# Common Value Auctions with Costly Entry

Pauli Murto* Juuso Välimäki†

March 3, 2017

**Abstract**

We consider a model where potential bidders consider paying an entry cost to participate in an auction. The value of the object sold depends on an unknown state of the world, and the bidders have conditionally i.i.d. signals on the state. We consider mostly the case where entry decisions are taken after observing the signal. We compare first- and second-price auction formats and show that for many symmetric equilibria of the game, first-price auction results in higher expected revenue to the seller.

JEL Classification: D44

# 1 Introduction

Entering an auction often entails different kinds of costs. In addition to the opportunity cost of time and effort spent on being physically present at an auction site, acquiring information about one's own valuation for the good is often costly. On top of this, the preparation of bids may be costly as a result of concerns for due diligence. Since the entry decision is taken by all potential bidders at a stage where the eventual number of participants in the auction is random, it is natural to consider the performance of various auction formats when the number of participating bidders is uncertain at the

*Aalto University School of Business, pauli.murto@aalto.fi
†Aalto University School of Business, juuso.valimaki@aalto.fi

bidding stage. In this paper, we show that accounting for this uncertainty leads to new types of bidding equilibria and to new revenue results in auctions with affiliated common values. In particular, we demonstrate the superior performance of first price auctions with an undisclosed number of bidders for relatively large entry costs.

The previous literature has focused on the case where the entry decision is taken at the first stage. In the second stage, all entering bidders see the realized number of participants and bid optimally in the ensuing auction. In this paper, we relax the assumption that the set of entering bidders is common knowledge at the bidding stage. We call an auction an informal auction if each individual bidder knows only the equilibrium entry strategy, but not the realized number of participating bidders at the time of choosing her bid. This auction format is meant to capture selling procedures such as an open first price bidding game where the highest bidder wins and pays her own bid or an open ascending auction where only the highest current standing bid is observed at the auction stage.

We consider the case of free entry in the sense that a large (infinite) pool of potential entrants decides whether to pay a positive entry cost $c$ to enter the bidding stage. Since the object for sale has a bounded value, only a finite number of bidders can enter the auction with positive probability in equilibrium. This means that any symmetric equilibrium of the game must have the feature that potential bidders are indifferent between entering and staying out. In other words, entry costs dissipate all bidder rents from the auction. As a result of this, the entire expected surplus generated in the entry game followed by the auction will be collected by the auctioneer as expected sales revenues. This allows us to compare the revenue properties of the auctions in tandem with their social efficiency.

In the existing literature, the entry decisions of the potential bidders are usually modeled as a simultaneous form game. If the agents learn their valuation for the object only upon entering, the participation game can be modeled is a normal form game where the payoffs are taken to be the equilibrium ex ante expected payoffs of the auction (as a function of the realized entry profile). The other alternative is to consider entry decisions at the interim stage where the private signals are already known to the bidders.

The entry game is now a Bayesian game where the payoffs are given by the interim expected payoffs (given the conditional distribution on the bidder valuations that depend on the equilibrium entry profile). We concentrate on the case of interim entry and in Appendix B (**??**), we show that most of our results extend to the case of ex ante entry decisions.

In order to keep the analysis of equilibrium bidding analytically tractable, we have kept the informational model as simple as possible.[1] At the beginning of the game, a large number of potential bidders decide whether to enter an auction for a single indivisible object or not. We consider the Poisson game model where the number of entering bidders is drawn from a Poisson distribution.[2] The true value of the object is a binary random variable and all potential bidders observe a conditionally i.i.d. binary signal on this value. Bidding is costly: in order to submit any bid, a positive cost $c$ has to be paid. Those bidders that have paid the cost proceed to submit a positive bid $b$. We concentrate on the informal auctions and compute the equilibria of this bidding game under first price and second price rules.

In our characterization of the symmetric bidding equilibria for the informal auctions, we find new qualitative features for both the first price and the second price auction. In the informal first price auction, we demonstrate the possibility that the bid supports of the two different bidder types are overlapping. This means that a bidder with a less favorable signal wins with positive probability over a bidder with a more favorable signal. This is never possible if the number of bidders is disclosed before the bidding stage (formal first price auction). In the formal version, bidders with low values submit a pure bid at the expected value of the object conditional on all bidders having the low signal. In the informal version, each bidder has a positive probability

[1] In the standard setting, the signals of the bidders are symmetrically distributed and from a continuous distribution. The analysis then concentrates on pure strategy equilibria of the Bayesian game between the bidders. In our model with unknown numbers of bidders, we do not have sufficient monotonicity in the best responses to guarantee the existence of a pure strategy equilibrium. The full analysis of mixed strategy equilibria with a continuum of types is not analytically tractable.

[2] The outcome (joint distribution of the entering types and bids) of the Poisson modelshould be thought of as an idealized model of symmetric entry decisions in a model with an infinite set of players.

of being the only entering bidder. This forces the low signal bidders also to use mixed strategies and the equilibria with overlapping supports are possible when the expected number of entering bidders is relatively small.

In the informal second price auction, bidders with low signals pool on a bid, but this bid is not uniquely determined as there are multiple equilibria with different pooling bids. This can be explained by equilibrium rationing in the model. In all of these equilibria, bidders with low signals never win against bidders with high signals. On the other hand, the probability of winning at the pooled bid is the highest when there are few low type entrants. In an affiliated common values model, this is evidence of a high value object (since we are conditioning on the event of no high signal entrants at the auction). This means that a slight deviation to a higher bid results in a loss if the pooling bid is the expected value of the object conditional on the (favorable) equilibrium rationing rule. A deviating bid wins with a uniform probability against all cases with no high signal bidders and results in an expected loss. We call this new effect resulting from the positive rationing at low bids the *winner's blessing*, and we see that the equilibrium pooling bid of the low bidders is not uniquely pinned down. Even though our finite signal structure is somewhat unconventional for the auctions literature, the pooling and rationing effects are present in models with continuous signals too. In Murto and Välimäki (2016), we have constructed an example of a partially pooling equilibrium in a second-price auction for a standard model with a continuum of signals.

A second failure of monotonicity arises from the differential entry intensities of the two types of bidders. Since bidders with high signals earn higher rents in the auction stage in models with independent signals, entry is monotonic in the sense that only bidders with high signals enter in models with an unlimited pool of potential bidders.[3] Affiliated values changes this conclusion. A high signal is indicative of a high value of the object, but also of a higher probability that the other players have high signals. Hence higher estimated value of the object comes with an associated increase in the degree of competition for the object. In the model with interim participation

[3]With a finite pool of potential entrants, low type bidders enter only if all high type bidders enter.

decisions, equilibrium entry rates of the two types of bidders balance these two effects.

The revenue ranking of the auctions differs considerably from the case of auctions with a known number of bidders. First of all, the informal FPA dominates the informal SPA in terms of the expected revenue to the seller.[4] Second, when the entry cost is high, the informal FPA dominates in terms of the expected revenue the formal auctions where the number of participants is disclosed at the bidding stage. This indicates that withholding information regarding the number of bidders may result in higher revenues. Both of these results seem to be in contradiction with the received wisdom on affiliated common values auctions.

Milgrom & Weber (1982) demonstrate the revenue superiority of the formal SPA over formal FPA for affiliated common value auctions. This result together with a set of results demonstrating the good revenue properties of public information disclosure are also known as the linkage principle. The key idea is that for a fixed own bid, any auction format that increases the linkage between own information and the perception of other players' bids increases the expected payment. To see how this principle fails in our informal auctions, consider an equilibrium in the informal SPA where low type bidders pool on a price strictly below the the bid support of the high type bidders. By placing a bid between the pooling bid and the lowest in the support of the high types, the bidder wins if and only if no high bidders participate in the auction. In this case, the payment is either the pooling bid if there is some competition, or zero if no other bidder participate. By affiliation, it is more likely that no bidders with a low signal participate if the value of the object is high. But this means that the expected payment of the high type bidder is lower than the expected payment of the low bidder. In Appendix A, we show that for the case of two potential bidders, this argument shows that the informal first price auction dominates the other auction formats in

[4] We compare here the expected revenue in the informal FPA and in the informal SPA where the bidders bid according to the bidder optimal symmetric equilibrium in the bidding stage. For low entry costs, we have shown that the revenue ranking is reversed in some equilibria of the SPA. This reversal depends crucially on the equilibrium rationing effects.

terms of the expected revenue for all parameter values of the model.

