshape`", and we use all partial derivatives in the same $(m_a \cdot m_y) \times 1$ vector array in order to express c^* and $\boldsymbol{\phi}^*$ according to equations (6) and (8).

1.3 Ensuring that consumption is above subsistence and treatment of borrowing constraints

The functional form of utility that we use satisfies an Inada condition as $c \to \chi$, which is obvious from (3). This is the reason that equation (6) holds. The RHS of (6) has a simple interpretation: as long as V^* is well-defined, it is guaranteed that $c > \chi$. Notice that the first-order condition given by (1) holds even if there is a borrowing constraint $a \geq b$. The presence of a borrowing constraint, $a \geq b$, does not affect (2) either. In order to implement $a \geq b$, all we need to do is to ensure that the deterministic part of the budget constraint is nonnegative when $a = b$, i.e.

$$\left[\boldsymbol{\phi}^* \mathbf{R}^T + \left(1 - \boldsymbol{\phi}^* \mathbf{1}^T\right) r_f\right] b + y - c^* \geq 0 \ . \tag{42}$$

Inserting (42) into (9) is achieved by the modified version of (9),

$$\begin{aligned} 1 = \ & \Bigg\{ \frac{\rho}{\theta(1-\gamma)} \left(c^* - \chi\right)^{\theta(1-\gamma)} \left[(1-\gamma) V^*\right]^{1-\theta} + \\ & + \max\left\{\left[\boldsymbol{\phi}^* \mathbf{R}^T + \left(1 - \boldsymbol{\phi}^* \mathbf{1}^T\right) r_f\right] a + y - c^* \ , \ 0\right\}\Big|_{a=b} \cdot V_a^* \\ & + \frac{1}{2} a^2 \boldsymbol{\phi}^* \boldsymbol{\sigma} \boldsymbol{\sigma}^T \left(\boldsymbol{\phi}^*\right)^T \cdot V_{aa}^* + \mu_y y \cdot V_y^* \\ & + \frac{1}{2} \left(\sigma_y y\right)^2 \cdot V_{yy}^* + \sigma_y a y \boldsymbol{\phi} \boldsymbol{\sigma} \boldsymbol{\rho}_y^T \cdot V_{ay}^* \Bigg\} \Big/ \left(\frac{\rho}{\theta} V^*\right) \ , \end{aligned} \tag{43}$$

using an indicator function in order to implement the conditionality operator $(\cdot)|_{a=b}$. The presence of the term $\max\left\{\left[\boldsymbol{\phi}^* \mathbf{R}^T + \left(1 - \boldsymbol{\phi}^* \mathbf{1}^T\right) r_f\right] a + y - c^* \ , \ 0\right\}\Big|_{a=b} \cdot V_a^*$ in (43) has not affected our results, as we had strictly positive saving rates in all our calibration exercises. Notably, our borrowing constraint is $b = \underline{a}$. As we have averaged across income groups of stockholders, $\underline{a}$ is well above 0, it amounts to USD $85,520$, which is a rather tight borrowing constraint. Yet, at this level of wealth ($\underline{a}$), and for all gridpoints for y, households chose interior solutions.

1.4 The recursive algorithm

Using the HJB equation (equation (9)), we perform iterations on $\mathbf{\Theta}$, using the Matlab command "`nlinfit`" which is designed in order to solve nonlinear minimum least-squares econometric models. The inputs of "`nlinfit`" are a (nonlinear econometric) model, a matrix of regressors, and a vector of model parameters that need to be estimated. In order to match the input structure of the "`nlinfit`" Matlab procedure, we compute all the above $(m_a \cdot m_y) \times 1$ vectors corresponding to equations (30) through (37) and also to (6) and (8), and we use equation (9) in order to produce a Matlab m-file "`HJB.m`" with inputs $\bar{\mathbf{a}}_{grid\_long}$, $\bar{\mathbf{y}}_{grid\_long}$ and $\mathbf{\Theta}_{vector} \equiv \mathtt{reshape}(\mathbf{\Theta}, \nu_a \cdot \nu_y, 1)$, which is an $(\nu_a \cdot \nu_y) \times 1$ vector resulting from stacking all columns of $\mathbf{\Theta}$. This "`HJB.m`" function defines the model to be estimated, and we also create an $(m_a m_y) \times 2$ matrix with columns consisting of vectors $\bar{\mathbf{a}}_{grid\_long}$ and $\bar{\mathbf{y}}_{grid\_long}$, which is the regressor matrix.

1.5 The importance of a good first guess

The initial guess is the $\hat{\mathbf{\Theta}}^{(0)}_{vector}$ which corresponds to the closed-form solution given by Proposition 1 in the paper, for the special case $\boldsymbol{\rho}_y \boldsymbol{\rho}_y^T = 1$. When $\boldsymbol{\rho}_y \boldsymbol{\rho}_y^T = 1$ holds, the performance of the algorithm is satisfactory, since $\hat{\mathbf{\Theta}}^{(0)}_{vector} = \hat{\mathbf{\Theta}}^{*}_{vector}$ in one iteration.

We perform iterations for the version of the model with two risky assets, for cases in which $\rho_{ys}^2 + \rho_{yb}^2 = 1$, and also for cases in which $\rho_{ys}^2 + \rho_{yb}^2 < 1$. We compute the decision rules of the model for $\rho_{ys}^2 + \rho_{yb}^2 \in \{0.5, 0.75\}$, taking gradual steps down from$\rho_{ys}^2 + \rho_{yb}^2 = 1$ to $\rho_{ys}^2 + \rho_{yb}^2 = 0.75$, and then from $\rho_{ys}^2 + \rho_{yb}^2 = 0.75$ to $\rho_{ys}^2 + \rho_{yb}^2 = 0.5$. In each case, we use the solution found in the previous step as a first guess in the "`nlinfit`" Matlab procedure, finding that this strategy is stable and efficient. Typically, setting $\nu_a = \nu_y = 3$, and $m_a = m_y = 20$, performs satisfactorily well, producing all results plotted in Figure 5 of the paper in about 13.5 seconds on a state-of-the art laptop.

1.6 Dealing with the nonlinear relationship between assets and income across income categories in the data

Panel C of Figure 4 shows that, after ranking households according to their after-tax adult-equivalent income, a and y are linked through a nonlinear relationship in the data. This nonlinear relationship is not reflected by the two grids, $\bar{\mathbf{a}}$ and $\bar{\mathbf{y}}$. This failure of reflecting the nonlinear relationship occurs because grids $\bar{\mathbf{a}}$ and $\bar{\mathbf{y}}$ should be consistent with Chebyshev nodes, in order to ensure that discrete-orthogonality conditions hold accurately. Discrete-orthogonality conditions are a necessary requirement for good performance of the Chebyshev approximation. The fact that grids $\bar{\mathbf{a}}$ and $\bar{\mathbf{y}}$ do not reflect the nonlinear relationship in the data means that we cannot directly select matrix elements from the resulting matrix

$$\bar{\mathbf{\Phi}} = \left[\bar{\phi}_{k,\ell}^*\right] = \left[\Phi\left(\bar{a}_k, \bar{y}_\ell\right)\right] = \left[\Phi\left(A\left(\bar{x}_{a,k}\right), Y\left(\bar{x}_{y,\ell}\right)\right)\right] \ , \quad k \in \{1, ..., m_a\}\,, \ \ell \in \{1, ..., m_y\}\,,$$

of the code in order to report them in Figure 5. In order to deal with this issue, we first interpolate the a_{data} and y_{data} data observations that correspond to the six income categories in panel C of Figure 4 in order to capture the nonlinear relationship in that figure, say

$$y_{data} = g\left(a_{data}\right) \ ,$$

using the "spline"-interpolation option of Matlab's "interp1" routine; specifically, we produce an $m_a \times 1$ vector, called $\mathbf{y}^{nl}$, that uses $\bar{\mathbf{a}}$ as the interpolation domain, so,

$$\mathbf{y}^{nl} = g\left(\bar{\mathbf{a}}\right) \ . \tag{44}$$

In order to produce Figure 5, the goal is to report portfolio shares which are consistent with

$$\phi^* = \Phi\left(\bar{\mathbf{a}}, g\left(\bar{\mathbf{a}}\right)\right) \ .$$

So, for all $k \in \{1, ..., m_a\}$, fix an $\bar{a}_k$ gridpoint and define the function,

$$\tilde{\phi}_k\left(y\right) \equiv \Phi\left(\bar{a}_k, y\right) \ ,$$

using the "spline"-interpolation option of Matlab's "interp1" routine, using gridpoints $\bar{\mathbf{y}}$ that correspond to the k-th row of matrix $\bar{\mathbf{\Phi}}$ as the domain, and the $m_y \times 1$ vector $\left[\Phi\left(\bar{a}_k, \bar{\mathbf{y}}\right)\right]^T$ as the image of function $\tilde{\phi}_k\left(y\right)$. So,

$$\phi_k^* \equiv \Phi\left(\bar{a}_k, g\left(\bar{a}_k\right)\right) = \left\{\Phi\left(\bar{a}_k, y\right) \ | \ \tilde{\phi}_k^{-1}\left(y\right) = y_k^{nl}\right\} \ ,$$

in which y_k^{nl} corresponds to the k-th element of vector $\mathbf{y}^{nl}$, defined by (44), fills in a new $m_a \times 1$ vector, $\boldsymbol{\phi}^*$. Vector $\boldsymbol{\phi}^*$ contains the values that we report in Figure 5, after interpolating the pair $\left(\mathbf{y}^{nl}, \boldsymbol{\phi}^*\right)$ and projecting this interpolation on the 6×1 vector $\mathbf{y}_{data}$, using the "spline"-interpolation option of Matlab's "interp1" routine.

