Do coalitions matter in designing institutions?*

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Abstract

In this paper, we re-examine the classical questions of implementation theory under complete information in a setting where coalitions are the fundamental behavioral units and the outcomes of their interactions are predicted by applying the solution concept of the core. The planner's exercise consists of designing a code of rights, which specifies the collection of coalitions that have the right to block one outcome by moving to another. A code of individual rights is a code of rights in which only unit coalitions may have blocking powers. We provide necessary and sufficient conditions for implementation (under core equilibria) by codes of rights as well as by codes of individual rights. We show that these two modes of implementation are not equivalent. This result is proven robust and extends to alternative notions of core, such as that of an externally stable core. Therefore, coalitions are shown to bring value added to institutional design. The characterization results address the limitations that restrict the relevance of existing implementation theory.

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1 Introduction

The challenge of implementation lies in designing a mechanism (i.e., game form) where the equilibrium behavior of agents always coincides with the recommendations given by a *social choice rule* (SCR). If such a mechanism exists, the SCR is said to be implementable.

As such, the key question is how to design an implementing mechanism such that its outcomes are predicted through the application of game theoretic solution concepts. Most early studies on implementation focused on noncooperative solution concepts, such as the Nash equilibrium and its refinements. However, one of the difficulties with this approach is that the canonical mechanisms are typically complex and difficult to explain in natural terms, as they rely on tail-chasing constructions, such as the integer game.

As demonstrated in the seminal paper by Koray and Yildiz (2018) [henceforth KY], an alternative to the noncooperative approach is to allow groups of agents to coordinate their behaviors in a mutually beneficial way. To move away from noncooperative modeling, the details of coalition formation are left unmodeled. Then, coalitions—not individuals—become basic decision making units. Here, the role of the solution concept is to explain why, when, and which coalition forms and what it can achieve.

More importantly, the chosen coalitional solution concept is independent of the physical structure under which coalition formation takes place (see, e.g., Chwe, 1994). This structure, often defined by an effectivity relationship, specifies which coalitions are permitted to form given the status quo outcome and what they can achieve when they form, that is, what new status quo outcomes they can induce. From the implementation viewpoint, the effectivity relationship is the design variable of the planner and plays the role of mechanism.

KY formalize this idea and study its implications. In their framework, SCR implementation is achieved by designing a generalization of the effectivity relationship, introduced by Sertel (2001), called a *rights structure*. A rights structure Γ consists of a state space S, an outcome function h that associates every state to an outcome, and a code of rights γ. A code of rights specifies, for each pair of distinct states (s, t), a collection of coalitions $\gamma(s, t)$ effective in moving from s to t. The rights structure is more flexible than the effectivity function, as it allows strategic options of coalitions to depend on how the status quo outcome is reached (i.e., on the current state).

As coalitional solution, KY adopt a version of the *core*, referred to as the $\Gamma-$*equilibrium*. A state s is an equilibrium state under a given rights structure and agents' preferences if no effective coalition can guarantee each of its members a utility level higher than the one they receive under s. Then, the implementation problem consists of designing a rights structure Γ, with the property that, for each profile of agents' preferences, its equilibrium outcomes under those preferences coincide with the outcomes a given SCR would select for that profile. If such a rights structure exists, the SCR is said to be *implementable by a rights structure*.

The implications of KY's approach are interesting. Any SCR that is implementable by a rights structure can also be implemented by a rights structure in which *only unit coalitions have blocking powers*, that is, an individual-based rights structure, and vice versa. A counterintuitive implication of this result is that coalition formation does not bring any value added to the implementation by the rights structure. As such, for all purposes, it is sufficient to focus on unit coalitions alone. The question is then why should institutions be designed based on coalition formation, as is often the case? For example, under a typical democratic constitution, a bill can only be passed by consent from a majority of individuals.

Consequently, the scope of this paper is analyzing the insights of KY's study. Specifically, we generalize their characterization results, and our characterizations, which are complete in each case, are easy to verify and provide comparative information on the restrictiveness of different implementation modes. More importantly, our conditions allow determining when and why coalitions matter under implementation by right structures.

We first provide a general characterization of the SCRs that are implementable by rights structures. We show that *(Maskin) monotonicity* and *unanimity*, both being restricted to a superset of the SCR image, are necessary and sufficient conditions for implementation.[1] More importantly, this result does not make any domain assumptions and, hence, the result generalizes KY's characterization, which assumes a full domain of preferences.

To answer the above question, we focus on the simple and natural restriction of rights structures already introduced by KY. We take the set of outcomes as our state space, that is, we assume that the implementation device is an *effectivity relationship* or, using the terminology of KY, a *code of rights.* Technically, we assume that outcome function h is the identity map and the implementation exercise by rights structures is reduced to the design of a code of rights γ, which specifies, for each pair of outcomes x and y, a collection of effective coalitions $\gamma(x, y)$ that can induce y from x. Under this framework, the equilibrium notion of KY is reduced to the familiar notion of core: an outcome x is an equilibrium under a given code of rights and a given profile of preferences if there is no coalition K that will find it beneficial to reject x and induce an outcome that renders all members of K better off. Therefore, the implementation problem consists of designing a code of rights γ with the property that, for each profile of agents' preferences, its equilibrium outcomes under those preferences coincide with the outcomes a given SCR would select for that profile. If such a code of rights exists, the SCR is said to be *implementable by a code of rights.*[2]

[1] Maskin's monotonicity condition is known in the social choice literature under the name strong positive association (e.g., Muller and Satterthwaite, 1977).

[2] Our approach is different from Peleg and Winter's (2002), who use the notion of effectivity by Moulin and Peleg (1982) to appeal to the notion of implementation, where the game form not only implements an SCR under Nash equilibrium but also induces the same distribution of power as that of the implemented SCR. See also Peleg et al. (2005). Moreover, Andjiga and Moulen (1988, 1989)'s analysis is a special case of ours, because in their model there is a simple game, rather than a code of rights, that specifies coalitions that have the right to block one outcome by moving to another.

Note that a code of rights captures the allocation of blocking powers in many real-life situations in a natural way. In most democratic legislatures, bills are proposed and amended under a process of deliberation until the final version is adopted or rejected by a majority of individuals (see, e.g., Ray and Vohra, 2014). This notion has some theoretical advantages, as the power of a coalition in a code of rights does not depend on *how* the state is reached but only what its physical implications are, that is, what is the status quo outcome to become implemented. Arguably, this type of distribution for coalitional power is rather in line with the standard approach of coalition formation, which abstracts from the details of the coalition formation process.

We demonstrate that the implementation by a code of rights is fundamentally dependent on nonsingleton coalitions. To this end, we identify two necessary conditions for implementability: one is the unanimity condition and the other we call *strong monotonicity*, being stronger than (Maskin) monotonicity (see Section 5). These two conditions are again different from the characterization of KY, which is only applicable under a full domain of preferences. We prove that the two conditions are also sufficient for implementation by codes of rights. More importantly, this full characterization result is without restrictions over the preference domain.

We also study an implementation by codes of rights under which only unit coalitions can induce new outcomes. We call this type of codes of rights *codes of individual rights*. Under this setting, we identify the necessary condition for implementability, which we call *singleton strong monotonicity*. This condition is also sufficient when combined with unanimity.

Singleton strong monotonicity implies strong monotonicity but, as we will demonstrate, they are *not* equivalent. Therefore, the key insight for the implementation by rights structures—that coalitions do not matter—does *not* extend to the implementation by codes of rights. The underlying reason for this observation is that implementation by an individual right structure requires a significant amount of information concerning the preferences of the agent permitted to move at a particular state. When the state space is coarsened, the needed information may no longer be conveyed by the underlying state. Since the preferences of a coalition are less volatile than those of an individual agent, coalitions may no longer be usefully replaceable by individuals. An example is the *majority solution*, which is implementable by a majority coalition-based code of rights but not by an individual-based one.

This conclusion is robust and can be extended to the implementation by codes of rights for alternative core definitions. Indeed, we add to the notion of core the requirement that blocking must be achievable through outcomes that are themselves in the core, meaning we also consider implementation by codes of rights of what is often referred to as an *externally stable core*. We call this type of implementation *externally stable implementation by codes of rights*. This externally stable implementation is a robust way of implementing outcomes, being more reliable than the implementation of core outcomes, since external stability guarantees that no outcome outside the core can be sustained. Moreover, an externally stable core is also more robust than the Von Neumann–Morgenstern (vNM) stable set or its derivatives, since it avoids indirect internal stability problems (i.e., the *Harsanyi critique*; Harsanyi, 1974).

We provide a full characterization of this type of implementation by showing that *strong winner monotonicity*—a strengthening of the strong monotonicity—unanimity, and the *no-simultaneous domination of* F are necessary and sufficient conditions for implementation. Again, these conditions are different from those introduced by KY in the context of externally stable implementation by a rights structure, which are only applicable under a full domain assumption. Instead, our characterization does not make any domain assumptions. More importantly, we show that an externally stable implementation by codes of rights is *not* equivalent to an externally stable implementation by codes of individual rights, thus providing further motivation to examine why coalitions should matter in implementation. A counterexample is the majority solution.

2 On monotonicity conditions and tail-chasing constructions

The solution concept of this paper—the core—can be viewed as a generalization of the Nash equilibrium. Indeed, as demonstrated by KY, every normal form game can be formulated as a code of rights and its Nash equilibrium as the core associated with the game. The same applies to *strong Nash implementation*, which also accounts for coalitional deviations (i.e., monotonicity conditions for implementation in strong Nash equilibrium are more stringent than those for Nash implementation; see Maskin, 1981; Korpela, 2013). Therefore, any (strongly) Nash implementable SCR is also implementable via a code of rights. Still, the implication does not extend in the other direction.

