Learning and Selfconfirming Equilibria in Network Games*

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Abstract

Consider a set of agents who play a network game repeatedly. Agents may not know the network. They may even be unaware that they are interacting with other agents in a network. Possibly, they just understand that their payoffs depend on an unknown state that in reality is an aggregate of the actions of their neighbors. Each time, every agent chooses an action that maximizes her subjective expected payoff and then updates her beliefs according to what she observes. In particular, we assume that each agent only observes her realized payoff. A steady state of such dynamic is a **selfconfirming equilibrium** given the assumed feedback. We characterize the structure of the set of selfconfirming equilibria in network games and we relate selfconfirming and Nash equilibria.

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1 Introduction

Social networks can be quite complex objects. Think about friendship networks, networks of people interacting online (as Twitter, Facebook, Instagram, ...), or even at networks of firms (input-output or R&D networks). These networks easily consist of thousands (or millions) of agents or firms interacting, and agents very rarely know how the network is shaped. In this paper we provide a novel approach to analyze how incomplete information about the network shapes behavior and learning processes. We propose a framework in which agents *may* ignore how the network affects their payoffs, how the network is shaped or, as extreme cases, even that they are interacting in a network. We analyze how agents use feedbacks they may receive to act as good as they can, to learn how to play, and we characterize behavior under different settings of local and global externalities.

1.1 Examples of applications of the model

To be more specific about our modeling approach, let us introduce an example that will guide us through the whole discussion. Imagine an online social network, like Twitter, with many users. Let us consider a simultaneous-moves game, in which each user i decides her level of activity $a_i \geq 0$ in the social network.[1] The payoff that agents get from their activity depends on the social interaction. In particular, active user i receives idiosyncratic externalities, that can be positive and negative, from the other users with whom she is in contact in the social network. The externality from user i to user j is proportional to the time that they both spend on the social network, a_i and a_j. Sticking to a quadratic specification, that allows for linear best replies, let us assume that the payoff of i from this game is[2]

$$u_i(a_i, \mathbf{a}_{-i}) = \alpha_i a_i - \frac{1}{2}a_i^2 + \sum_{j \in I \setminus \{i\}} z_{ij} a_i a_j. \tag{1}$$

In equation (1), I is the set of agents in the social network and a_i is the level of activity of $i \in I$, while α_i represents the individual pleasure of i from being active on the social network in isolation, which results in the *bliss point* of activity in autarchy. For each $j \in I \setminus \{i\}$, there is some exogenous level of externality from j to i denoted by z_{ij}. We say that j affects i, or that j is a **peer** of i, if $z_{ij} \neq 0$.

Later on, in this paper, we will also consider an extra **global** term in the payoff function

$$u_i(a_i, \mathbf{a}_{-i}) = \alpha a_i - \frac{1}{2}a_i^2 + \sum_{j \in I \setminus \{i\}} z_{ij} a_i a_j + \beta \sum_{k \in j \in I \setminus \{i\}} a_k. \tag{2}$$

[1] Even if online social networks have a dominant role in our societies, there is a very scarce literature based on game theory that models the incentive of people in these platforms. We are aware of some attempts by computer scientists, stemming from the early era of this form of interaction, as Fu *et al.* (2007). In the economic literature, the only paper that we know on this is Tarbush and Teytelboym (2017), which does not focus on the interaction between the activity of users, but on the endogenous formation of contacts.

[2] This is the class of *linear quadratic network* games originally analyzed in the economic literature by Ballester *et al.* (2006), as we discuss in the next section.

We can interpret this extra term as an additional pleasure that i gets from being member (even if not active) of an online social network that is overall *popular*.

In this paper, the network described by the matrix $\mathbf{Z}$ of all the z_{ij}'s is *exogenous*. As a first approximation, this fits a *directed* online social network like Twitter or Instagram, where users cannot decide who follows them. Under this interpretation, i receives positive or negative externalities from those who follow her, that are proportional to her activity. We assume that player i does not know any of the z_{ij}'s, either because she does not see who is following her,[3] or because she knows her followers but she does not know the sign of this externality (i.e. whether each of her followers likes or not what she writes). Player i acquires *popularity* from being active or not in the social network. Payoff represents what i can indirectly observe about her own popularity (i.e. *likes* that she receives, people congratulating with her in real world conversations, and so on...). We imagine that i cannot choose the style of what she writes, since she just follows her exogenous nature. In this interpretation, a_i represents the amount of *tweets* that i writes, and this can make her more or less popular for those who follow her, according to how her style combines with the (typically unobserved) tastes of each of her followers.

Since we are going to analyze learning dynamics and their steady states, we also have to specify what agents observe after their choices, because this affects how they update their beliefs. Twitter user i typically observes perfectly her own activity level a_i, but she may not observe the sign of the externalities and the activity of others. However, she gets indirect measures of her level of popularity that come from her conversations and experiences in the real world, where her popularity from Twitter affects her social and professional real life. If the players are small firms using Twitter for advertising, they will observe their actual profits. Players of this game may have wrong beliefs about the details of the game they are playing (e.g. the structure of the network, or the value of the parameters) and about the actions of other players. With this, they update their beliefs in response to the feedback they receive, which will be their (possibly indirectly measured) payoff. This updating process may lead to a *learning dynamic* that does not converge to a Nash equilibrium of the game.

We propose an online social network as our leading example, but there are other possibile cases in which incomplete information about the network is key. One case can be a network of firms, where the feedbacks are observed profits, and actions may be levels of production, setting prices, or R&D activities.[4] Many firms are competitors, experiencing local substitutabilities in their choices,

[3] Actually, both Twitter and Facebook have recently made it more difficult for users to access this information (see these two recent articles from the Verge, respectively about Twitter and Facebook). There are online social networks like Reddit, which do actually not provide this information at all to their users. Reddit, in particular, provides a measure to each user, that they call *karma*, which is apparently based on how many other people follow, and how much they like, what that user posts. However, the algorithm on which this measure is based is not public.

[4] These specific applications have been considered specifically in the literature, each with ad hoc assumptions and

some are *complementors*, for which local complemetarities are shown, and for some of them it may not be clear the kind of strategic interaction. Sometimes the firm does not have an idea of the set of all her competitors or complementors. Moreover firms often tend to hide their investment plans and R&D choices to some of their partners, while each firm observes its own profits. In this case each firms ignore important aspects of the network, and incomplete information plays a critical role and suboptimal choices are likely to be implemented.

1.2 Scope of the model

In light of the above mentioned examples, we propose a model in which agents *may* ignore any arbitrary piece of information about how the network is shaped and how it affects payoffs. We just require agents to know that their payoff depends on own actions and on some **payoff state** (that in turn depends on network and neighbors' actions, but agents may ignore it). Actions are based on conjectures about this payoff state and best respond to them. Still, agents receive some feedback, i.e. their profits, and good conjectures must be compatible with the feedback received, that is, they must be *confirmed* by the evidence agents get. Actions and conjectures profiles that satisfy these requirements constitutes a **selfconfirming equilibrium**. Notably, in a selfconfirming equilibrium agents best respond to conjectures that can be wrong, but still believed as true since confirmed by the evidence agents get.

In a framework in which just local externalities are at work (i.e. positive or negative peer effects), there exists a discontinuity in what agents learn from their feedback about their neighborhood depending on whether they are active (choosing a positive action) or inactive (choosing a null action). In details, active players are always able to exactly infer from the feedback the level of the payoff state affecting them, even if they may have a totally wrong conjecture about how many neighbors they have or what their neighbors choose. We say that they have correct *shallow* conjectures, but possibily wrong *deep* conjectures, also because, theoretically, they may ignore that they are playing on an network. On the other hand, inactive agents receive a totally uninformative feedback, that may induce them to persist in an *inactivity trap*, making wrong conjectures confirmed, and keeping them out of a best reply. This has strong consequences in terms of actions profile that can be played by agents who try to have conjectures confirmed. We find that, if the game displays just local complementarities, as in the example from (1), then the set of selfconfirming equilibria coincides with the set of Nash ones. However, if also local substitutabilities are at play, then any arbitrary set of agents can stay inactive and be confirmed in her conjecture. In this case

different approaches from ours. For example, Bimpikis *et al.* (2019) consider Cournot competition, while Nermuth *et al.* (2013), Lach and Moraga-González (2017) and Heijnen and Soetevent (2018) consider Bertrand competition on multiple markets, modelling the environment as a network with local externalities. This is the same approach that Westbrock (2010) and König *et al.* (2019) use to model R&D local interactions between firms.

inactivity is a best reply to own confirmed conjectures, but not to others' actions. Moreover, given the linearity of best replies, we can characterize the equilibrium action profiles as Nash equilibria of reduced games in which just an arbitrary set of agents is considered to stay active. We also discuss how the structure of the (unknown) network adjacency matrix determines the existence of these equilibria.

We then study what happens when agents display an explicit dynamic of conjectures, mimicking the fact that agents try to learn what is payoff relevant to them. We consider the case of best reply dynamics in which conjectures at a given period are the conjectures that would have been confirmed in the previous period. It must be noticed that, as discussed in the paper, any adaptive learning dynamics, if it converges, it does so to a selfconfirming equilibrium. We then provide conditions on the adjacency matrix for convergence and stability of such dynamics. Again, what we find is the possibility of inactivity traps. Consider, as a matter of example, the case of firms competition (or online social networks). If, for some unfortunate case, an agent experiences a negative payoff from interaction because some of her competitors (or some of her followers), among those providing negative externalities to her, played high actions (either being aggressive on the market or giving negative feedback online), then she may choose to abstain from interacting, even if conditions on the market (or on the platform) may improve. However, by abstaining, our agent, will never know that it could be profitable to turn active again.[5]

Models of games on network have mainly focused on the impact of local externalities, since global ones just change welfare without altering the first order conditions and the optimal choice. However, when agents get feedbacks, and when feedbacks involve payoffs, the presence of global externalities may impact the way in which conjectures are confirmed. In terms of our setting, a problem is not solely characterized by its first order condition, but also by the structure of the payoff function. For simplicity we consider just the case of positive local and global externalities. Even in this simple case, agents ignoring important features of the network may have many possible conjectures about the relative size of the two externalities. Most importantly, we find that active agents are no more able to perfectly induce the size of the local externality, and thus can rationally choose suboptimal acitons. We get an important result: by changing how agents conjecture how central they are in the network, they can play higher or lower levels of actions than predicted by a Nash equilibrium, and these wrong conjectures can be confirmed by their feedback. In the case of positive externalities, then, having agents thinking to be more central, than what they actually are, is welfare improving. If we consider the example of online social networks, this may help explaining why firms always try to send to their users messages to make them believe that they

[5] Actually, for the online application, this inactivity trap is perceived by the platforms, at the point that many of them, after some period of inactivity of agents, start sending emails about what is happening on the online social network to provide a positive signal and make agents more prone to be active again.

are very connected, so to increase their level of activity. At the same time, using the investing firms network, one can get an intuition of why firm under(over)–invest with respect to what would be the optimal plan, since they may under(over)–estimate what their neighbors do, without being corrected by the feedback that they receive.

The paper is structured as follows. In Section 2 we discuss the related literature. Section 3 presents our baseline model, while Section 4 discusses the maintained assumption of the model. We characterize the set of selfconfirming equilibria in Section 5, and we study the learning process in Section 6. In Section 7 we analyze a more general model that accounts for global externalities.

Section 8 concludes. We devote appendices to proofs and technical results. Appendix I analyzes properties of feedback and selfconfirming equilibria in a class of games including as special cases the linear quadratic network games that we consider. In Appendix J we study what happens when players have more information about the network structure introducing rationalizability. Appendix K reports existing and novel results in linear algebra, that we use to find sufficient conditions for reaching interior Nash equilibria in network games. Appendix L contains the proofs of our propositions.

2 Related literature

We model interaction with *linear quadratic network games*. We focus on this class because it has very well known properties, and it has been used for modeling a variety of different environments where strategic interaction is local and can be described by a network structure, as surveyed by Zenou (2016) and Bramoullé and Kranton (2016). Moreover, the property of this class of games put them in the broader class of *nice games* (Moulin, 1984), for which we provide in Appendix I some general results. Bramoullé *et al.* (2014) show that, if we focus only on best reply correspondences, then there are many other payoff structures that have the same Nash equilibria of linear quadratic network games. However, we focus on *selfconfirming equilibria* (SCE), and, as will be clear from the explanation in next section, we cannot abstract from the specific original payoff function of network games, as introduced in the economic literature by Ballester *et al.* (2006).

Before we relate to other papers, a **terminological clarification** is necessary. Following Battigalli *et al.* (2015), we call "selfconfirming equilibria" the steady states of learning processes when static or dynamic games are played recurrently, independently of what assumptions are made about feedback (monitoring) at the end of each one-period play (see also Battigalli *et al.* 1992). This, therefore, encompasses what used to be called "conjectural equilibrium," which describes (imperfect) feedback explicitly, as well as the original "selfconfirming equilibrium" of Fudenberg and Levine (1993), who implicitly maintain that the whole path of play is observed at the end of each period. In an SCE, agents best respond to confirmed conjectures that may be inconsistent with sophisticated strategic reasoning. The latter has been added to SCE relating it to rationalizability. Rationalizable SCE represents states at which it is true and commonly believed (given the interactive knowledge about the game) that agents are rational and their conjectures are confirmed (e.g., Esponda 2013). A weaker refinement of SCE just requires that agents best reply to confirmed conjectures consistent with common belief in rationality, but possibly inconsistent with common belief in the confirmation of conjectures (e.g., Battigalli 1987, Battigalli and Guaitoli 1997). This is SCE with rationalizable conjectures.[6] We mostly focus on SCE. We analyze SCE with rationalizable conjectures in Section Appendix J. Lipnowski and Sadler (2019) apply SCE and rationalizable SCE to games where feedback about the actions of others is described by a network topology: agents observe only the actions of their peers (neighbors), but their payoff may depend on everybody's actions and it is not observed ex post.We instead assume that agents observe (only) their realized payoff and that the network describes how the payoff of each agent is affected by the actions of other agents.[7] McBride (2006) applies SCE (called "conjectural equilibrium") to games of network formation with asymmetric information. In his model, agents observe (only) the private information of other agents they link to, and possibly of agents to whom they are indirectly linked. We instead assume that the network is exogenous and actions are activity levels. We allow for information incompleteness, but – except for Section Appendix J – we do not assume that agents are necessarily aware of states of nature (e.g., the possible network structures) or reason about them. Frick *et al.* (2018) apply a refinement of rationalizable SCE to analyze a model with asymmetric information and assortative matching. The refinement is obtained by assuming that agents neglect the assortativity of matching when they make inferences from feedback. Foerster *et al.* (2018) shares elements of Lipnowski and Sadler (2019) and of McBride (2006). Like the former, agents observe other agents to whom they are linked, but also observe public links. Like the latter, theirs is a model of network formation. They assume that beliefs satisfy a kind of rationalizable SCE condition. Unlike those papers, however, Foerster *et al.* (2018) do not explicitly analyze the equilibria of a non-cooperative game, but rather adopt a reduced-form notion of stability akin to Jackson and Wolinsky (1996).

3 Basic Framework

Consider a finite set of agents I, with cardinality $n = |I|$ and generic element i. Agents are located in a network $\mathbf{Z} \in \mathcal{Z} \subseteq \mathbb{R}^{I \times I}$, where $\mathcal{Z}$ is the *compact* set of all possibile networks, here expressed

[6]Unlike SCE, rationalizable SCE lacks a learning foundation. SCE in rationalizable actions, instead, can be justified as the result of convergent learning processes in repeated games played by myopic agents under common strong belief in rationality.

[7]There are other differences: unlike Lipnowski and Sadler (2019), we analyze static (one-period) games. Also, when we analyze strategic reasoning we just consider SCEs consistent with common belief in rationality, whereas rationalizable SCE also requires common belief in the confirmation of conjectures.

as adjacency matrices. Each agent $i \in I$ chooses an action a_i from a *compact interval* $A_i = [0, \bar{a}_i]$, where the upper bound $\bar{a}_i$ is "sufficiently large".[8] For each $i \in I$, $\mathbf{A}_{-i} := \times_{j \neq i} A_j$ denotes the set of feasible action profiles $\mathbf{a}_{-i} = (a_j)_{j \in I \setminus \{i\}}$ for players different from i. For each $i \in I$, we posit a *compact interval* $X_i := [\underline{x}_i, \bar{x}_i] \subset \mathbb{R}$ of **payoff states for** i, with the interpretation that i's payoff is determined by her action a_i and by her payoff state x_i according to a utility function

$$v_i : A_i \times X_i \to \mathbb{R}, \tag{3}$$

where v_i is *strictly quasi-concave* in a_i and *continuous.*[9] The payoff state x_i is in turn determined by the actions of i's neighbors and is unknown to i at the time of his choice. In details, for each agent $i \in I$, given the parameter space $\mathcal{Z}$, we consider a *continuous* parameterized **aggregator** of the co-players' actions

$$\ell_i : \mathbf{A}_{-i} \times \mathcal{Z} \to X_i \tag{4}$$

such that its *range* $\ell_i (\mathbf{A}_{-i} \times \mathcal{Z})$ is *connected.*[10] With this, we derive the **parameterized payoff function**

$$\begin{array}{rccc} u_i : & A_i \times \mathbf{A}_{-i} \times \mathcal{Z} & \to & \mathbb{R}, \\ & (a_i, \mathbf{a}_{-i}, \mathbf{Z}) & \mapsto & v_i (a_i, \ell_i (\mathbf{a}_{-i}, \mathbf{Z})). \end{array}$$

Thus, $x_i = \ell_i (\mathbf{a}_{-i}, \mathbf{Z})$ is the payoff-relevant state that i has to guess in order to choose a subjectively optimal action. With this, for each $\mathbf{Z} \in \mathcal{Z}$, $\langle I, (A_i, u_{i,\mathbf{Z}})_{i \in I} \rangle$ is a *nice game* (Moulin, 1979), and $\langle I, \mathcal{Z}, (A_i, u_i)_{i \in I} \rangle$ is a *parameterized nice game.* We let

$$\begin{array}{rccc} r_i : & X_i & \to & A_i \\ & x_i & \mapsto & \arg\max_{a_i \in A_i} v_i (a_i, x_i) \end{array}$$

denote the continuous **best reply function** of player $i \in I$. To choose an action, subjectively rational agents must have some deterministic or probabilistic conjecture about the payoff state x_i. We refer to conjectures about the state x_i as **shallow conjectures**, as opposed to **deep conjectures,** which concern the specific network topology $\mathbf{Z}$ and the actions of other players $\mathbf{a}_{-i}$. In our framework, given the continuous best-reply function and the connectedness of X_i, it is sufficient to focus on *deterministic shallow conjectures.* Indeed, for each $i \in I$, for every probabilistic

[8] Note that in the network literature it is common to assume $A_i = \mathbb{R}_+$. However, for the games we consider, we can always find an upper bound $\bar{a}$ on actions such that the problem is unchanged when actions are bounded above by $\bar{a}$. Even for the case of externalities with complementarities, we assume constraints on the parameters so that assuming an upper bound on actions is without loss of generality.

[9] That is, v_i is jointly continuous in (a_i, x_i) and, for each $x_i \in [\underline{x}_i, \bar{x}_i]$, the section $v_{i,x_i} : [0, \bar{a}_i] \to \mathbb{R}$ has a unique maximizer a_i^* (that typically depends on x_i), it is strictly increasing on $[0, a_i^*]$, and it is strictly decreasing on $[a_i^*, \bar{a}_i]$. Of course, the monotonicity requirement holds vacuously when the relevant subinterval is a singleton.

[10] Since the range of each section $\ell_{i,\mathbf{Z}}$ must be a closed interval, we require that the union of the closed intervals $\ell_{i,\mathbf{Z}} (\mathbf{A_{-i}})$ $(\mathbf{Z} \in \mathcal{Z})$ is also an interval, which must be closed because $\mathcal{Z}$ is compact and ℓ_i continuous.

conjecture $\mu_i \in \Delta(X_i)$, there exists a corresponding deterministic conjecture $\hat{x}_i \in X_i$ that justifies the same action a_i^* as the unique best reply [11], that is, $\hat{x}_i$ is such that

$$r_i(\hat{x}_i) = \underset{a_i \in A_i}{\operatorname{argmax}} \, \mathbb{E}_{\mu_i}[v_i(a_i, \cdot)]. \tag{5}$$

We assume that the game is repeatedly played by agents maximizing their instantaneous payoff. After each play agents get some feedback. Let $M \subseteq \mathbb{R}$ be an abstract set of "messages". The information obtained by agent $i \in I$ at the end of each period, after taking action a_i is described by a **feedback function** $f_i : A_i \times X_i \to M$.[12] We assume that each agent $i \in I$ knows how her payoff depends on her action and on her payoff state, that is, we assume that i knows function v_i, but we do *not* assume that i knows $\mathbf{Z}$. Actually, from the perspective of our analysis, agent i might even ignore how the payoff state x_i is formed and, as an extreme case, that a network may even play a role, because *we are not modeling how i reasons strategically.*[13] Assuming that i knows how her feedback is determined by the payoff state given her action, if she receives message m after choosing and recalling action a_i, she infers that the state x_i belongs to the "ex post information set"

$$f_{i,a_i}^{-1}(m) := \left\{x_i' \in X_i : f_i\left(a_i, x_i'\right) = m\right\}.$$

This completes the description of the object of our analysis. The structure

$$NG = \left\langle I, \mathcal{Z}, (A_i, X_i, v_i, \ell_i, f_i)_{i \in I} \right\rangle$$

is a (parameterized) **network game with feedback,** or simply **network game.** Our analysis depends on assumptions about the payoff functions and the feedback functions described in Section 4. Section Appendix I contains a more general analysis.

3.1 Selfconfirming equilibrium

We analyze a notion of equilibrium which is broader than Nash equilibrium. Recall that our approach allows for the possibility of agents who are unaware of the full game. In equilibrium—i.e., in a steady state—agents best respond to conjectures consistent with the feedback that they receive, which is not necessarily fully revealing.