2. Appendix B – Simulating the discrete-time model (2 risky assets)

2.1 Statement of the Problem

The household solves,

$$V\left(a_t, y_t\right) = \max_{\left(c_t, \phi_t\right)} \frac{\left\{\left(1-\beta\right)\left(c_t - \chi\right)^{1-\frac{1}{\eta}} + \beta\left\{\left(1-\gamma\right) E_t\left[V\left(R_{p,t+1} a_t + y_t - c_t \ , \ y_{t+1}\right)\right]\right\}^{\frac{1-\frac{1}{\eta}}{1-\gamma}}\right\}^{\frac{1-\gamma}{1-\frac{1}{\eta}}}}{1-\gamma} \ , \tag{45}$$

in which,

$$R_{p,t+1} \equiv \left(R^s_{t+1} - r_f\right)\phi^s_t + \left(R^b_{t+1} - r_f\right)\phi^b_t + r_f \ , \tag{46}$$

and with,

$$\ln\left(y_{t+1}\right) - \ln\left(y_t\right) = \mu_y + \varepsilon_{y,t+1} \ , \ \ \varepsilon_{y,t+1} \sim N\left(0, \sigma_y^2\right) \ , \tag{47}$$

$$\ln\left(P_{s,t+1}\right) - \ln\left(P_{s,t}\right) = R_s + \varepsilon_{s,t+1} \ , \ \ \varepsilon_{s,t+1} \sim N\left(0, \sigma_s^2\right) \ , \tag{48}$$

$$\ln\left(P_{b,t+1}\right) - \ln\left(P_{b,t}\right) = R_b + \varepsilon_{b,t+1} \ , \ \ \varepsilon_{b,t+1} \sim N\left(0, \sigma_b^2\right) \ , \tag{49}$$

where $P_{s,t}$ and $P_{b,t}$ denote the stock price and the business equity price in period t, while,

$$\frac{Cov\left(\varepsilon_{s,t+1}, \varepsilon_{y,t+1}\right)}{\sigma_s \sigma_y} = \rho_{ys} \ , \tag{50}$$

$$\frac{Cov\left(\varepsilon_{b,t+1}, \varepsilon_{y,t+1}\right)}{\sigma_b \sigma_y} = \rho_{yb} \ , \tag{51}$$

$$\frac{Cov\left(\varepsilon_{s,t+1}, \varepsilon_{b,t+1}\right)}{\sigma_s \sigma_b} = \rho_{sb} \ , \tag{52}$$

R^s_t in equation (46) is given by,

$$R^s_t = e^{R_s + \varepsilon_{s,t}} \ , \tag{53}$$

R^b_t in equation (46) is given by,

$$R^b_t = e^{R_b + \varepsilon_{b,t}} \ , \tag{54}$$

y_t is given by,

$$y_t = e^{\mu_y + \varepsilon_{y,t}} \ , \tag{55}$$

and (a_0, y_0, ϕ_0) are given.

The problem stated by (45) is the discrete-time analogue to the continuous-time version of the model with two risky assets (here we focus on stock-market portfolio holdings and business equity holdings). Notice that in this case of two risky assets, the condition for insurability (diversifiability) of labor-income risk becomes $\rho_{ys}^2 + \rho_{bs}^2 = 1$, so labor-income risk is uninsurable if $\rho_{ys}^2 + \rho_{bs}^2 \notin \{-1, 1\}$. Let,

$$\theta \equiv \frac{1-\frac{1}{\eta}}{1-\gamma} \ ,$$

which transforms (45) into,

$$V(a_t, y_t) = \max_{\left(c_t, \phi_{t+1}^s, \phi_{t+1}^b\right)} \frac{\left\{(1-\beta)(c_t-\chi)^{(1-\gamma)\theta} + \beta\left\{(1-\gamma)E_t\left[V\left(R_{p,t+1}a_t + y_t - c_t \ , \ y_{t+1}\right)\right]\right\}^{\theta}\right\}^{\frac{1}{\theta}}}{1-\gamma} \ . \tag{56}$$

2.2 Necessary conditions

Applying the envelope theorem on (56),

$$V_a(a_t, y_t) = \frac{\partial\left[RHS \text{ of eq. } (56)\right]}{\partial a_t} \ , \tag{57}$$

while the first-order conditions of (56) with respect to c_t give,

$$(1-\beta)(c_t-\chi)^{(1-\gamma)\theta-1}\left\{(1-\beta)(c_t-\chi)^{(1-\gamma)\theta} + \beta\left\{(1-\gamma)E_t\left[V\left(a_{t+1}, y_{t+1}\right)\right]\right\}^{\theta}\right\}^{\frac{1}{\theta}-1} =$$

$$= \frac{\partial\left[RHS \text{ of eq. } (56)\right]}{\partial a_t} \ . \tag{58}$$

In equilibrium, the optimal sequence $\left\{\left(c_t^*, \phi_{t+1}^{s*}, \phi_{t+1}^{b*}, x_{t+1}^*\right)\right\}_{t=0}^{\infty}$ satisfies the Bellman equation given by (56), so, after discarding the max operator, (56) gives,

$$\left\{(1-\beta)(c_t^*-\chi)^{(1-\gamma)\theta} + \beta\left\{(1-\gamma)E_t\left[V\left(a_{t+1}^*, y_{t+1}\right)\right]\right\}^{\theta}\right\} = \left[(1-\gamma)V(a_t, y_t)\right]^{\theta} \ . \tag{59}$$

Combining equations (59) and (58) we obtain,

$$(1-\beta)\left(c_t^*-\chi\right)^{(1-\gamma)\theta-1}\left[(1-\gamma)V\left(a_t,y_t\right)\right]^{1-\theta}=\frac{\partial\left[RHS \text{ of eq. } (56)\right]}{\partial a_t}\ . \tag{60}$$

So, combining (57) with (60) we obtain,

$$c_t^*=C\left(a_t,y_t\right)=\chi+\left\{\frac{1}{1-\beta}V_a\left(a_t,y_t\right)\left[(1-\gamma)V\left(a_t,y_t\right)\right]^{\theta-1}\right\}^{\frac{1}{(1-\gamma)\theta-1}}\ . \tag{61}$$

Equation (61) is crucial for solving the model numerically using value-function iteration. Equation (61) states that, once we have a guess for the value function, $V(a,y)$, we immediately have a closed-form solution for the decision rule, $C(a,y)$, which depends only on $V(a,y)$and $V_a(a,y)$. So, if we use a projection method for approximating $V(a,y)$, then we can immediately incorporate the formula given by (61) into the RHS of the Bellman equation. Most importantly, equation (61) helps in the direct computation of portfolio shares, directly from the first-order condition with respect to ϕ^s and ϕ^b.