However, there are striking differences between the two modes of implementation. At the heart of the various characterization results concerning Nash implementation are monotonicity conditions, which compare the alternatives chosen by a given SCR under pairs of preference profiles. Generally, this is an easy task and makes monotonicity an attractive condition to work with. Surprisingly, as shown by Maskin (1999, working paper 1977), this type of monotonicity is also almost sufficient for implementation with at least three agents, and monotonicity plus a condition of no veto power is sufficient. However, this result, as well as the full characterization of Moore and Repullo (1990), relies on the construction of mechanisms with unnatural features. On one hand, their strategy spaces are complex and difficult to interpret in natural terms and, on the other, the construction relies on tail-chasing procedures to eliminate unwanted equilibria (see Jackson, 1992 and Abreu and Matsushima, 1992).[3]

[3] One way to overcome these limitations is focusing on the implementation of specific SCRs, such as the Walrasian and Lindahl correspondences (Hurwicz, 1979; Walker, 1981; Tian, 1989; Corchón and Wilkie, 1996), as well as setting several desirable restrictions on mechanisms and identifying the class of SCRs that are implementable by the mechanisms satisfying those restrictions (Jackson, 1992; Dutta et al., 1995; Saijo et al., 1996; Thomson, 2005).

The mechanisms used in this paper or those of KY do not rely on any tail-chasing constructions. As the monotonicity condition that characterizes SCRs implementable by rights structures is just a version of (Maskin) monotonicity, it is worth asking how our canonical mechanism can eliminate unwanted equilibria. The reason lies within the flexibility of the rights structure. Unlike normal form games, where the strategy space associated with a player is independent of the strategies chosen by other players, a rights structure permits careful tailoring of the blocking coalitions for each state. Therefore, in normal form games, the formation of a new equilibrium after a deviation from untruthful reporting must be blocked by triggering a tail-chasing construction; in the context of rights structure, this can be achieved by simply finding a blocking coalition for the target state of the deviation. Thus, the additional degree of freedom in the design of the rights structure can be used to block unwanted equilibria ex post, that is, once the move has taken place. This implicit dynamic is a powerful tool and also leads to a simpler characterization of implementable SCRs than comparable ones in the context of Nash implementation (Moore and Repullo, 1990; Lombardi and Yoshihara, 2013).

However, new questions on the flexibility of rights structures and the embedded dynamics emerge. Blocking by a coalition under an untruthful equilibrium will not lead to a new one, since it is further blocked by another coalition. Hence, there is no need to worry whether the blocking would lead to a new equilibrium state. However, the question is where does the blocking process lead to? Further, can we be sure that the end state of the process creates an effective deterrence for the deviating coalition? These questions, which form the scope of an expanding stream of literature on farsighted coalition formation, do not have a clear answer in the current context.[4] As pointed out by KY, implementation via rights structures is a myopic concept.

However, it is noteworthy that strengthening the solution concept makes the problem of farsightedness less pronounced. This can be clearly seen with reference to

[4] See, e.g., Chwe (1994), Vartiainen (2011), Vohra and Ray (2017), and Dutta and Vohra (2017).

implementation under externally stable solution concepts by a code of rights, under which the mode of implementation is characterized by strong winner monotonicity and unanimity. Indeed, the mechanism associated with this characterization result allows deviations from any untruthful, non-desired equilibria so that they lead to a desired equilibrium in one step. Hence, the solution is consistent with all behaviors being farsighted. This is important because it implies that strong winner monotonicity (along with unanimity) characterizes a mode of implementation free both from tail-chasing procedures and myopicity. Furthermore, as the devised mechanism associated with the characterization of the class of SCRs that are externally stable implementable by a code of rights is the same as that constructed for implementation by a code of rights, the results also show that SCRs satisfying strong monotonicity and unanimity can be implemented in a way that they are free from both criticisms.[5]

One benefit of our characterizations is that the monotonicity conditions associated with the modes of implementation are progressively more stringent—and directly comparable—when we move towards more demanding solution concepts. Moreover, they make the comparisons between coalition- and individual-based solutions transparent.

The remainder of this paper is divided into five sections. Section 3 sets out the theoretical framework and outlines the basic model. Section 4 revisits the implementation by rights structures. Section 5 provides a novel characterization of the class of SCRs implementable via codes of rights, whereas Section 6 fully characterizes the class of SCRs that are externally stable implementable by codes of rights. Section 7 concludes the paper.

[5]KY also show that their canonical mechanism is externally stable in the context of implementation by rights structure.

3 Preliminaries

We consider a finite (nonempty) set of *agents*, denoted by $N = \{1, \cdots, n\}$, and a (nonempty) set of *outcomes*, denoted by Z. For every set A, the power set of A is denoted by $\mathcal{A}$ and $\mathcal{A}_0 \equiv \mathcal{A} - \{\varnothing\}$ is the set of all nonempty subsets of A. Each element K of $\mathcal{N}_0$ is called a *coalition*. A *preference ordering* R_i is a complete and transitive binary relation over Z. Each agent $i (\in N)$ has a preference ordering R_i over Z. The *asymmetric* part P_i of R_i is defined by xP_iy if and only if xR_iy and not yR_ix, while the *symmetric* part I_i of R_i is defined by xI_iy if and only if xR_iy and yR_ix. A *preference profile* is thus an n-tuple of preference orderings $R \equiv (R_i)_{i \in N}$. The *preference domain*, denoted by $\mathcal{R}$, consists of the set of admissible preference profiles.

For R and K, we write xR_Ky for xR_iy for all $i \in K$ and xP_Ky for xP_iy for all $i \in K$.

The goal of the designer is to implement an SCR F, defined by $F : \mathcal{R} \to \mathcal{Z}_0$. We refer to $x \in F(R)$ as an F-optimal outcome at R. The *range of* F is the set

$$F(\mathcal{R}) \equiv \{x \in Z | x \in F(R) \text{ for some } R \in \mathcal{R}\}.$$

Following KY, to implement F, the designer designs a *rights structure* Γ, which is a triplet (S, h, γ), where S is the *state space*, $h : S \to Z$ the *outcome function*, and γ a *code of rights*, which is a (possibly empty) correspondence $\gamma : S \times S \twoheadrightarrow \mathcal{N}$. Subsequently, a code of rights specifies, for each pair of distinct states (s, t), a family of coalitions $\gamma(s, t)$ entitled to approve a change from state s to t. A rights structure Γ is said to be an *individual-based rights structure* if, for each pair of distinct states (s, t), $\gamma(s, t)$ contains only unit coalitions if it is nonempty.

For any rights structure Γ and any preference profile R, a state $s \in S$ is an *equilibrium* at R if, for no $t \in S - \{s\}$, so that $h(s) \neq h(t)$ and no $K \in \gamma(s, t)$ is $h(t) P_K h(s)$. We express $C(\Gamma, R)$ for the set of Γ-equilibria at R.

Definition 1 *A rights structure* Γ implements F *if and only if* $F(R) = h \circ C(\Gamma, R)$

for all $R \in \mathcal{R}$*. If such a rights structure exists,* F *is* implementable by a rights structure*.*

Definition 2 F *is* implementable by an individual-based rights structure *if there exists an individual-based rights structure* Γ *so that* Γ *implements* F*.*

KY show that the following monotonicity condition is necessary and sufficient for an implementation by rights structures. We formalize the condition as follows. For any preference ordering R_i and outcome x, the *lower contour set of* R_i *at* x is defined by $L(x, R_i) = \{x' \in Z | x R_i x'\}$. Therefore,

Definition 3 (Koray and Yildiz, 2018) F *satisfies the condition of* image monotonicity *provided that, for all* $x \in Z$ *and all* $R, R' \in \mathcal{R}$*, if* $x \in F(R)$*,*

$$L(x, R_i) \bigcap F(\mathcal{R}) \subseteq L(x, R'_i) \text{ for all } i \in N,$$

then $x \in F(R')$*.*

A linear ordering of agent i, denoted by P_i, is a complete, transitive, and antisymmetric binary relation over Z. We denote by $\mathcal{P}_Z$ the collection of all profiles of linear orderings, that is, the *unrestricted domain of linear orderings.* Koray and Yildiz's (2018) equivalence result can be stated as follows.

Theorem 1 (Koray and Yildiz, 2018, p. 488) *Let* $F : \mathcal{P}_Z \to \mathcal{Z}_0$ *be any SCR. Then, the following statements are equivalent:*

(i) F *is* implementable by a rights structure*;*

(ii) F *satisfies the condition of* image monotonicity*;*

(iii) F *is* implementable by an individual-based rights structure*.*

4 Implementation by rights structures—Full characterization

We first revisit implementation by rights structures. To this end, we propose two conditions that are together necessary and sufficient for an SCR to be implemented by a rights structure. The first condition is the well-known (Maskin) *monotonicity* (Maskin, 1999). This condition states that, if an outcome x is F-optimal at the profile R and x does not strictly fall in the preference for anyone when the profile is changed to R', then x must remain an F-optimal outcome at R'. For subsequently presented reasons, we define below a stronger variant of the standard monotonicity condition.

Definition 4 *F is* (Maskin) monotonic w.r.t. *$W \subseteq Z$ provided that, for all $x \in W$ and all $R, R' \in \mathcal{R}$, if $x \in F(R)$ and*

$$L(x, R_i) \bigcap W \subseteq L(x, R'_i) \bigcap W \text{ for all } i \in N,$$

then $x \in F(R')$.

It can be easily verified that image monotonicity is equivalent to monotonicity w.r.t. the range of F.

The second condition is a variant of the familiar unanimity condition, which states that, if an outcome is at the top of the preferences of all agents, that outcome should be selected by the SCR. This can be defined as follows.

Definition 5 *F satisfies* unanimity w.r.t. *$Y (\subseteq Z)$ provided that $F(\mathcal{R}) \subseteq Y$ and that, for all $x \in Y$ and all $R \in \mathcal{R}$, if $Y \subseteq L(x, R_i)$ for all $i \in N$, then $x \in F(R)$.*

The theorem below generalizes Theorem 1 by relaxing the preference domain assumption. Indeed, our first main result is that only SCRs that are monotonic w.r.t. Y, as well as unanimous w.r.t. Y, are implementable by rights structures. Moreover, what can be implemented by a rights structure can also be implemented by an individual-based rights structure, and vice versa.

Theorem 2 *Take any F. The following statements are equivalent:*

(i) *F is* implementable by a rights structure*;*

(ii) *F satisfies* unanimity w.r.t. Y and monotonicity w.r.t. Y*;*

(iii) *F is* implementable by an individual-based rights structure*.*

Proof.