Definition 1. *A profile $(a_i^*, \hat{x}_i)_{i \in I} \in \times_{i \in I}(A_i \times X_i)$ of actions and (shallow) deterministic conjectures is a* ***selfconfirming equilibrium (SCE) at Z*** *if, for each $i \in I$,*

1. (subjective rationality) $a_i^* = r_i(\hat{x}_i)$,

[11] See the discussion in Section I.1

[12] Here the assumption that M is a set of real numbers is without loss of generality, because the same holds for the set of payoff states X_i.

[13] We relate to strategic reasoning in Section Appendix J.

2. (confirmed conjecture) $f_i(a_i^*, \hat{x}_i) = f_i\left(a_i^*, \ell_i\left(\mathbf{a}_{-i}^*, \mathbf{Z}\right)\right)$.

The two conditions require that 1) each agent best responds to her own conjecture; 2) the conjecture in equilibrium must belong to the ex-post information set, so that the expected feedback coincides with the actual feedback at $\ell_i\left(\mathbf{a}_{-i}^*, \mathbf{Z}\right)$. We say that $\mathbf{a}^* = (a_i^*)_{i \in I}$ is a **selfconfirming action profile** at $\mathbf{Z}$ if there exists a corresponding profile of conjectures $(\hat{x}_i)_{i\in I}$ such that $(a_i^*, \hat{x}_i)_{i\in I}$ is a selfconfirming equilibrium at $\mathbf{Z}$, and we let $\mathbf{A}_{\mathbf{Z}}^{SCE}$ denote the set of such action profiles. Also, for any adjacency matrix $\mathbf{Z} \in \mathcal{Z}$, we denote by $\mathbf{A}_{\mathbf{Z}}^{NE}$ the set of (pure) Nash equilibria of the (nice) game determined by $\mathbf{Z}$, that is,

$$\mathbf{A}_{\mathbf{Z}}^{NE} := \left\{\mathbf{a}^* \in \times_{i\in I} A_i : \forall i \in I, a_i^* = r_i\left(\ell_i\left(\mathbf{a}_{-i}^*, \mathbf{Z}\right)\right)\right\}.$$

Nice games satisfy all the standard assumptions for the existence of Nash equilibria.[14] Hence, we obtain the existence of selfconfirming equilibria for each $\mathbf{Z} \in \mathcal{Z}$. Indeed a Nash equilibrium is a selfconfirming equilibrium with correct conjectures. To summarize:

REMARK **1.** *For every* $\mathbf{Z}$*, there is at least one Nash equilibrium, and every Nash equilibrium is a selfconfirming profile of actions:*

$$\forall\ \mathbf{Z} \in \mathcal{Z},\ \emptyset \neq \mathbf{A}_{\mathbf{Z}}^{NE} \subseteq \mathbf{A}_{\mathbf{Z}}^{SCE}.$$

4 Maintained assumptions

We now present some maintained assumptions, we are going to use throughout the paper. All the (few) exceptions will be explicitly written in the text.

The network Recall that the network is characterized by an adjacency matrix $\mathbf{Z} \in \mathbb{R}^{I\times I}$, where entry z_{ij} specifies whether agent i is linked to agent $j \neq i$ and the weight of this link, and we let $z_{ii} = 0$ by convention. In what follows we consider the case of directed networks, so that, given $i, j \in I$, we allow $z_{ij} \neq 0$, and $z_{ji} = 0$. Local externality weights may be an unknown parameter of the model. We assume that there are commonly known upper and lower bounds $\bar{w}$ and $\underline{w}$ in the weighted local externalities, that can be positive or negative. We let $\mathcal{Z} \subseteq [\underline{w}, \bar{w}]^{I\times I}$ denote the compact set of possible weighted networks $\mathbf{Z}$. The network game is parameterized by $\mathbf{Z} \in \mathcal{Z}$.

Throughout the paper we play with different properties and specifications of matrix $\mathbf{Z}$. To simplify the notation we often decompose $\mathbf{Z}$ in a way that distinguishes between the actual links, that specify if there is an externality between two players, and the magnitude and the sign of this externality. We call $\mathbf{Z}_0 \in \{0,1\}^{I\times I}$ the basic underlying representation of the network, the

[14] Since the self-map $\mathbf{a} \mapsto (r_i(\mathbf{a}_{-i}, \mathbf{Z}))_{i\in I}$ is continuous on the convex and compact set $A = \times_{i\in I}[0, \bar{a}_i]$, by Brouwer's Theorem it has a fixed point.

adjacency matrix whose ij element specifies whether the action of j has an externality on i. We think of it as a link from i to j because j is one of i's peers. $\mathbf{Z}_0$ is a directed network.

On top of that we build $\mathbf{Z}$ adding weights on the links of $\mathbf{Z}_0$. This can be done in several ways, depending on how much heterogeneity we want to allow for. We will write $\mathbf{Z} = w\mathbf{Z}_0$ when all links bear the same level of externality $w \in [\underline{w}, \bar{w}]$. We will write $\mathbf{Z} = \mathbf{W}\mathbf{Z}_0$, where $\mathbf{W}$ is a diagonal matrix, when we want to specify that each player i is affected by the same weight $w_i \in [\underline{w}, \bar{w}]$ from all her peers, but these w_i's may be heterogeneous. We will also consider the case in which the existing links may have weights of different signs but same intensity. That is, we write $\mathbf{Z} = \mathbf{S} \odot \mathbf{Z}_0$ (in which the operator $\odot$ is the Hadamard product), for $w \in [0, \bar{w}]$, and $\mathbf{S} \in \{-w, w\}^{I \times I}$. Finally, when we write simply $\mathbf{Z}$, we consider the case of a generic directed weighted network $\mathbf{Z} \in \mathcal{Z}$. Many of our results will hold for this most general case.

The parametrized aggregator For each agent $i \in I$ and matrix $\mathbf{Z} \in \mathcal{Z}$, we model the parameterized **aggregator** of the co-players' actions $\ell_i : \mathbf{A}_{-i} \times \mathcal{Z} \to X_i$ such that the section of ℓ_i at $\mathbf{Z}$ is[15]

$$\begin{array}{rccl} \ell_{i,\mathbf{Z}}: & \mathbf{A}_{-i} & \to & X_i, \\ & \mathbf{a}_{-i} & \mapsto & \sum_{j \neq i} z_{ij} a_j. \end{array}$$

Thus, the *payoff-relevant state for i depends only on her neighbors' actions.* Since X_i is the codomain of ℓ_i, we are effectively assuming that, for every $\mathbf{Z} \in \mathcal{Z}$,

$$\underline{x}_i \leq \sum_{j \in N_i^-} z_{ij} \bar{a}_j, \; \bar{x}_i \geq \sum_{j \in N_i^+} z_{ij} \bar{a}_j,$$

where $N_i^- := \{j \in I : z_{ij} < 0\}$ denotes the set of neighbors of player i that have a negative effect on the payoff state of i, and $N_i^+ := \{j \in I : z_{ij} > 0\}$ denotes the set of neighbors of player i that have a positive effect on the payoff state of i.

In Section 7, we consider a second aggregator that is based, in a similar way, on the actions of all the other players, and is not based on a network structure. We will distinguish between a *local* aggregator ℓ_i (the one specified above) and a global aggregator g_i.

The payoff function Notice that, although the aggregator is linear, if the "proximate" best reply function $r_i : X_i \to A_i$ is non-linear,[16] then also the best reply $r_i(\ell_i(\mathbf{a}_{-i}, \mathbf{Z}))$ is non-linear in $\mathbf{a}_{-i}$. Linearity obtains if and only if v_i is quadratic in a_i and linear in x_i. Without substantial loss of generality, among such utility functions we consider the following form, generalizing equation (1)

[15] In principle we can allow for non–linear aggregators, as in Feri and Pin (2017). However, in this paper, we focus on the linear case.

[16] More precisely, not affine.

that we discussed earlier:

$$\begin{array}{rccl} v_i : & A_i \times X_i & \to & \mathbb{R}, \\ & (a_i, x_i) & \mapsto & \alpha_i a_i - \frac{1}{2} a_i^2 + a_i x_i. \end{array} \tag{6}$$

with $\alpha_i > 0$ for each $i \in I$,

DEFINITION **2.** *A network game with feedback NG is* ***linear-quadratic*** *if the utility function of each player has the linear-quadratic form (6), up to non strategic additional terms.*

In this case, the proximate best-reply function is

$$r_i(x_i) = \begin{cases} 0, & \text{if } x_i \leq -\alpha_i, \\ \alpha_i + x_i, & \text{if } -\alpha_i < x_i < \bar{a}_i - \alpha_i, \\ \bar{a}_i, & \text{if } x_i \geq \bar{a}_i - \alpha_i. \end{cases} \tag{7}$$

From this we can derive the best reply to the actions of others given $\mathbf{Z}$:

$$r_i(\ell_i(\mathbf{a}_{-i}, \mathbf{Z})) = \begin{cases} 0, & \text{if } \sum_{j \neq i} z_{ij} a_j \leq -\alpha_i, \\ \alpha_i + \sum_{j \neq i} z_{ij} a_j, & \text{if } -\alpha_i < \sum_{j \neq i} z_{ij} a_j < \bar{a}_i - \alpha_i, \\ \bar{a}_i, & \text{if } \sum_{j \neq i} z_{ij} a_j \geq \bar{a}_i - \alpha_i. \end{cases} \tag{8}$$

In Section 7 we will add the global aggregator in the payoff function, but in a way that will not change the best reply of the players.

The feedback We now discuss the most important properties of the feedback we assume to hold in our framework.

DEFINITION **3.** *Feedback f_i satisfies* ***observability if and only if player*** i ***is active*** *(OiffA) if section f_{i,a_i} is injective for each $a_i \in (0, \bar{a}_i]$ and constant for $a_i = 0$; f_i satisfies* ***just observable payoffs*** *(JOP) relative to v_i if there is a function $\bar{v}_i : A_i \times M \to \mathbb{R}$ such that*

$$\forall (a_i, x_i) \in A_i \times X_i,\ v_i(a_i, x_i) = \bar{v}_i(a_i, f_i(a_i, x_i))$$

and the section $\bar{v}_{i,a_i} : M \to \mathbb{R}$ is injective for each $a_i \in A_i$. A network game with feedback NG satisfies ***observability by active players*** *if feedback f_i satisfies OiffA, for each player $i \in I$, and it satisfies* ***just observable payoffs*** *if f_i satisfies JOP for each player $i \in I$.*

In a game with just observable payoffs, because of injectivity of the feedback function, agents infer their realized payoff from the message they get, but no more than that. That is, inferences about the payoff state can be obtained by looking at the preimages of the payoff function. For example, the feedback could be a total benefit, or revenue function

$$\begin{array}{rccl} f_i : & A_i \times X_i & \to & \mathbb{R}, \\ & (a_i, x_i) & \mapsto & \alpha_i a_i + a_i x_i, \end{array}$$

with the payoff given by the difference between benefit and activity cost $C_i(a_i)$:

$$\begin{array}{rccl} v_i: & A_i \times X_i & \to & \mathbb{R}, \\ & (a_i, x_i) & \mapsto & f_i(a_i, x_i) - C_i(a_i). \end{array}$$

Under the reasonable assumption that agent i knows her cost function, when she chooses a_i and then gets message m, she infers that her payoff is $\bar{v}_i(a_i, m) = m - C_i(a_i)$. Thus, each section $\bar{v}_{i,a_i}$ ($a_i \in A_i$) is indeed injective. If the feedback/benefit function is $f_i(a_i, x_i) = \alpha_i a_i + a_i x_i$, then it satisfies observability if and only if i is active. Until Section 7, we maintain the assumption that $f_i = v_i$ and that the game satisfies just observable payoffs.

Remark **2.** *If NG is* linear-quadratic *and satisfies* just observable payoffs*, then it satisfies observability by active players. If NG satisfies observability by active players, then*

$$f_{i,a_i}^{-1}(f_i(a_i, x_i)) = \begin{cases} X_i, & \text{if } a_i = 0, \\ \{x_i\}, & \text{if } a_i > 0 \end{cases} \tag{9}$$

for every agent $i \in I$ and action-state pair $(a_i, x_i) \in A_i \times X_i$.

Most of our analysis, up to Section 7, focuses on linear-quadratic network games with feedback and just observable payoffs (this is what we call *network game*). This implies that agents who are active get as feedback a message enabling them to perfectly determine the state. Conversely, inactive agents get a completely uninformative message.

5 A characterization of SCE

In this section we characterize the set $\mathbf{A}_{\mathbf{Z}}^{SCE}$ of selfconfirming equilibrium action profiles at $\mathbf{Z}$. All our proofs are derived from the results in Appendix I, which refers to the case of generic network games without the restriction to linear best replies, and from the results in Appendix K. The proofs are in Appendix L. We start with the simplest case in which every agent necessarily finds it subjectively optimal to be active (that is, being inactive is dominated – see Lemma A in Appendix I).

Proposition **1.** *Consider a network game such that, for every $i \in I$ and for every $\hat{x}_i \in X_i$, $r_i(\hat{x}_i) > 0$. Then, for each $\mathbf{Z} \in \mathcal{Z}$, $\mathbf{A}_{\mathbf{Z}}^{SCE} = \mathbf{A}_{\mathbf{Z}}^{NE}$.*[17]

Assume that $\mathbf{Z} = w\mathbf{Z}_0$, with $w > 0$ and that $\mathbf{Z}_0 \in \{0,1\}^{I \times I}$. In this context it is natural to assume that $\min X_i \geq 0$, hence that conjectures $\hat{x}_i$ are not negative. This represents the standard case of local complementarities studied by Ballester *et al.* (2006). If $w(n-1) < 1$ there is a unique

[17] Given the stated assumptions about feedback, the same result holds also for non linear and continuous aggregators ℓ_i and continuous and strictly quasi-concave utility functions v_i.

Nash equilibrium which is also interior. Our proposition states that, in this case, since being inactive is not justifiable as a best reply to any shallow conjecture, then there is only one selfconfirming equilibrium action profile, which necessarily coincides with the unique Nash equilibrium.

We now consider a more general case in which agents may be inactive. Let I_0 denote the **set of players for whom being inactive is justifiable**:[18]

$$I_0 = \{i \in I : \min r_i(X_i) = 0\}.$$

Also, for each $\mathbf{Z} \in \mathcal{Z}$ and a non–empty subset of players $J \subseteq I$, let $\mathbf{A}^{NE}_{J,\mathbf{Z}}$ denote the set of Nash equilibria of the auxiliary game with player set J obtained by imposing $a_i = 0$ for each $i \in I \backslash J$, that is,

$$\mathbf{A}^{NE}_{J,\mathbf{Z}} = \left\{\mathbf{a}^*_J \in \times_{j\in J} A_j : \forall j \in J, a^*_j = r_j\left(\ell_j\left(\mathbf{a}^*_{J\backslash\{j\}}, \mathbf{0}_{I\backslash J}, \mathbf{Z}\right)\right)\right\},$$

where $\mathbf{0}_{I\backslash J} \in \mathbb{R}^{I\backslash J}$ is the profile that assigns 0 to each $i \in I\backslash J$. If $J = \emptyset$, let $\mathbf{A}^{NE}_{J,\mathbf{Z}} = \{\varnothing\}$ by convention, where $\varnothing$ is the peudo-action profile such that $(\varnothing, \mathbf{0}_I) = \mathbf{0}_I$.[19] We relate the set of selfconfirming equilibria to the sets of Nash equilibria of such auxiliary games.

Proposition **2.** *Given a network game, for each $\mathbf{Z} \in \mathcal{Z}$, the set of selfconfirming action profiles is*

$$\mathbf{A}^{SCE}_{\mathbf{Z}} = \bigcup_{I\backslash J\subseteq I_0} \mathbf{A}^{NE}_{J,\mathbf{Z}} \times \left\{\mathbf{0}_{I\backslash J}\right\},$$

that is, in each SCE profile $\mathbf{a}^*$, a subset $I\backslash J$ of players for whom being inactive is justifiable choose 0*, and every other player chooses the best reply to the actions of her co-players. Therefore, in each SCE profile $\mathbf{a}^*$ and for each player $i \in I$,*

$$\begin{aligned} a^*_i &= 0 \Rightarrow \underline{x}_i \leq -\alpha_i, \\ a^*_i &> 0 \Rightarrow \left(\alpha_i + \sum_{j\in I} z_{ij}a^*_j > 0 \ \wedge a^*_i = \min\left\{\bar{a}_i, \alpha_i + \sum_{j\in I} z_{ij}a^*_j\right\}\right). \end{aligned} \tag{10}$$

In every SCE we can partition the set of agents in two subsets. Agents in $J \subseteq I$ are active, i.e., they choose a strictly positive action, agents in $I \setminus J$ instead choose the null action. Start considering the latter. Since they play $a^*_i = 0$, they get null payoff independently of others' actions. Since every conjecture is consistent with this payoff, their conjecture is (trivially) consistent with their feedback. As for agents in J, since they choose a strictly positive action $a^*_i > 0$, they receive a message that enables them to infer the true payoff state x^i; with this, they necessarily choose the

[18]This definition is consistent with Lemma A in Appendix I. In Section I.1 we discuss also the more general case of probabilistic conjectures and why not applying it to our context is without loss of generality.

[19]As we do in set theory with the empty set, when we consider functions whose domain is a subset of some index set I, it is convenient to have a symbol for the pseudo-function with empty domain. For example, if $I = \mathbb{N}$, such functions are (finite and countably infinite) sequences, or subsequences, and $\varnothing$ is the empty sequence.

objective best reply to their neighbours actions, whether or not they are aware of them. Note that, if being inactive is justifiable for every agent ($I_0 = I$), then $\mathbf{0}_I \in \mathbf{A}^{SCE}_{\mathbf{Z}}$ for every $\mathbf{Z} \in \mathcal{Z}$.

This implies that the set of selfconfirming equilibria can be characterized by means of the sets of Nash equilibria of the auxiliary games in which only active agents are considered. If, for example, there is a unique interior Nash equilibrium for the auxiliary game corresponding to every subset of active players, then $|\mathbf{A}^{SCE}_{\mathbf{Z}}| = 2^{|I|}$, that is, there are exactly 2^n SCE action profiles. If we allow for strategic substitutes, then the Nash equilibria for each auxiliary game, in which only agents in $J \subseteq I$ may be active, can be characterized as in Bramoullé *et al.* (2014). Note that in this case, some of the agents in J can be active and some inactive. Appendix I.3 discusses the equilibrium characterization for the generalized case of non linear-quadratic network games.

Example 1. Consider Figure 1, representing a network among 4 nodes. We set $\alpha_i = 0.1$ for each player i. Let us first assume that each arrow represents a positive externality of 0.2 (and arrows point to the source of the externality), but allow agents to believe that links may also be a source of negative externality. Then, agents may find it justifiable to be inactive. In this case we have one NE, but 16 possible SCEs, one for each subset of the players that we allow to be active. Table 1 reports the actions of players in each case (we omit redundant pairs and singletons). Note that player 3, when active, always plays the same action $a_3 = 0.1$, because she is not affected by any externality. Other players, instead, play differently when active, according to who else is active.

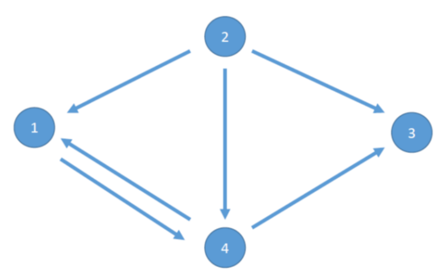

Figure 1: A network between 4 nodes. Every arrow is for an externality of equal magnitude and sign.

Consider now the same network, but assume that each arrow represents a negative externality of 0.6. In this case we have more NEs (there is no NE where all players are active, but there are 3 NEs), but less than 16 SCEs (there are 13), because for some subset J of players (such as $J = I = \{1, 2, 3, 4\}$) there is no SCE in which all its agents are active. Table 2 reports the actions of players in each case (we omit redundant pairs and singletons). ∎

	All	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	...	$\emptyset$
a_1	0.1292	0.1	0.125	0.1292	0	0.1	0.1	0.125		0
a_2	0.1750	0.14	0.15	0	0.144	0.12	0	0		0
a_3	0.1	0.1	0	0.1	0.1	0	0.1	0		0
a_4	0.1458	0	0.125	0.1458	0.12	0	0	0.125		0

Table 1: Self confirming equilibria of the network from Figure 1, with all positive externalities of 0.2. The unique Nash Equilibrium is in bold.

	$\{\mathbf{1},\mathbf{2},\mathbf{4}\}$	$\{\mathbf{2},\mathbf{3},\mathbf{4}\}$	$\{1,2\}$	$\{\mathbf{1},\mathbf{3}\}$	$\{1,4\}$	...	$\emptyset$
a_1	0.0625	0	0.1	0.1	0.0625		0
a_2	0.025	0.016	0.04	0	0		0
a_3	0	0.1	0	0.1	0		0
a_4	0.0625	0.04	0	0	0.0625		0

Table 2: Self confirming equilibria of the network from Figure 1, with all negative externalities of -0.6. Nash Equilibria are in bold.

5.1 Assumptions about the network

Next, we focus on the network $\mathbf{Z}$. We list below some properties of matrix $\mathbf{Z}$ that are *not* maintained assumptions. In different parts of the paper we will use some of these assumptions to have sufficient conditions for the existence and stability of selfconfirming equilibria. We refer to Appendix K for a deeper discussion on these assumptions and their implications.

Assumption **1.** *Matrix* $\mathbf{Z}$ *of size* n *has* bounded values, *i.e. for each* $i, j \in I$, $|z_{ij}| < \frac{1}{n}$.

Assumption **2.** *Matrix* $\mathbf{Z}$ *has the* same sign property *i.e., for each* $i, j \in I$, $sign(z_{ij}) = sign(z_{ji})$, *where the sign function can have values* -1, 0 *or* 1.[20]

Assumption **3.** *Matrix* $\mathbf{Z}$ *is* negative, *i.e. for each* $i, j \in I$, $z_{ij} < 0$.