The first-order condition with respect to ϕ^s implies,

$$E_t\left[V_a\left(R_{p,t+1}a_t+y_t-c_t\ ,\ y_{t+1}\right)\left(R_{t+1}^s-r_f\right)\right]=0\ ,$$

and the first-order condition with respect to ϕ^b implies,

$$E_t\left[V_a\left(R_{p,t+1}a_t+y_t-c_t\ ,\ y_{t+1}\right)\left(R_{t+1}^b-r_f\right)\right]=0\ ,$$

the detailed version of which is,

$$\begin{gathered}\underbrace{E_t\left\{V_a\left(\left[\left(R_{t+1}^s-r_f\right)\phi_t^s+\left(R_{t+1}^b-r_f\right)\phi_t^b+r_f\right]a_t+y_t-C\left(a_t,y_t\right)\ ,\ y_{t+1}\right)\left(R_{t+1}^s-r_f\right)\right\}}_{\overset{\shortparallel}{h_s\left(\phi_t^s,\phi_t^b,a_t,y_t\right)}}=0\ .\\ \underbrace{E_t\left\{V_a\left(\left[\left(R_{t+1}^s-r_f\right)\phi_t^s+\left(R_{t+1}^b-r_f\right)\phi_t^b+r_f\right]a_t+y_t-C\left(a_t,y_t\right)\ ,\ y_{t+1}\right)\left(R_{t+1}^b-r_f\right)\right\}}_{\overset{\shortparallel}{h_b\left(\phi_t^s,\phi_t^b,a_t,y_t\right)}}=0\ .\end{gathered} \tag{62}$$

So, based on (62), the decision rule for the portfolio share $\phi_t^{s*} = \Phi(a_t, y_t)$ and $\phi_t^{b*} = \Phi(a_t, y_t)$, is the implicit function that solve,

$$\begin{cases} h\left(\Phi^s(a,y), a, y\right) = 0 \ , \\ h\left(\Phi^b(a,y), a, y\right) = 0 \ . \end{cases} .$$

2.3 Algorithm: Value-Function Iteration

2.3.1 Overview

We use an initial guess on the value function V defined by (56), $V^{(0)}$. Then we utilize the contraction-mapping property of the Bellman equation described by the recursion,

$$V^{(j+1)}(a_t, y_t) = \max_{\left(c_t, \phi_{t+1}^s, \phi_{t+1}^b\right)} \frac{\left\{(1-\beta)(c_t-\chi)^{(1-\gamma)\theta} + \beta\left\{(1-\gamma) E_t\left[V^{(j)}\left(R_{p,t+1}a_t + y_t - c_t \ , \ y_{t+1}\right)\right]\right\}^{\theta}\right\}^{\frac{1}{\theta}}}{1-\gamma} . \tag{63}$$

in order to generate a Cauchy sequence $\left\{V^{(j)}\right\}_{j=0}^{\infty}$ with $V^{(j)} \to V^*$, which is a typical value-function iteration approach. The key issue in value-function iteration approaches is how one numerically implements the max operator on the right-hand side (RHS) of the Bellman equation. In order to perform maximization on the RHS of (63), we solve the first-order conditions given by (61) and (62), in each step of the recursive procedure, which relies on (the typically incorrect) value function $V^{(j)}$. For deriving the decision rule for consumption, $C^{(j)}(a, y)$ which is conditional upon the value function $V^{(j)}(a, y)$, equation (61) provides an explicit formula,

$$C^{(j)}(a_t, y_t) = \chi + \left\{\frac{1}{1-\beta} V_a^{(j)}(a_t, y_t)\left[(1-\gamma) V^{(j)}(a_t, y_t)\right]^{\theta-1}\right\}^{\frac{1}{(1-\gamma)\theta-1}} . \tag{64}$$

The formula $C^{(j)}(a_t, y_t)$ can be substituted directly into the RHS of (63), but we do have an analytical expression for the decision rule $\Phi_{i\in(s,b)}^{(j)}(a_t, y_t)$. In order to compute $\Phi_{i\in(s,b)}^{(j)}(a_t, y_t) \equiv$

$\left\{\phi_{i\in(s,b)} \mid h^{(j)}_{i\in(s,b)}\left(\phi,a,y\right)=0\right\}$, we need to numerically solve,

$$h^{(j)}_{i\in(s,b)}\left(\phi_{i\in(s,b)},a,y\right)=0 \ ,$$

in which,

$$h^{(j)}_{i\in(s,b)}\left(\phi_t^{i\in(s,b)},a_t,y_t\right)\equiv$$

$$E_tV_a^{(j)}\left(\left[\left(R^s_{t+1}-r_f\right)\phi^s_t+\left(R^b_{t+1}-r_f\right)\phi^b_t+r_f\right]a_t+y_t-C^{(j)}\left(a_t,y_t\right)\ ,\ y_{t+1}\right)\left(R^{i\in(s,b)}_{t+1}-r_f\right)\ . \tag{65}$$

Both in (65), and in RHS of (63), there is an expectations operator, $E_t(\cdot)$, that needs to be computed. This computation of the expectations operator is discussed in a separate subsection below.

Another technical necessity in (65) is how to compute $V_a^{(j)}(a,y)$, the partial derivative of the value function. In order to achieve this derivative computation, we employ a simple exponential-projection method which approximates functions using,

$$f\left(a,y\right)\simeq\hat{f}\left(a,y\right)\equiv e^{\sum_{i=0}^{\nu}\sum_{j=0}^{\nu}\xi_{ij}[\ln(a)]^i[\ln(y)]^j}\ . \tag{66}$$

An advantage of this $\hat{f}(x)$ approximation given by (66), is that we can take explicit derivatives, namely,

$$f_a\left(a,y\right)\simeq\hat{f}_a\left(a,y\right)=\hat{f}\left(a,y\right)\sum_{i=0}^{\nu}\sum_{j=0}^{\nu}i\cdot\xi_{ij}\frac{\left[\ln\left(a\right)\right]^{i-1}}{a}\left[\ln\left(y\right)\right]^j\ . \tag{67}$$

For values of parameter $\gamma>1$, the mapping $m(\cdot)=(\cdot)^{(1-\gamma)}/(1-\gamma)$, which is applied on the RHS of (63), is known to give negative values. This property, of having negative values for the the RHS of (63), is inherited by the value function on the LHS of (63) as well. Yet, the exponential-projection technique we suggest in (66), can only match positive values. In order to tackle this problem, we use the transformation,

$$V(a,y) = \frac{\left[\tilde{V}(a,y)\right]^{1-\gamma}}{1-\gamma} \Leftrightarrow \tilde{V}(a,y) = \left[(1-\gamma)V(a,y)\right]^{\frac{1}{1-\gamma}} \ . \tag{68}$$

A consequence of the transformation given by (68) is,

$$V_a(a,y) = \left[\tilde{V}(a,y)\right]^{-\gamma} \tilde{V}_a(a,y) \ . \tag{69}$$

So, we create a Matlab m-file, named "`Vtilde.m`", which implements the exponential approximation

$$\tilde{V}(a,y) \simeq e^{\sum_{i=0}^{\nu}\sum_{j=0}^{\nu}\xi_{ij}[\ln(a)]^i[\ln(y)]^j} \ , \tag{70}$$

on any grid for the state variables, a and y.

Using this projection approach, we take a first guess on the value function, $\tilde{V}^{(0)}$, and we obtain an estimate of the vector $\left\{\left\{\xi_{i,k}^{(0)}\right\}_{i=0}^{\nu}\right\}_{k=0}^{\nu}$ through the "`nlinfit`" command in Matlab. Our first guess, $\tilde{V}^{(0)}$, uses the calibrating parameters that we have found in continuous time, and the continuous-time functional form for the value function, $V(a,y)$ for the special case in which $\rho_{ys}^2 + \rho_{yb}^2 = 1$.

Using the recursive procedure described above, through (63) we generate a sequence of coefficients $\left\{\left\{\left\{\xi_{i,k}^{(j)}\right\}_{i=0}^{\nu}\right\}_{k=0}^{\nu}\right\}_{j=0}^{\infty}$, with $\lim_{j\to\infty}\left\{\left\{\xi_{i,k}^{(j)}\right\}_{i=0}^{\nu}\right\}_{k=0}^{\nu} = \left\{\left\{\xi_{i,k}^{*}\right\}_{i=0}^{\nu}\right\}_{k=0}^{\nu}$, in which

$$V^*(a,y) \simeq \frac{e^{(1-\gamma)\sum_{i=0}^{\nu}\sum_{j=0}^{\nu}\xi_{ij}^*[\ln(a)]^i[\ln(y)]^j}}{1-\gamma} \ ,$$

in which $V^*(a,y)$ solves (56).