(i) $\Longrightarrow$ ***(ii)***

Assume that Γ implements F. First, we show that F satisfies unanimity w.r.t. Y. We define Y as $Y \equiv \{x \in Z | x = h(s) \text{ for some } s \in S\}$. From the definition of Y, $F(\mathcal{R}) \subseteq Y$. Take any $x \in Y$ and any $R \in \mathcal{R}$. Then, $x = h(s)$ for some $s \in S$. Suppose that $Y \subseteq L(x, R_i)$ for all $i \in N$. Clearly, for no $t \in S - \{s\}$ so that $h(t) \neq x$ and no $K \in \gamma(s,t)$ is $h(t) P_K h(s)$, and thus $x \in F(R)$, by implementability.

Next, we show that F is monotonic w.r.t. the above set Y. To this end, take any $R \in \mathcal{R}$ and $x \in Z$ so that $x \in F(R) = h \circ C(\Gamma, R)$. Then, there exists $s \in S$ so that $h(s) = x$. Moreover, for no $t \in S - \{s\}$ so that $h(t) \neq x$ and no $K \in \gamma(s,t)$ is $h(t) P_K h(s)$. Take any $R' \in \mathcal{R}$. Suppose that $L(x, R_i) \cap Y \subseteq L(x, R_i') \cap Y$ for all $i \in N$. We then show that $x \in F(R') = h \circ C(\Gamma, R')$. Assume, to the contrary, that $x \notin F(R')$. Then, there exists $t \in S - \{s\}$ so that $h(t) \neq x$ and $K \in \gamma(s,t)$ so that $h(t) P_K' h(s)$. Since $L(x, R_i) \cap Y \subseteq L(x, R_i') \cap Y$ for all $i \in K$ and $h(t) P_K' h(s)$, it follows that $h(t) P_K h(s)$, which is a contradiction. Therefore, F is monotonic w.r.t. Y.

(ii) $\Longrightarrow$ ***(iii)***

Assume F is monotonic and satisfies unanimity w.r.t. Y. We define an individual-based rights structure $\Gamma = (S, h, \gamma)$ as follows. First, we define the set T as $T \equiv \{(x, R') \in Y \times \mathcal{R} | x \in F(R')\}$. Second, we define the state space S by $S = T \cup Y$.

We define the outcome function $h : S \to Z$ as follows: $h(x, R) = x$ for all $(x, R) \in T$ and $h(x) = x$ for all $x \in Y$. Finally, we define $\gamma : S \times S \twoheadrightarrow \mathcal{N}$ as follows. For all $i \in N$,

(a) For all $((x, R), (y, R')) \in T \times T$,

$$\{i\} \in \gamma((x, R), (y, R')) \iff xR_iy;$$

(b) For all $((x, R), y) \in T \times Y$, $\{i\} \in \gamma((x, R), y) \iff xR_iy$;

(c) For all $(x, (y, R')) \in Y \times T$, $\{i\} \in \gamma(x, (y, R'))$;

(d) For all $(x, y) \in Y \times Y$, $\{i\} \in \gamma(x, y)$.

We show that Γ implements F. To this end, fix any R.

Taking any $x \in F(R)$, we show that $x \in h \circ C(\Gamma, R)$. Since $x \in F(R)$, it follows that $(x, R) \in T$. We fix any $i \in N$ and any (y, R') or any $y \in S$. From part (a) of the definition of γ, we have $\{i\} \in \gamma((x, R), (y, R'))$ if xR_iy; otherwise, $\{i\} \notin \gamma((x, R), (y, R'))$. Moreover, from part (b) of the definition of γ, we have $\{i\} \in \gamma((x, R), y)$ if xR_iy; otherwise, $\{i\} \notin \gamma((x, R), y)$. Parts (c) and (d) of the definition γ never apply. Clearly, agent i is either not entitled to approve a change from state (x, R) to state (y, R') or state y, or does not have any incentives to do so. Since the choice of i, as well that of states (y, R') and y, is arbitrary, we have $x \in h \circ C(\Gamma, R)$.

Conversely, take any $s \in C(\Gamma, R)$. We show that $h(s) \in F(R)$. There are two possible cases as follows.

Let $s = x \in Y$ and fix any i. Since i is entitled to approve the change from state x to any $y \in Y$ and since $x \in C(\Gamma, R)$, it must be that xR_iy. Since the choice of $y \in Y$ is arbitrary, $Y \subseteq L(x, R_i)$. Since the choice of i is arbitrary, it follows that $Y \subseteq L(x, R_i)$ for all $i \in N$. Since F satisfies unanimity w.r.t. Y, $x \in F(R)$.

Let $s = (x, R') \in T$. It is obvious that only parts (a) and (b) of the definition of γ can apply. Assume, to the contrary, that $x \notin F(R)$. Monotonicity implies that there exist i and $y \in L(x, R'_i)$ so that yP_ix. Since $y \in L(x, R'_i)$, it follows from part (b) of the definition of γ that $\{i\} \in \gamma((x, R'), y)$. However, since yP_ix and $\{i\} \in \gamma((x, R'), y)$, we established that $(x, R') \notin C(\Gamma, R)$, which is a contradiction.

(iii) $\Longrightarrow$ ***(i)***

Clearly, F is implementable by a rights structure if it is also implementable by an individual-based rights structure. ■

Remark 1 When the preference domain is $\mathcal{P}_Z$, as in KY, Theorem 2 implies that the range of any implementable F coincides with the set of outcomes Y because F satisfies unanimity w.r.t. Y. Given that image monotonicity is equivalent to monotonicity w.r.t. $Y = F(\mathcal{P}_Z)$, it follows that Theorem 2 is equivalent to Theorem 1 when the preference domain is $\mathcal{P}_Z$.

5 Implementation via codes of rights

A natural candidate for the state space of a rights structure Γ is the set of outcomes Z. Arguably, this captures the most natural way of allocating blocking powers in real-life situations (see, e.g., Ray and Vohra, 2014). Therefore, by assuming that the outcome function h, defined over Z, is the identity map, the implementation exercise is reduced to the design of the *code of rights* $\gamma : Z \times Z \twoheadrightarrow \mathcal{N}$. For each pair of outcomes x and y, γ specifies a collection of coalitions $\gamma(x, y)$ that are effective for moving from x to y. If coalition K is an element of $\gamma(x, y)$ and if yP_Kx, we say that x is *blocked* and K is a *blocking coalition.* If there is no such blocking coalition, we say that x is *unblocked.* Here, we consider and analyze implementation exercises by codes of rights, which we call *implementation by codes of rights.*

If we choose the set of outcomes Z as the state space of a rights structure Γ and the identity function as the outcome function in the definition of Γ-equilibria, we revert to the familiar notion of core, which can be defined as follows.

Definition 6 *For any γ and any R, x is an* equilibrium *at R if for no $y \neq x$ and no $K \in \gamma(x,y)$ it is $yP_K x$.*

The blocking notion yields the concept of core at one preference profile R, which is the set of all unblocked outcomes.[6] Note that, in our definition of blocking, we require that every member of the blocking coalition is strictly better off. For any code of rights γ, we write $C(\gamma, R)$ for the set of the equilibria at R.

Definition 7 *A code of rights γ* implements *F if and only if $F(R) = C(\gamma, R)$ for all $R \in \mathcal{R}$. If such a code of rights exists, then F is* implementable by a code of rights.

One can easily verify that monotonicity w.r.t. Z is a necessary condition for the implementation via codes of rights. However, one also can check that it is not sufficient. We introduce below a new condition, called *strong monotonicity* using the following additional notation. For any coalition $K \in \mathcal{N}_0$, preference profile $R \in \mathcal{R}$, and outcome $x \in Z$, let

$$L(x, R_K) \equiv \bigcup_{i \in K} L(x, R_i),$$

$$F^{-1}(x) \equiv \{R \in \mathcal{R} | x \in F(R)\},$$

and

$$\Lambda_K^F(x) \equiv \bigcap_{R \in F^{-1}(x)} L(x, R_K).$$

We here present strong monotonicity from the viewpoint of necessity. To this end, assume an SCR F is implementable by a code of rights γ. Taking an outcome $x \in F(R)$ for some preference profile $R \in \mathcal{R}$, x must be an equilibrium at R. We fix any coalition K and denote by $\gamma(x, K)$ the set of outcomes for which coalition K is effective, that is, $\gamma(x, K) \equiv \{y \in Z | K \in \gamma(x, y)\}$. Since x is an equilibrium at R, it follows that x is unblocked, that is, for every outcome $y \neq x$, if coalition $K \in \gamma(x, y)$ is effective in moving from x to y, then coalition K cannot be a blocking one. This

[6] Ray (1989) shows that the credibility of blocking coalitions is implicit in the definition of the core.

entails that y must be an element of $L(x, R_K)$ if $y \in \gamma(x, K)$. Since the choice of outcome $y \in \gamma(x, K)$ is arbitrary, the set $\gamma(x, K)$ must be contained in $L(x, R_K)$.

For a canonical rights structure in which the set of states is $S \equiv \{(z, \bar{R}) | z \in F(\bar{R})$ for some $\bar{R} \in \mathcal{R}\}$, the designer can infer the lower contour set of R_i at x for member $i \in K$, that is, to infer the set $L(x, R_K)$.[7] However, in a setting for which the state space coincides with the set of outcomes, the designer cannot obtain information on member i's lower contour sets. Hence, the designer needs to consider any preference profile $\hat{R}$ satisfying $x \in F(\hat{R})$. We choose such a preference profile $\hat{R}$. Then, the preceding argument leads to the conclusion that the set of outcomes $\gamma(x, K)$ for which coalition K is effective must be contained in $L(x, \hat{R}_K)$. This condition should be satisfied for each admissible preference profile $\hat{R}$ satisfying $x \in F(\hat{R})$, that is, the set $\gamma(x, K)$ must be contained in the intersection $\Lambda_K^F(x)$.

Therefore, if at some preference profile R' the set $L(x, R'_K)$ contains $\Lambda_K^F(x)$ and thus the set of outcomes $\gamma(x, K)$ for which coalition K is effective, coalition K cannot be a blocking one. If this conclusion holds for all coalitions, x is unblocked, meaning it is an equilibrium at R'. It follows that x must be F-optimal for this profile by implementability. More formally, strong monotonicity can be stated as follows.[8]

Definition 8 *F is* strongly monotonic *provided that, for all $x \in Z$ and all $R, R' \in \mathcal{R}$, if $x \in F(R)$ and*

$$\Lambda_K^F(x) \subseteq L(x, R'_K) \text{ for all } K \in \mathcal{N}_0,$$

then $x \in F(R')$.