When comparing the informal auctions to formal auctions, a second consideration emerges: the price paid in the formal auction is determined by the realized number of participating bidders. Whenever a bidder is the sole participant, she gets the object for free. For a formal auction with $n-1$ other participants, the equilibrium bid by the low bidders is equal to the value of the object conditional on $n$ low signals. Observe that due to affiliation, this payment is decreasing in $n$. For low values of the entry cost, the probability of the event that no other bidders are present vanishes. Since a high type bidder assigns a higher probability to lower $n$, we see that the expected payment of the bidder is positively linked with the type and the usual linkage principle applies. This explains why formal SPA dominates informal FPA and by the earlier result also informal SPA.

We get our strongest results when the cost of entry is relatively high and the affiliation in the signals is strong. In this case, a bid of zero is in the support of the equilibrium bidding strategies of both types of players in the informal first price auction. Since bidding strategies cannot have atoms at zero, this means that the payoff of each type of bidder coincides with the value of the good conditional on being the only entrant. This private benefit is also the maximum social benefit from inducing additional entry when restricted to symmetric strategies. Hence we conclude that the symmetric equilibrium entry rates maximize social welfare in the class of symmetric entry strategies. Since we have large numbers of potential bidders, the expected payoff to bidders net of entry costs must be zero and as a result the seller receives the maximal symmetric surplus as her expected revenue in this class of auctions. For these parameter values we see then that the informal first price auction is the revenue maximizing mechanism in the class of symmetric mechanisms.

## 1.1 Related Literature

Endogenous entry into auctions has been modeled in two separate frameworks. In the first, entry decisions are taken at an ex ante stage where all bidders are identical. Potential bidders learn their private information only upon paying the entry cost. Hence these models can be though of as games with endogenous information acquisition.[5] French & McCormick (1984) gives the first analysis of an auction with an entry fee in the IPV case. Harstad (1990) and Levin & Smith (1994) analyze the affiliated interdependent values case. These papers show that due to business stealing, entry is excessive relative to social optimum. They also show that second-price auctions results in higher expected revenues than the first-price auction. All of these papers proceed under the assumption that the number of entering bidders is known at the moment when bids are submitted. What we call informal auctions are thus not covered at all in these papers.

In the other strand, bidders decide on entry only after knowing their own signals. Samuleson (1985) and Stegeman (1996) are early papers in the IPV setting where this question has been taken up. Due to revenue equivalence in the IPV case, comparisons across auction formats are not very interesting. To the best of our knowledge, common values auctions have not been analyzed in this setting.

Finally some recent papers have analyzed common values auctions with some similarities to our paper. Pekec & Tsetlin (2008) provides an example where informal FPA results in a higher expected revenue than an informal SPA. The distribution of the bidders in that paper is somewhat extreme and not derived from entry decisions. Lauermann & Wolinsky (2015) analyze first-price auctions where an informed seller chooses the number of bidders to invite to an auction. The bidders do not observe how many others were invited and hence the bidding stage analysis is as in our model with an exogenous entry rate. They find partially pooling equilibria in the first price auction due to effects similar to our winner's blessing effect. This paper does not compare revenues across different auction formats and since the distribution of entering bidders results from an optimal invitation decision by the seller, the analysis is quite different from our paper. Atakan & Ekmekci (2014) consider a common value auction where the winner in the auction has

[5] The equilibrium determination of information accuracy in common values auctions started with Matthews (1977) and Matthews (1984) and Persico (2000) extended this line of research to revenue comparisons for different auction formats. Since the number of bidders and equilibrium information acquisition decisions are deterministic, equilibrium bidding in these papers is still as in standard affiliated auctions models.

to take an additional action after winning the auction. This leads to a non-monotonicity in the value of winning the auction that has some resemblance to the forces in our model that lead to non-monotonic entry (i.e. bidders with both types of signals enter with positive probability).

Since we concentrate on symmetric equilibria of a game with a large number of potential entrants, our model has some similarities to the urn-ball models of matching. Similar to those models, our insistence on symmetric equilibria can be seen as a way of capturing a friction in the market that precludes coordinated asymmetric decisions. A recent example of such models is Kim & Kircher (2015) that studies matching with private values uncertainty. This approach has also been used in Jehiel & Lamy (2015) a procurement auction setting with private (asymmetric) values. These models have not covered the case of common values to the best of our knowledge.

The paper is structured as follows. We start with the analysis of bidding equilibria in informal auctions with an exogenously given distribution of participating bidders of low and high types respectively. We consider next the equilibrium determination of entry for the two types of entry model. In the ex ante entry case, we take all the potential bidders to be identical and to observe their signal only upon paying the entry cost. In the interim entry model, we assume that the potential bidders have already observed their signals when deciding whether to pay the entry cost. The first model can be thought of as costly information acquisition whereas the second model focuses on the differential selection of different types of bidders to the auction. We then consider the revenue implications of the two types of models.

# 2 Model

We consider a model where a single indivisible good is sold to an infinite population of potential bidders. The state of the world $\omega$ determines the value of the product. We assume that this state is binary with $\omega \in \{0, 1\}$. The prior probability of the event $\{\omega = 1\}$ is denoted by $q$. The potential buyers are risk-neutral and if they win the auction at bid $b$, their payoff from the auction is

$$v(\omega) - b,$$

where we assume that $v(1) > v(0)$. Notice that given $\omega$, every bidder's valuation for the good is the same. We consider only standard auctions where losing bidders make no payments and we normalize the outside option of not winning to 0.

If any individual buyer decides to bid for the object, she incurs a cost $c > 0$. This cost is sunk prior to entering the auction, and the cost represents a true economic cost, not a transfer to the seller as in the case of a participation fee. Information about the product is conveyed by an imperfectly informative binary signal $\theta \in \{l, h\}$. We call $\theta$ sometimes also the type of the bidder. The probabilities of the signals conditional on the state are denoted by $\alpha := \Pr(\theta = h \,|\omega = 1)$ and $\beta := \Pr(\theta = h \,|\omega = 0)$. We name the signals so that $0 < \beta < \alpha < 1$ and hence the high signal $h$ is more likely if the object has the higher value $v(1)$. The signals of the bidders are assumed to be conditionally i.i.d. given the value of the object $\omega$.

We consider four different auction formats. In all of these auctions, the bidders submit their bids simultaneously after having paid $c$ and after observing their signal $\theta$. In formal auctions, the bidders are told the number of potential bidders that paid $c$ and hence participate in the auction. In informal auctions, the bidders only know the equilibrium distribution of bidders in the auction. For each of these cases, we consider first price and second price auctions.

The decision to enter the auction is taken simultaneously by all the potential bidders. In the main body of this paper, we consider an interim entry stage where the potential bidders have already received their signals and hence the entry decisions are conditional on the signal. In Appendix A, we show that most of our results remain true for the alternative model where entry is decided prior to learning the signal realization.