2.3.2 Approximating the joint density for the stochastic process for the returns (stock equity and business equity) and the labor-income growth

In equations (47), (48), (49), (51) and (52) above, we have mentioned that the model's three shocks ε_s, ε_b and ε_y are distributed so that,

$$\begin{pmatrix} \varepsilon_y \\ \varepsilon_s \\ \varepsilon_b \end{pmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_y^2 & \rho_{ys}\sigma_s\sigma_y & \rho_{yb}\sigma_y\sigma_b \\ \rho_{ys}\sigma_s\sigma_y & \sigma_s^2 & \rho_{sb}\sigma_s\sigma_b \\ \rho_{yb}\sigma_y\sigma_b & \rho_{sb}\sigma_s\sigma_b & \sigma_b^2 \end{bmatrix} \right) \tag{71}$$

We want to compute a joint-probability matrix in order to describe the joint density of shocks,

$$s_{shock} \equiv R_s + \varepsilon_s,\ b_{shock} \equiv R_b + \varepsilon_b \text{ and } \ y_{shock} \equiv \mu_y + \varepsilon_y \ , \tag{72}$$

first, we creat a partition of (ε_y) and $(\varepsilon_s, \varepsilon_b)$ to reduce the above trivariate normal system into a conditional univariate normal and an unconditional bivariate normal system conveniently,

$$\begin{pmatrix} y_{shock} \\ [s_{shock}, b_{shock}]^T \end{pmatrix} \sim N\left(\begin{bmatrix} \mu_y \\ [R_s, R_b]^T \end{bmatrix}, \begin{bmatrix} \sigma_y^2 & [\rho_{ys}\sigma_s\sigma_y, \rho_{yb}\sigma_y\sigma_b] \\ [\rho_{ys}\sigma_s\sigma_y, \rho_{yb}\sigma_y\sigma_b]^T & \begin{bmatrix} \sigma_s^2 & \rho_{sb}\sigma_s\sigma_b \\ \rho_{sb}\sigma_s\sigma_b & \sigma_b^2 \end{bmatrix} \end{bmatrix}\right) \tag{73}$$

to simplify (73), we have

$$\begin{pmatrix} y_{shock} \\ \mathbf{R}_{shock} \end{pmatrix} \sim N\left(\begin{bmatrix} \mu_y \\ \mathbf{R} \end{bmatrix}, \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{bmatrix}\right) \tag{74}$$

where

$$\begin{aligned}
\mathbf{R}_{shock} &= [s_{shock}, b_{shock}]^T \\
\mathbf{R} &= [R_s, R_b]^T \\
\mathbf{\Sigma}_{11} &= \sigma_y^2 \\
\mathbf{\Sigma}_{12} &= [\rho_{ys}\sigma_s\sigma_y, \rho_{yb}\sigma_y\sigma_b] \\
\mathbf{\Sigma}_{21} &= [\rho_{ys}\sigma_s\sigma_y, \rho_{yb}\sigma_y\sigma_b]^T \\
\mathbf{\Sigma}_{22} &= \begin{bmatrix} \sigma_s^2 & \rho_{sb}\sigma_s\sigma_b \\ \rho_{sb}\sigma_s\sigma_b & \sigma_b^2 \end{bmatrix}
\end{aligned}$$

the joint density of $(y_{shock}, s_{shock}, b_{shock})$ could be written as the product of conditional density of y_{shock} on (s_{shock}, y_{shock}) and the unconditional joint density of (s_{shock}, y_{bshock}). For the conditional density of y_{shock} on (s_{shock}, y_{shock}), we have

$$\left(y_{shock}|\mathbf{R}_{shock} = [R_s^j, R_b^k]^T\right) \sim N\left(\bar{\mu}, \bar{\Sigma}\right) \tag{75}$$

where

$$\bar{\mu} = \mu_y + [\rho_{ys}\sigma_s\sigma_y, \rho_{yb}\sigma_y\sigma_b] \begin{bmatrix} \sigma_s^2 & \rho_{sb}\sigma_s\sigma_b \\ \rho_{sb}\sigma_s\sigma_b & \sigma_b^2 \end{bmatrix}^{-1} \left([R_s^j, R_b^k]^T - [R_s, R_b]^T\right) \tag{76}$$

$$\bar{\Sigma} = \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21} \tag{77}$$

the unconditional distribution is given by

$$\phi(y_{shock}, \mathbf{R}_{shock}) = \phi(y_{shock}|\mathbf{R}_{shock}) \cdot \phi(\mathbf{R}_{shock})$$

based on the stochastic structure given by (71). The joint density of (s_{shock}, b_{shock}) is this of a bivariate normal with,

$$\phi\left(s_{shock}, b_{shock}\right) = \frac{1}{2\pi\sigma_s\sigma_b\sqrt{1-\rho_{sb}^2}} \times$$

$$\times \exp\left\{-\frac{1}{2(1-\rho_{sb}^2)}\left[\frac{(s_{shock}-R_s)^2}{\sigma_s^2}+\frac{(b_{shock}-R_b)^2}{\sigma_b^2}-\frac{2\rho_{bs}(s_{shock}-R_s)(b_{shock}-R_b)}{\sigma_s\sigma_b}\right]\right\} \ . \tag{78}$$

s_{shock} conditional upon b_{shock} is also normally distributed with,

$$s_{shock}|_{b_{shock}} \sim N\left(R_s+\frac{\sigma_s}{\sigma_b}\rho_{sb}(b_{shock}-R_b),(1-\rho_{sb}^2)\sigma_s^2\right) \ ,$$

so,

$$\phi\left(s_{shock}\ ,\ b_{shock}\right)=\phi\left(s_{shock}\mid b_{shock}\right)\cdot\phi\left(b_{shock}\right) \ , \tag{79}$$

in which

$$\phi\left(b_{shock}\right)=\frac{1}{\sigma_b\sqrt{2\pi}}\exp\left[-\frac{(b_{shock}-R_b)^2}{2\sigma_b^2}\right] \ ,$$

Now,we have

$$\phi(y_{shock},\mathbf{R}_{shock})=\phi(y_{shock}|\mathbf{R}_{shock})\cdot\phi(\mathbf{R}_{shock})=\phi(y_{shock}|\mathbf{R}_{shock})\cdot\phi\left(s_{shock}\mid b_{shock}\right)\cdot\phi\left(b_{shock}\right) \tag{80}$$

since $b_{shock} \sim N\left(R_b,\sigma_b^2\right)$. In order to calculate $\phi(y_{shock}|\mathbf{R}_{shock}),\phi\left(s_{shock}\mid b_{shock}\right)$ and $\phi\left(b_{shock}\right)$, we use the fact that,

$$\begin{aligned}\frac{y_{shock}|_{\mathbf{R}_{shock}}-\bar{\mu}}{\bar{\Sigma}} &\sim N\left(0,1\right),\\ \frac{s_{shock}|_{b_{shock}}-\left[R_s+\frac{\sigma_s}{\sigma_b}\rho_{sb}(b_{shock}-R_b)\right]}{\sigma_b\sqrt{1-\rho_{sb}^2}} &\sim N\left(0,1\right) \ ,\\ \text{and } \frac{b_{shock}-R_b}{\sigma_b} &\sim N\left(0,1\right) \ ,\end{aligned}$$

and we then use (80) in order to compute $\phi(y_{shock},\mathbf{R}_{shock})$ in matrix form. So, if the grid for s_{shock} is an $m_s\times 1$ vector, for b_{shock} is an $m_b\times 1$ and the grid for y_{shock} is an $m_y\times 1$ vector, then let the joint-probability matrix

$$\underbrace{M_{sby}}_{m_s\times m_b\times m_y} \equiv [M_{sby,klm}]=[\phi\left(s_{shock,k}\ ,\ b_{shock,l}\ ,\ y_{shock,m}\right)] \ , \tag{81}$$

$$k \in \{1, ..., m_s\}, \, l \in \{1, ..., m_b\}, \, m \in \{1, ..., m_y\} \ .$$

For specifying the grids for s_{shock} , b_{shock} and y_{shock}, we split the continuum into equispaced intervals, and then we proceed to calculating the probabilities associated with the midpoint of each interval, using Matlab's built-in calculator for the normal density (the command "`normcdf`", which calculates cumulative probabilities for a standard normal).

Because the support of normally distributed variables is $(-\infty, \infty)$, we need to choose an upper and lower level of the support for s_{shock} , b_{shock} and y_{shock}. For a standard normal notice that, in Matlab, "`normcdf(-3)=0.0013`", "`normcdf(-10)=7.6199e-24`", "`normcdf(-12)=1.7765e-33`", with the latter being a negligible number. In order to avoid accumulating errors (numbers such as 10^{-33} tend to create this error-accumulation problem), for the lowest gridpoint of s_{shock} (same for b_{shock} and y_{shock}) called "$r_{\min}$", we use

$$s_{shock\_\min} = R_s + \sigma_s \cdot (-3.5) \ ,$$

and for the largest gridpoint we use

$$s_{shock\_\max} = R_s + \sigma_s \cdot (+3.5) \ ,$$

in which -3.5 is a calibrating parameter related to the standard normal, ensuring that the suppport of s_{shock} does not have probability kinks at its endpoints, or that there is no error-accumulation problem (after plotting both the joint density function of $\phi(y_{shock}, \mathbf{R}_{shock})$, and individual density functions, we have concluded that the value "`normcdf(-3.5)=2.3263e-04`" works best.