We recall here that the spectral radius $\rho(\mathbf{Z})$ of $\mathbf{Z}$ is the largest absolute value of its eigenvalues.

Assumption **4.** *Matrix* $\mathbf{Z}$ *is* limited, *i.e.* $\rho(\mathbf{Z}) < 1$.

In Section 4 we discussed how, in some cases, we can write $\mathbf{Z} = \mathbf{W}\mathbf{Z}_0$, where $\mathbf{W}$ is a diagonal matrix, and $\mathbf{Z}_0$ is the basic underlying topology of the network. When this is possible, matrix $\mathbf{Z}$ represents a basic network combined with an additional idiosyncratic effect by which every agent

[20]The sign condition is the one used in Bervoets *et al.* (2016) to prove convergence to Nash equilibria in network games, under a particular form of learning.

i weights the effects of the others on her. This effect is modeled by the parameter w_i.[21] The next assumption adds an additional condition on $\mathbf{Z}_0$.

Assumption **5.** *Matrix* $\mathbf{Z}$ *is* symmetrizable, *i.e. it can be written as* $\mathbf{Z} = \mathbf{W}\mathbf{Z}_0$, *with* $\mathbf{W}$ *diagonal and* $\mathbf{Z}_0$ *symmetric. Moreover,* $\mathbf{W}$ *has all strictly positive entries in the diagonal.*

Note that if $\mathbf{Z}$ is symmetrizable then all its eigenvalues are real. Moreover, since $\mathbf{W}$ has all positive entries, Assumption 5 implies that the sign condition (from Assumption 2) holds.
Our final assumption is discussed in Bramoullé *et al.* (2014) and combines Assumptions 4 and 5 above.

Assumption **6.** $\mathbf{Z} = \mathbf{W}\mathbf{Z}_0$ *is* symmetrizable-limited, *i.e.* $\mathbf{Z}$ *is symmetrizable and the matrix* $\bar{\mathbf{Z}}$*, defined for each* $i, j \in I$ *as* $\bar{z}_{ij} = z_{0,ij}\sqrt{w_i w_j}$*, is limited.*

Our previous results from Section 5, about the characterization of selfconfirming equilibria, state that we can choose any subset $J \subseteq I_0$ of agents and have them inactive in a SCE. However, we cannot ensure that the other agents are active, because their best response in the reduced game could be to stay inactive, since the Nash equilibrium of the reduced game in which only agents in $I \backslash J$ are considered may have both active and inactive agents. The next result goes in the direction of specifying under which sufficient conditions this does not happen. Given the matrix $\mathbf{Z}$, and given $J \subseteq I$, we call $\mathbf{Z}_J$ the submatrix which has only rows and columns corresponding to the elements of J.

Proposition **3.** *Consider a set* $J \subseteq I$*. Assume that, for every* $i \in I$ *there exists* $\hat{x}_i \in X_i$*, such that* $r_i(\hat{x}_i) = 0$*. Let us assume that* $\mathbf{Z}_J$ *satisfies at least one of the three conditions below:*

1. *it has bounded values (Assumption 1),*
2. *it is negative and limited (Assumptions 3 and 4),*
3. *or it is symmetrizable–limited (Assumption 6).*

Then, we have the two following results:

1. *the auxiliary game with player set* J *has a unique and strictly positive Nash equilibrium:* $\mathbf{A}^{NE}_{J,\mathbf{Z}} = \left\{\mathbf{a}^{NE}_J\right\}$ *with* $\mathbf{a}^{NE}_J > 0$*;*
2. $(\mathbf{a}^{NE}_J, \mathbf{0}_{I\backslash J})$ *is a selfconfirming equilibrium at* $\mathbf{Z}$*.*

[21] Then the payoff of $i \in I$ at a given profile $\mathbf{a}$ of the original game is

$$u_i(\mathbf{a}) = \alpha a_i - \frac{1}{2}a_i^2 + a_i w_i \sum_{j\in I} z_{0,ij} a_j = \alpha a_i - \frac{1}{2}a_i^2 + a_i \sum_{j\in I} z_{ij} a_j \ .$$

Proposition 3 provides sufficient conditions to have an arbitrary set of active and inactive players in a selfconfirming equilibrium. If any of the three conditions is satisfied for every subset of I, then the set of SCEs has the same cardinality as the power set 2^I, that is 2^n. The first sufficient condition in the statement is novel, while the other two were obtained respectively by Ballester *et al.* (2006) and Stańczak *et al.* (2006), and by Bramoullé *et al.* (2014).

We provide here below two examples, one with all positive externalities, the other with mixed externalities.

Example 2. Consider n players, and a random network between them, of the type $\mathbf{Z} = \mathbf{W}\mathbf{Z}_0$, obtained from the following generating process. $\mathbf{Z}_0$ is undirected, generated by an Erdos and Rényi (1960) process for which each link is i.i.d., and such that its expected number of overall links (i.e., counted in both directions) is $k \cdot n$, for some $k \in \mathbb{R}_+$. This means that the expected number of links for each player is k. It is well known that this model predicts, as n goes to infinity, that $\mathbf{Z}_0$ will have no clustering and, when $k \geq 2$, a connected giant component.

$\mathbf{W}$ is a diagonal matrix, such that each element w_i in the diagonal is positive and is generated by some i.i.d. random process with mean μ and variance σ^2. In this case, Füredi and Komlós (1981) prove that the expected highest eigenvalue of $\mathbf{Z}$, as n grows, is

$$E(\lambda_i) = k\mu + \frac{\sigma^2}{\mu} + O\left(\frac{1}{\sqrt{n}}\right) .$$

Under Assumption 6, as n tends to infinity, $\mathbf{Z}$ is symmetrizable–limited if $E(\lambda_i) < 1$, which implies that

$$\frac{\mu - \sigma^2}{\mu^2} > k .$$

Clearly, a necessary condition for previous inequality to hold is that $\mu > \sigma^2$. When this happens, as n grows to infinity, we will always have a unique NE of the game where all players are active, as stated by Proposition 3.

Note that, since the expected clustering of $\mathbf{Z}_0$ goes to 0, this limiting result excludes the possibility that there is a subset J of players, that have a dense sub–network between them, and a high realization of w_i's, such that there does not exist $\mathbf{a}^* \in \mathbf{A}_{\mathbf{Z}}^{SCE}$, for which $\mathbf{a}^* = \left\{\mathbf{a}_J^{NE}\right\} \times \left\{\mathbf{0}_{I\setminus J}\right\}$. In fact, if this was the case, since there are only positive externalities, we would not even have an all-active equilibrium for the whole population of n agents. ∎

Example 3. Proposition 3 provides alternative conditions, that are only sufficient, for interior NE in an auxiliary game in which only agents in J are considered. Figure 2 provides an example of game that do not satisfy any of them, but still has a unique interior NE. We set $\alpha_i = 0.1$ for each player i. Every blue arrow stands for a positive externality of 0.2 (so, the blue arrows represent just the first case from Example 1). The two red arrows stand for a negative externality of 0.2. This network game has a unique NE, and 16 SCE. Table 3 shows them all (redundant couples and singletons are omitted). ∎

	All	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	...	$\emptyset$
a_1	0.1257	0.1	0.125	0.128	0	0.1	0.1	0.125	0		0
a_2	0.1603	0.1346	0.15	0	0.144	0.12	0	0	0.1154		0
a_3	0.0412	0.731	0	0.720	0.1	0	0.1	0	0.0729		0
a_4	0.1336	0	0.125	0.14	0.12	0	0	0.125	0		0

Table 3: Self confirming equilibria of the network from Figure 2, with positive externalities of 0.2 and negative externalities of -0.2. The unique Nash Equilibrium is in bold.

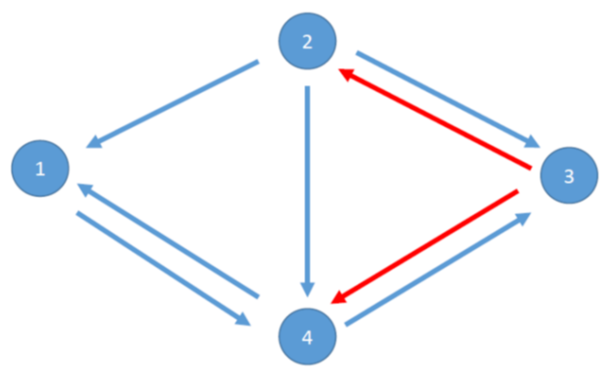

Figure 2: A network among 4 nodes. Blue arrows are for positive externalities, red arrows are for negative externalities.

6 Learning process

We have not considered any dynamics yet. Definition 1 of selfconfirming equilibrium, characterized also by the conditions stated in Proposition 2, identifies steady states: If agents happen to have selfconfirming conjectures and play accordingly, then they have no reason to move away from it. However, we may wonder how agents get to play SCE action profiles, and if these profiles are stable.

We first notice that SCE has solid learning foundations.[22] The following result is specifically relevant for this paper (see Gilli (1999) and Chapter 6 of Battigalli (2019)). Consider a temporal sequence of action profiles $(\mathbf{a}_t)_{t=0}^{\infty}$. Then, *if* $(\mathbf{a}_t)_{t=0}^{\infty}$ *is consistent with adaptive learning*[23] *and* $a_t \to a^*$, *it follows that* a^* *must be a selfconfirming equilibrium action profile.*

[22]See, for example, Battigalli *et al.* (1992), Battigalli *et al.* (2019), Fudenberg and Kreps (1995), and the references therein.

[23]In a *finite* game, a trajectory $(\mathbf{a}_t)_{t=0}^{\infty}$ is consistent with adaptive learning if for every $\hat{t}$, there exists some T such that, for every $t > \hat{t} + T$ and $i \in I$, $a_{i,t}$ is a best reply to some *deep* conjecture μ_i that assigns probability 1 to the set of action profiles $\mathbf{a}_{-i}$ consistent with the feedback received from $\hat{t}$ through $t-1$. The definition for compact, continuous games is a bit more complex (see Milgrom and Roberts (1990)), who assume perfect feedback).

Of course, the limit of the trajectory may or may not be a Nash equilibrium. Let us now consider a best response dynamics. This generates trajectories that—by construction—are consistent with adaptive learning. With this, we prove convergence (under reasonable assumptions), hence convergence to an SCE.

To ease the analysis we consider best reply dynamics for shallow conjectures. For each network $\mathbf{Z}$, each period $t \in \mathbb{N}$, and each agent $i \in I$, $a_{i,t} = r(\hat{x}_{i,t})$ is the best reply to $\hat{x}_{i,t}$. After actions are chosen, given the feedback received, agents update their conjectures. If conjectures are confirmed then an agent keeps her previous conjecture, otherwise she updates using as new conjecture the conjecture that would have been correct in the previous period. In details,

$$\hat{x}_{i,t+1} = \begin{cases} \hat{x}_{i,t} & \text{if } a_{i,t} = 0, \\ \ell_i(\mathbf{a}_{-i,t}, \mathbf{Z}) & \text{if } a_{i,t} > 0, \end{cases} \tag{11}$$

and, from (7) (considering that the upper bound $\bar{a}_i$ is set so that it is never reached) we have simply

$$a_{i,t+1} = r_i(\hat{x}_{i,t+1}) = \begin{cases} 0, & \text{if } \hat{x}_{i,t} \leq -\alpha_i, \\ \alpha_i + \hat{x}_{i,t+1}, & \text{if } \hat{x}_{i,t} > -\alpha_i. \end{cases}$$

Coherently with the previous analysis, this update rule states that if an agent i at time t is inactive ($a_{i,t} = 0$), past conjectures are confirmed and thus kept. In this case, that satisfies observability if and only if a player **is active** (OiffA – Definition 3), the set of inactive agents is absorbing, meaning that if an agent is inactive at time t she will remain so also at time $t+1$. If instead the agent is active ($a_{i,t} > 0$), feedback is such that agents can perfectly infer the payoff state $x_{i,t} = \ell_i(\mathbf{a}_{-i,t}, \mathbf{Z})$, and so they update conjectures according to (11). This is an adaptive learning dynamics. The result cited above implies that if the dynamics described above converges, then it must converge to a selfconfirming equilibrium, i.e., a rest point where players keep repeating their choices.

In this section we analyze the stability of such rest points in the simplest possible case of robustness to small perturbations, as in Bramoullé and Kranton (2007). However, we will not consider perturbations to the strategy profile, but perturbations on the profile of conjectures.

Definition 4 (Learning process). *Each player $i \in I$ starts at time 0 with a belief, and beliefs are represented by a vector of shallow deterministic conjectures $\hat{\mathbf{x}}_0 = (\hat{x}_{i,0})_{i \in I}$. In each period t players best reply to their conjectures: for each $i \in I$, $a_{i,t} = \max\{\alpha_i + \hat{x}_{i,t}, 0\}$.*
At the beginning of each period $t+1$ each player i keeps her t-period shallow conjecture if she was inactive, and updates her conjecture to period-t revealed payoff state if she was active, that is, $\hat{x}_{i,t+1} = \frac{u_i(\mathbf{a}_t)}{a_{i,t}} - \alpha_i + \frac{1}{2}a_{i,t}$.

Even if we consider the case of linear best replies, from equations (10) and (11), the system is not linear because

$$\hat{x}_{i,t+1} = \begin{cases} \hat{x}_{i,t} & \text{if} \quad \hat{x}_{i,t} \leq -\alpha_i \ , \\ \sum_{j \in I} z_{ij} a_{j,t} & \text{if} \quad \hat{x}_{i,t} > -\alpha_i \ , \end{cases}$$

and for every other player j, we have that $a_{j,t} = \max\{\alpha_j + \hat{x}_{j,t}, 0\}$.

Clearly an SCE of the game, as defined in the beginning of Section 5, is always a rest point of this learning dynamic. We now consider the stability of such rest points $\mathbf{a}^*$. Say that a profile of conjectures $\hat{\mathbf{x}}$ is **justifies** $\mathbf{a}^*$ if, for each $i \in I$, $a_i^* = r_i(\hat{x}_i)$ ..

Definition **5.** *A selfconfirming action profile* $\mathbf{a}^* \in \mathbf{A}_{\mathbf{Z}}^{SCE}$ *is* locally stable *if there are a profile of conjectures* $\hat{\mathbf{x}}$ *justifying* $\mathbf{a}^*$ *and an* $\epsilon > 0$ *such that the learning dynamic starting from any* $\hat{\mathbf{x}}'$ *with* $\|\hat{\mathbf{x}}' - \hat{\mathbf{x}}\| < \epsilon$ *converges back to* $\hat{\mathbf{x}}$*.*

Notice that our notion of stability with respect to conjectures relates to the standard notion of stability with respect to actions in the following way. First of all, since played actions are justified by some conjectures, the only reason for these actions to change is a perturbation of the surrounding conjectures, but this is not a sufficient condition. If all agents are active, the two definitions have the same consequences in terms of stability, since a perturbation with respect to actions happens if and only if every agent's conjecture is perturbed. However, if a SCE has inactive agents, then inactive agents who play a corner solution do not show perturbation in actions when their conjectures are perturbed. This implies that if an action profile is stable with respect to actions perturbations, then it is also stable under conjectures perturbations, but the converse does not hold.

6.1 Results

Each SCE is characterized by a set of active agents. So, given a strategy profile $\mathbf{a} = (a_i)_{i \in I}$, let $I_{\mathbf{a}} = \{i \in I : a_i > 0\}$ denote the set of active players at profile $\mathbf{a}$. For each action profile $\mathbf{a}$, $\mathbf{Z}_{I_{\mathbf{a}}}$ denotes the submatrix with rows and columns corresponding to players who are active in $\mathbf{a}$. This allows us to characterize locally stable selfconfirming equilibria.

Proposition **4.** *Consider a selfconfirming equilibrium* $\mathbf{a}^* \in \mathbf{A}_{\mathbf{Z}}^{SCE}$*. Profile* $\mathbf{a}^*$ *is locally stable if*

- *assumption 4 holds for matrix* $\mathbf{Z}_{I_{\mathbf{a}^*}}$*;*
- *for some profile of conjectures* $\hat{\mathbf{x}}$ *justifying* $\mathbf{a}^*$*, and for each* $i \in I \backslash I_{\mathbf{a}^*}$*,* $\alpha_i + \hat{x}_i < 0$*.*

Intuitively, consider a sufficiently small perturbation of players' conjectures. The first condition ensures that active players keep being active and their actions converge back to the Nash equilibrium of the auxiliary game with player set $I_{\mathbf{a}^*}$. The second condition ensures that inactive players keep being inactive. Next, we provide alternative sufficient conditions that allow to characterize the subsets of active agents associated to SCEs.

Proposition **5.** *Consider a selfconfirming profile* $\mathbf{a}^* \in \mathbf{A}_{\mathbf{Z}}^{SCE}$*. If* $\mathbf{Z}_{I_{\mathbf{a}^*}}$ *satisfies at least one of the three conditions below:*

1. *it has bounded values (Assumption 1),*

2. it is negative and limited (Assumptions 3 and 4),

3. or it is symmetrizable–limited (Assumption 6),

then $\mathbf{a}^*$ *is locally stable and, for every* $J \subseteq I_{\mathbf{a}^*}$, $\mathbf{a}^{**} = (\mathbf{a}_J^{NE}, \mathbf{0}_{I \setminus J})$ *is a locally stable SCE, where* $\mathbf{a}_J^{NE}$ *is the unique interior Nash equilibrium action profile of the game restricted to J.*

The proof is based on results from linear algebra. In fact, if an adjacency matrix satisfies one of the conditions from Proposition 5, then also every submatrix of that matrix satisfies that property.

We know that there may be SCEs that are not Nash equilibria, because some agents are inactive even if this is not a best response to the actions of the others. Proposition 5 tells us two additional things. Under the stated conditions, for any given SCE $\mathbf{a}^*$ with set of active agents $I_{\mathbf{a}^*}$, any subset $J \subseteq I_{\mathbf{a}^*}$ of those agents is associated to a stable SCE where all agents in J are active, and the other agents are inactive. Second, since the empty subset of agents is trivially associated to the stable SCE where every agent is inactive, for every network game there is always a subset J of agents associated to a stable SCE where all and only the agents in J are active.

6.2 Examples

The following example shows that we can reach SCEs that are not NE also if the initial beliefs induce all positive actions at the beginning of the learning dynamic.

Example 4. Consider the case of 4 players, with the network matrix $\mathbf{Z} \in \{-0.2, 0, 0.2\}^{I \times I}$ shown in Figure 2, and, for every i, $\alpha_i = 0.1$. This is a case of general externalities, that can be positive or negative. Figure 3 shows the learning dynamics of actions and beliefs that start from different initial conditions. In one case (left panels) the system converges to the unique Nash equilibrium of this game (the dotted lines), in the other (right panels) the learning dynamics put, after 2 rounds, one player out from the active agents, and the remaining 3 converge to a selfconfirming equilibrium which is not Nash. ∎

The next example (which also does not satisfy the local stability conditions of Proposition 5) shows that convergence may not occur even in a simple case of positive externalities.

Example 5. Now consider again the network from Example 1 (Figure 1), with 4 nodes. Even if there are only positive externalities, convergence depends on the magnitude of w. If $w < 1$, there is convergence. If instead $w \geq 1$ there is divergence. Figure 4 shows two cases, with $w = 0.9$ and $w = 1$ respectively, starting from the same initial beliefs. Note that, nodes/players 1 and 4 reinforce each other, and this gives rise to an oscillating behavior of their beliefs. ∎

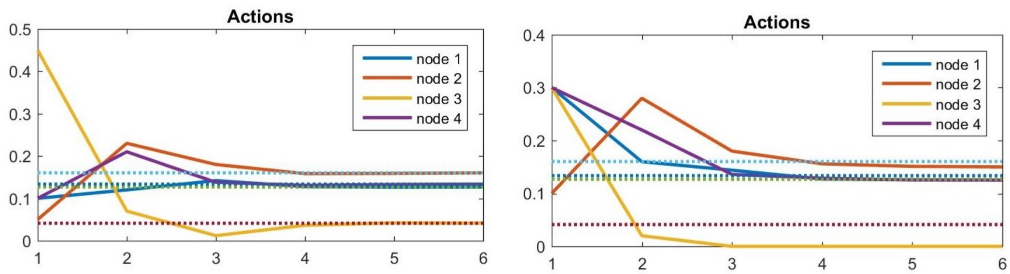

Figure 3: General strategic externalities. Starting from different beliefs on the same network (from Figure 2), the learning process may converge to the unique Nash equilibrium (left panels) or to a SCE which is not a Nash equilibrium (right panels). Note that at each time steps beliefs are just a downward translation of actions, by the quantity α.

7 Local and Global externalities

In most applications the feedback that players receive is not enough to find the optimal best response, even for active players. Users of online platforms may not understand what is the best response to others' activity, and a firm in a complex market may not infer optimal investment plans just observing prices. In our context, this means that the assumption of **observability if and only if a player is active** (OiffA) (Definition 3) may not hold. This analysis will bring us to two important considerations. First, players may be more active if they (possibly erroneously) think that they are more central in the network than they are actually are. When positive externalities are at play this can be welafre improving for the whole society. However, and this is the second point, if we consider adaptive learning processes, agents with too high perceived centrality may induce the learning dynamics to become unstable.

We consider here a simple variation to our model. Starting from equation (6), we add a global externality term with no strategic effects. For each $i \in I$, we posit an interval $Y_i = [\underline{y}_i, \bar{y}_i]$, a coefficient $\beta \in \mathbb{R}$, and we consider the following new aggregator:[24]

$$\begin{array}{rccc} g_{i,\beta}: & \mathbf{A}_{-i} & \to & Y_i \\ & \mathbf{a}_{-i} & \mapsto & \beta \sum_{j \neq i} a_j \end{array}.$$

We assume that every agent $i \in I$ knows Y_i. Then, we let $y_i = g_i(\mathbf{a}_{-i}, \beta)$ and we maintain the

[24]This aggregator g sums up the actions of all the agents in the network except agent i. We could have considered agent i as well, but we opted for this specification so as not to change the first order condition with respect to the case with just local externalities. If the action of agent i is considered in the global externality, then results do not change up to a rescaling of the coefficient α_i.