We model entry by assuming that the number of entrants is a Poisson random variable with an *endogeneously* determined parameter. For a discussion of the Poisson entry model in the context of a procurement auction with private values, see Jehiel & Lamy (2015). Modeling entry by a Poisson random variable can be understood as the infinite population limit of a symmetric mixed strategy entry in a finite population. To see this, assume for a moment that there are $N$ potential entrants. Each entrant has obtained signal $\theta = h$

with probability $\alpha$ if $\omega = 1$ and with probability $\beta$ if $\omega = 0$. If each entrant with signal $\theta$ enters with probability $\pi_\theta(N)$, then the number of high type entrants is a binomial random variable with parameters $(\alpha\pi_h(N), N)$ and $(\beta\pi_h(N), N)$ in states $\omega = 1$ and $\omega = 0$, respectively. Similarly, the number of low type entrans is a binomial with parameters $((1-\alpha)\pi_l(N), N)$ and $((1-\beta)\pi_l(N), N)$ in states $\omega = 1$ and $\omega = 0$, respectively. By letting $N$ increase towards infinity and letting $\pi_\theta(N) \cdot N \to \pi_\theta$ while keeping the number of entrants bounded ensures that in the limit the number of entrants with signal $\theta$ given state $\omega$ is a Poisson random variable $N_{\theta,\omega}$ with parameter $\lambda_{\omega,\theta}$, where:

$$\begin{aligned} \lambda_{1,h} &= \alpha\pi_h, \\ \lambda_{0,h} &= \beta\pi_h, \\ \lambda_{1,l} &= (1-\alpha)\pi_l, \\ \lambda_{0,l} &= (1-\beta)\pi_l\,, \end{aligned} \tag{1}$$

and where $N_{h,\omega}$ and $N_{l,\omega}$ are independent. Moreover, since each individual player enters with a vanishingly small probability, each actual entrant perceives the number of *other* players to be distributed by the same Poisson distribution.

Motivated by this limiting argument, we model the entry game directly as a Poisson game model, where $\pi_l$ and $\pi_h$ are endogenously determined parameters. Each potential bidder perceives the number of other participants of type $\theta$ to be given by a Poisson random variable $N_{\theta,\omega}$ with mean $\lambda_{\theta,\omega}$ that depends on $\pi_l$ and $\pi_h$ according to (1). By a symmetric equilibrium, we mean a pair $(b, \pi)$ where $b : \{h, l\} \to \Delta(\mathbb{R}_+)$ is the equilibrium bidding strategy and $\pi : \{h, l\} \to \mathbb{R}_+$ is the equilibrium entry strategy. The equilibrium condition for entry profile $\pi$ is given by the requirement that given $(b, \pi)$, each potential bidder is indifferent between entering and not entering the auction.

# 3 Equilibrium bidding with exogenous random entry

We start by analyzing the bidding stage in the case where the number of entrants is determined by an exogenously given distribution parametrized by $\pi_h$ and $\pi_l$. A bidder with signal $\theta$ chooses her optimal bid in informal auctions depending on her updated probability on the state $q_\theta := \Pr\{\omega = 1 \,|\theta\}$ and the conditional distribution $Q\left(n^h, n^l \,|\omega\right)$, where $n^\theta$ is the realized number of (other) bidders with signal $\theta$. The number of bidders with signal $\theta$ in state $\omega$ is a Poisson random variable $N_\omega^\theta$, where the parameter $\lambda_{\omega,\theta}$ depends on $\pi_\theta$ through (1). While parameters $\pi_h$ and $\pi_l$ are treated here exogeneous, they are made endogeneous in the next section. Since we consider here the bidding behavior of an individual bidder, we use $N_\omega^\theta$ to denote the number of bidders *excluding the bidder under consideration.*[6]

Note also that the random number of entrants with signal $h$ is independent of the number of entrants with signal $l$ conditional on $\omega$. This is a standard property of Poisson games and is easily verified by compounding the Poisson distribution of total number of entrants (conditional on state) with the binomial distribution of the resulting signals.

## 3.1 Pooling bids and rationing effects

With a random number of bidders, the analysis of pooling bids (i.e. bids that at least one of the types chooses with a strictly positive probability) is more delicate than in the case of a fixed number of bidders. We call a bid $p$ a pooling bid if at least one bidder type submits that bid with positive probability. With uniform tie-breaking for tied high bids, a bidder submitting a pooling bid is more likely to win if the number of tying bids is small. The number of tying bids contains information on the realization of $\left(N^h, N^l\right)$. Since this information is in turn informative on the state of the world and hence of the value of the object, the additional information contained in the

[6]Note that in a Poisson model, an individual bidder sees the number of other bidders distributed according to the same distribution as an outsider sees the number of all bidders in the game (conditional on state).

event of winning the auction must be accounted for when calculating the optimal bid. We call this effect the rationing effect of winning.

If only high bidders pool on a bid $p$, then the rationing effect is negative. By bidding $p$, a win is more likely when there are few tied bidders. But if only high type bidders bid $p$, then winning with a tied bid decreases the posterior on $\{\omega = 1\}$ and the value of the object conditional on winning is lower than the value conditional on the event that the bidder is tied for the highest bid. By bidding $p + \varepsilon$, for a small enough $\varepsilon$, the bidder wins in the event of a tied bid without any rationing and hence makes a positive gain from the deviation. Hence there cannot be pooling bids that are submitted only by high type bidders.

Pooling by low bidders is quite different. Consider the case that is relevant for our analysis, where all the low type bidders submit the same bid $p$, and all the high type bidders bid strictly above $p$. In this case, we can compute the probability of the event $A(p)$ where a bidder submitting the common low bid $p$ wins the object, conditional on state and conditional on no high types being present:

$$
\begin{aligned}
& \Pr\left(A(p)\left|\omega, N^h = 0\right.\right) \\
= \;& 1e^{-\lambda_{\omega,l}} + \frac{1}{2}\lambda_{\omega,l}e^{-\lambda_{\omega,l}} + \frac{1}{3}\frac{(\lambda_{\omega,l})^2}{2!}e^{-\lambda_{\omega,l}} + \frac{1}{4}\frac{(\lambda_{\omega,l})^3}{3!}e^{-\lambda_{\omega,l}} + \ldots \\
= \;& \frac{1}{\lambda_{\omega,l}}\left(-e^{-\lambda_{\omega,l}} + e^{-\lambda_{\omega,l}} + \lambda_{\omega,l}e^{-\lambda_{\omega,l}} + \frac{(\lambda_{\omega,l})^2}{2!}e^{-\lambda_{\omega,l}} + \frac{(\lambda_{\omega,l})^3}{3!}e^{-\lambda_{\omega,l}} + \ldots\right) \\
= \;& \frac{1}{\lambda_{\omega,l}}\left(1 - e^{-\lambda_{\omega,l}}\right).
\end{aligned}
$$

We call the event where a low type bidder is tied with another bidder for the highest bid at $p$ the *pivotal event* $B(p) = \left\{A(p) \cap \left\{N^l \geq 1\right\}\right\}$. To compute the value of of the object conditional on the pivotal event, we must first compute the probability of the pivotal event:

$$
\begin{aligned}
& \Pr\{B(p)\left|\omega, N^h = 0\right.\} \\
= \;& \frac{1}{\lambda_{\omega,l}}\left(1 - e^{-\lambda_{\omega,l}}\right) - e^{-\lambda_{\omega,l}} = \frac{1}{\lambda_{\omega,l}}\left(1 - e^{-\lambda_{\omega,l}} - \lambda_{\omega,l}e^{-\lambda_{\omega,l}}\right).
\end{aligned}
$$

Using this probability, we can define $L^{win}(p)$ as the likelihood ratio of the two states for a low type player who wins in the pivotal event:

$$\begin{aligned} L^{win}(p) &= \frac{\Pr\{\omega = 1 \,|\theta = l, N^h = 0, B(p)\}}{\Pr\{\omega = 0 \,|\theta = l, N^h = 0, B(p)\}} \\ &= \frac{q}{1-q}\frac{1-\alpha}{1-\beta}\frac{e^{-\lambda_{1,h}}}{e^{-\lambda_{0,h}}}\frac{\frac{1}{\lambda_{1,l}}\left(1 - e^{-\lambda_{1,l}} - \lambda_{1,l}e^{-\lambda_{1,l}}\right)}{\frac{1}{\lambda_{0,l}}\left(1 - e^{-\lambda_{0,l}} - \lambda_{0,l}e^{-\lambda_{0,l}}\right)} \\ &= \frac{q}{1-q}\frac{e^{-\lambda_{1,h}}}{e^{-\lambda_{0,h}}}\frac{1 - e^{-\lambda_{1,l}} - \lambda_{1,l}e^{-\lambda_{1,l}}}{1 - e^{-\lambda_{0,l}} - \lambda_{0,l}e^{-\lambda_{0,l}}} \\ &=: \frac{q^{win}}{1-q^{win}}, \end{aligned}$$

where we have used

$$\frac{q_l}{1-q_l} = \frac{1-\alpha}{1-\beta}\frac{q}{1-q} = \frac{\lambda_{1,l}}{\lambda_{0,l}}\frac{q}{1-q}.$$