2.4 Computing the portfolio share that satisfies the first-order conditions: applying the expectations operator

First, we choose grids for a and y calculated in accordance with the nonlinear relationship between a and y in the data (see Panel C in Figure 4 and the expression $y_{data} = g\,(a_{data})$

given by (44)). So, we generate two $n\times 1$ vectors, a_{grid} and y_{grid}, that satisfy $y_{grid}=g\left(a_{grid}\right)$. Consider that we are at the j-th iteration of the value-function iteration method, using $V^{(j)}$ for all calculations. At this stage we want to compute the function $h^{(j)}\left(\phi_t^s,a_t,y_t\right)$ based on (62), and a concern is how to apply the expectations operator in that function(similar also for $h^{(j)}\left(\phi_t^b,a_t,y_t\right)$). Using a loop, for each $i\in\{1,...,n\}$, we express function $h^{(j)}\left(\phi_t^s,a_t,y_t\right)$ in equation (62) as,

$$\sum_{k=1}^{m_s}\sum_{l=1}^{m_b}\sum_{m=1}^{m_y}M_{sby,k\ell m}\left\{V_a^{(j)}\left(\left[\left(\underbrace{e^{s_{shock,k}}}_{\substack{\shortparallel\\ R_{t+1}^s}}-r_f\right)\phi_t^s+\left(\underbrace{e^{b_{shock,l}}}_{\substack{\shortparallel\\ R_{t+1}^b}}-r_f\right)\phi_t^b+r_f\right]\underbrace{a_{grid,i}}_{\substack{\shortparallel\\ a_t}}+\underbrace{y_{grid,i}}_{\substack{\shortparallel\\ y_t}}\right.\right.$$
$$\left.\left.-\ C\left(\underbrace{a_{grid,i}}_{\substack{\shortparallel\\ a_t}},\ \underbrace{y_{grid,i}}_{\substack{\shortparallel\\ y_t}}\right),\ \underbrace{y_{grid,i}\cdot e^{y_{shock,m}}}_{\substack{\shortparallel\\ y_{t+1}}}\right)\left(\underbrace{e^{s_{shock,k}}}_{\substack{\shortparallel\\ R_{t+1}}}-r_f\right)\right\}=0\ , \tag{82}$$

function $h^{(j)}\left(\phi_t^b,a_t,y_t\right)$ in equation (62) as,

$$\sum_{k=1}^{m_s}\sum_{l=1}^{m_b}\sum_{m=1}^{m_y}M_{sby,k\ell m}\left\{V_a^{(j)}\left(\left[\left(\underbrace{e^{s_{shock,k}}}_{\substack{\shortparallel\\ R_{t+1}^s}}-r_f\right)\phi_t^s+\left(\underbrace{e^{b_{shock,l}}}_{\substack{\shortparallel\\ R_{t+1}^b}}-r_f\right)\phi_t^b+r_f\right]\underbrace{a_{grid,i}}_{\substack{\shortparallel\\ a_t}}+\underbrace{y_{grid,i}}_{\substack{\shortparallel\\ y_t}}\right.\right.$$
$$\left.\left.-\ C\left(\underbrace{a_{grid,i}}_{\substack{\shortparallel\\ a_t}},\ \underbrace{y_{grid,i}}_{\substack{\shortparallel\\ y_t}}\right),\ \underbrace{y_{grid,i}\cdot e^{y_{shock,m}}}_{\substack{\shortparallel\\ y_{t+1}}}\right)\left(\underbrace{e^{b_{shock,k}}}_{\substack{\shortparallel\\ R_{t+1}}}-r_f\right)\right\}=0\ , \tag{83}$$

in which $V_a^{(j)}(\cdot)$ is given by (69) for a given vector of coefficients $\left\{\left\{\xi_{i,k}^{(j)}\right\}_{i=0}^{\nu}\right\}_{k=0}^{\nu}$. The expression given by (82) and (83) defines a system of two functions $h^{(j)}\left(\phi^s,a_{grid,i},y_{grid,i}\right)$ and $h^{(j)}\left(\phi^b,a_{grid,i},y_{grid,i}\right)$, for each $i\in\{1,...,n\}$. We use Matlab's "`fsolve`" routine in order to solve the nonlinear equation system $h^{(j)}\left(\phi^s,a_{grid,i},y_{grid,i}\right)=0$ and $h^{(j)}\left(\phi^b,a_{grid,i},y_{grid,i}\right)=0$, so,

$$\phi^{s(j)}\left(a_{grid,i},y_{grid,i}\right)=\left\{\phi^{s}\mid h^{(j)}\left(\phi^{s},\phi^{b},a_{grid,i},y_{grid,i}\right)=0\right\}\quad\text{for all } i\in\{1,...,n\}\ , \tag{84}$$

and

$$\phi^{b(j)}\left(a_{grid,i},y_{grid,i}\right)=\left\{\phi^{b}\mid h^{(j)}\left(\phi^{s},\phi^{b},a_{grid,i},y_{grid,i}\right)=0\right\}\quad\text{for all } i\in\{1,...,n\}\ . \tag{85}$$

2.5 Performing value-function iteration

Here we use the Bellman equation given by (63) in order to perform value function iteration. We use (84),(85) and (61) in order to incorporate $\phi^{s(j)}(a_t,y_t)$, $\phi^{b(j)}(a_t,y_t)$ and $C^{(j)}(a_t,y_t)$ into the RHS of (63). One difficulty is the computation of the expectations term on the RHS of (63). We use,

$$\begin{aligned}&E_t\left[V^{(j)}\left(R_{p,t+1}a_t+y_t-c_t\ ,\ y_{t+1}\right)\right]=\\&\sum_{k=1}^{m_s}\sum_{\ell=1}^{m_b}\sum_{m=1}^{m_y}M_{sby,k\ell m}\left\{V^{(j)}\left(\left[\left(\underbrace{e^{s_{shock,k}}}_{\overset{\shortparallel}{R^s_{t+1}}}-r_f\right)\phi^s_t+\left(\underbrace{e^{b_{shock,k}}}_{\overset{\shortparallel}{R^b_{t+1}}}-r_f\right)\phi^b_t+r_f\right]\underbrace{a_{grid,i}}_{\overset{\shortparallel}{a_t}}\right.\right.\\&\left.\left.+\underbrace{y_{grid,i}}_{\overset{\shortparallel}{y_t}}-C\left(\underbrace{a_{grid,i}}_{\overset{\shortparallel}{a_t}},\underbrace{y_{grid,i}}_{\overset{\shortparallel}{y_t}}\right),\underbrace{y_{grid,i}\cdot e^{y_{shock,\ell}}}_{\overset{\shortparallel}{y_{t+1}}}\right)\left(\underbrace{e^{s_{shock,k}}}_{\overset{\shortparallel}{R^s_{t+1}}}-r_f\right)\right\}\ ,\end{aligned} \tag{86}$$

$$\begin{aligned}&E_t\left[V^{(j)}\left(R_{p,t+1}a_t+y_t-c_t\ ,\ y_{t+1}\right)\right]=\\&\sum_{k=1}^{m_s}\sum_{\ell=1}^{m_b}\sum_{m=1}^{m_y}M_{sby,k\ell m}\left\{V^{(j)}\left(\left[\left(\underbrace{e^{s_{shock,k}}}_{\overset{\shortparallel}{R^s_{t+1}}}-r_f\right)\phi^s_t+\left(\underbrace{e^{b_{shock,k}}}_{\overset{\shortparallel}{R^b_{t+1}}}-r_f\right)\phi^b_t+r_f\right]\underbrace{a_{grid,i}}_{\overset{\shortparallel}{a_t}}\right.\right.\\&\left.\left.+\underbrace{y_{grid,i}}_{\overset{\shortparallel}{y_t}}-C\left(\underbrace{a_{grid,i}}_{\overset{\shortparallel}{a_t}},\underbrace{y_{grid,i}}_{\overset{\shortparallel}{y_t}}\right),\underbrace{y_{grid,i}\cdot e^{y_{shock,\ell}}}_{\overset{\shortparallel}{y_{t+1}}}\right)\left(\underbrace{e^{b_{shock,k}}}_{\overset{\shortparallel}{R^b_{t+1}}}-r_f\right)\right\}\ .\end{aligned} \tag{87}$$

Because the curvature of the value function is more profound at low income/wealth levels, we adjust the grids for a and y so that they are more dense at low income/wealth levels. This strategy allows us to obtain efficient approximations even with 35 gridpoints for a_{grid} and y_{grid} in total (e.g., raising the number of gridpoints to 150 does not make an essential difference). Convergence in value function and/or coefficients $\left\{\left\{\xi_{i,k}\right\}_{i=0}^{\nu}\right\}_{k=0}^{\nu}$, is usually achieved in about 5 minutes for each model parameterization in Figure 6. Producing all graphs in Figure 6 takes about 30 minutes on a state-of-the art laptop.