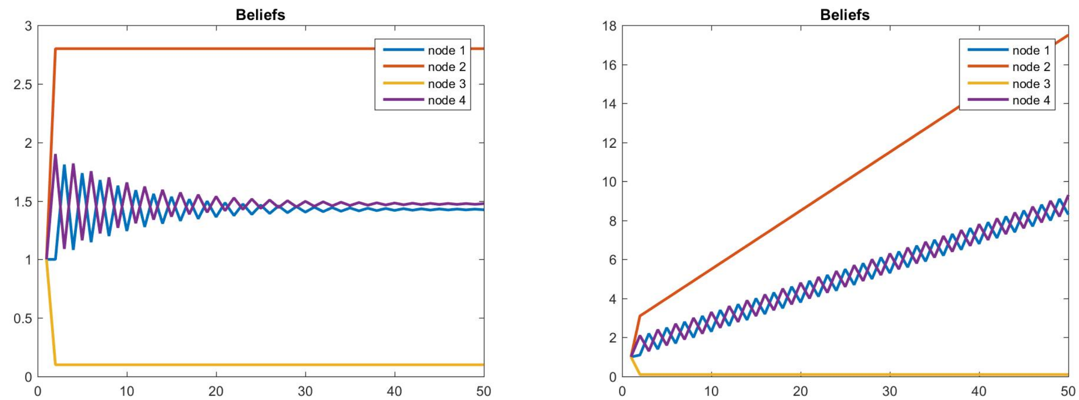

Figure 4: Only positive externalities. Starting from the same beliefs on the same network structure (from Figure 1), the learning process may converge or not depending on the size of w: $w = 0.9$ in the left panel; $w = 1$ in the right panel. Actions are just an upward translation of beliefs, of amount α.

assumption that $x_i = \ell_i(\mathbf{a}_{-i}, \mathbf{Z})$. The new parameterized utility function is

$$\begin{array}{llll} v_i: & A_i \times X_i \times Y_i & \to & \mathbb{R} \\ & (a_i, x_i, y_i) & \mapsto & \alpha_i a_i - \frac{1}{2}a_i^2 + a_i x_i + y_i \end{array}, \tag{12}$$

where both x_i and y_i are unknown. The general form of the feedback function is

$$f_i : A_i \times X_i \times Y_i \to M.$$

Deterministic shallow conjectures for each $i \in I$ are now determined by the pair $(\hat{x}_i, \hat{y}_i) \in X_i \times Y_i$. Note that the best reply depends only on $\hat{x}_i$, but $\hat{y}_i$ matters for the ex post inference made by i once he has observed his feedback. We provide now the definition of selfconfirming equilibrium for games with global externalities.

Definition **6.** *A profile* $(a_i^*, \hat{x}_i, \hat{y}_i)_{i \in I} \in \times_{i \in I}(A_i \times X_i \times Y_i)$ *of actions and (shallow) deterministic conjectures is a* ***selfconfirming equilibrium*** *at* $\mathbf{Z}$ *and* β *of a network game with global externalities if, for each* $i \in I$*,*

1. (subjective rationality) $a_i^* = r_i(\hat{x}_i)$,
2. (confirmed conjecture) $f_i(a_i^*, \hat{x}_i, \hat{y}_i) = f_i\left(a_i^*, \ell_i\left(\mathbf{a}_{-i}^*, \mathbf{Z}\right), g_i\left(\mathbf{a}_{-i}^*, \beta\right)\right)$.

Notice that the rationality condition is unchanged with respect to the case of only local externalities since best-reply conditions are not affected by the global externality term. To compare this game with the linear-quadratic network game with only local externalities, we focus on the property of *just observable payoffs*. Then, without loss of generality we can assume that $f_i = v_i$ for every $i \in I$. With this, we can characterize the SCE set as follows:

Proposition **6.** *Fix $\mathbf{Z} \in \mathcal{Z}$ and β. Every selfconfirming equilibrium profile $\left(a_i^*, \hat{x}^i, \hat{y}^i\right)_{i\in I} \in \times_{i\in I}\left(A_i \times X_i \times Y_i\right)$ of a network game with global externalities is such that, for every $i \in I$,*

1. *if $a_i^* = 0$, then $\hat{x}_i \in [\underline{x}_i, -\alpha_i]$ and $\hat{y}_i = y_i$;*
2. *if $a_i^* > 0$, then $a_i^* = \alpha_i + \hat{x}_i$ and $\hat{y}_i = y_i + a_i^*\left(x_i - \hat{x}_i\right)$.*

We discuss how the presence of the global externality term in the utility function changes radically the characterization of selfconfirming equilibria. As before, we assume that players observe their own realized payoffs. Yet, when global externalities are present, *observability by active players* does not hold anymore. Inactive players have correct conjectures about the global externality, but may have incorrect conjectures about the local externality. Active players, on the other hand, are not able to determine precisely the relative magnitude of the local effects with respect to the global effects. Given any strictly positive action a_i^*, the confirmed conjectures condition yields $(\hat{y}_i - y_i) = a_i^*\left(x_i - \hat{x}_i\right)$. Then, in equilibrium, if agent i overestimates (underestimates) the local externality, she must compensate this error by underestimating (overestimating) the global externality. Then, compared to the case of only local externalities, we have that: (i) active agents choose a best response to a (typically) wrong conjecture about x; thus, (ii) it is not possible to characterize SCE by means of Nash equilibria of the auxiliary games restricted to the active players.

Before moving on to the analysis of learning dynamics, we derive a very general result that is independent of the actual network of peer effects. We consider the functional form of u_i when $\mathbf{Z} = \mathbf{w}\mathbf{Z}_0$, with $w > 0$ and $\mathbf{Z_0}$ unweighted network, which means that there is a common positive externality w between all connected players:

$$u_i(a_i, \mathbf{a}_{-i}) = \alpha a_i - \frac{1}{2}a_i^2 + a_i \underbrace{w \sum_{j\in I\setminus\{i\}} z_{0,ij}a_j}_{\ell_i\left(\mathbf{a}^*_{-i},\mathbf{Z}\right)} + \underbrace{\beta \sum_{k\in j\in I\setminus\{i\}} a_k}_{g_i\left(\mathbf{a}^*_{-i},\beta\right)} \quad . \tag{13}$$

The following result applies to any possible network structure, and shows what can happen when players know the parameters of the payoff function, w and β, but not the network structure $\mathbf{Z_0}$.

Proposition **7.** *Consider a game with payoffs given by Equation (13), played on any network $\mathbf{Z_0}$. Suppose also that parameters $w > 0$ and $\beta \geq 0$ are common knowledge, with $w < \frac{1}{n-1}$, but the conjecture of every player is that she is connected with every other player. Then, the strategy profile of each player is increasing in β, it is equal to the Nash equilibrium of network $\mathbf{Z_0}$ when $\beta = 0$, while, as $\beta \to \infty$, the strategy profile of the unique SCE approaches the unique NE of the game with the same payoffs, played on a complete network.*

So, no matter what the original network $\mathbf{Z_0}$ is, if all players believe to be central and β is *large*, then the strategy profile of players approaches what they would play in the NE of the complete network. This result is interesting because a hypothetical learning dynamic is self–reinforcing. Players infer $\ell_i\left(\mathbf{a}^*_{-i}, \mathbf{Z}\right)$, from the payoff that they receive as feedback, using (13). This implies that, converging to a SCE, as they increase their own action they infer a higher $\ell_i\left(\mathbf{a}^*_{-i}, \mathbf{Z}\right)$ and a lower $g_i\left(\mathbf{a}^*_{-i}, \beta\right)$, to which they will respond with an even higher action. Nevertheless, this process does not explode to infinity, but it reaches the NE that would be played on the complete network.

Proposition 7 is a limiting result. However, for some networks where NE and SCE can be easily computed analytically, we can show that the actions of the SCE strategy profiles converge rapidly to the actions of the NE for the complete network. Figure 5 shows how this happens for a regular network and for a star network.

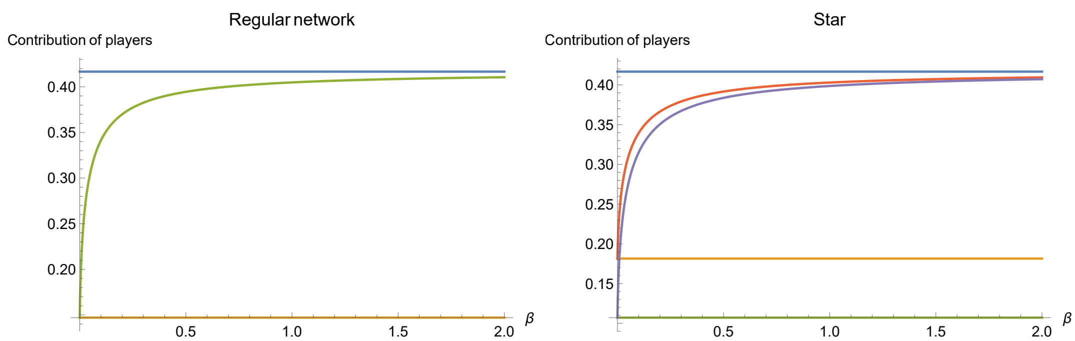

Figure 5: An example on how players actions change, when they all think they are connected to every other player, as parameter β grows. Both cases have parameters $\alpha = .1$, $w = .04$ and $n = 20$. The left panel is for the regular network with common degree 8: in blue we have the action that would be played in the NE of the complete network; in yellow the NE of the regular network; in green the SCE. The right panel is for the star network: in blue we have the actin that would be played in the NE of the complete network; in yellow and green the NE profile for the center and the spokes in the star network; in red and purple the SCE profile for the center and the spokes.

Finally, let us note that Proposition 7 is based on the assumption that players know the values of β and w. However, if they have wrong beliefs about β, overestimating it, their actions would even exceed those of the NE of the complete network. This is shown in the next example, where agents do not know the true value of β and may overestimate their centrality even above the centrality that they would have in a complete network.

Example 6. Consider three agents in a line network. Let agent 2 be at the center of the line. Then, for every $(\mathbf{a}^*, \mathbf{Z}, \beta)$, $\ell_2\left(\mathbf{a}^*_{-2}, \mathbf{Z}\right)$ is proportional to $g_2\left(\mathbf{a}^*_{-2}, \beta\right)$, always with the same ratio $\frac{\beta}{w}$, while this is not true for agents 1 and 3. We assume that each agent thinks she is playing in a complete network, so every $i \in I$ thinks that $\ell_i\left(\mathbf{a}^*_{-i}, \mathbf{Z}\right)$ is always proportional to $g_i\left(\mathbf{a}^*_{-i}, \beta\right)$, with the same ratio. In this case agents 1 and 3 believe to be more *central* than they actually are. Table 4 provides the Nash equilibria for the actual network and for the complete network, and the selfconfirming equilibrium actions for the case described above.

	Line NE	Complete NE	SCE
a_1	0.130	0.167	1.569
a_2	0.152	0.167	1.679
a_3	0.130	0.167	1.569

Table 4: Simulations for the case of $\alpha = 0.1$, $w = 0.2$, and $\beta = 1$. Columns refer to 1) Nash Equilibrium of the line network; 2) Nash equilibrium of complete network; 3) SCE in the line network in which each $i \in I$ believes that $\ell_i\left(\mathbf{a}^*_{-i}, \mathbf{Z}\right) = \frac{\beta}{w} g_i\left(\mathbf{a}^*_{-i}, \beta\right)$.

This numerical exercise shows that if agents overestimate the impact of local externalities this generates a *multiplier* effect that makes equilibrium actions increase at a level even larger that what would be predicted in a complete network by Nash equilibrium. This is the result of how agents misinterpret their feedbacks. In details, thinking to be in a complete network makes agents 1 and 2 overestimate local externalities. Take for instance agent 1. Given any $\mathbf{a}_{-1}$, she chooses a best reply higher than the Nash equilibrium one since she overestimates the local externality. This high action has the effect of increasing the global externality term for agent 3. Agent 3, by overestimating local externality, partly attributes this higher global externality to the local externality term, and chooses an action larger than predicted by Nash equilibrium. The choice of agent 3 increases in turns the global externality perceived by agent 1, and so on. At the same time agent 2, as neighbors choose higher actions, increases her own action level. This effect goes on and a multiplier effect seems to be at place. In the limit, selfconfirming equilibrium actions are almost ten times larger than the complete network Nash equilibrium. ∎

7.1 Learning with Global Externalities

We now consider the learning process that originates from an adaptive updating of conjectures, as we did for the case of only local externalities. Consider the payoff function that depends on players' actions, with the time index and specifying $x_{i,t}$ and $y_{i,t}$ as functions of co-players' actions:

$$u_{i,t}(a_{i,t}, \mathbf{a}_{-i,t}) = \alpha a_{i,t} - \frac{1}{2} a_{i,t}^2 + a_{i,t} \underbrace{\sum_{j \in I \setminus \{i\}} z_{ij} a_{j,t}}_{x_{i,t}} + \beta \underbrace{\sum_{k \in I \setminus \{i\}} a_{k,t}}_{y_{i,t}} .$$

To ease the analysis, *we assume the same parameter α for each player* and we focus on the case of *strictly positive justifiable actions.* We obtain this by *assuming that $\alpha > 0$ and that all the elements of $\mathbf{Z}$ are nonnegative.* At each time, there are infinitely many profiles of feasible pairs $(\hat{x}_{i,t}, \hat{y}_{i,t})_{i \in I}$

consistent with feedback. For each $i \in I$, and each time $t \in \mathbb{N}$, let $m_{i,t} = f_i(a_{i,t}, x_{i,t}, y_{i,t}) = u_i(a_{i,t}, \mathbf{a}_{-i,t})$ be the message agent i receives. Then, given message $m_{i,t-1}$, and considering that agents perfectly recall their past actions, $\hat{y}_{i,t}$ is uniquely determined as a function of $\hat{x}_{i,t}$. In details, at each time period, agent i's conjecture is a pair $(\hat{x}_{i,t}, \hat{y}_{i,t})$ consistent with the message received at the previous period. We obtain

$$\hat{y}_{i,t+1} = m_{i,t} - \alpha a_{i,t} + \frac{1}{2}(a_{i,t})^2 - a_{i,t}\hat{x}_{i,t+1} \ .$$

Then, we can just focus on the dynamics of $\hat{x}_{i,t}$, given by

$$\hat{x}_{i,t+1} = \frac{m_{i,t} - \hat{y}_{i,t+1}}{a_{i,t}} - \alpha + \frac{1}{2}a_{i,t} \tag{14}$$

This case does not satisfy observability if and only if a player **is active** (OiffA), bacause players check a two–dimensional conjecture with a feedback, the payoff, that has a single dimension. This leaves also freedom on the updating rule that players use. To avoid bifurcations at each time period, we need to use simplifying assumptions on conjectures. We define

$$c_{i,t} := \frac{\hat{x}_{i,t}}{\hat{y}_{i,t}}.\text{[25]} \tag{15}$$

Then,

Assumption **7.** *For each $i \in I$ and for each $t \in \mathbb{N}$, $c_{i,t} = c_{i,t+1} = c_i$.*

We call c_i the **perceived centrality** of player i. For each player, this parameter describes what she thinks to be the share of the activity in her neighborhood with respect to the sum of all the actions of the population. This perceived share has a strong relationship with the Bonacich centrality. In the unique Nash equilibrium $\mathbf{a}^*$ of the game, where all actions are positive, we have

$$a_i^* = \alpha + x_i = \alpha + \sum_{j \in I \setminus \{i\}} z_{ij} a_j^* \ .$$

The profile of **Bonacich centrality measures b** is the unique solution of the linear system[26]

$$b_i = \alpha + \sum_{j \in I \setminus \{i\}} z_{ij} b_j \ .$$

So, when beliefs are correct, as in the Nash equilibrium, we have $b_i = a_i$ and $c_i = \frac{b_i - \alpha}{y_i}$. Now, in the Nash equilibrium we have also $\frac{1}{y_i} - \frac{1}{y_j} = \beta\frac{a_j - a_i}{y_i y_j}$. If the number of players is large, we have $y_i \gg a_i$ and $y_j \gg a_j$, which implies $\frac{1}{y_i} \simeq \frac{1}{y_j}$, and so every c_i is roughly the same linear rescaling of b_i.

[25] In doing so, we implicitly assume that players think that not all the other players play the null action $a_{k,t} = 0$. This is actually a reasonable assumption, because under positive externalities any best response $a_{k,t}$ should be at least α.

[26] In general, independently of any game defined on the network, Bonacich centrality is a network centrality measure that depends on a paramater $\alpha > 0$. It is defined exactly as the solution of that same linear system. For a detailed discussion on this see Dequiet and Zenou (2017)

From equation (14), and expressing the message as the observed payoff, we get the following learning dynamic

$$\hat{x}_{i,t+1} = x_{i,t} + \frac{y_{i,t}}{a_{i,t}} - \frac{\hat{y}_{i,t+1}}{a_{i,t}} \ . \tag{16}$$

Plugging in $c_{i,t} = \frac{\hat{x}_{i,t}}{\hat{y}_{i,t}}$ we get

$$\hat{x}_{i,t+1} = \frac{c_{i,t}}{1 + c_{i,t}a_{i,t}} \left(a_{i,t}x_{i,t} + y_{i,t}\right) \ . \tag{17}$$

We define the **true centrality** of player i at time t as

$$c'_{i,t} := \frac{x_{i,t}}{y_{i,t}} \quad .$$

Note that $c'_{i,t} \in \left[0, \frac{\sum_{j\neq i} z_{ij}}{\beta}\right]$. For this reason, we also assume that the perceived centrality of each player i is such that $c_i \in \left(0, \frac{\sum_{j\neq i} z_{ij}}{\beta}\right]$, and this specifies the set of all admissible perceived centralities. The dynamic, then, can be written as

$$\hat{x}_{i,t+1} = c_i y_{i,t} \frac{a^*_{i,t} c'_{i,t} + 1}{a^*_{i,t} c_i + 1} \quad ,$$

which implies that the conjecture $\hat{x}_{i,t+1}$ is correct only when $c_i = c'_{i,t}$.

We look at best responses $a_{i,t+1} = \alpha + \hat{x}_{i,t+1}$, and study existence and characterization of the steady state of this learning process. Recall that $y_{i,t} = \beta \sum_{j\neq i} a_{j,t}$. To find a fixed point we look at the system of n equations

$$H_i(\mathbf{a}^*, \mathbf{c}, \beta, \mathbf{Z}) := \alpha + c_i \left(\beta \sum_{j\neq i} a^*_j\right) \frac{a^*_i c'_i + 1}{a^*_i c_i + 1} - a^*_i = 0 \ . \tag{18}$$

For comparison, we also study the system of equations that provide the Nash Equilibrium of this network game, namely:

$$F_i(\mathbf{a}^*, \beta, \mathbf{Z}) := \alpha + \sum_{j\in I} z_{ij} a^*_j - a^*_i = 0 \ . \tag{19}$$

Let $\mathcal{A} \subset [\alpha, \infty)^I$ denote the set of the solutions of the system (18). We have the following result.

Proposition 8. *If the system defined by* (19) *admits a solution, then for each vector* $\mathbf{c}$ *of perceived centralities also the system defined by* (18) *admits a solution. Moreover, the system implies a homeomorphism* Φ *between all profiles* $\mathbf{c}$ *and* $\mathcal{A}$*. Homeomorphism* Φ *is monotone with respect to the lattice order of the two sets.*

The previous result provides information only on the steady states of our dynamical system. Note however that the homeomorphism is implied by the particular learning dynamic that we are assuming, which is based on constant belief centralities. Here below we show a result that provides sufficient conditions for convergence of the learning dynamic. We impose as a sufficient condition that local and global externalities are not too large.

Proposition **9.** *If, for each player $i \in I$, $0 < c_i\beta(n-1) < \sum_{j\neq i} z_{ij} < 2$, then the dynamic defined by the learning process (18) always converges to its unique solution, which is stable.*

It should be noted that the assumptions of Proposition 9 imply that $|\sum_{j\neq i} z_{ij}| < 1$, which in turn implies that Assumption 4 holds and hence the learning dynamics of the corresponding game with only local externalities should converge. However, if for some player the perceived centralities are too high, still the learning dynamics defined by (18) for the global game may not converge.