Since the posterior on $\{\omega = 1\}$ conditional on winning is

$$q^{win}(p) = \frac{L^{win}(p)}{1 + L^{win}(p)},$$

we can write the expected value of the object conditional on winning in the pivotal event as $\overline{v}$:

$$\overline{v} = \frac{L^{win}(p)}{1 + L^{win}(p)} v(1) + \frac{1}{1 + L^{win}(p)} v(0). \tag{2}$$

By bidding $\overline{v}$ and winning the auction, low types make zero gains in the pivotal event. As a result, there is no incentive to out-bid or undercut the opponents. If the low type bidders pool on some $p$ slightly below $\overline{v}$, each bidder is then strictly better off getting the object (in the pivotal event) than not getting the object. Despite this, there is still no incentive to out-bid the others since by doing so one would miss out on the positive effects of rationing in this auction. We call this effect arising from the favorable rationing at the low bids the *winner's blessing* effect. Due to this effect there exists an interval of possible equilibrium pooling bids where out-bidding is not profitable and $\overline{v}$ is the upper bound of this interval.

To find the lower bound of this interval, consider the probability of not winning in the pivotal event in state $\omega$:

$$1 - e^{-\lambda_{\omega,l}} - \frac{1 - e^{-\lambda_{\omega,l}} - \lambda_{\omega,l} e^{-\lambda_{\omega,l}}}{\lambda_{\omega,l}}.$$

The likelihood ratio of the states from the viewpoint of a low type bidder conditional on losing the rationing in the pivotal event $B(p)$ and conditional on $N^h(p) = 0$ is:

$$L^{lose}(p) = \frac{q_l}{1 - q_l} \frac{e^{-\lambda_{1,h}(1 - F_h(p_-))}}{e^{-\lambda_{0,h}(1 - F_h(p_-))}} \frac{\left(1 - \frac{1 - e^{-\lambda_{1,l}}}{\lambda_{1,l}}\right)}{\left(1 - \frac{1 - e^{-\lambda_{0,l}}}{\lambda_{0,l}}\right)}.$$

We can then compute the value of the object conditional on losing in the pivotal event as:

$$\underline{v} = \frac{L^{lose}(p)}{1 + L^{lose}(p)} v(1) + \frac{1}{1 + L^{lose}(p)} v(0). \tag{3}$$

To see that $\underline{v}$ is the lower bound of the inerval of possible equilibrium pooling bids, suppose that low types pool at $\underline{v}$. By bidding $\underline{v}$, one either wins or loses a rationing if there are many low type bidders. Deviating and bidding slightly above $\underline{v}$ makes a difference only in those events, where one would lose such a rationing. But since price to be paid is exactly equal to the expected value conditional on losing rationing, one is indifferent between getting the object in such a case. Therefore, a bidder is indifferent about the deviation. For any price lower than $\underline{v}$, deviation would be strictly profitable. For any price higher than $\underline{v}$, deviation would be strictly non-profitable.

We use this analysis of pooled bids and rationing to show that pooling is not possible in symmetric equilibria of the informal first price auction while all equilibria of the informal second price auction display some pooling. We also show that pooling leads naturally to multiplicity even in the class of symmetric equilibria.

## 3.2 Symmetric bidding equilibrium in informal first price auction

In this section, we consider the informal first price auction with an exogenouely given distribution of bidders. The bidder submitting the highest bid $p \geq 0$ wins the object and pays her bid, any ties are broken symmetrically between highest bidders. Bidders that do not win make no payments.

At the bidding stage, each bidder, i.e. each potential bidder who has entered the auction, forms a subjective probability distribution of the number and types of other players that are present. Conditional on state $\omega$, each bidder believes that the number of other bidders with signal $\theta$ is a random variable

$$N_\omega^\theta \sim Poisson\left(\lambda_{\omega,\theta}\right),$$

where $\lambda_{\omega,\theta}$ are derived from (here exogenous) entry rates $(\pi_h, \pi_l)$ through (1). Note that $\alpha > \beta$ implies that $\lambda_{1,h} > \lambda_{0,h}$ and $\lambda_{0,l} > \lambda_{1,l}$.

The difference in the beliefs of the two types over the types and numbers of other players is fully captured by their different posteriors on state $\omega$. We denote by $q_h$ and $q_l$ the posterior belief that $\omega = 1$ for the high and low type, respectively. These are obtained from Bayes' rule:

$$\begin{aligned} q_h &= \frac{\alpha q}{\alpha q + \beta\left(1-q\right)} > \\ q_l &= \frac{\left(1-\alpha\right) q}{\left(1-\alpha\right) q + \left(1-\beta\right)\left(1-q\right)}. \end{aligned}$$

Anticipating that the relevant equilibria for our model are in atomless mixed strategies, we denote by $F_\theta\left(p\right)$ the (continuous) c.d.f. of the bids by a bidder with signal $\theta$. Since the number of entrants is determined from independent Poisson distributions and since the randomizations over bids are independent, the probability that no competing bid exceeds $p$ is given by

$$\begin{aligned} & \sum_{n^h \geq 0} \sum_{n^l \geq 0} \Pr\left(N^h = n^h \left|\omega\right.\right) F_h\left(p\right)^{n^h} \Pr\left(N^l = n^l \left|\omega\right.\right) F_l\left(p\right)^{n^l} \\ = & \sum_{n^h \geq 0} \frac{\left(\lambda_{\omega,l}\right)^{n^h} e^{-\lambda_{\omega,h}}}{n^h!} F_h\left(p\right)^{n^h} \sum_{n^l \geq 0} \frac{\left(\lambda_{\omega,l}\right)^{n^l} e^{-\lambda_{\omega,l}}}{n^l!} F_l\left(p\right)^{n^l} \\ = & e^{-\lambda_{\omega,h}\left(1-F_h(p)\right)} e^{-\lambda_{\omega,l}\left(1-F_l(p)\right)}. \end{aligned}$$

With this we can compute the expected payoff from bid $p$ to a bidder of type $\theta$ when the other players bid according to the profile $(F_h(p), F_l(p))$ as follows:

$$\begin{aligned} U_\theta(p) &= q_\theta e^{-\lambda_{1,h}(1-F_h(p))} e^{-\lambda_{1,l}(1-F_l(p))} (v(1) - p) \\ &\quad + (1 - q_\theta) e^{-\lambda_{0,h}} e^{-\lambda_{0,l}(1-F_l(p))} (v(0) - p). \end{aligned}$$

Our first result gives a characterization of bidding behavior in the first price auction. Singling out the case where bidding zero is in the support of both bidder types turns out to be extremely useful for our later revenue comparisons.

**Proposition 1** *The first price auction has a unique symmetric equilibrium in atomless bidding strategies. If*

$$\frac{1 - e^{-\lambda_{0,l}}}{1 - e^{-\lambda_{1,l}}} < \frac{v(1)}{v(0)},$$

*then the bid supports are non-overlapping intervals with a single point in common. If*

$$\frac{1 - e^{-\lambda_{0,l}}}{1 - e^{-\lambda_{1,l}}} > \frac{v(1)}{v(0)},$$

*then the intersection of bid supports contains an interval of positive length, and* 0 *is contained in the support of both types.*

It is not possible to rule out pooling equilibria in general. In Section 4, we show that with interim entry decisions pooling equilibria do not exist in informal first price auctions and hence the symmetric equilibrium is unique in that case. In general, the informal FPA with exogenous entry distributions can have pooling equilibria as well. In a slightly different model, Lauermann & Wolinsky (2015) have shown the existence of a pooling equilibrium in FPA.[7]

[7] In Lauermann & Wolinsky (2015), the signals are continuously distributed and the numbers of entrants are determined by a seller that has perfect information on the binary value of the object.