Note that strong monotonicity implies monotonicity (w.r.t. Z). Conversely, for an example of a monotonic SCR that is not strongly monotonic, see the example below.[9]

[7] See KY (proof of Proposition 1, p. 489).

[8] Our strong monotonicity condition must not be confused with the strong monotonicity condition of Peleg and Winter (2002). The latter condition is now widely referred to as essential monotonicity (Danilov, 1992).

[9] It can also be shown that the (constrained) Walrasian solution violates strong monotonicity in

Example 1 (Walrasian solution) Assume three commodities and two agents. Let $\mathcal{U}_i$ be the class of utility functions admissible for agent i, each assumed to be continuous, quasi-concave, and strictly monotonic. Let $\mathcal{U} \equiv \mathcal{U}_1 \times \mathcal{U}_2$ be the class of profiles of admissible utility functions and let $\mathcal{U}^{CD}$ denote the class of profiles of Cobb-Douglas utility functions.[10] Assume that $\mathcal{U}^{CD} \subseteq \mathcal{U}$. Suppose agent 1's endowment is $e_1 = (1, 2, 0)$ and agent 2's endowment $e_2 = (1, 0, 2)$. Let Z be the set of all feasible allocations, that is, $Z \equiv \{(x_1, x_2) \,|x_1 + x_2 = e_1 + e_2\}$. The *Walrasian solution*, denoted by W, can be defined as follows. For each $u \in \mathcal{U}$ and $x \in Z$,

$x \in W(u)$ if and only if there is a price vector $p \in \Delta$ such that for all $i \in N$, $p \cdot x_i = p \cdot e_i$ and for all $y_i \in \mathbb{R}^3_+$, if $p \cdot y_i \leq p \cdot x_i$, then $u_i(y_i) \leq u_i(x_i)$,

where $\Delta \equiv \left\{p \in \mathbb{R}^3_+ | \sum_{\ell=1}^{3} p_\ell = 1\right\}$.[11] Let us assume that, for all $u \in \mathcal{U}$ and $x \in W(u)$, it holds that $x_{i\ell} > 0$ for all agents i and commodity ℓ.[12] Let $u^0 \in \mathcal{U}^{CD}$ be so that $u^0_i(x) = x^2_{i1} x_{i2} x_{i3}$ for each agent i. Then, $\mathbf{1} \equiv (1, 1, 1)$ is a Walrasian allocation at u^0 and the Walrasian equilibrium prices are $p^0 \equiv \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$. Let $u^1 \in \mathcal{U}^{CD}$ be so that $u^1_i(x) = x_{i1} x_{i2} x_{i3}$ for each agent i. Again, $\mathbf{1}$ is a Walrasian allocation at u^1 generated by $p^1 \equiv \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. It follows that $u^0, u^1 \in W^{-1}(\mathbf{1})$. Finally, let $u^2 \in \mathcal{U}^{CD}$ be so that $u^2_1 = u^0_1$ and $u^2_2 = u^1_2$. W is not strongly monotonic since $\mathbf{1} \in W(u^0)$, $\Lambda^W_K(\mathbf{1}) \subseteq L(\mathbf{1}, u^2_K)$ for all $K \in \mathcal{N}_0$ and yet $\mathbf{1} \notin W(u^2)$. However, W is monotonic.

As we formally show below, strong monotonicity is a necessary condition for implementation via codes of rights, but it is not sufficient on its own for implementing

an exchange economy with more than three agents. The details are available upon request from the authors.

[10] Although Cobb-Douglas utility functions are not strictly monotonic on the boundary of consumption set $\mathbb{R}^3_+$, only strict monotonicity on the interior of $\mathbb{R}^3_+$ is necessary to obtain our result.

[11] p_ℓ denotes the price of commodity ℓ.

[12] Under this assumption, the Walrasian solution coincides with the constrained Walrasian solution (Hurwicz et al., 1995), which is known to be monotonic.

an SCR by a code of rights. The sufficiency is obtained by designing a code of rights. The problem for the designer is how to allocate blocking powers when we start from an equilibrium outcome x at R that does not belong to the range of F. We solve this problem by making every agent i effective in moving from x to y, for every feasible $y \in Z$. In other words, we allow the set of outcomes $\gamma(x, \{i\})$ for which agent i is effective to coincide with set Z. Given that x is an equilibrium at R, it must be that x is a top outcome for agent i according to his/her preference ordering R_i. Then, to make x an F-optimal outcome at R, we require the familiar condition of unanimity, which states that, if an outcome is at the top of the preferences of all agents, then that outcome should be selected by the SCR. The condition can be stated as follows.

Definition 9 *F satisfies* unanimity *provided that, for all $x \in Z$ and all $R \in \mathcal{R}$, if $Z \subseteq L(x, R_i)$ for all $i \in N$, then $x \in F(R)$.*

We show that strong monotonicity and unanimity are necessary and sufficient for an implementation by codes of rights.

Theorem 3 *F is* implementable by a code of rights *if and only if F satisfies the conditions of* strong monotonicity *and* unanimity.

Proof. "*Only If*": Assume that code of rights γ implements F. Since it is obvious that F satisfies unanimity, we show that F satisfies strong monotonicity. Take any R and x so that $x \in F(R)$. Furthermore, take any R' so that $\Lambda_K^F(x) \subseteq L(x, R'_K)$ for all K. We show that $x \in F(R') = C(\gamma, R')$.

Assume, to the contrary, that $x \notin C(\gamma, R')$. Then, there exist $y \neq x$ and $K \in \gamma(x, y)$ so that $yP'_K x$. It follows that $y \notin L(x, R'_K)$. Take any $\bar{R} \in F^{-1}(x)$. Since $x \in C\left(\gamma, \bar{R}\right)$ and since $K \in \gamma(x, y)$, it follows that $y \in L\left(x, \bar{R}_K\right)$. Since the choice of $\bar{R} \in F^{-1}(x)$ is arbitrary, we have $y \in \Lambda_K^F(x)$. By our initial assumption that $\Lambda_K^F(x) \subseteq L(x, R'_K)$, it follows that $y \in L(x, R'_K)$, which is a contradiction. Therefore, F is strongly monotonic.

"*If*": Assume that F satisfies the conditions of strong monotonicity and unanimity. Let us define a code of rights $\gamma : Z \times Z \twoheadrightarrow \mathcal{N}$ as follows. For all K,

(a) For all $x \in F(\mathcal{R})$ and all $y \in Z$,

$$K \in \gamma(x, y) \iff y \in \Lambda_K^F(x);$$

(b) For all $x \in Z - F(\mathcal{R})$ and all $y \in Z$, $K \in \gamma(x, y)$.

We show that γ implements F. Fix any R.

Let $x \in F(R)$. We show that $x \in C(\gamma, R)$. Since $x \in F(R)$, it follows that $x \in F(\mathcal{R})$. Then, only part (a) of the definition of γ applies. We fix any K and y. Assume that $y \in \Lambda_K^F(x)$. Then, by the definition of γ, it follows that $K \in \gamma(x, y)$. However, since $y \in \Lambda_K^F(x)$, it also follows that $y \in L(x, R_K)$. Moreover, by the definition of γ, it holds that $K \notin \gamma(x, y)$ if $y \notin \Lambda_K^F(x)$. Then, either $K \notin \gamma(x, y)$ if $y \notin \Lambda_K^F(x)$ or $K \in \gamma(x, y)$ and no yP_Kx if $y \in \Lambda_K^F(x)$. Since the choices of K and y are arbitrary, we conclude that $x \in C(\gamma, R)$.

Conversely, we take any $x \in C(\gamma, R)$. We proceed according to whether $x \in F(\mathcal{R})$ or not.

Assume that $x \in Z - F(\mathcal{R})$. Then, only part (b) of the definition of γ applies. Fix any i. Since $i \in \gamma(x, y)$ for all y and $x \in C(\gamma, R)$, xR_iy for all y, and so $Z \subseteq L(x, R_i)$. Since the choice of i is arbitrary, it follows that $Z \subseteq L(x, R_i)$ for all i. Since F satisfies the condition of unanimity, we have $x \in F(R)$.

Suppose that $x \in F(\mathcal{R})$. Then, $F^{-1}(x)$ is not empty, meaning only part (a) of the definition of γ applies. Assume, to the contrary, that $x \notin F(R)$. Then, strong monotonicity implies that there exist K and $y \in \Lambda_K^F(x)$ so that yP_Kx. Since $y \in \Lambda_K^F(x)$, $K \in \gamma(x, y)$ by definition of γ. Therefore, there exists $y \neq x$ so that yP_Kx, for some $K \in \gamma(x, y)$ and so $x \notin C(\gamma, R)$, which is a contradiction. ■

KY (p. 495) provide a characterization of the class of SCRs implementable via codes of rights. This result is given in terms of monotonicity (w.r.t. Z), as well as of a condition called *binary consistency*. The necessity for binary consistency relies on the fact that they focus on the unrestricted domain of linear orderings, $\mathcal{P}_Z$. To introduce this condition, we need the following additional notations. Take any $P \in \mathcal{P}_Z$ and any $a, b \in Z$. a is said to be *not Pareto dominated by b at P* if aP_ib for some agent i. Take any $P \in \mathcal{P}_Z$ and any $x, y \in Z$ so that y is not Pareto dominated by x at P. Let P^{xy} be the profile obtained from P, in which x and y are the two most preferred outcomes for all agents and, for all $z \in Z - \{x, y\}$ and $i \in N$, it holds that $xP^{xy}z$ and $yP^{xy}z$; further, $\{i \in N | yP^{xy}x\} = \{i \in N | yPx\}$.

Definition 10 *F is* binary consistent *provided that, for all $P \in \mathcal{P}_Z$ and $x \in Z$ if for all $y \in Z$ that are not Pareto dominated by x at P, it holds that $x \in F(P^{xy})$, then $x \in F(P)$.*

Theorem 3 is logically equivalent to the characterization provided by KY. Unfortunately, we were unable to find a direct, intuitive bridge between the implementing conditions in these theorems, although the formal arguments of their equivalence are available from the authors upon request—see the Addendum..

Although strong monotonicity is a demanding monotonicity-type condition, we present below prominent SCRs that are implementable by a code of rights.