2.6 Ensuring that consumption is above subsistence and treatment of borrowing constraints

The utility function we use satisfies an Inada condition as $c \to \chi$, which is obvious from (58). The RHS of (61) has the interpretation that, as long as V^* is well-defined, $c > \chi$ is guaranteed. In order to implement a borrowing constraint of the form $a_{t+1} \geq b$ we modify (63) as,

$$V^{(j+1)}\left(a_t, y_t\right) =$$

$$\max_{\left(c_t, \phi_{t+1}\right)} \frac{\left\{(1-\beta)\left(c_t-\chi\right)^{(1-\gamma)\theta}+\beta\left\{(1-\gamma) E_t\left[V^{(j)}\left(\max\left\{R_{p,t+1} a_t+y_t-c_t, b\right\}\Big|_{a_{t+1} \geq b}\ ,\ y_{t+1}\right)\right]\right\}^{\theta}\right\}^{\frac{1}{\theta}}}{1-\gamma}. \tag{88}$$

using an indicator function in order to implement the conditionality operator $\left.(\cdot)\right|_{a_{t+1} \geq b}$. As in our continuous-time analysis, the presence of the borrowing constraint has not affected our results. For our borrowing constraint $b \geq \underline{a}$, at this level of wealth ($\underline{a}$), and for all gridpoints for y, households chose interior solutions.

3. Calculating the correlation coefficient between risky-asset returns and labor-income growth

3.1 Labor-income dynamics: PSID 1970-2009

We use data from the Panel Study of Income Dynamics (PSID) between 1970 - 2009 in order to estimate the labor-income growth component that cannot be explained by household-demographic characteristics such as age, marital status, household composition, and some other (perhaps unobservable) family characteristics, such as cultural background, peer effects, etc. This labor-income growth component is our data proxy for variable y_{shock}, as defined by (47) and (72). The main estimation procedure follows Cocco, Gomes and Maenhout (2005). Cocco, Gomes and Maenhout (2005) restrict their sample to households headed by males. Unlike them, we keep households with both males and females as a household head, since we focus on explaining stockholding data from the Survey of Consumer Finances (SCF), in which we have not distinguished the gender of household heads. To single out the retirement behavior, which is abstract away from our model, we keep a subsample by eliminating retirees, nonrespondents and students.

Our definition of labor income is relatively inclusive in terms of fiscal transfers and government benefits, in order to focus on the pure absence of self-insuring potential against labor-income risk. We define labor income as total reported labor income plus unemployment compensation, workers compensation, social security, supplemental social security, other welfare, child support, and total transfers (mainly help from relatives). These calculations have been made for both the head of household and if a spouse is present we drop zero-income observations. We also deflate labor income using the Consumer Price Index, with 1992 as the base year.

We regress the logarithm of labor income on dummies for age, family fixed effects, marital status, and household composition. Using fixed-effect estimation, the econometric-model specification is,

$$y_{i,t} = \alpha + \mu_i + \mathbf{X}_{i,t}\boldsymbol{\beta} + \varepsilon_{i,t} \ , \quad y_{i,t} \equiv \ln(Y_{i,t}) \ , \tag{89}$$

in which $\mathbf{X}_{i,t}$ is the set of control variables. In order to explore the error structure further, we generate the residual from the above fitted model (89),

$$\widehat{\varepsilon_{i,t}} = y_{i,t} - \widehat{y_{i,t}} \ . \tag{90}$$

Combining (89) and (90), we formulate the cross-sectional mean of the unexplained part of the labor-income growth rate $\overline{\Delta\widehat{y_t}}$, as

$$y_{shock} \equiv \overline{\Delta\widehat{y_t}} = \frac{\sum\limits_{i=1}^{N} \Delta\widehat{y_{i,t}}}{N} = \frac{\sum\limits_{i=1}^{N} \widehat{\varepsilon_{i,t}} - \sum\limits_{i=1}^{N} \widehat{\varepsilon_{i,t-1}}}{N} \ , \tag{91}$$

which y_{shock} is the labor-income-shock concept that we use in the theoretical model.

3.2 Risky-asset returns

For generating the time series of risky-asset returns, we use the Standard and Poor's (S&P) stock-market index from 1970 to 2009, and calculate S&P-index returns as annual averages. The formula of the variable proxying s_{shock} in our theoretical model is,

$$s_{shock} \equiv \frac{\text{S\&P Index}_t}{\text{S\&P Index}_{t-1}} - 1 \ . \tag{92}$$

3.3 Correlation coefficient between risky-asset returns and labor-income growth

Table A.1 gives the correlation coefficient between y_{shock} and s_{shock}.

	Full Sample	**College Graduates**
Sample Period	1970 - 2009	1970 - 2009
$corr(y_{shock}, s_{shock})$	31.89%	50.78%

Table A.1

For the full sample, the correlation coefficient is about 32%. Because stockholders tend to have higher educational level, we also focus on college graduates by restricting the PSID-sample to college graduates, finding a higher number, which is about 51%.

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Online Data Appendix A: European Household Finance and Consumption Survey 2013

1. Data

The Eurosystem Household Finance and Consumption Survey (HFCS) is a joint project of all central banks of the Eurosystem. HFCS includes detailed household-level data on various aspects of household balance sheets and related economic and demographic variables, including income, pensions, employment, gifts and measures of consumption.

HFCS provides country-representative data, which have been collected in 15 euro area members for a sample of more than 62,000 households. These 15 countries are Belgium, Germany, Greece, Spain, France, Italy, Cyprus, Luxembourg, Malta, Netherlands, Austria, Portugal, Slovenia, Slovakia, and Finland.

For each country we consider only household heads between age 25 and 65 years old, which retained 42,553 households from the original sample. In addition, we also dropped households with zero income (215 observations).

The HFCS survey uses a multiple stochastic imputation strategy to recover the missing value or the non-responding households. It provides five imputed values (replicates) for every missing value corresponding to a variable.[1] We calculate the multiple imputed mean and standard deviation of our targeted variables (gross income and portfolio share on stocks) in

[1] A detailed description of the imputation procedure applied in the HFCS is given in chapter 6 of the Eurosystem Household Finance and Consumption Survey methodological report for the first wave. (https://www.ecb.europa.eu/pub/pdf/other)

Table 1. In Table 2, we calculate the mean of portfolio share on stocks for all Eurosystem countries, classifying by income category across the income distribution.

2. Definition of Variables

1. Stock Equity (direct and indirect stockholding excluding any pension accounts.)

    - Publicly Traded Stocks.
    - Mutual Funds: it includes funds predominately in equity, bonds, money market instruments, real estate, hedge funds and other fund types. The share of stock holding is adjusted conditional on fund types.[2]

2. Total Financial Assets: it includes deposits (sight accounts, saving accounts), investments in mutual funds, bonds, investments held in non-self-employment private businesses, publicly traded shares, managed investment accounts, money owed to households as private loans, other financial assets (options, futures, index certificates, precious metals, oil and gas leases, future proceeds from a lawsuit or estate that is being settled, royalties or any other), private pension plans and whole life insurance policies. However, current value of public and occupational pension plans is not included.

3. Total Income: it is measured as gross income and is defined as the sum of labor and non-labor income for all household members. Labor income is collected for all household members aged 16 and older, other income sources are collected at the household level. In some countries, as gross income is not well known by respondents it is computed from the net income given by the respondent. Specifically, the measure for gross income includes the following components: employee income, self-employment income,

[2] Note: stockholding from any public and occupational pension plans or individual retirement accounts are not included in our calculation.

income from pensions, regular social transfers, regular private transfers, income from real estate property (income received from renting a property or land after deducting costs such as mortgage interest repayments, minor repairs, maintenance, insurance and other charges), income from financial investments (interest and dividends received from publicly traded companies and the amount of interest from assets such as bank accounts, certificates of deposit, bonds, publicly traded shares etc. received during the income reference period less expenses incurred), income from private business and partnerships and other non-specified sources of income.[3]

4. Weight: weights are assigned in order to normalize the sample to representative-sampling standards. [4]

5. Income Percentiles: they are generated from the variable "total income".

3. Portfolio Share of Stockholding

We define the portfolio share of stockholding for income group j of country k as,

$$\phi_j^k = \frac{\sum_{n=1}^{N_j^k} \frac{\text{Stock}_{i,j,k}}{\text{Total Financial Assets}_{i,j,k}}}{N_j^k}$$

where N_j^k is the amount of households within income group j of country k.