## 3.3 Symmetric bidding equilibria in informal second price auction

Our main result in this subsection is that there is a continuum of symmetric equilibria where the high types mix their bids over an interval, while low types pool on a single bid. The key feature of these equilibria is that in the case of pooling, there is a positive probability of a tie for the highest bid. This induces random rationing amongst the high bidders. Since the number of players pooling on that bid is a random variable whose distribution depends on the value of the good, the outcome of rationing contains information about the state. In particular, winning the auction is more likely when there are few other players pooling at the same bid. The entire set of symmetric equilibria is in general hard to characterize, but with endogenous entry decisions, the set of possible equilibrium bidding equilibria is reduced as we show in the next two sections.

### 3.3.1 Bidding by high types

We start by considering high type bidders. There are no equilibria where the high type biddders pool on a bid that is not chosen with positive probability by the low types. To see this, recall that in an affiliated values model, tying with a large number of other high types is good news about the value of the object. Hence winning at the pooling bid is relatively more likely when the object is less valuable and hence a deviation to a higher bid (so that rationing is no longer an issue) is profitable.

This argument does not rule out pooling bids for the high types. With exogenous entry rates, it is possible to have equilibria where all bidders pool on the same price. This happens whenever the entry rate of low types is large and hence the rationing effect is positive. If high signals are relative uninformative about the state of the world, then an equilibrium exists where all bidders pool on the same single bid. In the next section, we show that endogenous entry decisions reduce the number of possible equilibrium configurations. We shall focus on the bidder optimal equilibrium of the game and in this case, the bids by the high types are strictly larger than the bids of the low types. As a consequence, the high types bid in atomless mixed strategies.

In an atomless mixed strategy equilibrium, a high type must be indifferent between winning at her current bid and losing, conditional on tying the highest bid with another player (since the probability of tying with more than a single opponent conditional on the event of a tie is zero). Therefore, if $b$ is in the support of high type bidders, the bidding distribution $F_h(p)$ must satisfy:

$$b = v(0) + (v(1) - v(0)) \cdot \frac{\lambda_{1,h}(1 - F_h(p))\, e^{-\lambda_{1,h}(1-F_h(p))} q_h}{\lambda_{1,h}(1 - F_h(p))\, e^{-\lambda_{1,h}(1-F_h(p))} q_h + \lambda_{0,h}(1 - F_h(p))\, e^{-\lambda_{0,h}(1-F_h(p))}(1 - q_h)} \tag{4}$$

which gives:

$$F_h(p) = 1 - \frac{1}{\lambda_{1,h} - \lambda_{0,h}} \log\left(\frac{q_h}{1 - q_h} \frac{v(1) - p}{p - v(0)} \frac{\lambda_{1,h}}{\lambda_{0,h}}\right). \tag{5}$$

We find the lower and upper bounds of the distribution, $p'$ and $p''$, by setting $F_h(b) = 0$ and $F_h(p) = 1$ in (5):

$$p' = v(0) + \frac{\lambda_{1,h} e^{-\lambda_{1,h}} q_h}{\lambda_{1,h} e^{-\lambda_{1,h}} q_h + \lambda_{0,h} e^{-\lambda_{0,h}}(1 - q_h)} (v(1) - v(0)) \tag{6}$$

$$= \mathbb{E}\left(v \,\middle|\, \theta = h, N^h = 1\right), \tag{7}$$

$$p'' = v(0) + \frac{\lambda_{1,h} q_h}{\lambda_{1,h} q_h + \lambda_{0,h}(1 - q_h)} (v(1) - v(0)) \tag{8}$$

$$= \mathbb{E}\left(v \,\middle|\, \theta = h, N^h \geq 1\right). \tag{9}$$

### 3.3.2 Bidding by low types

We turn our attention to the low type bidders next. The informal second price auction suffers from a severe multiplicity of equilibria even with endogenous entry because pooling by the low bidders is the only possible equilibrium behavior. In all symmetric equilibria of the informal second price auction, low type bidders pool on a discrete set of bids.

To see that continuous mixing is not possible for the low types, observe first that it is not possible that the supports of the two types of bidders overlap. This would contradict the requirement that each type of bidder be indifferent between winning and losing in the event of a tie at the current bid. If only low types bid between $p'$ and $p'' > p'$, then the expected value of the object conditional on winning at $p''$ is lower than conditional on winning at $p'$. But this results in a contradiction to the requirement that the bid be equal to the value conditional on a tie at the bid.

The informal SPA still has a large number of possible equilibrium configurations. All low type bidders may pool on the same bid, they may pool on a finite set of bids, possibly with some pooling by the high type bidders at the same bids. While we can show that the possible pooling bids must be between $\underline{v}$ and $\overline{v}$, it is not easy to give a full characterization of the symmetric equilibrium set. In the section with endogenous entry, we pay particular attention to the bidder optimal equilibrium. This equilibrium maximizes the expected payoff to both types of bidders simultaneously.

The following proposition shows that this equilibrium is the one where the low type bidders pool on as small a bid as possible. It should be noted that this equilibrium exists for the model with endogenous entry below (and also for the case of ex ante entry covered in Appendix B). Hence this bidder optimal equilibrium is a natural candidate for a representative equilibrium of the informal second price auction when comparing auction formats.

**Proposition 2** *The bidder optimal symmetric equilibrium of the informal second price auction is characterized by a pooling bid at $\underline{v}$ given by (3) for the low type bidders and the atomless bidding distribution given in (5) for the high type bidders.*

**Proof.** Pooling on a single bid maximizes the winner's blessing. Pooling on the lowest possible price maximizes gains from winning. By definition of $\underline{v}$, the low types are indifferent between pooling and a slightly higher bid. Since the high-type bidder is more optimistic about the value of the object, she strictly prefers the deviation and hence there cannot be pooling by both types on the same bid. ■

## 3.4 Symmetric bidding equilibria in formal auctions

For completeness, we compute the symmetric equilibria for auctions where the number of bidders $n$ is known to the bidders, i.e. the formal auctions.

**Proposition 3** *In the formal first-price auction, there is a unique symmetric equilibrium. Given that there are $n \geq 2$ participants, low type bidders pool at bid*

$$\underline{p}(n) = E\left[v \left| \theta = l,\ N^l = n-1,\ N^h = 0\right.\right] \text{ for } n \geq 2$$

*and high types have an atomless bidding support $\left[\underline{p}(n), \overline{p}(n)\right]$ with some $\overline{p}(n) > \underline{p}(n)$. With $n = 1$, the only participant bids zero.*

*In the formal second-price auction, there is a unique symmetric equilibrium. In this equilibrium, low types pool at bid*

$$\underline{p}(n) = E\left[v \left| \theta = l,\ N^l = n-1,\ N^h = 0\right.\right]$$

*and high types have an atomless bidding support $[p'(n), p''(n)]$, where*

$$\begin{aligned} p'(n) &= \mathbb{E}\left[v \left| \theta = l,\ N^l = n-2,\ N^h = 1\right.\right], \\ p''(n) &= \mathbb{E}\left[v \left| \theta = l,\ N^l \leq n-2,\ N^h \geq 1\right.\right]. \end{aligned}$$

*With $n = 1$, the only participant bids zero.*

It is thus easy to compute the expected payoff at the entry stage if the bidding stage is in a formal auction. In the next two sections, we compare the overall revenues in the games where we account for both the bidding stage and the costly entry stage.

# 4 Endogenous entry

We incorporate entry under the assumption that at the moment of choosing entry, potential bidders have already observed a signal on the state of the world. This specification is particularly relevant for cases where differential selection of the bidders plays a key role. The interesting situation is the one where both types enter at a positive rate. The equilibrium trades off two forces in a way that makes entry viable for both types. High signal types are more optimistic about the value of the object. Low signal types on the other hand find a low level of competition more likely. The differences between the auctions stem from the different way in which an auction format balances this tradeoff in comparison to how a social planner solves it. This feeds into different revenue properties of the acutions as we show in this section.