Example 2 (Pareto solution) The *(weak) Pareto solution* denoted by Po selects all weak Pareto optima corresponding to a given profile $R \in \mathcal{R}$:

$$Po(R) \equiv \{x \in Z | \text{for all } y \in Z - \{x\} \text{ there exists } i \in N \text{ such that } xR_iy\}.$$

To verify that this solution satisfies strong monotonicity, assume that $x \in Po(R)$ for some R. Moreover, suppose that, for some R', it holds that $\Lambda_K^{Po}(x) \subseteq L(x, R'_K)$ for all $K \in \mathcal{N}_0$. We show that $x \in Po(R')$. Assume, to the contrary, that $x \notin Po(R')$. Then, there exists outcome $y \in Z - \{x\}$ so that yP'_ix for every agent $i \in N$, that is, y Pareto dominates x at R'. Now, we fix coalition $N \in \mathcal{N}_0$. Next, we take any $\bar{R} \in Po^{-1}(x)$ that is not empty, since $x \in Po(R)$. Since $x \in Po(\bar{R})$, it follows from the definition of the Pareto solution that $x\bar{R}_i y$ for some $i \in N$. Since the choice of $\bar{R} \in Po^{-1}(x)$ is arbitrary, we have $y \in \Lambda_N^{Po}(x)$. Since, by our initial assumption, it holds that $\Lambda_N^{Po}(x) \subseteq L(x, R'_N)$, it follows that y cannot Pareto dominate x at the profile R', which is a contradiction. Since the Pareto solution satisfies unanimity, it follows that it *is implementable by a code of rights.*

Example 3 (Condorcet solution) The *Condorcet solution* denoted by CON selects outcomes that are (weakly) majority preferred to any other outcome. Formally, this solution can be defined as follows. For all $P \in \mathcal{P}$:

$$CON(P) \equiv \{x \in Z | \text{for all } y \in Z : |\{i \in N | xP_i y\}| \geq |\{i \in N | yP_i x\}|\},$$

where $\mathcal{P}$ is a (nonempty) set of profiles of linear orderings for which the solution is well-defined. This SCR *is implementable by a code of rights.* Clearly, it satisfies the condition of unanimity. Then, we also show that it satisfies strong monotonicity. To this end, let $x \in CON(P)$ for some $P \in \mathcal{P}$. Moreover, we assume that, for some profile P', it holds that $\Lambda_K^{CON}(x) \subseteq L(x, P'_K)$ for all $K \in \mathcal{N}_0$. We show that $x \in CON(P')$. To obtain a contradiction, let us assume that $x \notin CON(P')$. Then, there exists y so that $|\{i \in N | yP'_i x\}| > |\{i \in N | xP'_i y\}|$. Let us denote by K the set $\{i \in N | yP'_i x\}$. Note that $|\{i \in N | yP'_i x\}| > \frac{n}{2}$. Take any $\bar{P} \in CON^{-1}(x)$, which is not empty, since $x \in CON(P)$. Since $x \in CON(\bar{P})$, it follows that x is majority preferred to y at $\bar{P}$, that is, $|\{i \in N | x\bar{P}_i y\}| \geq |\{i \in N | y\bar{P}_i x\}|$. Since $|\{i \in N | yP'_i x\}| > \frac{n}{2}$, it follows that $\{i \in N | x\bar{P}_i y\} \cap K$ is not empty. This implies that $y \in L(x, \bar{P}_K)$. Since the choice of $\bar{P} \in CON^{-1}(x)$ is arbitrary, we have $y \in \Lambda_K^{CON}(x)$. As by our initial assumption $\Lambda_K^{CON}(x) \subseteq L(x, P'_K)$, it thus follows that $xP'_i y$ for some $i \in K$, which is a contradiction.

Example 4 (Individually rational solution) The *individually rational solution*, denoted by Ir, with respect to some outcome $a^0 \in Z$ can be defined as follows: for all $R \in \mathcal{R}$,

$$Ir(R) \equiv \{x \in Z | xR_i a^0 \text{ for all } i \in N\}.$$

This SCR *is implementable by a code of rights.* To determine it, observe that this solution satisfies the condition of unanimity. To show that this solution is strongly monotonic, assume that $x \in Ir\,(R)$ for some profile $R \in \mathcal{R}$. Furthermore, let us assume that, for some profile R', it holds that $\Lambda_K^{Ir}(x) \subseteq L(x, R'_K)$ for all $K \in \mathcal{N}_0$. We show that $x \in Ir\,(R')$. Take any $\bar{R} \in Ir^{-1}(x)$, which is not empty, since $x \in Ir\,(R)$. Since $x \in Ir\left(\bar{R}\right)$, it follows from the definition of Ir that $x\bar{R}_i a^0$, for all $i \in N$. Since the choice of $\bar{R} \in Ir^{-1}(x)$ is arbitrary, we have $a^0 \in \Lambda_{\{i\}}^{Ir}(x)$ for all $i \in N$. As $\Lambda_{\{i\}}^{Ir}(x) \subseteq L(x, R'_i)$ for all i holds according to our initial assumption, it follows that $xR'_i a^0$ for all i, and so Ir is strongly monotonic.

Example 5 (No-envy solution; Foley, 1967) Let the set of outcomes Z be $\mathbb{R}_+^n$. The *no-envy solution* denoted by N can be defined for each $R \in \mathcal{R}$, by

$$N\,(R) \equiv \{x \in Z | x_i R_i x_j \text{ for all } i, j \in N\}\,.$$

We assume that preference domain $\mathcal{R}$ consists of selfish preferences R_i, being defined over $\mathbb{R}_+$. This solution is *implementable by a code of rights.* We omit the straightforward proof that N satisfies the condition of unanimity. Let us show that it is strongly monotonic. To this end, let $x \in N\,(R)$ for some feasible R. Additionally, assume that, for some feasible profile R', it holds that $\Lambda_K^N(x) \subseteq L(x, R'_K)$ for all $K \in \mathcal{N}_0$. Assume, to the contrary, that $x \notin N\,(R')$. Then, there exists an agent i who finds his/her assignment to be worse than the bundle assigned to agent $j \neq i$, that is, $x_j P'_i x_i$. Let us denote by x^* a permutation of x so that $x_i^* = x_j$, $x_j^* = x_i$ and $x_k^* = x_k$ for every other agent $k \in N - \{i, j\}$. Then, by construction and the selfishness of preferences, $x^* P'_i x$. Next, take any $\bar{R} \in N^{-1}(x)$, which is not empty, since, by our initial assumption, $x \in N\,(R)$. Since $x \in N\left(\bar{R}\right)$, it follows from the definition of N that $x_i \bar{R}_i x_j$, and so $x\bar{R}_i x^*$. Since the choice of $\bar{R} \in N^{-1}(x)$ is arbitrary, we have $x^* \in \Lambda_{\{i\}}^N(x)$. Since, by our initial assumption it holds that $\Lambda_{\{i\}}^N(x) \subseteq L(x, R'_i)$, it follows that $xR'_i x^*$, which is a contradiction.

Application to matching problems: Implementation of the stable solution

A *matching problem* is a quadruplet $(M, W, P, \mathcal{M})$ so that:

- M is a finite nonempty set of men with m as a typical element;
- W is a finite nonempty set of women with w as a typical element;
- $P \in \mathcal{P}$ is a profile of linear orderings so that (i) every man $m \in M$'s preference relation is represented by a linear ordering P_m over $W \cup \{m\}$ and (ii) every woman $w \in W$'s preference relation is represented by a linear ordering P_w over $M \cup \{w\}$.
- $\mathcal{M}$ is a collection of all matchings, with μ as a typical element. $\mu : M \cup W \to M \cup W$ is a bijective function, matching every agent $i \in M \cup W$ either to a partner of the opposite sex or with himself/herself. If an agent i is matched with himself/herself, we say that this i is *single* under μ.

We refer to $(M, W, \mathcal{P}, \mathcal{M})$ as a *class of matching problems*, with $(M, W, P, \mathcal{M})$ as a typical matching problem. Note that $Z = \mathcal{M}$ and $M \cup W = N$.

To apply Theorem 3 to matching problems, we extend the linear ordering P_m of a man $m \in M$ to the preference ordering R_m on $\mathcal{M}$ as follows: for every $\mu, \mu' \in \mathcal{M}$,

$$\mu R_m(\theta)\, \mu' \Leftrightarrow \text{either } \mu(m)\, P_m(\theta)\, \mu'(m) \text{ or } \mu(m) = \mu'(m).$$

Similarly, this can be done for every woman $w \in W$. Let $\mathcal{R}$ denote the preference domain over $\mathcal{M}$, obtained by a collection $\mathcal{P}$ of profiles of linear orderings.

A matching μ is *blocked by agent i at* $R \in \mathcal{R}$ if $i P_i \mu(i)$. A matching μ is *blocked by a pair* $(m, w) \in M \times W$ *at* $R \in \mathcal{R}$ if $m P_w \mu(w)$ and $w P_m \mu(m)$. A matching μ is *stable at* $R \in \mathcal{R}$ if it is not blocked by any agent or any pair of a man and a woman at R. Given a matching problem, the *stable solution*, denoted by St, can be defined, for each $R \in \mathcal{R}$, by

$$St(R) \equiv \{\mu \in \mathcal{M} | \mu \text{ is stable at } R\}.$$

Theorem 4 *Let $(M, W, \mathcal{P}, \mathcal{M})$ be any class of matching problems. The* stable solution*, St, defined over $\mathcal{R}$,* is implementable by a code of rights.

Proof. In light of Theorem 3, we need to show that St satisfies the conditions of unanimity and strong monotonicity. We omit the straightforward proof that St satisfies the condition of unanimity. Then, we need to show that this solution satisfies strong monotonicity as well.

Take any two matching problems $(M, W, P, \mathcal{M})$ and $(M, W, P', \mathcal{M})$. Assume that $\mu \in St(R)$. Moreover, suppose that $\Lambda_K^{St}(\mu) \subseteq L(\mu, R'_K)$ for all $K \in \mathcal{N}_0$. We show that $\mu \in St(R')$. To obtain a contradiction, we assume that $\mu \notin St(R')$.