[3] See section 9.2.4 of the Methodological Report in details on the collection of income variables in various countries.

[4] All statistics in this document are calculated using the household weight provided. Within each country, the sum of estimation weights equals the total number of households in the country, so that the sum of weights in the whole dataset equals the total number of households in the 15 countries participating in the 1st wave of the survey.

REFERENCES

ECB (2013): "The Eurosystem Household Finance and Consumption Survey - Methodological Report for the First Wave," Eurosystem Household Finance and Consumption Network, Statistics Paper Series, NO I/April 2013.

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Table 1: Portfolio Share on Stocks by Income Percentile

Country Code	Income Percentiles	Gross Income (EUR)	Gross Income(s.e.)	Portfolio Share on Stocks	Portfolio Share on Stocks(s.e.)
AT	20	12,976.54	256	1.02%	0.0022
AT	40	24,967.37	404	1.08%	0.0028
AT	60	37,117.68	632	1.96%	0.0071
AT	80	53,357.09	1874	2.41%	0.0045
AT	100	109,326.63	15025	3.22%	0.0047
BE	20	9,281.07	54	2.06%	0.0003
BE	40	24,812.93	253	2.27%	0.0011
BE	60	40,320.70	354	2.96%	0.0020
BE	80	62,584.55	277	4.06%	0.0021
BE	100	138,496.66	3590	7.22%	0.0041
CY	20	12,187.64	152	5.55%	0.0081
CY	40	26,616.23	237	5.92%	0.0124
CY	60	37,687.29	372	6.93%	0.0077
CY	80	54,505.03	739	8.38%	0.0113
CY	100	115,287.60	3738	10.89%	0.0152
DE	20	11,943.90	226	0.53%	0.0013
DE	40	26,100.98	215	1.01%	0.0016
DE	60	39,488.18	192	1.81%	0.0012
DE	80	56,594.20	123	2.27%	0.0011
DE	100	115,300.16	632	4.02%	0.0011
ES	20	10,789.32	40	1.56%	0.0005
ES	40	20,483.62	25	2.14%	0.0011
ES	60	28,516.08	61	2.19%	0.0023
ES	80	39,324.07	176	2.55%	0.0011
ES	100	79,661.73	367	5.96%	0.0021
FI	20	15,486.37	0	2.63%	0.0000
FI	40	30,490.31	0	6.19%	0.0000
FI	60	44,594.16	0	6.47%	0.0000
FI	80	61,658.27	0	8.06%	0.0000
FI	100	105,894.09	0	16.87%	0.0000
FR	20	13,363.59	0	1.30%	0.0002
FR	40	23,818.85	0	2.45%	0.0003
FR	60	33,383.88	0	3.13%	0.0011
FR	80	45,058.05	0	4.24%	0.0004
FR	100	87,867.95	0	8.78%	0.0008
GR	20	9,343.90	53	0.62%	0.0009
GR	40	18,067.57	56	0.28%	0.0028
GR	60	25,897.98	68	0.84%	0.0043
GR	80	36,438.48	67	2.67%	0.0058
GR	100	68,567.06	758	1.75%	0.0040
IT	20	10,963.57	0	0.18%	0.0000
IT	40	21,951.38	0	0.99%	0.0000
IT	60	31,572.36	0	2.03%	0.0000
IT	80	44,861.28	0	1.29%	0.0000
IT	100	84,829.13	0	4.81%	0.0000
LU	20	23,090.13	330	1.50%	0.0071
LU	40	45,705.58	529	1.39%	0.0070
LU	60	68,371.82	227	2.36%	0.0036

LU	80	99,800.81	633	3.91%	0.0067
LU	100	210,510.78	1895	10.32%	0.0055
MT	20	7,749.43	70	2.81%	0.0040
MT	40	14,410.10	32	4.03%	0.0027
MT	60	21,843.21	96	3.55%	0.0029
MT	80	32,611.60	91	3.13%	0.0025
MT	100	55,681.82	329	6.91%	0.0012
NL	20	15,827.96	638	0.69%	0.0024
NL	40	32,431.63	698	0.72%	0.0061
NL	60	43,275.19	617	1.51%	0.0080
NL	80	57,456.25	604	1.21%	0.0040
NL	100	91,322.43	1316	1.35%	0.0008
PT	20	5,634.04	93	0.01%	0.0001
PT	40	11,801.17	66	0.28%	0.0003
PT	60	16,916.69	92	0.50%	0.0005
PT	80	24,892.49	142	1.03%	0.0004
PT	100	55,466.72	233	4.17%	0.0014
SI	20	2,976.53	121	6.20%	0.0009
SI	40	12,617.03	175	5.82%	0.0042
SI	60	22,103.94	146	5.93%	0.0063
SI	80	31,954.13	573	5.86%	0.0263
SI	100	60,898.29	957	7.27%	0.0187
SK	20	5,215.67	69	0.05%	0.0002
SK	40	9,139.51	77	0.05%	0.0003
SK	60	12,591.09	26	0.23%	0.0004
SK	80	16,646.84	56	0.25%	0.0009
SK	100	30,152.51	256	0.32%	0.0002

Note: s.e. stands for the multiple imputed standard errors.

Source: European Household Finance and Consumption Survey 2013

Table 2: Portfolio Share on Stocks by Income Percentile (EU mean)

Region	Income Percentiles	Portfolio Share on Stocks
Euro System Countries	20	1.779%
	40	2.308%
	60	2.828%
	80	3.422%
	100	6.258%

Source: European Household Finance and Consumption Survey 2013

Online Data Appendix B: Chinese Household Finance Survey – 1st Wave 2013

1. Data

The China Household Finance Survey (CHFS) is conducted by the survey and research center for China Household Finance, which is based at Southwestern University of Finance and Economics. This survey is the only nationally representative survey in China that has detailed information about household finance and assets, including housing, business assets, financial assets, and other household assets. In addition, the survey also provides information about income and expenditures, social and commercial insurance, and more.

We use the 1st survey that was conducted in summer 2011 with a sample size of 8,438 households and 29,500 individuals,which covers 21 provinces (including the autonomous regions) and 4 Municipalities (Beijing, Shanghai, Tianjin and Chongqing). This survey employs a stratified 3-stage probability proportion to size (PPS) random sample design, which is necessary to ensure that the survey is nationally representative[1] .

We consider only household heads between age 25 years old and 65 years old, which retained 6,952 households. In addition, we have dropped households with zero income (226 observations).

2. Definition of Variables

1. Stock Equity (direct and indirect stockholding excluding any pension account)

[1] Details about the sampling design could refer http://www.chfsdata.org/detail-14,15.html.

- Publicly Traded Stocks.
- Non Publicly Traded Stocks.
- Mutual Funds: it includes funds predominatly in equity, bonds, money market instruments, also includes mixed stratergy funds and other types.
- Financial Products (categorized as Wealth Management Products)

2. Total Financial Assets: it comprises total balance of demand deposits, total balance of time deposits, stocks (public traded and non-public traded), bonds, mutual funds, derivatives, warrants, other financial derivatives, financial products, foreign currency assets, gold, cash at home and other type of liquid assets.
3. Total Income: it includes income from all sources (salary, interest, dividend, compensations, transfers etc).
4. Weight: weights are assigned in order to normalize the sample to representative-sampling standards, the weight variable in the data is "swgt".
5. Income Percentiles: they are generated from variable "total income".

3. Portfolio Share of Stockholding

We define the portfolio share of stockholding for income group j as,

$$\phi_j = \frac{\sum_{n=1}^{N_j} \frac{\text{Stock}_{i,j}}{\text{Total Financial Assets}_{i,j}}}{N_j}$$

where N_j is the amount of households within income group j. Table 1 shows the detailed information of portfolio share on stocks across the income distribution, together with inforamtion on total asset holding and total financial-asset holding.

REFERENCES

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Gan, Li, Zhichao Yin, Nan Jia, Shu Xu, Shuang Ma, and Lu Zheng (2013): "Data you need to know about China: Research Report of China Household Finance Survey." 2014, XV, 172 p., Springer.