We analyze the alterative timing where entry is chosen prior to signal realizations in Appendix B, and we show thare that most of our results hold for this case as well.

## 4.1 Symmetric planner's optimum

Even though we are ultimately interested in the equilibrium determination of $(\pi_h, \pi_l)$, we start with the surplus maximizing benchmark where a social planner chooses $(\pi_h, \pi_l)$ to maximize the expected gain from allocating the object net of the expected entry cost.[8] Notice that with a free entry condition, we guarantee that the bidders do not earn a positive expected rent in the game. Hence the entire social surplus generated in the game is collected by the seller in expected revenues. This allows for an easy revenue comparison of the various auction mechanisms. The mechanism that yields entry rates with the highest social surplus generates the highest revenues.

Since our method of analysis compares the equilibrium rates to the planner's solution, we explain the connection between private and social entry incentives in detail in this section. The planner's objective is to

$$\max_{\pi_l, \pi_h \geq 0} W(\pi_h, \pi_l),$$

where $W(\pi_h, \pi_l)$ is the expected total surplus with a given entry profile:

$$\begin{aligned} W(\pi_h, \pi_l) &= q[v(1)\left(1 - e^{-\alpha\pi_h - (1-\alpha)\pi_l}\right) - (\alpha\pi_h + (1-\alpha)\pi_l)c] \\ &+ (1-q)[v(0)\left(1 - e^{-\beta\pi_h - (1-\beta)\pi_l}\right) - (\beta\pi_h + (1-\beta)\pi_l)c]. \end{aligned}$$

[8] Notice that we are restricting the planner to use the same pair of instruments $(\pi_h, \pi_l)$ that determines the outcome of our symmetric bidding games. Hence the planner's problem already incorporates the restriction to symmetric decision rules across potential bidders.

The first order conditions for interior solutions are given by:

$$\begin{aligned} \alpha q \left( v(1) e^{-\alpha\pi_h^{opt} - (1-\alpha)\pi_l^{opt}} - c \right) + \beta (1-q) \left( v(0) e^{-\beta\pi_h^{opt} - (1-\beta)\pi_l^{opt}} - c \right) &= 0, \\ (1-\alpha) q \left( v(1) e^{-\alpha\pi_h^{opt} - (1-\alpha)\pi_l^{opt}} - c \right) + (1-\beta)(1-q) \left( v(0) e^{-\beta\pi_h^{opt} - (1-\beta)\pi_l^{opt}} - c \right) &= 0. \end{aligned}$$

Since $\alpha > \beta$, this homogenous linear system of equations with the net benefit from an additional entrant in each state,

$$\left( v(1) e^{-\alpha\pi_h^{opt} - (1-\alpha)\pi_l^{opt}} - c \right), \left( v(0) e^{-\beta\pi_h^{opt} - (1-\beta)\pi_l^{opt}} - c \right),$$

as the variables, has a unique solution at

$$\begin{aligned} v(1) e^{-\alpha\pi_h^{opt} - (1-\alpha)\pi_l^{opt}} &= c, \\ v(0) e^{-\beta\pi_h^{opt} - (1-\beta)\pi_l^{opt}} &= c. \end{aligned} \tag{10}$$

Since $\alpha\pi_h + (1-\alpha)\pi_l$ is the entry rate in state $\omega = 1$ and $\beta\pi_h + (1-\beta)\pi_l$ is the entry rate in state $\omega = 0$, we see from equation (10) that in the planner's optimal solution, the entry rates are optimally chosen state by state. The two entry rates provide a sufficient set of instruments to obtain this equalization of marginal benefits from entry state by state if the optimal entry rates are strictly positive.

To see when the interior solution is valid, solve first ignoring the non-negativity constraint to get:

$$\begin{aligned} \pi_h^{opt} &= \frac{1}{\alpha - \beta} \left( (1-\beta) \log \left( \frac{v(1)}{c} \right) - (1-\alpha) \log \left( \frac{v(0)}{c} \right) \right), \\ \pi_l^{opt} &= \frac{1}{\alpha - \beta} \left( -\beta \log \left( \frac{v(1)}{c} \right) + \alpha \log \left( \frac{v(0)}{c} \right) \right). \end{aligned}$$

Notice that we have immediately the result that $\pi_h^{opt} > \pi_l^{opt}$ since

$$\pi_h^{opt} - \pi_l^{opt} = \frac{1}{\alpha - \beta} \log \left( \frac{v(1)}{v(0)} \right).$$

This simply reflects the intuitive requirement that under an optimal entry profile, there must be more entry in the states where the object has a larger value.

For the first-order condition to yield a valid solution, we must have $\pi_l^{opt} > 0$ so that the condition for interior solution in terms of model parameters is:

$$\alpha \log \left( \frac{v(0)}{c} \right) > \beta \log \left( \frac{v(1)}{c} \right).$$

Setting this as an equality and solving for $c$ we find a threshold $\overline{c}$

$$\overline{c} = e^{\frac{\alpha \log(v(0)) - \beta \log(v(1))}{\alpha - \beta}} > 0$$

such that for $c \in (0, \overline{c})$ we have an interior solution with $\pi_h^{opt} > \pi_l^{opt} > 0$.

When $c \geq \overline{c}$, we have a corner solution where $\pi_l^{opt} = 0$. In that case, the optimal entry rate for the high types is solved from the first line in the first order conditions (10) with $\pi_l^{opt} = 0$, or:

$$\alpha q \left( v(1) e^{-\alpha \pi_h^{opt}} - c \right) + \beta (1-q) \left( v(0) e^{-\beta \pi_h^{opt}} - c \right) = 0.$$

This gives an interior solution $\pi_h^{opt} > 0$ as long as $c < \overline{\overline{c}}$, where

$$\overline{\overline{c}} = \frac{\alpha q}{\alpha q + \beta (1-q)} v(1) + \frac{\beta (1-q)}{\alpha q + \beta (1-q)} v(0).$$

To summarize, for low entry cost $c < \overline{c}$ the socially optimal entry profile features both types entering with positive rate, for intermediate entry cost $c \in \left[\overline{c}, \overline{\overline{c}}\right)$ only high type enters with positive rate, and for high entry cost $c \geq \overline{\overline{c}}$ there is no entry. Note that the threshold $\overline{\overline{c}}$ is the expected value of the object for a player that has observed a high signal. This is intuitive: as long as entry cost $c$ is below the expected value of the object for the most optimistic potential entrant, it is sosially optimal to have at least some entry.

To understand the connection between private and social inventives, let us now consider the private entry incentives to an individual entrant. Before going to the actual auction mechanisms in the next Section, let us consider a hypothetical situation, where the object is given for free to the entrant if there is only one entrant. If there are two or more entrants, the object is withdrawn from the market and no entrant gets any compensation for the sunk entry cost. Denote by $V_\theta^0 (\pi_h, \pi_l)$ the expected value of a player with signal $\theta$ who enters in such a situation, when the entry profile is given by $(\pi_h, \pi_l)$:

$$\begin{aligned} V_h^0(\pi_h, \pi_l) &= \frac{q\alpha}{q\alpha + (1-q)\beta} e^{-\alpha\pi_h - (1-\alpha)\pi_l} v(1) \\ &\quad + \frac{(1-q)\beta}{q\alpha + (1-q)\beta} e^{-\beta\pi_h - (1-\beta)\pi_l} v(1), \\ V_l^0(\pi_h, \pi_l) &= \frac{q(1-\alpha)}{q(1-\alpha) + (1-q)(1-\beta)} e^{-\alpha\pi_h - (1-\alpha)\pi_l} v(1) \\ &\quad + \frac{(1-q)(1-\beta)}{q(1-\alpha) + (1-q)(1-\beta)} e^{-\beta\pi_h - (1-\beta)\pi_l} v(1). \end{aligned}$$

By finding $(\pi_h, \pi_l)$ that give $V_h^0(\pi_h, \pi_l) = V_l^0(\pi_h, \pi_l) = c$ in the above equations guarantees that every entrant is indifferent between entering and staying out. But this leads exactly to the same condition as in (10), and hence the private entry incentives are socially optimal. To understand why this is the case, note that the hypothetical situation that we are considering here gives every entrant exactly her individual contribution to the social welfare as her reward for entry: since the value of the object is common to all the players, an entrant contributes to the total surplus if and only if she is the only entrant and getting the object for free in such a situation rewards her exactly by her social contribution. Therefore, entry incentives coincide exactly with the social value of entry. As we show below, actual auctions will give a too high entry incentive at least for the high type entrants, which is the key mechanism in our revenue results.