Suppose that μ is blocked by agent i at R', that is, $iP'_i\mu(i)$. Then, $\hat{\mu} \notin L(\mu, R'_i)$ for all $\hat{\mu} \in \mathcal{M}$ so that $\hat{\mu}(i) = i$. Next, take any $\bar{R} \in St^{-1}(\mu)$, which exists since $\mu \in St(R)$. Since $\mu \in St(\bar{R})$, it follows that μ is not blocked by agent i at $\bar{R}$, that is, either $\mu(i)\bar{P}_i i$ or $\mu(i) = i$. Then, for all $\hat{\mu} \in \mathcal{M}$, if $\hat{\mu}(i) = i$, then $\hat{\mu} \in L(\mu, \bar{R}_i)$. Since the choice of $\bar{R} \in St^{-1}(\mu)$ is arbitrary, we have $\hat{\mu} \in \Lambda_{\{i\}}^{St}(\mu)$ for all $\hat{\mu} \in \mathcal{M}$ so that $\hat{\mu}(i) = i$. As by our initial assumption, it holds that $\Lambda_{\{i\}}^{St}(\mu) \subseteq L(\mu, R'_i)$, it follows that $\mu R'_i \hat{\mu}$ for $\hat{\mu} \in \mathcal{M}$ so that $\hat{\mu}(i) = i$, which is a contradiction.

Assume that μ is blocked by a pair $(m, w) \in M \times W$ at R', that is, $mP'_w\mu(w)$ and $wP'_m\mu(m)$. Then, $\hat{\mu} \notin L\left(\mu, R'_{\{m,w\}}\right)$ for all $\hat{\mu} \in \mathcal{M}$ so that $\hat{\mu}(w) = m$ and $\hat{\mu}(m) = w$. Next, take any $\bar{R} \in St^{-1}(\mu)$, which exists since $\mu \in St(R)$. Since $\mu \in St(\bar{R})$, it follows that μ is not blocked by (m, w) at $\bar{R}$, that is, not $m\bar{P}_w\mu(w)$ or not $w\bar{P}_m\mu(m)$. For all $\hat{\mu} \in \mathcal{M}$, if $\hat{\mu}(w) = m$ and $\hat{\mu}(m) = w$, $\hat{\mu} \in L(\mu, \bar{R}_{\{m,w\}})$. Since the choice of $\bar{R} \in St^{-1}(\mu)$ is arbitrary, we have $\hat{\mu} \in \Lambda_{\{m,w\}}^{St}(\mu)$ for all $\hat{\mu} \in \mathcal{M}$ so that $\hat{\mu}(w) = m$ and $\hat{\mu}(m) = w$. As by our initial assumption, it holds that $\Lambda_{\{m,w\}}^{St}(\mu) \subseteq L(\mu, R'_{\{m,w\}})$, it follows that $\mu R'_w \hat{\mu}$ or $\mu R'_m \hat{\mu}$ for $\hat{\mu} \in \mathcal{M}$ so that $\hat{\mu}(w) = m$ and $\hat{\mu}(m) = w$, which is a contradiction. ■

5.1 Implementation via codes of individual rights

A code of rights γ is said to be a *code of individual rights* if, for each pair of distinct outcomes x and y, $\gamma(x, y)$ contains only unit coalitions if it is not empty, that is, it contains only coalitions of size one. Here, we study implementation exercises in which the designer can devise only codes of individual rights, which we call *implementation by codes of individual rights.*

Definition 11 *F is* implementable by a code of individual rights *if there exists a code of individual rights γ so that γ implements F.*

Although strong monotonicity is still a necessary condition for implementation by codes of individual rights, one can easily verify that it is not sufficient. We introduce below a stronger variant of strong monotonicity, called *singleton strong monotonicity*, which is shown to be necessary and sufficient for implementation by a code of individual rights when combined with the unanimity condition. To introduce this condition, we need additional notation as follows. For any outcome x and agent i, let

$$\Lambda_i^F(x) \equiv \bigcap_{R \in F^{-1}(x)} L(x, R_i).$$

Therefore,

Definition 12 *F is* singleton strongly monotonic *provided that, for all $x \in Z$ and all $R, R' \in \mathcal{R}$, if $x \in F(R)$ and*

$$\Lambda_i^F(x) \subseteq L(x, R_i') \text{ for all } i \in N,$$

then $x \in F(R')$.

One can easily verify that the above condition is nothing more than the condition of strong monotonicity restricted to unit coalitions. Roughly, the intuitions of the two conditions are the same. The above condition appears in Saijo et al. (1996; p. 955) under the name Condition W^*. The theorem below characterizes the class of SCRs implementable by codes of individual rights.

Theorem 5 *F is* implementable by a code of individual rights *if and only if F satisfies the conditions of* singleton strong monotonicity *and* unanimity*.*

Proof. Let us define a code of individual rights $\gamma : Z \times Z \twoheadrightarrow \mathcal{N}$ as follows. For all i,

(a) For all $x \in F(\mathcal{R})$ and all $y \in Z$,

$$\{i\} \in \gamma(x,y) \iff y \in \Lambda_i^F(x);$$

(b) For all $x \in Z - F(\mathcal{R})$ and all $y \in Z$, $\{i\} \in \gamma(x,y)$.

We omit the proof, which uses the above γ to prove the "*If*" part of the statement and similar arguments to the proof of Theorem 3. ■

The class of SCRs implementable by a code of individual rights is not empty. The reason is that the individually rational solution and no-envy solution are singleton strongly monotonic. Since the arguments respectively presented in examples 4 and 5 suffice, we omit them here.

5.2 Non-equivalence

Coalition formation does not bring any value added to the implementation of core equilibria by rights structures; for all purposes, it is sufficient to focus on unit coalitions. We now show that, once one focuses on the allocation of blocking powers via the design of codes of rights, the implementation of core equilibria may require non-singleton coalitions.

By Theorems 3 and 5, it is not evident whether the class of SCRs implementable via codes of rights is equal to that of SCRs implementable by codes of individual rights. We find they are not identical. An example is the Condorcet solution defined in example 3. This result can be stated as follows.

Theorem 6 *.*

(i) Singleton strong monotonicity *implies* strong monotonicity.

(ii) Strong monotonicity *does not imply* singleton strong monotonicity.

Proof. The proof of part (i) is obvious and thus omitted. Let us now demonstrate part (ii) by assuming that $N \equiv \{1, 2, 3\}$ and $Z \equiv \{x, y\}$ with $x \neq y$.[13] Let $\mathcal{P}_Z$ be the set of all profiles of linear orderings over Z. Moreover, let CON be the Condorcet solution.

One can verify that there are four profiles $\{P^0, P^3, P^2, P^1\} \subseteq \mathcal{P}_Z$ at each x being CON-optimal, since x is preferred to y by every agent at profile P^0 and by everyone except agent $j \in N$ at profile P^j. Similarly, let $\left\{\hat{P}^0, \hat{P}^3, \hat{P}^2, \hat{P}^1\right\} \subseteq \mathcal{P}_Z$ be the profiles at which y is CON-optimal, since y is preferred to x by everyone at profile $\hat{P}^0$ and by everyone except agent $j \in N$ at profile $\hat{P}^j$. By example 3, we already know that CON is strongly monotonic. To complete the proof, we need only to show that CON violates singleton strong monotonicity. One can verify that

$$\Lambda_i^{CON}(x) = \{x\} \text{ and } \Lambda_i^{CON}(y) = \{y\}, \text{ for all } i \in N.$$

Then, by construction, one can verify that, for each $j = 0, 1, 2, 3$ and $i \in N$, it holds that

$$\Lambda_i^{CON}(x) \subseteq L\left(x, P_i^j\right) \text{ and } \Lambda_i^{CON}(x) \subseteq L\left(x, \hat{P}_i^j\right),$$
$$\Lambda_i^{CON}(y) \subseteq L\left(y, P_i^j\right) \text{ and } \Lambda_i^{CON}(y) \subseteq L\left(y, \hat{P}_i^j\right).$$

Now, to determine whether CON violates the condition of singleton strong monotonicity, it suffices to observe that, for any $j = 0, 1, 2, 3$, we have $x \in CON\left(P^j\right) - CON\left(\hat{P}^j\right)$ but $\Lambda_i^{CON}(x) \subseteq L\left(x, \hat{P}_i^j\right)$ for each $i \in N$, in *violation* of singleton strong monotonicity. ■

The example constructed to prove part (ii) of Theorem 6 can be used to show

[13] For simplicity, we prove the claim by assuming $n = 3$. The proof will be similar for $n > 3$.

that the Pareto solution is not singleton strongly monotonic.[14] In light of Theorems 3 and 5, the main implication of Theorem 6 can be formally stated as follows.

Corollary 1 Implementation by codes of rights *is not equivalent to* implementation by codes of individual rights.

6 Externally stable implementation by codes of rights

We achieved the above results by focusing on the traditional notion of core, which has been criticized for not being symmetric (Greenberg, 1990). Indeed, from its original definition, an outcome x is not a core point if it is blocked, that is, if there is a coalition that would reject x and move to another outcome y that would be preferred by all its members. However, outcome y itself can, in turn, be blocked. Then, if we require that an outcome be immune against blocking, symmetry would require that the same should hold for y. That is, we should require that blocking be done through outcomes that are themselves unblocked. This consistency requirement leads to the concept widely referred to as an externally stable core.

Definition 13 *For any code of rights γ and any preference profile R, a set $Z^* \subseteq Z$ of outcomes is* externally stable *at R if, for all $y \in Z - Z^*$, there is $x \in Z^*$ so that xP_Ky for some $K \in \gamma(y, x)$.*

Externally stable equilibria at R are denoted by $EC(\gamma, R)$. The set of equilibria in $C(\gamma, R)$ is unique and, hence, $EC(\gamma, R)$ is unique when it exists.

Here, we consider and analyze the implementation of an externally stable core by codes of rights, which we call *externally stable implementation by codes of rights.*

[14] To this end, observe that $x \in Po\left(P^0\right)$, $\Lambda_i^{Po}(x) = \{x\} \subseteq L\left(x, \hat{P}_i^0\right)$ for each agent i but yet $x \notin Po\left(\hat{P}^0\right)$.

Definition 14 *A code of rights* γ externally stable implements F *if* $F(R) = EC(\gamma, R)$ *for all* $R \in \mathcal{R}$*. If such a code of rights exists,* F *is* externally stable implementable by a code of rights*.* F *is* externally stable implementable by a code of individual rights *if there exists a code of individual rights* γ *so that* γ *externally stable implements* F*.*

Externally stable implementation is a robust way of implementing optimal outcomes, particularly being more reliable than implementation, since external stability guarantees that no outcome outside the core can be sustained. An externally stable core is also more robust than the vNM stable set or its derivatives, since it avoids problems with indirect internal stability (i.e., the *Harsanyi critique*).