Example 7. Under the conditions of Proposition 9, we use equation (17) to run dynamical systems converging to the SCE implicitely defined by (18). This allows us to provide a graphical illustration of Proposition 8, for the case of three nodes. We do this for the case of a line network (where each of the two links is bidirectional), and for the case of a complete network. We take the case from equation (13), with $\beta = 1$ and $w = .2$. Figure 6 shows the results. We can start from any pattern of perceived centralities for the three nodes. The left panel shows the profile of perceived centralities when at least one node has maximal perceived centrality (the three faces of the cube have different colors, according to which node has the maximal centrality). The central panel shows the corresponding SCE conjectures $\hat{\mathbf{x}}$ when the network is a line (the node that has perceived centrality 1 in the red dots is the central node). The right panel shows the corresponding SCE beliefs $\hat{\mathbf{x}}$ when the network is a complete triangle. The figure suggests that homeomorphism Φ (from Proposition 8) is highly non linear, because of the self reinforcement process in beliefs that we discussed in Example 6. The figure also shows that, as stated by Proposition 8, homeomorphism Φ respects the lattice order on the two sets. ∎

Proposition 8 tells us that a non-negative shift in each perceived centrality will always result in a non-negative shift in each agent's action in the resulting SCE. However, Proposition 9 gives an implicit warning. Too high perceived centralities may imply that the sufficient conditions for stability are lost, and convergence to the corresponding SCE may be lost. Note also that, summing up equation (2) for all the players, the aggregate welfare is maximized if the vector of actions satisfies the linear system

$$a_i^* = \alpha + (n-1)\beta + \sum_{j\in I\setminus\{i\}} (z_{ij} + z_{ij})a_j^* \ .$$

This aspect deserves a comment, relatively to the online social networks application of our model. Social platforms like Facebook and Twitter often provide information to users about the activity of their peers. The social platform Reddit actually do not show followers to their users, but only a measure of popularity that is called *karma*. A rationale for this marketing strategy can be that these companies want to change the beliefs of players, making them feel more important (i.e. central) in the social network. Even a benevolent social planner may want to set the perceived centralities to the level for which the social optimum is achieved. However, according to our model,

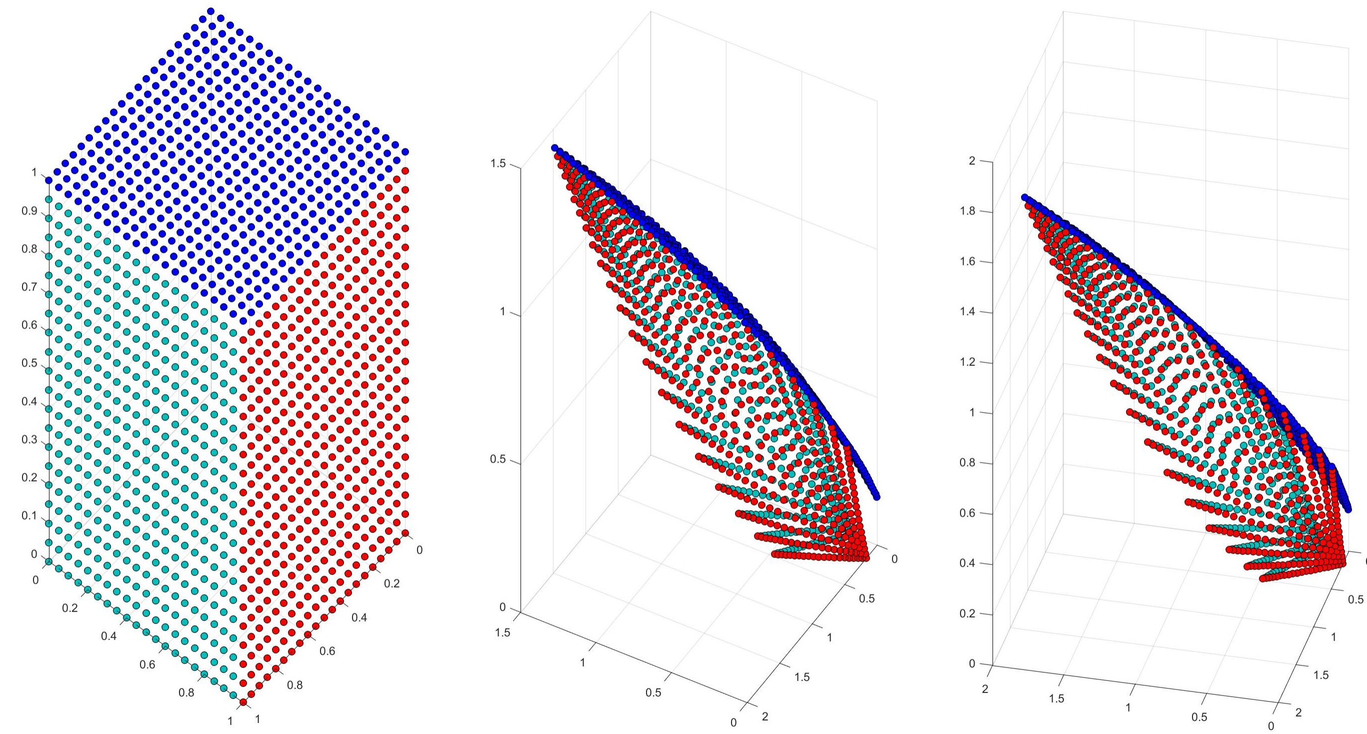

Figure 6: Simulations showing the homeomorphism of Proposition 9 for the case of 3 nodes, as discussed in Example 7. The left panel shows vectors of preceived centralities. The central panel shows the corresponding SCE beliefs $\hat{x}$ when the network is a line (the node that has perceived centrality 1 in the red dots is the central node). The right panel shows the corresponding SCE beliefs $\hat{x}$ when the network is a complete triangle.

if perceived centralities are too high, the system may become unstable. This is shown in the following example.

Example 8. We replicate the same exercise that we did in Example 7, only for the case of the complete triangle. However we do it for a wider range of perceived centralities. Figure 7 shows that in this case it can happen that there are combinations of perceived centralities which do not allow the laearning dynamics to converge. ∎

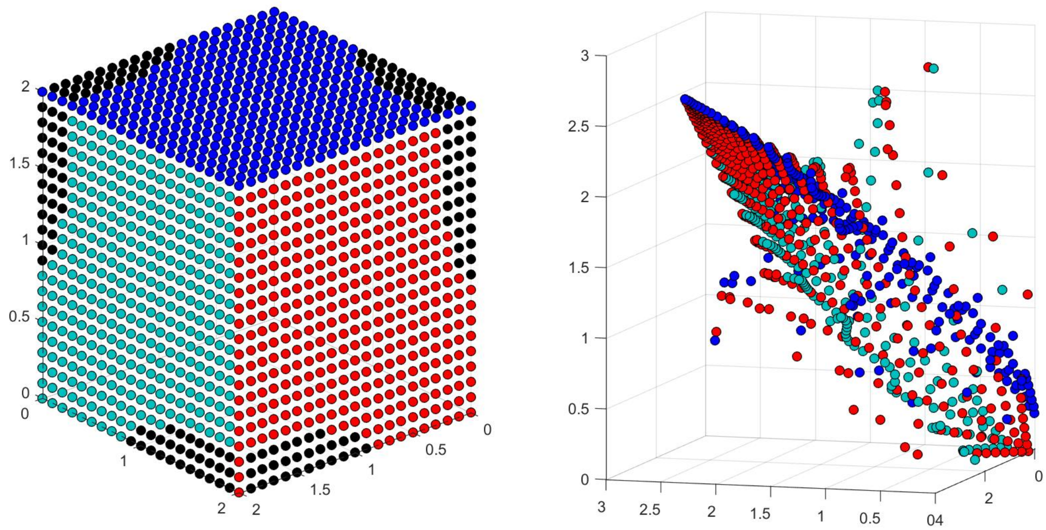

Figure 7: Simulations showing the homeomorphism of Proposition 9 for the case of 3 nodes, as discussed in Example 8. The left panel shows vectors of preceived centralities. With respect to Figure 6, we allow for higher values of perceived centralities. Black dots represent cases for which the learning dynamics do not converge. The right panel shows the corresponding SCE beliefs $\hat{x}$ when the network is a complete triangle. and when the learning dynamics is converging.

8 Conclusion

In this paper we lay the basis for a novel approach to network games. Many of the applications of those games mimic large societies with million of nodes and non regular distribution of connections. It is natural to assume that players are not aware of the complete structure of the network; thus, they do not perform sophisticated strategic reasoning possibly leading to a Nash equilibrium, but just best–respond to some to subjective beliefs affected by information feedback they receive. We analyze simple adaptive dynamics and show that in some cases they converge to stable Nash equilibria. However, we characterize also those situations in which feasible stable outcomes are not Nash equilibria, but rather selfconfirming equilibria in which some (if not *all*) players have wrong beliefs and yet the feedback they receive is consistent with such beliefs. We also show that simple biases in the perception of own centrality in the network may lead players to play action profiles that are very far from the unique Nash equilibrium of the game.

One natural application of this approach is to online social platforms like Facebook and Twitter. Using a linear quadratic structure for the payoff function we have also laid the ground for a tractable welfare analysis of the model. However, policy implications are not straightforward if we want to consider the long–run benefits of connections and not only about the instantaneous payoffs of the users of those platforms.

Our analysis does not account for the strategic reasoning that agents can perform given some commonly know features of the network. For example, known results about rationalizability imply that, if the (nice) network game has strategic complementarities and is common knowledge, then sophisticated strategic reasoning leads to Nash equilibrium.[27] If only some aspects of the network game are commonly known, then both strategic reasoning and learning affect the long-run outcome, which is a kind of rationalizable self-confirming equilibrium.

[27] On nice games with strategic complementaries see, e.g., Chapter 5 of Battigalli (2019) and the references therein. We address this analysis also in Appendix J.

appendix

Appendix I Selfconfirming equilibria in parameterized nice games with aggregators

In this section we develop a more general analysis of selfconfirming equilibria in a class of games that contains the linear-quadratic network games with feedback. To ease reading, we make this section self-contained, repeating some definitions from the main text.

A **parameterized nice game with aggregators and feedback** is a structure

$$G = \left\langle I, \mathcal{Z}, (A_i, \ell_i, v_i, f_i)_{i \in I} \right\rangle$$

where

- I is the finite **players set**, with cardinality $n = |I|$ and generic element i.
- $\mathcal{Z} \subseteq \mathbb{R}^m$ is a *compact* **parameter space**.
- $A_i = [0, \bar{a}_i] \subseteq \mathbb{R}_+$, a *closed interval*, is the **action space** of player i with generic element $a_i \in A_i$.
- $X_i = [\underline{x}_i, \bar{x}_i] \subseteq \mathbb{R}$, a *closed interval*, is the a **space** of **payoff states** for i.
- $\ell_i : \mathbf{A}_{-i} \times \mathcal{Z} \to X_i$ (where $\mathbf{A}_{-i} = \times_{j \in I \setminus \{i\}} A_j$) is a *continuous* parameterized **aggregator** of the actions of i's co-players such that its *range* $\ell_i(\mathbf{A}_{-i} \times \mathcal{Z})$ is *connected*.[28]
- $v_i : A_i \times X_i \to \mathbb{R}$ is the **payoff (utility) function** of player i, which is *strictly quasi-concave* in a_i and *continuous*,[29] and from which we derive the **parameterized payoff function**

  $$\begin{array}{rccl} u_i : & A_i \times \mathbf{A}_{-i} \times \mathcal{Z} & \to & \mathbb{R}, \\ & (a_i, \mathbf{a}_{-i}, \mathbf{Z}) & \mapsto & v_i(a_i, \ell_i(\mathbf{a}_{-i}, \mathbf{Z})). \end{array}$$

  Thus, $x_i = \ell_i(\mathbf{a}_{-i}, \mathbf{Z})$ is the payoff relevant state that i has to guess in order to choose a subjectively optimal action. With this, for each $\mathbf{Z} \in \mathcal{Z}$, $\left\langle I, (A_i, u_{i,\mathbf{Z}})_{i \in I} \right\rangle$ is a nice game (Moulin, 1979), and $\left\langle I, \mathcal{Z}, (A_i, u_i)_{i \in I} \right\rangle$ is a parameterized nice game. We let

  $$\begin{array}{rccl} r_i : & X_i & \to & A_i \\ & x_i & \mapsto & \arg\max_{a_i \in A_i} v_i(a_i, x_i) \end{array}$$

[28] Since the range of each section $\ell_{i,\mathbf{Z}}$ must be a closed interval, we require that the union of the closed intervals $\ell_{i,\mathbf{Z}}(\mathbf{A_{-i}})$ ($\mathbf{Z} \in \mathcal{Z}$) is also an interval, which must be closed because $\mathcal{Z}$ is compact and ℓ_i continuous.

[29] That is, v_i is jointly continuous in (a_i, x_i) and, for each $x_i \in [\underline{x}_i, \bar{x}_i]$, the section $v_{i,x_i} : [0, \bar{a}_i] \to \mathbb{R}$ has a unique maximizer a_i^* (that typically depends on x_i), it is strictly increasing on $[0, a_i^*]$, and it is strictly decreasing on $[a_i^*, \bar{a}_i]$. Of course, the monotonicity requirement holds vacuously when the relevant subinterval is a singleton.

denote the **best reply function** of player i. The Maximum theorem implies that r_i is continuous.

- Let $M \subseteq \mathbb{R}$ be a set of "messages," $f_i : A_i \times X_i \to M$ is a **feedback function** that describes what i observes (a "message," e.g., a monetary outcome) after taking any action a_i given any payoff state x_i.

On top of the formal assumptions stated above, we maintain the following *informal assumption* about players' knowledge of the game:

- Each player i *knows* v_i *and* f_i.

Unless we explicitly say otherwise, we instead do not assume that i knows $\mathbf{Z}$, or function ℓ_i, or even that i understands that his payoff is affected by the actions of other players. However, since i knows the feedback function $f_i : A_i \times X_i \to M$ and the action he takes, what i infers about the payoff state x_i after he has taken action a_i and observed message m is that

$$x_i \in f_{i,a_i}^{-1}(m) := \left\{x'_i : f_i\left(a_i, x'_i\right) = m\right\}.$$

I.1 Conjectures

Definition **A.** *A* ***shallow conjecture*** *for i is a probability measure $\mu_i \in \Delta(X_i)$. A (deep)* ***conjecture*** *for i is a probability measure $\bar{\mu}_i \in \Delta(\mathbf{A}_{-i} \times \mathcal{Z})$. An action a_i^* is* ***justifiable*** *if there exists a shallow conjecture μ_i such that*

$$a_i^* \in \arg\max_{a_i \in A_i} \int_{X_i} v_i(a_i, x_i)\, \mu_i(\mathrm{d}x_i);$$

in this case we say that μ_i ***justifies*** *a_i^*. Similarly, we say that (deep) conjecture $\bar{\mu}_i \in \Delta(\mathbf{A}_{-i} \times \mathcal{Z})$* ***justifies*** *$a_i^*$ if the shallow conjecture induced by $\bar{\mu}_i$ ($\mu_i = \bar{\mu}_i \circ \ell_i^{-1} \in \Delta(X_i)$) justifies a_i^*.*

Remark **A.** *If $a_i \mapsto v_i(a_i, x_i)$ is* strictly concave *for each x_i, then also $a_i \mapsto \int_{X_i} v_i(a_i, x_i)\, \mu_i(\mathrm{d}x_i)$ is strictly concave and the map*

$$\mu_i \mapsto \arg\max_{a_i \in A_i} \int_{X_i} v_i(a_i, x_i)\, \mu_i(\mathrm{d}x_i)$$

is a continuous function.[30]

The following lemma summarizes well known results about nice games (see, e.g., Battigalli 2019) and some straightforward consequences for the more structured class of nice games with aggregators considered here. We include the proof to make the exposition self-contained.

[30] When $\Delta(X_i)$ is endowed with the topology of weak convergence of measures.

Lemma **A.** *The best reply function $r_i : X_i \to A_i$ is continuous, hence its range $r_i(X_i)$ is a closed interval, just like X_i. Furthermore, for each given $a_i^* \in A_i$, the following are equivalent:*

- *a_i^* is justifiable,*
- *$a_i^* \in r_i(X_i)$ (that is, a_i^* is justified by a deterministic shallow conjecture),*
- *there is no a_i such that $v_i(a_i^*, x_i) < v_i(a_i, x_i)$ for all $x_i \in X_i$ (that is, a_i^* is not dominated by any other pure action).*

Proof. With a slight abuse of notation, we let $r_i(\mu_i)$ denote set set of best replies to (shallow) conjecture μ_i:

$$r_i(\mu_i) := \arg\max_{a_i \in A_i} \int_{X_i} v_i(a_i, x_i)\, \mu_i(\mathrm{d}x_i).$$

By the Maximum theorem $\mu_i \mapsto r_i(\mu_i)$ has a closed graph, which—under the stated assumptions—is equivalent to upper hemicontinuity. By strict quasi-concavity, the restriction of the best reply correspondence to the domain X_i of deterministic conjcetures is single-valued; hence, it must be a continuous function.

Fix any *closed* sub-*interval* $C \subseteq X_i$. Let $ND_{i,p}(C)$ denote the set of **actions that are not strictly dominated by other pure actions.** By inspection of the defitions, it holds that

$$r_i(C) \subseteq r_i(\Delta(C)) \subseteq ND_{i,p}(C).$$

We prove that $ND_{i,p}(C) \subseteq r_i(C)$, that is, $A_i \backslash r_i(C) \subseteq A_i \backslash ND_{i,p}(C)$, which therefore implies the thesis. Since r_i is a continuous function on C, which is compact and connected, $r_i(C)$ is compact and connected as well, hence, it is a compact interval. Therefore, it is enough to show that all the actions below $\min r_i(C)$ or above $\max r_i(C)$ are dominated. Fix any $a_i < \min r_i(C)$, by strict quasi-concavity,

$$\forall x_i \in C,\ v_i(a_i, x_i) < v_i(\min r_i(C), x_i) \leq v_i(r_i(x_i), x_i).$$

Therefore, every $a_i < \min r_i(C)$ is strictly dominated by $r_i(C)$. A similar argument shows that every $a_i > \max r_i(C)$ is strictly dominated by $\max r_i(C)$. Since there are no other actions outside $r_i(C)$, this concludes the proof. ■

Corollary **1.** *Suppose that the aggregator ℓ_i is* onto*. Then, an action of player i is justifiable if an only if it is justified by a deep conjecture.*

Proof. The "if" part is trivial. For the "only if" part, fix a justifiable action a_i^* arbitrarily. By Lemma A, there is some $x_i \in X_i$ such that $a_i^* = r_i(x_i)$. Since the aggregator ℓ_i is onto, there is some $(\mathbf{a}_{-i}, \mathbf{Z}) \in \ell_i^{-1}(x_i)$ such that

$$a_i^* \in \arg\max_{a_i \in A_i} u_i(a_i, \mathbf{a}_{-i}, \mathbf{Z}).$$

Hence a_i^* is justified the deep conjecture $\delta_{(\mathbf{a}_{-i},\mathbf{Z})}$, that is, the Dirac measure supported by $(\mathbf{a}_{-i},\mathbf{Z})$. ■

With this, from now on we mostly restrict our attention to (shallow, or deep) *deterministic conjectures*.

I.2 Feedback properties

Definition **B.** *Feedback f_i satisfies* ***observable payoffs*** *(OP) relative to v_i if there is a function $\bar{v}_i : A_i \times M \to \mathbb{R}$ such that*

$$v_i\left(a_i, x_i\right) = \bar{v}_i\left(a_i, f_i\left(a_i, x_i\right)\right)$$

for all $(a_i, x_i) \in A_i \times X_i$; if the section $\bar{v}_{i,a_i}$ is injective for each $a_i \in A_i$, then we say that f_i satisfies ***just observable payoffs*** *(JOP) relative to v_i. Game G satisfies (just) observable payoffs if, for each player $i \in I$, feedback f_i satisfies (J)OP relative to v_i.*

If f_i satisfies JOP, we may assume without loss of generality that $f_i = v_i$, because, for each action a_i, the partitions of X_i induced by the preimages of v_{i,a_i} and f_{i,a_i} coincide:

Remark **B.** *Feedback f_i satisfies JOP relative to v_i if and only if*

$$\forall a_i \in A_i,\ \left\{v_{i,a_i}^{-1}\left(u\right)\right\}_{u \in v_{i,a_i}(X_i)} = \left\{f_{i,a_i}^{-1}\left(m\right)\right\}_{m \in f_{i,a_i}(X_i)}. \tag{a}$$

Proof. (Only if) Fix $a_i \in A_i$. Since f_i satisfies JOP relative to v_i, $v_{i,a_i}\left(X_i\right) = \left(\bar{v}_{i,a_i} \circ f_{i,a_i}\right)\left(X_i\right)$ (by OP), for each $u \in v_{i,a_i}\left(X_i\right)$ there is a unique message $m_{a_i,u} = \bar{v}_{i,a_i}^{-1}\left(u\right)$ (by injectivity of $\bar{v}_{i,a_i}$), and

$$\begin{aligned} v_{i,a_i}^{-1}\left(u\right) &= \left\{x_i \in X_i : v_i\left(a_i, x_i\right) = u\right\} \\ &= \left\{x_i \in X_i : \bar{v}_i\left(a_i, f_i\left(a_i, x_i\right)\right) = u\right\} \\ &= \left\{x_i \in X_i : f_i\left(a_i, x_i\right) = m_{a_i,u}\right\} = f_{i,a_i}^{-1}\left(m_{a_i,u}\right), \end{aligned}$$

which implies eq. (a).

(If) Suppose that eq. (a) holds. For every $a_i \in A_i$ and $m \in f_{i,a_i}\left(X_i\right)$ select some $\xi_i\left(a_i, m\right) \in f_{i,a_i}^{-1}\left(m\right)$. Let

$$D := \bigcup_{a_i \in A_i} \left\{a_i\right\} \times f_{i,a_i}\left(X_i\right)$$

With this,

$$\xi_i : D \to X_i$$

is a well defined function. Domain D is the set of action-message pairs for which the definition of $\bar{v}_i$ matters. Define $\bar{v}_i$ as follows:

$$\bar{v}_i\left(a_i, m\right) = \begin{cases} v_i\left(a_i, \xi_i\left(a_i, m\right)\right) & \text{if } \left(a_i, m\right) \in D, \\ 0 & \text{otherwise.} \end{cases}$$

By construction, eq. (a) implies that

$$\forall (a_i, x_i) \in A_i \times X_i,\ \bar{v}_i (a_i, f_i (a_i, x_i)) = v_i (a_i, x_i) .$$

Hence, OP holds. Furthermore, for all $a_i \in A_i$, $m', m'' \in f_{a_i} (X_i)$,

$$\begin{aligned} m' &\neq m'' \Rightarrow \xi_i \left(a_i, m'\right) \neq \xi_i \left(a_i, m'\right) \\ &\Rightarrow v_i \left(a_i, \xi_i \left(a_i, m'\right)\right) \neq v_i \left(a_i, \xi_i \left(a_i, m''\right)\right) \\ &\Rightarrow \bar{v}_i \left(a_i, m'\right) \neq \bar{v}_i \left(a_i, m'\right) \end{aligned}$$

where the first and the second implications follow from eq. (a) ($\xi_i (a_i, m')$ and $\xi_i (a_i, m')$ belong to different cells of the coincident partitions, hence yield different utilities), and the third holds by construction. Therefore, $\bar{v}_{i,a_i}$ is injective for every a_i, which means the JOP holds. ■

Definition **C.** *Feedback f_i satisfies* ***observability if and only if** i **is active*** *(OiffA) if section f_{i,a_i} is injective for each $a_i > 0$ and constant for $a_i = 0$. Game G satisfies* ***observability by active players*** *if OiffA holds for each i.*

Remark **C.** *If a network game is* linear-quadratic *and satisfies* just observable payoffs*, then it satisfies observability by active players.*

Proof. By Remark B JOP implies that, for each $a_i \in A_i$,

$$\left\{v_{i,a_i}^{-1} (u)\right\}_{u \in v_{i,a_i}(X_i)} = \left\{f_{i,a_i}^{-1} (m)\right\}_{m \in f_{i,a_i}(X_i)} .$$

The linear-quadratic form of v_i implies that, for every $x_i \in X_i$,

$$v_{i,0}^{-1} (v_{i,0} (x_i)) = X_i$$

$$\forall a_i > 0, v_{i,a_i}^{-1} (v_{i,a_i} (x_i)) = \{x_i\} .$$

These equalities imply that $f_{i,0}$ is constant and f_{i,a_i} is injective for $a_i > 0$, that is, NG satisfies observability by active players. ■

Definition **D.** *Feedback f_i satisfies* ***own-action independence*** *(OAI) of feedback about the state if, for all* justifiable *actions a_i^*, a_i^o and all payoff states $\hat{x}_i, x_i$,*

$$f_i (a_i^*, \hat{x}_i) = f_i (a_i^*, x_i) \Rightarrow f_i (a_i^o, \hat{x}_i) = f_i (a_i^o, x_i) .$$

Game G satisfies own-action independence of feedback about the state if, for each player $i \in I$, feedback f_i satisfies OAI.