In Figure 1 we illustrate the social optimum in the interior case by drawing the planner's reaction curves

$$\begin{aligned} \pi_l^*(\pi_h) &:= \arg\max_{\pi_l \geq 0} W(\pi_l, \pi_h), \\ \pi_h^*(\pi_l) &:= \arg\max_{\pi_h \geq 0} W(\pi_l, \pi_h), \end{aligned}$$

in the $(\pi_h, \pi_l)$ −plane. The social optimum is at the intersection of these curves. As shown above, these curves are also the indifference contours $V_h^0(\pi_h, \pi_l) = c$ and $V_l^0(\pi_h, \pi_l) = c$ for a potential entrant of type $h$ and $l$, respectively, who gets her social contribution as expected payoff.

We will see that many of the auctions will give an additional reward to potential entrants on top of their social contribution, which distorts the entry rates from socially optimal levels. Nevertheless, in most cases the optimality of the low type entry rate will be retained even when high type entry rate is distorted so that the distorted entry profile stays on the social planner's reaction curve $\pi_l^* (\pi_h)$. For our revenue comparisons it is useful to analyze how such a distortion changes the total surplus. The following Lemma confirms that the further $\pi_h$ is distorted away from $\pi_h^{opt}$ while $\pi_l$ adjusts to maximize surplus, the lower the total surplus. This is illustrated in Figure 1 by arrows along $\pi_l^* (\pi_h)$ that point towards increasing social surplus.

**Lemma 1** *Let $\pi_h^{opt} > 0$ denote the socially optimal high type entry rate. We have*

$$\frac{dW^* (\pi_h)}{d\pi_h} \begin{cases} > 0 \text{ for } \pi_h < \pi_h^{opt} \\ = 0 \text{ for } \pi_h = \pi_h^{opt} \\ < 0 \text{ for } \pi_h > \pi_h^{opt} \end{cases}. \tag{11}$$

## 4.2 Equilibrium with interim entry decisions

Let us now move to the equilibrium analysis. We first clear out the less interesting case, where $c \geq \overline{c}$ and the planner's solution has $\pi_l^{opt} = 0$. Since all the entering bidders have observed a high signal, it is easy to see that in all the auction formats that we consider, the equilibrium payoff to the entering bidders is given by the expected value of the object given that no other bidder entered. Since the updated belief of a high type bidder on $\{\omega = 1\}$ is given by $\frac{q\alpha}{q\alpha+(1-q)\beta}$, equating the expected benefit from entry to the cost of entry gives:

$$\frac{q\alpha}{q\alpha + (1-q)\beta} e^{-\alpha\pi_h} v(1) + \frac{(1-q)\beta}{q\alpha + (1-q)\beta} e^{-\beta\pi_h^{opt}} v(0) = c.$$

This concides with the planner's optimality condition. Hence the symmetric equilibrium entry profile in all of the auction formats that we consider coincides with the planner's optimal solution. The key reason for this is the lack of heterogeneity in the bidders' information.

The bidders are indifferent between entering and not entering. Hence their expected payoff must be at their outside option of 0. Since the auctions generate maximal social surplus (under the restriction to symmetric entry profiles) in this case and since bidders expected payoff is at zero, the seller collects the entire expected social surplus in expected revenues. Hence all these auction formats are also revenue maximizing within the class of symmetric mechanisms (where we require symmetry at the entry stage as well as at the bidding stage).

**Proposition 4** *If $\bar{c} \leq c < \bar{\bar{c}}$, then only the high type enters and all the auction formats are efficient and hence revenue equivalent. If $c \geq \bar{\bar{c}}$, then there is no entry in the planner's solution nor in any auction format.*

We move next to the more interesting case where the planner's solution induces entry by both types. In this case, we see immediately that at least the second price auctions, whether informal or formal, lead to suboptimal entry decisions. This conclusions follows from a very simple argument showing that a bidder with a high signal earns a higher private benefit in the auction stage than their social contribution. In a model with common values, additional entry is socially valuable only if no other bidder participates in the auction. In a second price auction, the bidder with a high type gets the social benefit, but she also receives an extra information rent when bidding against bidders with low signals. This is an immediate consequence of the usual logic in models with affiliated values. Hence models with a second price auction format in the bidding stage feature excessive entry by the high types relative to the planner's solution.

We now analyze formally equilibria with endogenous entry. We start with the informal first-price auction, and show that there is a unique equilibrium. This result narrows down the possible set of equilibria in two ways. First, we show that with endogenous entry, there cannot exist equilibria with pooling in the bidding stage. This follows from the fact that the bidders are indifferent between entering and not entering at the interim stage, and hence we know that the expected payoff given equilibrium entry rates must be $c$ for both types of bidders. With this additional piece of information, we can show that the atomless bidding equilibrium given in Proposition 1 is the unique symmetric equilibrium in the bidding stage. Second, denoting by $V_{\theta}^{FPA}(\pi_h, \pi_l)$ the ex-ante payoff in the bidding stage of a bidder with signal $\theta$, when bidding follows the unique bidding equilibrium given entry rates $\pi_h, \pi_l$, we show that there is a unique pair $\left(\pi_h^{FPA}, \pi_l^{FPA}\right)$ such that $V_h^{FPA}\left(\pi_h^{FPA}, \pi_l^{FPA}\right) = V_l^{FPA}\left(\pi_h^{FPA}, \pi_l^{FPA}\right) = c$.

In addition to the uniqueness, the proposition also contains a qualitative statement about the equilibrium. A low type entrant gets a payoff in the bidding stage that is exactly her social contribution:

$$V_l^{FPA}(\pi_h, \pi_l) = V_l^0(\pi_h, \pi_l),$$

which means that equilibrium entry point $\left(\pi_h^{FPA}, \pi_l^{FPA}\right)$ must be along the planner's reaction curve $\pi_l^{FPA} = \pi_l^*\left(\pi_h^{FPA}\right)$. A high type may get more, so that to dissipate excess rent of the high type, we must have $\pi_h^{FPA} \geq \pi_l^*\left(\pi_l^{FPA}\right)$. As a result, the equilibrium entry rate must be (weakly) too high for the high type and (weakly) too low for the low type:

**Proposition 5** *The informal first price auction with interim entry has a unique symmetric equilibrium with entry rates* $\pi_h^{FPA} \geq \pi_h^{opt}$ *and* $\pi_l^{FPA} = \pi_l^*\left(\pi_h^{FPA}\right) \leq \pi_l^{opt}$. *All entering bidder types use atomless bidding strategies. Zero bids are always in the support of the low type bidders and the upper bound of the high type bidder support is given by*

$$b_h := \frac{q\alpha}{q\alpha + (1-q)\beta} v(1) + \frac{(1-q)\beta}{q\alpha + (1-q)\beta} v(0) - c.$$

We next consider the informal second-price auction. We know that SPA has multiple equilibria in the bidding stage. We will concentrate on the bidder-optimal equilibrium, where all the low type bidders pool on the single bid $\underline{v}$ as defined in equation (3). The reason for focusing on this equilibrium is that at the bidding stage the equilibrium expected payoff is maximized for both bidder types in this equilibrium. Hence if the bidders can coordinate on their most preferred equilibrium, pooling at $\underline{v}$ is a natural equilibrium selection. We denote by $V_\theta^{SPA}(\pi_h, \pi_l)$ the expected payoff of an entrant with signal $\theta$, $\theta = h, l$, when entry profile is given by $(\pi_h, \pi_l)$.