We propose two conditions that are together necessary and sufficient, when combined with unanimity, for an SCR to be externally stable implemented by a code of rights. The first condition is a variant of strong monotonicity, called *strong winner monotonicity.*

Definition 15 F *is* strongly winner monotonic *provided that, for all* $x \in Z$ *and all* $R, R' \in \mathcal{R}$*, if* $x \in F(R)$ *and*

$$\Lambda_K^F(x) \bigcap F(R') \subseteq L(x, R'_K) \text{ for all } K \in \mathcal{N}_0,$$

then $x \in F(R')$*.*

In other words, this condition implies that, whenever x is F-optimal at one profile R and for every coalition K, x is a maximal element in $\Lambda_K^F(x) \cap F(R')$ according to the preferences of coalition K at R'_K, it should be F-optimal at R'. The intuition is straightforward. Assume that x is F-optimal at R. Since the externally stable core at this R is a subset of the core at R, we know from Theorem 3 that whenever preferences change from R to R' and $\Lambda_K^F(x) \subseteq L(x, R'_K)$ for each coalition $K \in \mathcal{N}_0$, x must remain F-optimal at new profile R'. However, by the requirement of external stability, if x were not F-optimal at R', there should exist a coalition K that can, and wants to, reject x and move to an F-optimal outcome at R'. This means that it is not the entire set $\Lambda_K^F(x)$ that matters to remove x as an equilibrium outcome when preferences change from R to R', but it is set $\Lambda_K^F(x) \bigcap F(R')$ that matters.

The condition is stronger than strong monotonicity, since $\Lambda_K^F(x) \cap F(R') \subseteq \Lambda_K^F(x)$ for all $R' \in \mathcal{R}$ and $K \in \mathcal{N}_0$. Conversely, for an example of a strongly monotonic (and monotonic) SCR that is not strongly winner monotonic, see the example below.

Example 6 (Pareto solution with veto power) There are three players in $N \equiv \{1,2,3\}$, and two profiles of linear orderings P and P' over set $Z \equiv \{v,x,y\}$. Preferences are represented in the table below,

P			P'		
1	2	3	1	2	3
x	v	x	x	v	x
v	x	v	y	x	v
y	y	y	v	y	y

where, as usual, $\substack{a \\ b}$ for agent i means that he/she strictly prefers a to b. Let F be so that $F(P) = \{v,x\}$ and $F(P') = \{x\}$. F is not strongly winner monotonic, since $v \in F(P)$, $\Lambda_K^F(v) \cap F(P') \subseteq L(v, P'_K)$ for all $K \in \mathcal{N}_0$ and yet $v \notin F(P')$. However, one can verify that F is strongly monotonic (and thus monotonic).

The second condition can be stated as follows.

Definition 16 *F satisfies the* no-simultaneous domination of F *provided that, for all $x \in Z$ and all $R \in \mathcal{R}$, if $x \in Z - F(R)$, then for some $i \in N$, yP_ix for some $y \in F(R)$.*

The condition simply states that, if outcome x is not F-optimal at R, it cannot be that this x dominates every outcome in the range of F at R in the sense that x is at least as good as every F-optimal outcome at R for every agent $i \in N$. When preference domain $\mathcal{R}$ is the domain of linear orderings, the condition implies that an outcome x that is not F-optimal at R cannot Pareto dominate every F-optimal outcome at R.[15]

Our next result is that the class of SCRs externally stable implementable by codes of rights coincides with the class of SCRs that satisfy strong winner monotonicity, unanimity, and no-simultaneous domination of F.

Theorem 7 *F is* externally stable implementable by a code of rights *if and only if F satisfies the conditions of* strong winner monotonicity*,* unanimity*, and* no-simultaneous domination of F*.*

Proof. "*Only If*": Assume that γ externally stable implements F. Since $EC(\gamma, R') \subseteq C(\gamma, R')$ for every $R' \in \mathcal{R}$, it is obvious that F satisfies unanimity and the no-simultaneous domination of F. Then, we only show that F satisfies strong winner monotonicity. Take any R and x so that $x \in F(R)$. Take any R' so that $\Lambda_K^F(x) \cap F(R') \subseteq L(x, R'_K)$ for all $K \in \mathcal{N}_0$. We show that $x \in F(R')$. Assume, to the contrary, that $x \notin F(R') = EC(\gamma, R')$. Then, there exist $y \in EC(\gamma, R')$ and $K \in \gamma(x, y)$ so that yP'_Kx. Considering implementability, $y \in F(R')$. Take any $\bar{R} \in F^{-1}(x)$. Since $x \in EC(\gamma, \bar{R})$ and $K \in \gamma(x, y)$, it follows that $y \in L(x, \bar{R}_K)$; otherwise, $x \notin C(\Gamma, \bar{R})$, which is a contradiction. Since the choice of $\bar{R} \in F^{-1}(x)$ is arbitrary, we have $y \in \Lambda_K^F(x)$. By our initial assumption that $\Lambda_K^F(x) \cap F(R') \subseteq L(x, R'_K)$, it follows that $y \in L(x, R'_K)$, which is a contradiction. Therefore, F is strongly winner monotonic.

"*If*": Assume that F satisfies strong winner monotonicity, unanimity, and no-simultaneous domination of F. Let us define $\gamma : Z \times Z \twoheadrightarrow \mathcal{N}$ as in the proof of Theorem 3. We show that γ externally stable implements F. We fix any R.

Since strong winner monotonicity implies strong monotonicity, Theorem 3 implies that $F(R) = C(\gamma, R)$. We complete the proof by showing that $C(\gamma, R)$ is an externally stable set.

[15] For any profile R, we say that outcome x *Pareto dominates* y if xP_iy for all $i \in N$.

Conversely, assume that $C(\gamma, R)$ is not externally stable. Then, there exists $y \in Z - C(\gamma, R)$ so that, for all $x \in C(\gamma, R)$, it holds that yR_Kx or $K \notin \gamma(y, x)$ for all K. We fix one of outcomes y. We now proceed according to whether $y \in F(\mathcal{R})$ or not.

Case 1: $y \in F(\mathcal{R})$

Then, $y \in F(R')$ for some R'. Since $y \notin F(R) = C(\gamma, R)$, strong winner monotonicity implies that there exist K and x so that $x \in \Lambda_K^F(y) \cap F(R)$ and xP_Ky. Since $x \in \Lambda_K^F(y)$, it follows that $K \in \gamma(y, x)$ from part (a) of the definition of γ. Then, there is an outcome $x \in C(\gamma, R)$ so that xP_Ky for some $K \in \gamma(y, x)$, which is a contradiction.

Case 2: $y \notin F(\mathcal{R})$

Then, $y \in Z - F(\mathcal{R})$. Further, from part (b) of the definition of γ, $K \in \gamma(y, x)$ for all $x \in F(R)$ and all K. Since $C(\gamma, R)$ is not externally stable and $y \in Z - C(\gamma, R)$, it follows that yR_Kx for all K, meaning y simultaneously dominates the range of F, which is a contradiction. ■

One may wonder whether the non-equivalence result of Corollary 1 extends to this notion of equilibrium. The answer is yes. To this end, we define below a variant of strong winner monotonicity, which we call *singleton strong winner monotonicity*.

Definition 17 *F is* singleton strongly winner monotonic *provided that, for all $x \in Z$ and all $R, R' \in \mathcal{R}$, if $x \in F(R)$ and*

$$\Lambda_i^F(x) \bigcap F(R') \subseteq L(x, R_i') \text{ for all } i \in N,$$

then $x \in F(R')$.

One can easily verify that singleton strong winner monotonicity is necessary for the externally stable implementation by a code of individual rights. Additionally, this condition is sufficient for the externally stable implementation by a code of individual rights when combined with unanimity and the no-simultaneous domination of F. Indeed, we have the following theorem.

Theorem 8 F *is* externally stable implementable by a code of individual rights *if and only if it satisfies the conditions of* singleton strong winner monotonicity*,* unanimity*, and* no-simultaneous domination of F*.*

Proof. Consider the code of individual rights γ in the proof of Theorem 5. Since the proof readily follows from arguments similar to those used in the proof of Theorem 7, we omit it here. ■

The fact that the externally stable implementation by codes of rights is not equivalent to that by codes of individual rights readily follows from the facts that singleton strong winner monotonicity implies singleton strong monotonicity, the Condorcet solution is externally stable implementable when the number of voters is odd (see example 8 below), and the Condorcet solution is not singleton strongly monotonic from the proof of part (b) of Theorem 6.

Corollary 2 Externally stable implementation by codes of rights *is not equivalent to* externally stable implementation by codes of individual rights*.*

Finally, the following examples show that the class of SCRs externally stable implementable by a code of rights is not empty. The formal definitions of the SCRs below have been given in Section 5.

Example 7 *The Pareto solution, Po, is externally stable implementable.* Since it is straightforward to verify that Po satisfies no-simultaneous domination of F and unanimity, we omit the proofs here. To check whether Po is strong winner monotonic, assume that $x \in Po\,(R)$ for some R. Moreover, assume that, for some R', it holds that $\Lambda_K^{Po}\,(x) \cap Po\,(R') \subseteq L(x, R'_K)$ for all $K \in \mathcal{N}_0$. We show that $x \in Po\,(R')$. Assume, to the contrary, that $x \notin Po\,(R')$. Then, there exists an outcome $y \in Z - \{x\}$ so that y Pareto dominates x at R'. We fix coalition $N \in \mathcal{N}_0$. Take any $\bar{R} \in Po^{-1}(x)$, which is not empty, since $x \in Po(R)$. Assume that $z \notin L(x, \bar{R}_N)$ for some $z \in Po(R')$. Then, z Pareto dominates x at $\bar{R}$, which is a contradiction. This means that $Po(R') \subseteq L(x, \bar{R}_N)$. Since the choice of $\bar{R} \in Po^{-1}(x)$ is arbitrary, we have $Po(R') \subseteq \Lambda_N^{Po}(x)$. Finally, take any $\bar{R} \in Po^{-1}(x)$. Then, $x\bar{R}_i y$ for some $i \in N$. Since the choice of $\bar{R} \in Po^{-1}(x)$ is arbitrary, we have $y \in \Lambda_N^{Po}(x)$. Since, by the above discussion, it holds that $\Lambda_N^{Po}(x) \subseteq L(x, R'_N)$, it follows that y cannot Pareto dominate x at R', which is a contradiction.