In other words, OAI says that if player i cannot distinguish between two payoff states $\hat{x}_i$ and x_i when he chooses some given justifiable action a_i^*, then he cannot distinguish between these two states when he chooses any other justifiable action a_i^o. This is equivalent to requiring that the partitions of X_i of the form $\left\{f_{i,a_i}^{-1}(m)\right\}_{m\in f_{i,a_i}(X_i)}$ coincide across justifiable actions, i.e., across actions $a_i \in r_i(X_i)$ (see Lemma A).

The following lemma—which holds for any game, not just nice games—states that, under payoff observability and own-action independence, an action is justified by a confirmed conjecture if and only if it is a best reply to the actual payoff state:

Lemma **B.** *If f_i satisfies* payoff observability *relative to v_i and* own-action independence *of feedback about the state, then for all $(a_i^*, x_i) \in A_i \times X_i$ the following are equivalent:*

1. *there is some $\hat{x}_i \in X_i$ such that $a_i^* \in \arg\max_{a_i\in A_i} v_i(a_i, \hat{x}_i)$ and $f_i(a_i^*, \hat{x}_i) = f_i(a_i^*, x_i)$,*

2. *$a_i^* \in \arg\max_{a_i\in A_i} v_i(a_i, x_i)$.*

Proof.(Cf. Battigalli *et al.* 2015, Battigalli 2019) It is obvious that (2) implies (1) independently of the properties of f_i. To prove that (1) implies (2), suppose that f_i satisfies OP-OAI and let $\hat{x}_i$ be such that (1) holds. Let a_i^o be a best reply to the actual state x_i. We must show that also a_i^* is a best reply to x_i. Note that both a_i^* and a_i^o are justifiable; hence, by OAI, $f_i(a_i^*, \hat{x}_i) = f_i(a_i^*, x_i)$ implies $f_i(a_i^o, \hat{x}_i) = f_i(a_i^o, x_i)$. Using OP, condition (1), and OAI as shown in the following chain of equalities and inequalities, we obtain

$$v_i(a_i^*, x_i) \overset{(\mathrm{OP})}{=} \bar{v}_i(a_i^*, f_i(a_i^*, x_i)) \overset{(1)}{=} \bar{v}_i(a_i^*, f_i(a_i^*, \hat{x}_i)) \overset{(\mathrm{OP})}{=} v_i(a_i^*, \hat{x}_i) \overset{(1)}{\geq}$$
$$v_i(a_i^o, \hat{x}_i) \overset{(\mathrm{OP})}{=} \bar{v}_i(a_i^o, f_i(a_i^o, \hat{x}_i)) \overset{(1,\mathrm{OAI})}{=} \bar{v}_i(a_i^o, f_i(a_i^o, x_i)) \overset{(\mathrm{OP})}{=} v_i(a_i^o, x_i).$$

Since a^o is a best reply to x_i and $v_i(a_i^*, x_i) \geq v_i(a_i^o, x_i)$, it must be the case that also a_i^* is a best reply to x_i. ■

Corollary **2.** *Suppose that G satisfies* payoff observability *and* own-action independence *of feedback about the state, then the sets of selfconfirming action profiles and Nash equilibrium action profiles coincide for each* $\mathbf{Z}$:

$$\forall \mathbf{Z} \in \mathcal{Z},\ \mathbf{A}_{\mathbf{Z}}^{SCE} = \mathbf{A}_{\mathbf{Z}}^{NE}.$$

Proof By Remark 1, we only have to show that $\mathbf{A}_{\mathbf{Z}}^{SCE} \subseteq \mathbf{A}_{\mathbf{Z}}^{NE}$. Fix any $\mathbf{a}^* = (a_i^*)_{i\in I} \in \mathbf{A}_{\mathbf{Z}}^{SCE}$ and any player i. By definition of SCE, there is some $\hat{x}_i \in X_i$ such that $a_i^* \in r_i(\hat{x}_i^*)$ and $f_i(a_i^*, \hat{x}_i) = f_i\left(a_i^*, \ell_i\left(\mathbf{a}_{-i}^*, \mathbf{Z}\right)\right)$. By Lemma B $a_i^* \in r_i\left(\ell_i\left(\mathbf{a}_{-i}^*, \mathbf{Z}\right)\right)$. This holds for each i, hence $\mathbf{a}^* \in \mathbf{A}_{\mathbf{Z}}^{NE}$. ■

Corollary 2 provides sufficient conditions for the equivalence between SCE and NE. Next, we give sufficient conditions that allow a characterization of $\mathbf{A}_{\mathbf{Z}}^{SCE}$ by means of Nash equilibria of auxiliary games.

I.3 Equilibrium Characterization

If $a_i \in [0, \bar{a}_i]$ is interpreted as an activity level (e.g., effort) by player i, then it makes sense to say that i is **active** if $a_i > 0$ and **inactive** otherwise. Let I_0 denote the **set of players for whom being inactive is justifiable**. Note that, by Lemma A,

$$I_0 = \{i \in I : \min r_i (X_i) = 0\}.$$

Also, for each $\mathbf{Z} \in \mathcal{Z}$ and nonempty subset of players $J \subseteq I$, let $\mathbf{A}^{NE}_{J,\mathbf{Z}}$ denote the set of Nash equilibria of the auxiliary game with players set J obtained by letting $a_i = 0$ for each $i \in I \backslash J$, that is,

$$\mathbf{A}^{NE}_{J,\mathbf{Z}} = \left\{ \mathbf{a}^*_J \in \times_{j \in J} A_j : \forall j \in J, a^*_j = r_j \left(\ell_j \left(\mathbf{a}^*_{J \backslash \{j\}}, \mathbf{0}_{I \backslash J}, \mathbf{Z} \right) \right) \right\},$$

where $\mathbf{0}_{I \backslash J} \in \mathbb{R}^{I \backslash J}$ is the profile that assigns 0 to each $i \in I \backslash J$. If $J = \emptyset$, let $\mathbf{A}^{NE}_{J,\mathbf{Z}} = \{\varnothing\}$ by convention, where $\varnothing$ is the peudo-action profile such that $(\varnothing, \mathbf{0}_I) = \mathbf{0}_I$.

Lemma **C.** *Suppose that the parameterized nice game with aggregators and feedback G satisfies* observability by active players. *Then, for each $\mathbf{Z}$, the set of selfconfirming action profiles is*

$$\mathbf{A}^{SCE}_{\mathbf{Z}} = \bigcup_{I \backslash J \subseteq I_0} \mathbf{A}^{NE}_{J,\mathbf{Z}} \times \left\{ \mathbf{0}_{I \backslash J} \right\}.$$

Proof Fix $\mathbf{a}^*$ and let J be the set of players i such that $a^*_i > 0$. Fix $\mathbf{Z} \in \mathcal{Z}$ arbitrarily. Suppose that $\mathbf{a}^* \in \mathbf{A}^{SCE}_{\mathbf{Z}}$ and fix any $i \in I$. If $a^*_i = 0$, then 0 is justifiable for i, that is $i \in I_0$. If $a^*_i > 0$, OiifA implies that f_{i,a^*_i} is injective, that is, action a^*_i reveals the payoff state, hence the (shallow) conjecture justifying a^*_i is correct: $a^*_i = r_i \left(\ell_i \left(\mathbf{a}^*_{-i}, \mathbf{Z} \right) \right)$. Thus, $\mathbf{a}^* = \left(\mathbf{a}^*_J, \mathbf{a}^*_{I \backslash J} \right)$ is such that $a^*_i = 0$ for each $i \in I \backslash J \subseteq I_0$, and $a^*_j = r_j \left(\ell_j \left(\mathbf{a}^*_{J \backslash \{j\}}, \mathbf{0}_{I \backslash J}, \mathbf{Z} \right) \right) > 0$ for each $j \in J$. Hence,

$$\mathbf{a}^* = \left(\mathbf{a}^*_J, \mathbf{a}^*_{I \backslash J} \right) \in \mathbf{A}^{NE}_{J,\mathbf{Z}} \times \left\{ \mathbf{0}_{I \backslash J} \right\} \text{ with } I \backslash J \subseteq I_0.$$

Let $I \backslash J \subseteq I_0$ and $\left(\mathbf{a}^*_J, \mathbf{a}^*_{I \backslash J} \right) \in \mathbf{A}^{NE}_{\mathbf{Z}} \times \left\{ \mathbf{0}_{I \backslash J} \right\}$. Since G satisfies OiffA, for each $i \in I \backslash J$, any conjecture justifying $a^*_i = 0$ (any $\hat{x}_i \in r^{-1}_i (0)$) is trivially confirmed. For each $j \in J$, $a^*_j > 0$ is by assumption the best reply to the correct, hence confirmed, conjecture $x^*_j = \ell_i \left(\mathbf{a}^*_{J \backslash \{j\}}, \mathbf{0}_{I \backslash J}, \mathbf{Z} \right)$. Hence, $\left(\mathbf{a}^*_J, \mathbf{a}^*_{I \backslash J} \right) = \left(\mathbf{a}^*_J, \mathbf{0}_{I \backslash J} \right) \in \mathbf{A}^{SCE}_{\mathbf{Z}}$. ■

Appendix J Knowledge of the network and iterated strategic reasoning

The SCE concept does not rely, explicitly or implicitly, on strategic reasoning. Thus, some SCEs may be supported by confirmed conjectures that are inconsistent with the assumption that other agents are rational and think strategically. In this section we consider what happens when agents use the information they have about the network to perform iterated strategic reasoning and, at the same time, to form conjectures about the unknowns that must be confirmed in a SCE. Thus we analyze which SCEs are consistent with common belief in rationality, which may help in selecting some SCEs when there is a multiplicity of equilibria. More specifically, when agents have some information about the network, it is reasonable to assume that they use it to determine how they should act. Indeed, using one step of reasoning, every agent may try to infer which actions her direct neighbors may play, shape own conjectures accordingly, and depending on the knowledge about the strategic interaction she is exposed to, determine the set of her own actions that are best replies to such conjectures. Going further, she can take into account that her neighbors actions should be best replies to conjectures consistent with the rationality her neighbors' neighbors, and so on. This yields a notion of rationalizability of conjectures, and a corresponding definition of **selfconfirming equilibrium with rationalizable conjectures**, which is the object of our analysis in this section. We obtain results for the cases analyzed in the previous sections of the paper, that is positive local externalities, unconstrained local externalities, and positive local externalities joint with positive global externalities.[31]

Knowledge and Deep Conjectures As defined in the previous sections, $\mathcal{Z} \subseteq [\underline{w}, \bar{w}]^{I \times I}$ is the set of possible weighted networks and it represents the uncertainty space. We maintain the assumption that $\mathcal{Z}$ is common knowledge, and that there is common knowledge of the parameterized payoff functions. For the purposes of this analysis, we consider three possible cases. i) $\mathcal{Z} = \{\mathbf{Z}\}$, so that the network is common knowledge; ii) $\mathcal{Z} = [0, \bar{w}]^{I \times I}$, so that the network $\mathbf{Z}$ is unknown, but it is common knowledge that links must be non-negative and bounded, so that only positive local externalities are possibile; iii) $\mathcal{Z} = [\underline{w}, \bar{w}]^{I \times I}$, that means that $\mathbf{Z}$ is unknown, but it is common knowledge that each link of the network $\mathbf{Z}$ is in the interval $[\underline{w}, \bar{w}]$. Besides common knowledge of $\mathcal{Z}$, we allow each agent to have **deep conjectures**, that is, conjectures about the network $\mathbf{Z}$ and the actions of other agents in the network. For each agent $i \in I$, deep conjectures are defined as probability measures $\mu_i \in \Delta(\mathbf{A}_{-i} \times \mathcal{Z})$ (see definition A in I.1). Notice that, if $\mathcal{Z}$ is a singleton, the only uncertainty agents have is about others' actions.

Rationalizability Given common knowledge of the parameterized game $\langle I, \mathcal{Z}, (A_i, u_i)_{i \in I} \rangle$, we can characterize the behavioral implications of *rationality and common belief in rationality (RCBR)*,

[31] Theoretically, we can distinguish among different elements that can be the object of knowledge: i) the pure topological structure of the network (who is linked with whom); ii) the kind of interaction (complementarity or substitution) that operates on each link; iii) the intensity of this interaction. Here we focus on two extreme cases, common knowledge of the network $\mathbf{Z}$, and just common knowledge of the uncertainty space $\mathcal{Z}$, which may satisfy some properties, such as positive local externalities. Thus, we ignore other intermediate cases that could be analyzed within our framework. In particular, we ignore the possibility that agents have private information about the network, which simplifies the analysis.

i.e., the set of action profiles consistent with these (so called) **epistemic assumptions**. A formal expression of these epistemic assumptions and a characterization of their behavioral implications in a class of games that contains those considered here is given, for example, in Battigalli and Tebaldi (2019) and in Battigalli (2019). In our setting, an action profile is consistent with RCBR if and only if, given $\mathcal{Z}$, for every $i \in I$, it survives the following procedure of iterated elimination of non best replies:

- $A_i^0 = A_i$,
- $A_i^{n+1} = \left\{a_i^* \in A_i : \exists \mu_i \in \Delta(\mathbf{A}_{-i}^n \times \mathcal{Z}),\ a_i^* \in \arg\max_{a_i \in A_i} \mathbb{E}_{\mu_i}\left[u_i(a_i, \cdot)\right]\right\}$,
- $A_i^\infty = \bigcap\limits_{n \in \mathbb{N}} A_i^n$.

Definition **E.** *An action a_i of player i is* rationalizable *if $a_i \in A_i^\infty$. A conjecture μ_i of player i is* rationalizable *if $\mu_i \in \Delta(\mathbf{A}_{-i}^\infty \times \mathcal{Z})$.*

As we did for the case of shallow conjectures, for each agent $i \in I$, we can restrict our attention to *deterministic* deep conjectures $(\hat{a}_{-i}, \hat{\mathbf{Z}}_i) \in \mathbf{A}_{-i} \times \mathcal{Z}$. We are allowed to use deterministic deep conjectures because since $\mathbf{A}_{-i}$ and $\mathcal{Z}$ and are compact and connected and thus, given the continuity of u_i and strict quasi-concavity of each section $u_{i,a_{-i},\mathbf{Z}}$, for every probabilistic deep conjecture there exists a deterministic deep conjecture that delivers the same best reply (see Appendix I and Battigalli 2019). This implies that if $\mathbf{A}_{-i}^n$ is compact and connected, than A_i^{n+1} is the compact interval of best replies to deterministic conjectures (see Lemma A). By induction, this holds for every step. To see this in detail, let $\mathcal{C} \subseteq 2^{\mathbf{A}}$ denote the collection of compact Cartesian subsets of $\mathbf{A}$. It is convenient to define the following selfmap

$$\begin{array}{rccl} \rho: & \mathcal{C} & \to & \mathcal{C}, \\ & \mathbf{C} & \mapsto & \times_{i \in I} r_i\left(\ell_i\left(\mathbf{C}_{-i} \times \mathcal{Z}\right)\right). \end{array}$$

This is a selfmap because each best reply map $r_i \circ \ell_i$ is a continuous function, hence $\times_{i \in I} r_i\left(\ell_i\left(\mathbf{C}_{-i} \times \mathcal{Z}\right)\right)$ is a product of compact sets whenever $\mathbf{C}$ is. In words, $\rho\left(\mathbf{C}\right)$ is the set of profiles of best replies to deterministic (deep) conjectures such that each i is certain that the co-players choose in $\mathbf{C}_{-i}$. Let $\rho^n = \rho \circ \rho^{n-1}$ denote the nth iteration of ρ. With this, the following result follows from Lemma A and a straightforward induction argument:

Theorem **A.** *In a parametrized nice game with aggregators*

$$\mathbf{A}^n = \rho^n\left(\mathbf{A}\right) = \times_{i \in I}\left[\min A_i^n, \max A_i^n\right]$$

for all $n \in \mathbb{N} \cup \{\infty\}$.

Selfconfirming equilibrium with rationalizable conjectures Assuming observability of payoffs, we extend the definition of selfconfirming equilibrium to incorporate also the requirements of rationalizability.

Definition **F.** *A profile* $\left(a_i^*, \hat{a}_{-i}, \hat{\mathbf{Z}}_i\right)_{i\in I} \in \times_{i\in I}\left(A_i \times \mathbf{A}_{-i} \times \mathcal{Z}\right)$ *of actions and deterministic deep conjectures is a* ***selfconfirming equilibrium at*** $\mathbf{Z}$ ***with rationalizable conjectures*** *(SCER) if, for each player* $i \in I$*,*

1. (best reply) $a_i^* \in r_i\left(\hat{a}_{-i}, \hat{\mathbf{Z}}_i\right)$,

2. (confirmed conjectures, given observable payoffs) $u_i\left(a_i^*, \hat{a}_{-i}, \hat{\mathbf{Z}}_i\right) = u_i\left(a_i^*, \mathbf{a}_{-i}^*, \mathbf{Z}\right)$.

3. (rationalizable conjectures) $(\hat{a}_{-i}, \hat{\mathbf{Z}}_i) \in \mathbf{A}_{-i}^{\infty} \times \mathcal{Z}$,

We denote by $\mathbf{A}_{\mathcal{Z}}^{SCER}$ the sets of SCE actions profiles justified by rationalizable confirmed conjectures, given the commonly known parameter space $\mathcal{Z}$. Note, this is the set of action profiles consistent with the following assumptions: (a) player are rational, (b) players' conjectures are confirmed, and (c) there is common belief of (a). A stronger notion of "rationalizable selfconfirming equilibrium" due to Rubinstein and Wolinsky (1994) is based on the following assumptions: (a) player are rational, (b) players' conjectures are confirmed, and (c*) there is common belief of (a) and (b). We limit our analysis to the weaker SCER concept for two reasons: (i) it is simpler; (ii) to our knowledge, there is no learning foundation of rationalizable SCE à la Rubinstein and Wolinsky, whereas one can justify our concept by considering learning dynamics like those analyzed in this paper, assuming that players always hold rationalizable conjectures because there is common belief in rationality. Note that such belief cannot ever be falsified by what players observe, given that they best respond to rationalizable conjectures, and therefore always choose rationalizable actions.

We now discuss how SCER actions are shaped depending on the type of strategic interaction at work in the network.

Local Complementarities The first case analyzed in the previous sections of the paper is when there are local complementarities or mild substitutions. For simplicity of exposition, we consider just the case of positive local externalities. This is to say that when the actual network is unknown, then $\mathcal{Z} = [0, \bar{w}]^{I\times I}$, while when the network is common knowledge then $\mathcal{Z} = \{\mathbf{Z}\}$ with $\mathbf{Z} \in [0, \bar{w}]^{I\times I}$. Letting $X_i = \ell_i\left(\mathbf{A}_{-i} \times \mathcal{Z}\right)$, the hypothesis of Proposition 1 is satisfied, because $\underline{x}_i = 0$ and $\min r_i\left(X_i\right) = r_i\left(0\right) = \alpha_i > 0$. Thus, in the case of positive local externalities, the set of SCE action profiles is a singleton that coincides with the unique (interior) Nash equilibrium. Consequently, adding rationalizability on top of the SCE requirements does not change the result. Indeed, the Nash equilibrium action profile is always rationalizable.