Table 1: Portfolio Share on Stocks by Income Percentile

Country Code	Income Percentiles	Portfolio Share on Stocks (%)	Gross Income (CNY)	Total Assets (CNY)	Total Financial Assets (CNY)
China	20	1.443	4,220.57	396,191.75	23,482.43
	40	1.273	16,367.13	275,096.63	16,263.76
	60	3.150	30,631.76	440,228.28	27,103.34
	80	5.414	52,124.18	594,769.38	37,879.68
	100	11.860	208,030.73	1,810,124.50	167,894.25

Source: Chinese Household Finance Survey 1st Wave 2013

Online Data Appendix C: US Survey of Consumer Finances (SCF) 2007

1 Description of Variables (Source: Survey of Consumer Finances (SCF) 2007)

1. `Stock Equity (Direct and Indirect Stockholding)`:

   (a) **Direct stockholding**
      - Publicly Traded Stocks.

   (b) **Stockholding through mutual funds**
      - Saving and Money Market Accounts.
      - Mutual Funds.
      - Annuities, Trusts and Managed Investment Accounts.

   (c) **Stockholding through Retirement Accounts**
      - IRA/KEOGH Accounts.
      - Past Pension Accounts.
      - Current Benefits and Future Benefits from Pensions.

2. `Business Equity`:
   - Actively Managed Business.
   - Non-Actively Managed Business.

3. `Total Assets`: Assets of all categories covered in the SCF 2007 database (stocks, business equity, bonds, saving and checking accounts, retirement accounts, life insurance, primary residence, and other residential real estate, nonresidential real estate, vehicles, artwork, jewelry, etc.).

4. `Total Income`: Income from all sources (salary, interest, dividend, compensations, transfers etc).

5. `Weight`: Weights are assigned in order to normalize the sample to representative-sampling standards (see the section "Analysis Weights" in the "Codebook for the 2007 Survey of Consumer Finances").[1]

6. `Income Percentiles`: Benchmark value from Bucks et al. (2009a, Table A.2, p. A53).

7. `Equivalence Scales`: The equivalence scale is $\sqrt{n}$ in which n is the number of household members. This equivalence-scale measure approximates the standard OECD equivalance scales.

Table 1: **Income Percentiles**

Percentile	Total Labor Income
20	20,600
40	36,500
60	59,600
80	98,200
90	140,900

Notes: Full sample in 2007 USD. Data in the survey is in 2006 USD, which is adjusted according to the CPI-U table (U.S. Department of Labor Bureau of Labor Statistics, Consumer Price Index). The 2006-2007 average to average change is 2.84%.

[1] The "Codebook for the 2007 Survey of Consumer Finances" is downloadable from `http://federalreserve.gov/econresdata/scf/scf_2007documentation.htm`

2 Matching Data with Descriptive Statistics in the SCF 2007 Chartbook

To show that our database is constructed in a reliable way, we compare key statistics with those reported in the SCF2007 Chartbook. Our robustness checks are:

- **Matching median values of key variables in the SCF 2007 chartbook**:
  The reason for choosing medians instead of means in order to perform a robustness check is that median values capture more information regarding a variable's distribution. In addition, mean values can be substantially affected by outliers. Indeed, our database matches median values in the SCF2007 chartbook.

- **Matching median values of each income group in the SCF 2007 chartbook**:
  Our database generated should match the income benchmark in small differences by income quintile or decile, which is a more demanding task. Our results are listed in the following tables demonstrate that the matching is satisfactory.

Table 2: **Median Values of Key Variables**

Variables	SCF2007 Chartbook	Our Data
Total Asset	221.5	221.9
Total Income	47.3	46.5
Stock Equity	35.0	34.8
Business Equity	100.5	80.6

Notes: Full sample. Values in thousands of 2007 US dollars.

Table 3: **Median Values of Pre-Tax Family Income for All Families, Classified by Income Percentile**

Income Percentile	SCF2007 Chartbook	Our Data
Less than 20%	12.3	12.3
20%-39.9%	28.8	28.8
40%-59.9%	47.3	47.1
60%-79.9%	75.1	74.9
80%-89.9%	114.0	114.8
90%-100%	206.9	209.0

Notes: Full sample, in thousands of 2007 US dollars. Data in the survey are in 2006 US dollars. We adjusted them according to the CPI-U table (U.S. Department of Labor Bureau of Labor Statistics, Consumer Price Index). 2006-2007 Average to Average change is 2.84% .

Table 4: **Median Values of Total Assets for Families with Positive Asset Holdings, Classified by Income Percentile**

Income Percentile	SCF2007 Chartbook	Our Data
Less than 20%	23.5	26.1
20%-39.9%	84.9	90.1
40%-59.9%	183.5	182.2
60%-79.9%	343.1	345.6
80%-89.9%	567.5	561.2
90%-100%	1358.4	1355.5

Notes: Full sample, in thousands of 2007 US dollars.

Table 5: **Median Values of Different Asset Types for Families with Positive Asset Holdings by Income Percentile**

	SCF2007 Chartbook		Our Data	
Income Percentile	**Stocks**	**Business**	**Stocks**	**Business**
Less than 20%	6.5	50.0	7.3	43.6
20%-39.9%	8.4	19.5	10.0	20.4
40%-59.9%	17.7	30.8	17.4	33.8
60%-79.9%	34.2	55.1	33.8	41.6
80%-89.9%	62.0	72.1	59.1	70.6
90%-100%	219.6	379.5	223.8	381.0

Notes: Full sample, in thousands of 2007 US dollars.

3 Portfolio Shares of Risky Assets

Portfolio shares of risky assets are calculated by income groups. For each income group we have the formula,

$$SHARE_i = \frac{\sum_k \frac{\sum_n SHARE_{obs(n)}}{N}}{K} \ ,$$

in which n is the observation number, k is the imputation number and i is the risky-asset type. Final results are shown in the following tables. SCF weights are not shown in the above formula but have been included in the calculation. The comparison between Tables 6 and 7 justifies why we did not restrict the full sample into a particular age range such as household heads aged between 25-59 years old. Demographic or life-cycle biases seem to play a rather mild role, so we have chosen to utilize the entirety of the infomation provided by the SCF 2007 database in our calibration exercises.

Table 6: **Portfolio Share on Risky Assets by Income Percentile (Full Sample per Equivalent Adult)**

	Risky Assets (%)		General Information			Tax Information	
Income Percentile	**Stocks**	**Business**	**Total Income**	**Total Assets**	**Income/Asset (%)**	**Effective Marginal Tax Rate**	**After-tax Income**
Generated Data							
Less than 20%	2.44	3.24	9.03	85.52	10.56	-1.83%	9.19
20%-39.9%	5.84	1.84	19.42	139.82	13.89	2.78%	18.88
40%-59.9%	7.72	3.97	32.20	210.93	15.27	6.47%	30.11
60%-79.9%	12.44	4.51	49.84	327.22	15.23	14.28%	42.72
80%-89.9%	15.96	6.14	74.61	511.32	14.60	22.63%	57.73
90%-100%	20.53	24.55	252.12	2452.22	10.28	29.27%	178.33

Notes: Full sample, in thousands of 2007 US dollars.

Table 7: **Portfolio Share on Risky Assets by Income Percentile (Age group 25-59 per Equivalent Adult)**

	Risky Assets (%)		General Information			Tax Information	
Income Percentile	**Stocks**	**Business**	**Total Income**	**Total Assets**	**Income/Asset (%)**	**Effective Marginal Tax Rate**	**After-tax Income**
Generated Data							
Less than 20%	2.78	3.35	10.15	73.74	13.76	-1.83%	10.33
20%-39.9%	5.05	3.35	22.93	99.79	23.00	2.78%	22.30
40%-59.9%	8.32	3.73	37.60	188.48	19.95	6.47%	35.17
60%-79.9%	12.90	4.74	54.49	303.89	17.93	14.28%	46.71
80%-89.9%	14.89	7.26	78.69	475.10	16.57	22.63%	60.88
90%-100%	18.52	25.31	242.57	1908.87	12.71	29.27%	171.57

Notes: Age group 25-59, in thousands of 2007 US dollars.

References

[1] Bucks, Brian K., Arthur B. Kennickell, Traci L. Mach and Kevin B. Moore (2009a): "Changes in U.S. Family Finances from 2004 to 2007: Evidence from the Survey of Consumer Finances," Survey of Consumer Finances, Board of Governors of the Federal Reserve System.

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