Example 8 *The Condorcet solution, CON, is externally stable implementable.* Let us allow the cardinality of voters, n, to range over all odd natural numbers. One can easily verify that this solution satisfies the conditions of unanimity and no-simultaneous domination of F. To check whether it also satisfies strong winner monotonicity, assume that $x \in CON(P)$ for some $P \in \mathcal{P}$.[16] Moreover, assume that, for some profile P', it holds that $\Lambda_K^{CON}(x) \cap CON(P') \subseteq L(x, P'_K)$ for all $K \in \mathcal{N}_0$. We show that $x \in CON(P')$. Take any coalition K whose cardinality is not lower than $\frac{n+1}{2}$. Take any $y \in F(P')$. Let us show that $y \in \Lambda_K^{CON}(x)$. To this end, take any $\bar{P} \in CON^{-1}(x)$. Since $x \in CON(\bar{P})$, it follows that there is a voter $i \in K$ so that $x\bar{P}_i y$, and so $y \in L(x, \bar{P}_K)$. Since the choice of $\bar{P}$ is arbitrary, as is that of outcome y, it follows that $F(P') \subseteq \Lambda_K^{CON}(x)$. Next, assume, to the contrary, that $x \notin CON(P')$. Therefore, there exists $y \in F(P')$ so that $|\{i \in N | yP'_i x\}| > |\{i \in N | xP'_i y\}|$. Let $K \equiv \{i \in N | yP'_i x\}$ and note that its cardinality is at least $\frac{n+1}{2}$. Thus, $y \notin L(x, P'_K)$. However, since we know that $F(P') \subseteq \Lambda_K^{CON}(x) \subseteq L(x, P'_K)$, it follows that $y \in L(x, P'_K)$, which is a contradiction.

Example 9 *The individually rational solution, Ir, is externally stable implementable.* Clearly, Ir satisfies the conditions of unanimity and no-simultaneous domination of

[16] Recall that $\mathcal{P}$ is a (nonempty) set of profiles of linear orderings, for which the solution is well-defined.

F. To determine whether this solution is also strongly winner monotonic, assume that $x \in Ir(R)$ for some profile $R \in \mathcal{R}$. Furthermore, let us assume that, for some profile R', it holds that $\Lambda_K^{Ir}(x) \cap Ir(R') \subseteq L(x, R'_K)$ for all $K \in \mathcal{N}_0$. We show that $x \in Ir(R')$. By the same arguments used in example 4, we know that $a^0 \in \Lambda_{\{i\}}^{Ir}(x)$ for all $i \in N$. Moreover, from the definition of Ir, it also holds that $a^0 \in Ir(R')$. Since, by our initial assumption, it holds that $\Lambda_{\{i\}}^{Ir}(x) \cap Ir(R') \subseteq L(x, R'_i)$ for all $i \in N$, it follows that $xR'_i a^0$ for all $i \in N$, meaning Ir is strongly winner monotonic.[17]

7 Concluding remarks

Since the seminal contribution of Maskin (1999), economists have been interested in understanding how to circumvent the limitations imposed by Maskin monotonicity by exploring the possibilities offered by approximate (as opposed to exact) implementation (Abreu and Matsushima, 1992; Abreu and Sen, 1991), as well as by implementation under the refinements of Nash equilibria (Moore and Repullo, 1988; Abreu and Sen, 1990; Palfrey and Srivastava, 1991; Jackson, 1992; Vartiainen, 2007) and repeated implementation (Kalai and Ledyard, 1998; Lee and Sabourian, 2011; Mezzetti and Renou, 2017).

Coalitional implementation does not quite fit any of these literature strands, as the coalitional approach relies on a certain degree of coordination by agents within a coalition. The theory is thus silent on how this will take place but assumes it will when the conditions for cooperation are appropriate. Due to this abstraction from the details of interaction, a powerful mode of implementation becomes feasible.

We provide a full characterization of the class of SCRs implementable by codes of rights, as well as of the class of SCRs implementable by codes of individual rights. In contrast to implementation by rights structures, the specialization of the state

[17] Since the same arguments also show that the individually rational solution is externally stable implementable by a code of individual rights, it follows that the class of SCRs characterized by Theorem 8 is not empty.

space in the set of outcomes has a rather intuitive implication, as we prove that, to implement an SCR by a code of rights, blocking powers need to be allocated to non-singleton coalitions. This is the case if we want to implement the Pareto or Condorcet solutions, for example. We have also shown that this insight is robust and extends to the implementation by codes of rights for alternative definitions of the core, such as an externally stable core.

A persistent criticism of the theory of implementation is that the mechanisms used in the constructive proofs have unnatural features (Abreu and Matsushima, 1992; Jackson, 1992, 2001). The reason is that the devised mechanisms rely on tail-chasing constructions such as the integer or modulo games to ensure that undesired strategy combinations do not form an equilibrium. KY have shown that implementation by rights structures, as well as by codes of rights, do not suffer from this criticism. They have achieved this important result by focusing on the unrestricted domain of linear orderings, for which the class of (Maskin) monotonic SCRs is "small" and the implementing mechanisms are relatively simpler (Saijo, 1987; Dasgupta et al., 1979; Muller and Satterthwaite, 1977). However, it remains unclear whether this important result generalizes to other preference domains or hinges upon their domain assumption. In this regard, the characterization results presented in this paper remove any doubt, by showing that the result generalizes to any type of preference domain, even preference domains admitting indifference.

We believe that the implementation framework in this paper is simple and intuitively appealing, and may thus have an important bearing on mechanism design. The developed methodology is thus likely to prove useful in the important task of analyzing environments, where agents have incomplete information, particularly on the preferences of the other agents they are facing, as well as on incorporating elements of farsightedness. Therefore, this is a fruitful area for future research.

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Addendum: not for publication

Theorem 9 *Take any SCR F defined on $\mathcal{P}_Z$. Then, F is* monotonic *(w.r.t. Z) and* binary consistent *if and only if F satisfies* strong monotonicity *and* unanimity*.*

Proof. "*Only If*": Suppose that F, defined over $\mathcal{P}_Z$, is monotonic and binary consistent.

Fix any $P \in \mathcal{P}_Z$. Suppose that $Z \subseteq L(x, R_i)$ for all $i \in N$. Assume, to the contrary, that $x \notin F(P)$. Binary consistency implies that there exists $y \in Z$, not Pareto dominated by x at P, such that $x \notin F(P^{xy})$, which contradicts our supposition $Z \subseteq L(x, R_i)$ for all $i \in N$. Thus, F satisfies unanimity.

Let $x \in F(P)$ for some $P \in \mathcal{P}_Z$. Moreover, assume that for some profile $\bar{P} \in \mathcal{P}_Z$, it holds that $\Lambda_K^F(x) \subseteq L(x, \bar{P}_K)$ for all $K \in \mathcal{N}_0$. We show that $x \in F(\bar{P})$. To obtain a contradiction, let us suppose that $x \notin F(\bar{P})$. Binary consistency implies that there exists $y \in Z$, not Pareto dominated by x at $\bar{P}$, such that $x \notin F(\bar{P}^{xy})$. Define the set $K = \{i \in N | y\bar{P}_i x\}$, which is not empty by binary consistency. Also, note that $y \notin L(x, \bar{P}_K)$, by construction.

Now, take any profile of linear orderings $\tilde{P}$ such that $\{i \in N | y\tilde{P}_i x\} = K$. Suppose that $x \in F(\tilde{P})$. Then, by construction of the profile $\bar{P}^{x,y}$, it follows that $K = \{i | y\bar{P}^{xy} x\}$. Moreover, $x\bar{P}^{xy} z$ for all $z \in Z - \{x, y\}$, for all $i \in N$. Then, since $L(x, \tilde{P}_i) \subseteq L(x, \bar{P}_i^{xy})$ for all $i \in N$, monotonicity implies that $x \in F(\bar{P}^{xy})$, which is a contradiction. We conclude that for every $\tilde{P} \in \mathcal{P}_Z$, if $\{i \in N | y\tilde{P}_i x\} = K$, then $x \notin F(\tilde{P})$.

Take any $\tilde{P} \in F^{-1}(x)$, which is not empty since $x \in F(P)$, by our initial supposition. Since $x \in F(\tilde{P})$, it follows that $\{i \in N | y\tilde{P}_i x\} \neq K$. This implies that there exists an agent $i \in K$ such that $x\tilde{P}_i y$, and so $y \in L(x, \tilde{P}_K)$. Since the choice of $\tilde{P} \in F^{-1}(x)$ is arbitrary, it follows that $y \in \Lambda_K^F(x)$. Since by our initial supposition it holds that $\Lambda_K^F(x) \subseteq L(x, \bar{P}_K)$, it follows that $y \in L(x, \bar{P}_K)$, which is a contradiction.

"*If*": Suppose that F, defined over $\mathcal{P}_Z$, satisfies strong monotonic and unanimity. Since it is straightforward to verify that F is monotonic, we omit its proof here. Let us show that F is binary consistent. Fix any $P \in \mathcal{P}_Z$ and any $x \in Z$. Suppose that for each $y \in Z$ that is not Pareto dominated by x at P, it holds that $x \in F(P^{xy})$. We show that $x \in F(P)$. Assume, to the contrary, that $x \notin F(P)$.

Unanimity implies that y cannot be Pareto dominated by x at P. Then, by our initial supposition, $x \in F(P^{xy})$. Strong monotonicity implies that there exist $y \in \Lambda_K^F(x)$ and $K \in \mathcal{N}_0$ such that yP_Kx. By construction of the profile P^{xy}, it follows that $K \subseteq \{i \in N | yP_i^{xy}x\} = \{i \in N | yP_ix\}$. Next, take any $\tilde{P} \in F^{-1}(x)$. Since $y \in \Lambda_K^F(x)$, it follows that $y \in L\left(x, \tilde{P}_K\right)$. Since the choice of $\tilde{P} \in F^{-1}(x)$ is arbitrary, it follows from $x \in F(P^{xy})$ that $y \in L(x, P_K^{xy})$, which is a contradiction. ■