Corollary **3.** *In any network game, for every* $\mathcal{Z} \subseteq [0, \bar{w}^{I \times I}]$ *and for all* $\mathbf{Z} \in \mathcal{Z}$, $\mathbf{A}^{SCE}_{\mathbf{Z}} = \mathbf{A}^{SCER}_{\mathcal{Z}} = \mathbf{A}^{NE}_{\mathbf{Z}}$.[32]

Even if with positive local externalities rationalizability does not change the set of SCE, it is still interesting to understand how rationalizability works in a linear quadratic network game, and more generally in nice games with strategic complementarities.
Given the finite index set I, the vector space $\mathbb{R}^I$ is endowed with the standard partial order: $\mathbf{v}' \leq \mathbf{v}''$ if and only if $v'_i \leq v''_i$ for each $i \in I$. With this, our assumptions imply that $\mathcal{Z} \subseteq \mathbb{R}^{I \times I}$ is a *complete lattice*, which implies that also $\mathbf{A} \times \mathcal{Z}$ is a complete lattice. We let $\underline{\mathbf{Z}}$ and $\bar{\mathbf{Z}}$ respectively denote the smallest and largest elements of $\mathcal{Z}$. Let $Y_i = \mathbf{A}_{-i} \times \mathcal{Z}$. A payoff function $u_i : A_i \times Y_i \to \mathbb{R}$ has **increasing differences** if, for all $a'_i, a''_i \in A_i$, $x'_i, x''_i \in X_i$ such that $a'_i \leq a''_i$ and $x'_i \leq x''_i$

$$u_i\left(a''_i, x'_i\right) - u_i\left(a'_i, x'_i\right) \leq u_i\left(a''_i, x''_i\right) - u_i\left(a'_i, x''_i\right).$$

Definition **G.** *A network game has* ***strategic complementarities*** *if* $\mathcal{Z} \subseteq [0, \bar{w}]^{I \times I}$ *is a complete lattice and, for each* $i \in I$, v_i *has increasing differences.*

Remark **G.** *If network game has strategic complementarities, then each game* $\left\langle I, (A_i, u_{i,\mathbf{Z}})_{i \in I} \right\rangle$ *with* $\mathbf{Z} \in \mathcal{Z}$ *is supermodular.*

It is well known that the set of Nash equilibria of a supermodular game is a complete lattice (e.g. Milgrom and Roberts, 1990). With this, for any network game with strategic complementarities, we let $\underline{\mathbf{a}}^{NE}_{\underline{\mathbf{Z}}}$ and $\bar{\mathbf{a}}^{NE}_{\bar{\mathbf{Z}}}$ respectively denote the smallest Nash equilibrium of game $\left\langle I, \left(A_i, u_{i,\underline{\mathbf{Z}}}\right)_{i \in I} \right\rangle$ and the largest Nash equilibrium of game $\left\langle I, \left(A_i, u_{i,\bar{\mathbf{Z}}}\right)_{i \in I} \right\rangle$. The "box," or order-interval in $\mathbb{R}^I$ determined by $\underline{\mathbf{a}}^{NE}_{\underline{\mathbf{Z}}}$ and $\bar{\mathbf{a}}^{NE}_{\bar{\mathbf{Z}}}$ is

$$\left[\underline{\mathbf{a}}^{NE}_{\underline{\mathbf{Z}}}, \bar{\mathbf{a}}^{NE}_{\bar{\mathbf{Z}}}\right] := \times_{i \in I} \left[\underline{a}^{NE}_{i,\underline{\mathbf{Z}}}, \bar{a}^{NE}_{i,\bar{\mathbf{Z}}}\right].$$

Proposition **A.** *Consider a network game with strategic complementarities. The set of rationalizable action profiles is* $\mathbf{A}^{\infty} = \left[\underline{\mathbf{a}}^{NE}_{\underline{\mathbf{Z}}}, \bar{\mathbf{a}}^{NE}_{\bar{\mathbf{Z}}}\right]$*, that is, the set of rationalizable actions of each player is the interval between the lowest Nash equilibrium action in the game determined by the lowest parameter* $\underline{\mathbf{Z}}$ *and the highest Nash equilibrium action in the game determined by the highest parameter* $\bar{\mathbf{Z}}$.

Proof Consider an auxiliary game $\hat{G}$ where an indifferent pseudo-player chooses $\mathbf{Z} \in \mathcal{Z}$, and the action sets and payoff functions of each $i \in I$ are those specified in the network game NG given $\mathbf{Z}$. It is easy to verify that the auxiliary game $\hat{G}$ is supermodular and every $\mathbf{Z} \in \mathcal{Z}$ is a Nash equilibrium action for the indifferent pseudo-player, that is, the set of Nash equilibria of $\hat{G}$ is

$$\bigcup_{\mathbf{Z} \in \mathcal{Z}} \mathbf{A}^{NE}_{\mathbf{Z}} \times \{\mathbf{Z}\}.$$

[32] As we noted for Proposition 1, the same result holds also for a non linear and continuous aggregator ℓ_i and a generic continuous and strictly quasi-concave utility function v_i.

It is also easy to check that the set of rationalizable profiles of $\hat{G}$ is $\mathbf{A}^{\infty} \times \mathcal{Z}$, and Theorem A implies that $\mathbf{A}^{\infty}$ is an order-interval. Finally, Theorem 5 in Milgrom and Roberts (1990) implies that the smallest element of $\mathbf{A}^{\infty} \times \mathcal{Z}$ is $\left(\underline{\mathbf{a}}^{NE}_{\underline{\mathbf{Z}}}, \underline{\mathbf{Z}}\right)$ and the largest element of $\mathbf{A}^{\infty} \times \mathcal{Z}$ is $\left(\bar{\mathbf{a}}^{NE}_{\bar{\mathbf{Z}}}, \bar{\mathbf{Z}}\right)$; therefore, $\mathbf{A}^{\infty} = \left[\underline{\mathbf{a}}^{NE}_{\underline{\mathbf{Z}}}, \bar{\mathbf{a}}^{NE}_{\bar{\mathbf{Z}}}\right]$.

Proposition A characterizes the set of rationalizable action profiles for a generic complete lattice $\mathcal{Z}$. It is straightforward to see that if the network $\mathbf{Z}$ is common knowledge, $\bar{\mathbf{Z}} = \underline{\mathbf{Z}}$, then $\underline{\mathbf{a}}^{NE}_{\underline{\mathbf{Z}}} = \bar{\mathbf{a}}^{NE}_{\bar{\mathbf{Z}}}$, since there exists a unique Nash equilibrium $\mathbf{a}^{NE}_{\mathbf{Z}}$, and $\mathbf{A}^{\infty} = \left\{\mathbf{a}^{NE}_{\mathbf{Z}}\right\}$, that is, rationalizability yields the unique NE.

Unconstrained local externalities We consider now the case in which a network allows for negative weights, so that $\mathcal{Z} = [\underline{w}, \bar{w}]^{I \times I}$ with $\underline{w} < 0$ and $\bar{w} > 0$. The SCE analysis for this case performed in Section 5 shows that a selfconfirming equilibrium with shallow conjectures may allow any arbitrary set of agents to be inactive. Here we show that having knowledge of the network, and using strategic iterated reasoning, may help in selecting some SCEs, even if we do not necessarily get rid of all the non-Nash ones. The most intuitive reason for this result is that when strategic substitutabilities are at work, the set of rationalizable action profiles is generically larger than the set of Nash equilibria. Here, we characterize the set of SCE that survive strategic iterated reasoning.

Consider first two matrices $\mathbf{Z}_{-} < 0$ and $\mathbf{Z}_{+} > 0$ such that $\mathbf{Z} = \mathbf{Z}_{-} + \mathbf{Z}_{+}$. $\mathbf{Z}_{-}$ is the matrix reporting just the negative links of $\mathbf{Z}$, and $\mathbf{Z}_{+}$ is the matrix reporting just the positive links of $\mathbf{Z}$. Define a sequence of pair of action profiles $(\underline{\mathbf{a}}^n, \bar{\mathbf{a}}^n)_{n \in \mathbb{N}}$; $\underline{\mathbf{a}}^n$ and $\bar{\mathbf{a}}^n$ are, respectively, the lower and the upper bound on action profiles that survive n steps of iterated deletion of non best replies. $(\underline{\mathbf{a}}^n, \bar{\mathbf{a}}^n)_{n \in \mathbb{N}}$ is such that $\underline{\mathbf{a}}^0 = \mathbf{0}$, $\bar{\mathbf{a}}^0 = \bar{a} \cdot \mathbf{1}$, and for every $n \in \mathbb{N}$, $\underline{\mathbf{a}}^n = \boldsymbol{\alpha} + \mathbf{Z}_{+}\underline{\mathbf{a}}^{n-1} + \mathbf{Z}_{-}\bar{\mathbf{a}}^{n-1}$ and $\bar{\mathbf{a}}^n = \boldsymbol{\alpha} + \mathbf{Z}_{+}\bar{\mathbf{a}}^{n-1} + \mathbf{Z}_{-}\underline{\mathbf{a}}^{n-1}$. Then, at the n^{th} step of iterated deletion of non dominated strategies, the interval of actions agent $i \in I$ can play is $\mathbf{A}^n_i = [\underline{a}^n_i, \bar{a}^n_i]$. Indeed, for each $i \in I$, $\underline{a}^n_i$ is given by a best reply to i) the highest possibile actions of neighbors towards whom i experiences strategic substitution that can be rationalized after n steps of reasoning, and ii) the smallest possibile actions of neighbors towards whom i experience strategic complementarities that can be rationalized after n steps of reasoning. Similarly, $\bar{a}^n_i$ is build considering the highest possibile actions for neighbors who shows complementarity, and the smallest possible action for neighbors who shows substitutability. Define $I^*_0 := \{i \in I : lim_{n \to \infty} \underline{a}^n_i = 0\}$. This is the set of agents for whom being inactive is rationalizable. We can then characterize the set $\mathbf{A}^{SCER}_{\mathcal{Z}}$. We first consider the case of common knowledge of the network. Recall that, in this case, for each $i \in I$, deep conjectures are just conjectures about $\mathbf{A}_{-i}$.

PROPOSITION **B.** *Suppose that there is common knowledge of the network. Then, for all* $\mathbf{Z} \in \mathcal{Z}$

$$\mathbf{A}_{\mathbf{Z}}^{SCER} = \mathbf{A}_{\mathbf{Z}}^{SCE} \setminus \left(\bigcup_{J : J \cap (I \setminus I_0^*) \neq \emptyset} \mathbf{A}_{I \setminus J, \mathbf{Z}}^{NE} \times \{\mathbf{0}_J\} \right) = \bigcup_{J : J \subseteq I_0^*} \mathbf{A}_{I \setminus J, \mathbf{Z}}^{NE} \times \{\mathbf{0}_J\}.$$

Proof. Let $\bar{\mathbf{A}}_{\mathbf{Z}} := \left(\bigcup_{J : J \cap (I \setminus I_0) \neq \emptyset} \mathbf{A}_{I \setminus J, \mathbf{Z}}^{NE} \times \{\mathbf{0}_J\} \right)$. We first prove that $\bar{\mathbf{A}}_{\mathbf{Z}} \cap \mathbf{A}_{\mathcal{Z}}^{SCER} = \emptyset$. By inspection of the definition of $\bar{\mathbf{A}}_{\mathbf{Z}}$, for each $\bar{\mathbf{a}}_{\mathbf{Z}} \in \bar{\mathbf{A}}_{\mathbf{Z}}$, there exists an $i \in I$ such that i) $\bar{a}_{\mathbf{Z},i} = 0$ and ii) $i \in I \setminus I_0$, so that $\bar{a}_{\mathbf{Z},i} = 0$ is not rationalizable. Then $\bar{\mathbf{a}}_{\mathbf{Z}} \notin \mathbf{A}_{\mathcal{Z}}^{SCER}$. Since this holds for all $\bar{\mathbf{a}}_{\mathbf{Z}} \in \bar{\mathbf{A}}_{\mathbf{Z}}$, then $\bar{\mathbf{A}}_{\mathbf{Z}} \cap \mathbf{A}_{\mathcal{Z}}^{SCER} = \emptyset$.

We now prove that all the action profiles in $\mathbf{A}_{\mathbf{Z}}^{SCE} \setminus \bar{\mathbf{A}}_{\mathbf{Z}}$ are part of $\mathbf{A}_{\mathcal{Z}}^{SCER}$. Take a profile $\mathbf{a} \in \mathbf{A}_{\mathbf{Z}}^{SCE} \setminus \bar{\mathbf{A}}_{\mathbf{Z}}$. Consider an $i \in I$ such that $a_i = 0$. Then $i \in I_0$ and, by the definition of I_0, $a_i \in A_i^{\infty}$. Then $\mathbf{a}_{I_0} \in \mathbf{A}_{I_0}^{\infty}$.
Define now $J := \{i \in I : a_i > 0\}$. By Proposition 2, $\mathbf{a}_J \in \mathbf{A}_{J,\mathbf{Z}}^{NE}$, so that $\mathbf{a}_J \in \mathbf{A}_{J,\mathbf{Z}}^{\infty}$. Given linearity of best reply functions, for each $i \in J$

$$r(\mathbf{a}_{J,-i}) = r(\mathbf{a}_{J,-i}, \mathbf{0}) = r(\mathbf{a}_{J,-i}, \mathbf{a}_{I_0}). \tag{b}$$

Given (b), and recalling that $\mathbf{a}_{I_0} \in \mathbf{A}_{I_0}^{\infty}$, $\mathbf{a} \in \mathbf{A}^{\infty}$. ■

The characterization of selfconfirming equilibria with rationalizable conjectures under common knowledge of the network makes it clear how rationalizability can refine the predictions of SCE. Proposition B states that from the set of SCEs action profiles, we have to drop those in which there are inactive agents forn whom being inactive is not a best reply to a rationalizable conjecture. Indeed, subsets of the form $J \cap (I \setminus I_0) \neq \emptyset$ are those containing at least one agent for whom being inactive is not rationalizable. Then $\bigcup_{J : J \cap (I \setminus I_0) \neq \emptyset} \mathbf{A}_{I \setminus J, \mathbf{Z}}^{NE} \times \{\mathbf{0}_J\}$ identifies, among all the SCEs, the ones in which at least one inactive agent plays a not rationalizable action. Once we drop these action profiles, we characterize $\mathbf{A}_{\mathcal{Z}}^{SCER}$ as the SCE profiles where inactive agents are best responding to a rationalizable conjecture, which is trivially confirmed.

We now provide two examples. The first one shows two cases in which rationalizability exactly selects the unique Nash equilibrium. The second one shows that rationalizability restricts the set of SCEs without converging to the Nash equilibria set.

Example G. Consider the network in Figure 2. This is a particularly simple network since each agent experiences just strategic complementarities or just strategic substitutions. We now analyze how the procedure of iterated elimination of non best replies selects a specific SCE out of all the 16 possible SCEs. As in Figure 2, we first consider the case in which the magnitude of local externalities and substitutabilities is 0.2, so that a unique interior Nash equilibrium exists which is given by $(0.1267, 0.1603, 0.0412, 0.1335)$. We now study how the vector of lower and upper bounds ($\underline{a}$ and $\bar{a}$) is shaped for each agent at each step of the iterative reasoning.

$$\underline{a}^0 = (0, 0, 0, 0) \qquad \bar{a}^0 = (1, 1, 1, 1)$$
$$\underline{a}^1 = (0.1, 0.1, 0, 0.1) \qquad \bar{a}^1 = (0.3, 0.7, 0.1, 0.5)$$
$$\underline{a}^2 = (0.12, 0.14, 0, 0.12) \qquad \bar{a}^2 = (0.2, 0.28, 0.06, 0.18)$$
$$\underline{a}^3 = (0.124, 0.148, 0.008, 0.124) \qquad \bar{a}^3 = (0.136, 0.19, 0.048, 0.15)$$
$$\underline{a}^4 = (0.1248, 0.151, 0.032, 0.126) \qquad \bar{a}^4 = (0.130, 0.17, 0.045, 0.136)$$
$$\underline{a}^5 = (0.1252, 0.156, 0.039, 0.131) \qquad \bar{a}^5 = (0.127, 0.162, 0.044, 0.135)$$

In this case we get convergence to the unique Nash equilibrium. Notice that, in the first two rounds, agents think that agent 3 may reasonably stay inactive because of substitutability. It takes 3 rounds to understand that, given what others can choose, these substitutabilities are not very strong, and agent 3's neighbors' largest actions are not high enough, to make agent 3 inactive.
We now consider the case of a higher level of substitution. Let the magnitude of local externalities and substitutabilities be $w = 0.5$ and keep the network topology fixed. In this case the unique Nash equilibrium is given by $(0.2, 0.3, 0, 0.2)$.

$$\underline{a}^0 = (0, 0, 0, 0) \qquad \bar{a}^0 = (1, 1, 1, 1)$$
$$\underline{a}^1 = (0.1, 0.1, 0, 0.1) \qquad \bar{a}^1 = (0.6, 1.6, 0.1, 1.1)$$
$$\underline{a}^2 = (0.15, 0.2, 0, 0.15) \qquad \bar{a}^2 = (0.65, 1., 0, 0.45)$$
$$\underline{a}^3 = (0.175, 0.25, 0, 0.175) \qquad \bar{a}^3 = (0.32, 0.65, 0, 0.42)$$
$$\underline{a}^4 = (0.187, 0.27, 0, 0.187) \qquad \bar{a}^4 = (0.31, 0.47, 0, 0.26)$$
$$\underline{a}^5 = (0.193, 0.28, 0, 0.193) \qquad \bar{a}^5 = (0.23, 0.38, 0, 0.25)$$
$$\underline{a}^6 = (0.196, 0.293, 0, 0.196) \qquad \bar{a}^6 = (0.22, 0.34, 0, 0.215)$$
$$\underline{a}^7 = (0.198, 0.296, 0, 0.198) \qquad \bar{a}^7 = (0.207, 0.32, 0, 0.214)$$

Again there is convergence to the unique Nash equilibrium. Notice that in this case strategic substitution is so strong that the only rationalizable action for agent 3 is to be inactive. ∎

In the previous example we show how rationalizability selects exactly one equilibrium (the Nash one) out of the 16 SCEs. These ones were easy cases since each agent just experiences only strategic complementarities or only strategic substitutions. We now consider a modified network in which agent 3 have links of different signs to give an example of a network with more heterogeneity in strategic relations, in which iterated strategic reasoning does not converge to a unique action profile.

Example G. Consider again the network in Figure 2, in which the sign of the link from agent

3 to agent 2 is changed from negative to positive. Let the magnitude of local externalities and substitutabilities be 0.5. In this case the iterative process is as follows

$$
\begin{aligned}
\underline{a}^0 &= (0,0,0,0) & \bar{a}^0 &= (1,1,1,1) \\
\underline{a}^1 &= (0.1,0.1,0,0.1) & \bar{a}^1 &= (0.6,1.6,0.6,1.1) \\
\underline{a}^2 &= (0.15,0.2,0,0.15) & \bar{a}^2 &= (0.65,1.25,0.85,0.7) \\
\underline{a}^3 &= (0.17,0.25,0,0.17) & \bar{a}^3 &= (0.45,1.2,0.65,0.85) \\
\underline{a}^4 &= (0.18,0.27,0,0.18) & \bar{a}^4 &= (0.52,1.075,0.61,0.65) \\
\underline{a}^5 &= (0.193,0.28,0,0.193) & \bar{a}^5 &= (0.42,0.993,0.54,0.668) \\
\underline{a}^6 &= (0.196,0.293,0,0.196) & \bar{a}^6 &= (0.43,0.918,0.5,0.584) \\
\underline{a}^7 &= (0.198,0.296,0,0.198) & \bar{a}^7 &= (0.39,0.859,0.46,0.567) \\
\underline{a}^8 &= (0.199,0.298,0,0.199) & \bar{a}^8 &= (0.383,0.81,0.43,0.526)
\end{aligned}
$$

The iteration continues not far from the last values reported. By intersecting the interval of each agent's A_i^∞ with the set of SCE, there are just two SCEs with rationalizable conjectures. In details, the equilibrium in which every agent is active, that is the Nash equilibrium $(0.26, 0.48, 0.18, 0.32)$, and the equilibrium in which only 3 is inactive, that is $(0.2, 0.3, 0, 0.2)$, that is not a Nash equilibrium. ∎

Finally, we note that when it is common knowledge that complementarities and substitutabilities are mild then there is a unique SCER, which—necessarily—coincides with the unique Nash equilibrium. This is the case, for example, if $-\frac{\alpha}{(n-1)\bar{a}} < \underline{w}$ and $\bar{w} < \frac{\bar{a}-\alpha}{(n-1)\alpha}$, and this is common knowledge, then rationalizability yields to the unique interior Nash equilibrium. One can get intermediate results by changing the threshold for just one of $\underline{w}$ and $\bar{w}$.

Local and global externalities We consider now the case of both local and global externalities. As discussed in Section 7, we restrict our attention to situations in which local externalities are positive. In this case, there is a continuum of SCEs, one for each vector of perceived centralities. We now study whether iterated strategic reasoning helps in selecting some SCEs. The main result is that, if there is common knowledge of the network, iterated strategic reasoning selects the unique interior Nash equilibrium among the infinite possible SCEs.

Proposition **C.** *Consider a network game with positive local externalities, global externalities, common knowledge of the network ($\mathcal{Z} = \{\mathbf{Z}\}$), and a unique Nash equilibrium that is interior. Then* $\mathbf{A}_{\mathcal{Z}}^{SCER} = \mathbf{A}_{\mathbf{Z}}^{NE}$

Proof. The result follows from Theorem A. Indeed the game we are considering has strategic complementarities. Then, $\mathbf{A}^{\infty} = \left[\underline{\mathbf{a}}_{\mathbf{Z}}^{NE}, \bar{\mathbf{a}}_{\mathbf{Z}}^{NE}\right]$. Since $\mathbf{Z}$ is common knowledge, and there exists a unique interior Nash equilibrium, it follows that $\mathbf{A}_{\mathbf{Z}}^{\infty} = \left\{\mathbf{a}_{\mathbf{Z}}^{NE}\right\}$. Then, $\mathbf{A}_{\mathbf{Z}}^{SCER} = \left\{\mathbf{a}_{\mathbf{Z}}^{NE}\right\}$. ■

We can alternatively prove it by showing how iterative reasoning works in this case. Recall that if network is common knowledge and there are just strategic complementarities, then agents can only have positive justifiable actions. Consider $\underline{\mathbf{a}}^0 = \boldsymbol{\alpha}$ and $\bar{\mathbf{a}}^0 = \bar{\mathbf{a}}$. If the network is common knowledge, then

$$
\begin{aligned}
\underline{\mathbf{a}}^1 &= \boldsymbol{\alpha} + \mathbf{Z}\boldsymbol{\alpha}, & \bar{\mathbf{a}}^1 &= \boldsymbol{\alpha} + \mathbf{Z}\bar{\mathbf{a}} \\
\underline{\mathbf{a}}^2 &= \boldsymbol{\alpha} + \mathbf{Z}(\boldsymbol{\alpha} + \mathbf{Z}\boldsymbol{\alpha}), & \bar{\mathbf{a}}^1 &= \boldsymbol{\alpha} + \mathbf{Z}(\boldsymbol{\alpha} + \mathbf{Z}\bar{\mathbf{a}}) \\
&= \boldsymbol{\alpha} + \mathbf{Z}\boldsymbol{\alpha} + \mathbf{Z}^2\boldsymbol{\alpha}, & &= \boldsymbol{\alpha} + \mathbf{Z}\boldsymbol{\alpha} + \mathbf{Z}^2\bar{\mathbf{a}} \\
\underline{\mathbf{a}}^3 &= \boldsymbol{\alpha} + \mathbf{Z}\boldsymbol{\alpha} + \mathbf{Z}^2\boldsymbol{\alpha} + \mathbf{Z}^3\boldsymbol{\alpha}, & \bar{\mathbf{a}}^3 &= \boldsymbol{\alpha} + \mathbf{Z}\boldsymbol{\alpha} + \mathbf{Z}^2\boldsymbol{\alpha} + \mathbf{Z}^3\bar{\mathbf{a}} \\
&\dots & &\dots \\
\underline{\mathbf{a}}^n &= \alpha \sum_{t=0}^{n} \mathbf{Z}^t, & \bar{\mathbf{a}}^n &= \alpha \sum_{t=0}^{n-1} \mathbf{Z}^t + \mathbf{Z}^n\bar{\mathbf{a}}
\end{aligned}
$$

Since the game is assumed to have an unique Nash equilibrium that is also interior, then $\lim_{n\to\infty} \sum_{t=0}^{n} \mathbf{Z}^n$ exists and it is finite, and $\lim_{n\to\infty} \mathbf{Z}^n = \mathbf{0}$. Then $\underline{\mathbf{a}}^{\infty} = \bar{\mathbf{a}}^{\infty} = \mathbf{a}_i^{NE}$. Then, since $\mathbf{A}_{\mathbf{Z}}^{\infty} = \mathbf{A}_{\mathbf{Z}}^{NE} = \{\mathbf{a}^{NE}\} \supseteq \mathbf{A}_{\mathbf{Z}}^{SCE}$, it follows that $\mathbf{A}_{\mathbf{Z}}^{SCER} = \mathbf{A}_{\mathbf{Z}}^{\infty} \cap \mathbf{A}_{\mathbf{Z}}^{SCE} = \mathbf{A}_{\mathbf{Z}}^{NE}$.

Appendix K Interior Nash equilibria

Proposition 2 in Section 5 show that in our framework there exists an equivalence between any selfconfirming equilibrium and the Nash equilibrium of a reduced game in which only active agents are considered and there is also $OiffA$. Moreover, we can set any subset of agents to be inactive. We now provide some results about existence of these selfconfirming equilibria, that will be useful in proving Proposition 3 in Section 5. We first present sufficient conditions that are present in the literature for the existence of interior Nash equilibria, then we provide some original results.

In this appendix we formulate the problem as a linear algebra problem. We consider a square matrix $\mathbf{Z} \in \mathbb{R}^{n\times n}$ such that $z_{ii} = 0$ for all $i \in \{1, \dots, n\}$. We call $\mathbf{I}$ the identity matrix, $\lambda_{max}(\mathbf{Z})$ the maximal eigenvalue of $\mathbf{Z}$, $\rho(\mathbf{Z})$ the spectral radius of $\mathbf{Z}$ (i.e. the largest absolute value of its eigenvalues), $\mathbf{1}$ is the vector of all 1's, $\mathbf{0}$ is the vector of all 0's, and $\gg$ is the strict partial ordering between vectors (meaning that all the elements in the first vector are pairwise strictly greater than the elements in the second vector).

Proposition **D.** *If for all i, $z_{ii} = 0$, for all $j \neq i$, $z_{ij} \leq 0$, and if $\rho(\mathbf{Z}) < 1$, then* $(\mathbf{I} - \mathbf{Z})^{-1}\,\mathbf{1} \gg \mathbf{0}$.[33]

[33]This is Theorem 1 in Ballester *et al.* (2006). The same result is in Appendix A in Stańczak *et al.* (2006).

There are also results when the sign of the externalities are mixed. Recall that matrix $\mathbf{Z}$ is symmetrizable if there exists a diagonal matrix $\mathbf{W}$ and a symmetric matrix $\mathbf{Z}_0$ such that $\mathbf{Z} = \mathbf{W}\mathbf{Z}_0$. Note that if $\mathbf{Z}$ is symmetrizable then all its eigenvalues are real. If for all i, $z_{ii} = 0$, and $\mathbf{Z}$ is symmetrizable, we define the symmetric matrix $\tilde{\mathbf{Z}}$ to be such that $\tilde{z}_{ij} = z_{ij}\sqrt{w_i w_j}$.

PROPOSITION **E.** *If for all i, $z_{ii} = 0$, $\mathbf{Z}$ is symmetrizable, and if $|\lambda_{max}(\tilde{\mathbf{Z}})| < 1$, then $(\mathbf{I} - \mathbf{Z})^{-1}\mathbf{1} \gg \mathbf{0}$.*[34]

We provide here below an alternative condition, which does also guarantee all positive solutions.

PROPOSITION **F.** *Consider a square matrix $\mathbf{Z} \in \mathbb{R}^{n\times n}$ such that:*

- $z_{ii} = 0$ *for all* $i \in \{1, \dots, n\}$;
- $|z_{ij}| < \frac{1}{n}$ *for all* $i, j \in \{1, \dots, n\}$.

Then $(\mathbf{I} - \mathbf{Z})^{-1}\mathbf{1} \gg \mathbf{0}$.

Proof: Call $\mathbf{B} = (\mathbf{I} - \mathbf{Z})$. First of all, by *Gershgorin circle theorem*, $\mathbf{B}$ has all eigenvalues, possibly complex, with real part strictly between 0 and 2, so $det(\mathbf{B}) \neq 0$.
Consider the n vectors $\mathbf{b}^1, \dots, \mathbf{b}^n$ given by the n rows of $\mathbf{B}$, and take the hyperplane in $\mathbb{R}^n$ passing by those n points:

$$H = \{\mathbf{h} \in \mathbb{R}^n : \exists \boldsymbol{\alpha} \in \mathbb{R}^n \text{ with } \boldsymbol{\alpha}' \cdot \mathbf{1} = 1 \text{ and } \mathbf{h} = \mathbf{B}'\boldsymbol{\alpha}\} \ .$$

Now, consider the following vector

$$\mathbf{v} = \mathbf{B}^{-1}\mathbf{1} \ .$$

v_i is exactly the sum of the elements in i^{th} row of $\mathbf{B}^{-1}$. However, $\mathbf{v}$ is also a vector perpendicular to H. That is because for any $\mathbf{h} \in H$ we have

$$\begin{aligned} \mathbf{h} \cdot \mathbf{v} &= \left(\mathbf{B}'\boldsymbol{\alpha}\right)' \cdot \mathbf{B}^{-1}\mathbf{1} \\ &= \boldsymbol{\alpha}'\mathbf{1} \\ &= \sum_{i=1}^{n} \alpha_i = 1 \ , \end{aligned}$$

which is a constant.

Now, we want to show that H does not pass through the convex region of vectors with all non-postitive elements: $H \cap (-\infty, 0]^n = \emptyset$. In fact, it is impossible to find $\mathbf{w} \in \mathbb{R}^n$, such that

[34] See Section VI of Bramoullé *et al.* (2014), generalizing Proposition 2 therein. Note that in their payoff specification externalities have a *minus* sign, while in (6) we have a *plus* sign: this is why we have a condition on the maximal eigenvalue and not on the minimal eigenvalue.

$\mathbf{w}' \cdot \mathbf{1} = 1$ and $\mathbf{B}'\mathbf{w} \ll \mathbf{0}$. If it was the case, by absurdum, we could take $k = \arg\max_{i \in \{1,\ldots,n\}}\{w_i\}$ ($\alpha^k > 0$ because $\sum_{i=1}^n w_i = 1$), and write

$$w\mathbf{b}^k = w_k + \sum_{j \neq k} w_j b_{jk} > w_k - \sum_{j \neq k} |w_j||z_{jk}| > w_k \left(1 - \sum_{j \neq k} |z_{jk}|\right) > 0 \ ,$$

which would be a contradiction.

Finally, we show that if an hyperplane H satisfies $H \cap (-\infty, 0]^n = \emptyset$, then its perpendicular vector from the origin has all positive elements, and this would close the proof .
We do so by induction on n.

1. $n = 2$: This is easy to show graphically. In the Cartesian plane the hyperplane is a line. Not passing by $(-\infty, 0]^2$, it will cross both axis in their strictly positive part: call these intersection points A and B. So, the segment that from the origin crosses this line perpendicularly will cross it in a point C that on the line lies between A and B.

2. **Induction hypothesis**: Suppose it is true for $n = m - 1$.

3. **Induction step**: a vector $\mathbf{v} \in \mathbb{R}^m$ from the origin which is perpendicular to an hyperplane H not passing through the origin can be obtained in the following way. For each dimension $i \in \{1, \ldots, m\}$ take $V_{\neg i} = \{\mathbf{v} \in \mathbb{R}^m : v_i = 0\}$. Call $H_{\neg i}$ the intersection of H with $V_{\neg i}$, and take a vector $\mathbf{v}_{\neg i} \in V_{\neg i}$ from the origin that is perpendicular to $H_{\neg i}$. By the induction hypothesis $\mathbf{v}_{\neg i}$ has all positive elements. We can obtain the vector $\mathbf{v}$ from the origin that is perpendicular to H by rescaling each $\mathbf{v}_{\neg i}$, such that $\mathbf{v}_{\neg i}$ is the projection of $\mathbf{v}$ on $H_{\neg i}$. By construction, $\mathbf{v}$ will have all positive elements.

Notice that, if $\mathbf{Z}$ satisfies the conditions of Proposition F, then it must also hold that $|\lambda_{max}(\mathbf{Z})| < 1$, because of *Gershgorin circle theorem*. However, the condition that $|\lambda_{max}(\mathbf{Z})| < 1$ is in general not sufficient to guarantee that $(\mathbf{I} - \mathbf{Z})^{-1}\mathbf{1} \gg \mathbf{0}$. ■

Appendix L Proofs

Proof of Proposition 1

Proof. Since every agent is active, **state observability by active players** implies *own action independence of the feedback about the state*. Then, the result derives from Corollary 2 in Appendix I. ■

Proof of Proposition 2

Proof. By Remark C, *NG* satisfies observability by active players. Hence, Lemma C in Appendix I and the best reply equation (8) yield the result. ■

Proof of Proposition 3

Proof. Condition (i), (ii) and (iii) correspond, respectively, to the conditions in Propositions F, D and E from Appendix K. ■

Proof of Proposition 4

Proof. If for every $i \in I \backslash I_{\mathbf{a}^*}$ we have that $\alpha + \hat{x}_i < 0$, then perturbing $\hat{x}_i$ such that the inequality is still strict, it will not make i become active.

So, let us focus on the subset $I_{\mathbf{a}^*}$ of active agents. For each $i \in I_{\mathbf{a}^*}$, a perturbation in $\hat{x}_i$ induces a change in the corresponding a_i. Assumption 4 guarantees that the discrete dynamic system defined for actions by (10) and (11) is stable. So, the perturbation of beliefs can always be small enough such that all actions of agents in $I_{\mathbf{a}^*}$ remain strictly positive;

we are in a neighborhood of $\mathbf{a}^*$ in the actions' space, such that the discrete dynamical system defined for actions by (10) and (11) converges back to $\mathbf{a}^*$. ■

Proof of Proposition 5

Proof. When we remove elements from $J_{\mathbf{a}}$ and set them to 0, it is as if we delete corresponding rows and columns in the $\mathbf{Z}_{J_{\mathbf{a}}}$ matrix. By the Cauchy interlace theorem applied to symmetrizable matrices (see Kouachi 2016) we know that the eigenvalues of the new matrix are between the minimal and the maximal eigenvalues of the old matrix. ■

Proof of Proposition 6

Proof. A selfconfirming equilibrium is such that, for all $i \in I$, rationality implies

$$a_i^* = \max\{0, \alpha_i + \hat{x}_i\} \ .$$

Each agent then thinks that

$$m^* = \alpha_i a_i^* - \frac{1}{2} \left(a_i^*\right)^2 + a_i^* \hat{x}_i + \hat{y}_i \ ,$$

so that

$$\hat{y}_i = m^* - \alpha_i a_i^* + \frac{1}{2} \left(a_i^*\right)^2 - a_i^* \hat{x}_i \ .$$

Substituting the expression of the true payoff function

$$m^* = \alpha_i a_i^* - \frac{1}{2} \left(a_i^*\right)^2 + a_i^* x_i + y_i$$

into it, we get the dependence between $\hat{y}_i$ and $\hat{x}_i$:

$$\hat{y}_i = y_i + a_i^* \left(x_i - \hat{x}_i\right) \ .$$

The first and second items in the proposition are derived, respectively, if $a_i^* = 0$ or $a_i^* > 0$. ■

Proof of Proposition 7

Proof. The Nash equilibrium of the game with payoff function (13) played on a complete network, is such that each player $i \in I$ best responds to $w \sum_{k \neq i} a_k$. Because of symmetry, each player $i \in I$ plays $a_i = \frac{\alpha}{1-(n-1)w}$.
For each profile $\mathbf{a}$, each player i, by perfect recall of her own action, can correctly infer the value of

$$a_i w \sum_{j \in I \setminus \{i\}} z_{0,ij} a_j + \beta \sum_{k \neq i} a_k \quad . \tag{c}$$

At the same time, since each player i thinks to be the central player of the line, she thinks that what she observes is

$$(a_i w + \beta) \sum_{k \in I \setminus \{i\}} a_k \quad . \tag{d}$$

So, she would extrapolate v , dividing (c) by $(a_i w + \beta)$, because this would be the correct procedure if (d) was true. This means that, for her, $\ell_i\left(\mathbf{a}^*_{-i}, \mathbf{Z}\right)$ is

$$w \frac{a_i w \sum_{j \in I \setminus \{i\}} z_{0,ij} a_j + \beta \sum\limits_{k \neq i} a_k}{a_i w + \beta} \quad .$$

This quantity is equal to the NE of the correct network $\mathbf{Z_0}$ when $\beta = 0$. It grows with β. Finally, as $\beta \to \infty$ this quantity converges to $w \sum_{k \neq i} a_k$ for every player i. ■

Proof of Proposition 8

Proof. First, we derive some properties. Each equation in the system given by (18) can be written as a parabola $b_1 a_i^2 + b_2 a_i + b_3 = 0$, in the following way

$$\begin{aligned} H_i(\mathbf{a}, \mathbf{c}, \mathbf{Z}) \quad &= \quad \underbrace{c_i}_{:=b_1} a_i^2 + \underbrace{\left(1 - \alpha c_i - c_i \left(\sum_{j \in I} z_{ij} a_{j,t}\right)\right)}_{:=b_2} a_i \\ &\quad - \underbrace{\left(1 + c_i \left(\beta \sum_{j \neq i} a_{j,t}\right)\right)}_{:=b_3} = 0 \; . \end{aligned} \tag{e}$$

So, for each $i \in I$, the solution a_i^* is such that $\ell_i(\mathbf{a}, \mathbf{c}, \mathbf{Z}) = 0$ lays in the right–arm of an upward parabola, where $\left.\frac{d\ell_i}{da_i}\right|_{a_i = a_i^*} > 0$. Each $\ell_i(\mathbf{a}, \mathbf{c}, \mathbf{Z})$ is linear in c_i.
Note also that each a_i is bounded in the interval

$$\alpha < a_i < \alpha + \left(\sum_{j \in N_i} z_{ij} a_j\right) + \beta \frac{\sum_{k \neq i} a_k}{a_i} \; .$$

Considering that a_i^* is increasing in b_3 and decreasing in b_2, it is easy to see that each a_i^* increases in each a_j, with $j \neq i$.

Second, we show that there is a homeomorphism. There is a continuous function defined from each $\mathbf{c} \in [0,1]^n$ to an element $\mathbf{a} \in \mathcal{A}$, that is because

- either $c_i = 0$ and then $a_i^* = \alpha$;
- or $c_i > 0$ and then each a_i^* is continuously increasing in each x_j with $j \neq i$.

$$\lim_{c_i \to 0} a_i^* = \alpha \ .$$

a_i^* is bounded above by

$$\alpha + \left(\sum_{j \in N_i} z_{ij} a_j \right) + \beta \frac{\sum_{j \neq i} a_j}{a_i^*} \ .$$

Since the system defined by (19) admits a solution, also this system has a finite solution.

This function is one–to–one and invertible, because for each $\mathbf{a} \in \mathcal{A}$, we obtain a unique vector $\mathbf{c} \in [0,1]^n$, and since we obtain it from a linear system of equations, also the inverse function from $\mathcal{A}$ to $[0,1]^n$ is continuous.

To analyze the relation between $\mathbf{a}^*$ and $\mathbf{c}$, we can apply the implicit function theorem to $F_i(\mathbf{a}, \mathbf{c}, \mathbf{Z})$.

We can compute

$$\frac{dF_i}{dc_i} = \frac{\beta \sum_{j \neq i} a_{j,t}}{(a_i c_i + 1)^2}$$

Now, since

$$\ell_i(\mathbf{a}, \mathbf{c}, \mathbf{Z}) = -(a_i c_i + 1) F_i(\mathbf{a}, \mathbf{c}, \mathbf{Z}) \ \ ,$$

we have that $\ell_i(\mathbf{a}, \mathbf{c}, \mathbf{Z})$, with respect to a_i, has the same zeros as $F_i(\mathbf{a}, \mathbf{c}, \mathbf{Z})$, and that, for each a_i, $\ell_i(\mathbf{a}, \mathbf{c}, \mathbf{Z})$ is negative if and only if $F_i(\mathbf{a}, \mathbf{c}, \mathbf{Z})$ is positive. As they are both continuous functions, this means that since $\frac{d\ell_i}{da_i}\Big|_{a_i = a_i^*} > 0$, we have $\frac{dF_i}{da_i}\Big|_{a_i = a_i^*} < 0$. So, we obtain that

$$\frac{da_i}{dv_i}\Big|_{a_i = a_i^*} = - \frac{\partial F_i / \partial c_i}{\partial F_i / \partial a_i}\Big|_{a_i = a_i^*} > 0 \ . \tag{f}$$

This shows that a_i^* is increasing with v_i, and vice versa. ■

Proof of Proposition 9

Proof. We consider the system (18)

$$F_i(\mathbf{a}, \mathbf{v}, \mathbf{Z}) = \alpha + c_i \left(\beta \sum_{j \neq i} a_{j,t} \right) \frac{a_i c'_i + 1}{a_i c_i + 1} - a_i = 0 \quad ,$$

with $c'_{i,t} = \frac{\sum_{j \in I} z_{ij} a_{j,t}}{\beta \sum_{j \neq i} a_{j,t}}$. We can compute its Jacobian, with respect to $\mathbf{a}$, and check that each row of the Jacobian sum to less than 1, so that the process is always a contraction. The Jacobian J is such that, for each $i, j \in I$:

$$\begin{cases} J_{ij} &= \frac{v_i}{a_i c_i + 1} (\beta + a_i z_{ij}) \\ J_{ii} &= c_i \left(\beta \sum_{j \neq i} a_j \right) \left(\frac{c'_i}{a_i c_i + 1} - c_i \frac{a_i c'_i + 1}{(a_i c_i + 1)^2} \right) - 1 \end{cases}$$

The sum of each row of the Jacobian is

$$\sum_{j \in I} J_{ij} = \frac{c_i}{a_i c_i + 1} \left(\beta \left(\sum_{j \neq i} a_j \right) \left(c'_i - c_i \frac{a_i c'_i + 1}{a_i c_i + 1} \right) + a_i \left(\sum_{j \neq i} z_{i,j} \right) + \beta(n-1) \right) - 1 \tag{g}$$

Let us analyze expression (g) with respect to a_i, for any $a_i \geq 0$.
First note that

$$\lim_{a_i \to \infty} \sum_{j \in I} J_{ij} = \sum_{j \neq i} z_{i,j} - 1 \quad , \tag{h}$$

whose absolute value is less than one by assumption.
Moreover,

$$\lim_{a_i \to 0} \sum_{j \in I} J_{ij} = c_i \beta \left(\left(\sum_{j \neq i} a_j \right) (c'_i - c_i) + (n-1) \right) - 1 \quad . \tag{i}$$

An interior maximum or minimum of the numerical expression (g), with respect to a_i, must satisfy first order condition

$$\begin{aligned} -\left(\frac{c_i}{a_i c_i + 1} \right)^2 \left(\beta \left(\sum_{j \neq i} a_j \right) \left(c'_i - c_i \frac{a_i c'_i + 1}{a_i c_i + 1} \right) + a_i \left(\sum_{j \neq i} z_{i,j} \right) + \beta(n-1) \right) & \\ + \frac{c_i}{a_i c_i + 1} \left(\beta \left(\sum_{j \neq i} a_j \right) \left(\frac{c_i}{a_i c_i + 1} \right) \left(c'_i - c_i \frac{a_i c'_i + 1}{a_i c_i + 1} \right) + \left(\sum_{j \neq i} z_{i,j} \right) \right) &= 0 \end{aligned}$$

Last expression can be simplified and results in

$$v_i \beta (n-1) = \sum_{j \neq i} z_{i,j} \quad ,$$

which is independent on a_i. So, the only candidates for being minima or maxima for espression (g) are its value in the extrema, namely (h) and (i).

Also, the sign of the first derivative of (g) with respect to a_i is equal to the sign of $\sum_{j\neq i} z_{i,j} - c_i\beta(n-1)$. So, if $c_i\beta(n-1) < \sum_{j\neq i} z_{i,j}$ we have that (g) is strictly increasing in a_i, and then (h) is strictly greater than (i).

The value of (h) is between -1 and 1, by assumption, because $0 < \sum_{j\neq i} z_{i,j} < 2$.

The quantity in (i) is minimized by $v_i \to 0$; and $c_i' \to 0$. In this case (i) goes to -1 from the right, and for any $c_i > 0$ it will be greater than -1. This complete the proof, because we have shown that any row of the Jacobian J sums to a number between -1 an 1. ■

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