With endogenous entry, a bidder optimal equilibrium consists of entry profile $\left(\pi_h^{SPA}, \pi_l^{SPA}\right)$ and bidding strategy $(b_h, b_l)$ such that 1) the bidding strategy is the bidder-optimal equilibrium of the bidding stage given entry rates $\left(\pi_h^{SPA}, \pi_l^{SPA}\right)$, and 2) bidders are indifferent between entering and not:

$$V_h^{SPA}\left(\pi_h^{SPA}, \pi_l^{SPA}\right) = V_l^{SPA}\left(\pi_h^{SPA}, \pi_l^{SPA}\right) = c.$$

In the following Proposition we show that informal SPA has a unique bidder-optimal equilibrium. The content of the proof is to show that for any $c$, there is a single crossing point for the indifference curves $V_h^{SPA}(\pi_h, \pi_l) = c$ and $V_h^{SPA}(\pi_h, \pi_l) = c$ in the $(\pi_h, \pi_l)$ −space. There is a qualitative difference between entry profiles in informal FPA and informal SPA: whereas in informal FPA the low type entry rate is on the planner's reaction curve $\pi_l^*(\pi_h)$, in informal SPA also a low type bidder gets more than her social contribution, i.e. $V_l^{SPA}(\pi_h, \pi_l) > V_l^0(\pi_h, \pi_l)$. Hence, in equilibrium entry rates of both types are distorted upwards from planners' reaction curves.

**Proposition 6** *Assume that $c < \bar{c}$. There is a unique bidder-optimal equilibrium with entry profile $\left(\pi_h^{SPA}, \pi_l^{SPA}\right)$, where $\pi_h^{SPA} > \pi_l^*\left(\pi_l^{SPA}\right)$ and $\pi_l^{SPA} > \pi_l^*\left(\pi_h^{SPA}\right)$.*

## 4.3 Revenue comparisons

We compare here the expected revenue across auction formats. We assume throughout this section that $c < \bar{c}$. If $c \geq \bar{c}$, then all the auction formats are revenue equivalent as already stated in Proposition 4.

Our first revenue comparison result shows that for some parameter values, the informal first price auction gives the entire (symmetric) social surplus to the seller in expected revenues. Hence the informal first price auction maximizes the seller's expected revenue in the class of symmetric mechanisms. Since both types of formal auctions and the informal second price auction fall short of this revenue, we establish the strict superiority of the informal first price auction for this case.

**Proposition 7** *If*

$$\frac{1-\beta}{1-\alpha} > \frac{v(1)}{v(0)},$$

*then there is a $c' < \overline{c}$ such that for $c \in (c', \overline{c})$, entry is efficient in the informal FPA and the expected revenue is strictly higher than in the three other auction formats.*

Our next result compares the informal FPA to the informal SPA. The comparison is unambiguous when we select the bidder optimal equilibrium in the informal SPA.

**Proposition 8** *The unique equilibrium in informal FPA generates a higher expected revenue than the bidder optimal equilibrium of the informal SPA.*

Finally, we compare informal FPA with the formal auctions. As shown in Section 3.4, FPA and SPA generate the same expected revenue when the number of players is observed at the bidding stage. This implies that the formal FPA and the formal SPA are equivalent in terms of revenues also with endogeneous entry in our environment. We show in the proof of Proposition 9 below that if entry rates are not very high, then the informal FPA doiminates the formal auctions. Tranlated into the exogenous parameters of the model, this means that whenever entry cost is sufficiently high, informal FPA raises more revenue than the formal auctions:

**Proposition 9** *When entry cost is high enough, informal FPA generates a higher expected revenue than formal auctions.*

# 5 Conclusion

This paper demonstrates that the revenue rankings for affiliated common value auctions can be overturned when the number of bidders is not disclosed to the bidders in an auction. Randomness in the number of bidders results from a costly entry decision where bidders randomize between participating and not participating.

The best symmetric equilibrium outcome is somethimes achievable through a simple mechanism: all bidders submit a sealed bid and the highest bidder wins and pays her own bid. We view this as a natural model for an informal bidding contest such as competitive takeover bidding. In general, first-price auctions without disclosure of the number of bidders results in a relatively high expected revenue to the seller whenever the expected number of participants is not too high.

# 6 Appendix A: Two Potential Entrants

In this Appendix, we show that for the case with two potential bidders, the informal FPA dominates the other three auction formats in terms of expected revenue. We start again by analyzing the bidding stage of FPA and SPA when there is uncertainty about whether a competitor exists. Let $(\pi_l, \pi_h)$ be the probabilities with which a competitor of type $\theta \in \{l, h\}$ enters. As before, we let

$$\begin{aligned}\lambda_{1,h} &= \alpha\pi_h,\\ \lambda_{0,h} &= \beta\pi_h,\\ \lambda_{1,l} &= (1-\alpha)\,\pi_l,\\ \lambda_{0,l} &= (1-\beta)\,\pi_l\,.\end{aligned}$$

## 6.1 Informal SPA

The informal SPA has a unique symmetric equilibrium where bidder of type $\theta$ submits a bid $b_\theta$ as follows:

$$b_l = \mathbb{E}\left(v\,|\theta_1 = \theta_2 = l\,\right) = \frac{q\,(1-\alpha)^2}{q\,(1-\alpha)^2 + (1-q)\,(1-\beta)^2}v\,(1) + \frac{(1-q)\,(1-\beta)^2}{q\,(1-\alpha)^2 + (1-q)\,(1-\beta)^2}v\,(0)\,,$$

and

$$b_h = \mathbb{E}\left(v\,|\theta_1 = \theta_2 = h\,\right) = \frac{q\alpha^2}{q\alpha^2 + (1-q)\,\beta^2}v\,(1) + \frac{(1-q)\,\beta^2}{q\alpha^2 + (1-q)\,\beta^2}v\,(0)\,.$$

Hence the low type bidders' payoff is equal to her payoff if she gets the object at prize zero if and only if she is the only bidder. Notice that this is also the social value of entry. The high bidder earns an information rent on top of this social value since her payment in the event that a competitor with a low type has entered is below the expected value of the object.

## 6.2 Informal FPA

In this subsection, we show that whenever if $0 < \pi_l \leq \pi_h < 1$, the informal FPA has a unique symmetric equilibrium and that this equilibrium is in atomless mixed strategies. [9] As in the main text, the key result for our analytical results is the characterization of the supports of the bidding distributions of the two bidder types. We show again that if the supports overlap on an interval, then zero bid is in the support of the equilibrium bid distributions for both bidder types. If this is not the case, then there exists a $p' > 0$ and a $p'' > p'$ such that $\text{supp}F_l(\cdot) = [0, p']$ and $\text{supp}F_h(\cdot) = [p', p'']$. Denote by $p^{\max}$ the largest bid in the union of the two supports.

**Proposition 10** *Assume that* $0 < \pi_l \leq \pi_h < 1$.*The informal FPA with two potential bidders has a unique symmetric equilibrium. Both types of bidders use atomless mixed strategies The supports of the two bid distributions* $F_\theta(\cdot)$ *for* $\theta \in \{l, h\}$ *satisfy*

1. $0 \in \text{supp}F_l(\cdot), p^{\max} \in \text{supp}F_h(\cdot)$, $\text{supp}F_l(\cdot) \cup \text{supp}F_h(\cdot) = [0, p^{\max}]$.
2. *Either* $0 \in \text{supp}F_h(\cdot)$ *or there exists a* $p' > 0$ *such that* $\text{supp}F_l(\cdot) = [0, p']$ *and* $\text{supp}F_h(\cdot) = [p', p^{\max}]$.

**Proof.** The proof is almost identical to the proof of the characterization result Proposition 1 in the main text. Consider again the payoff to a bidder with the signal $\theta$ from a bid of $p$.

$$\begin{aligned} U_\theta(p) &= q_\theta (1 - \lambda_{1,h}(1 - F_h(p)) - \lambda_{1,l}(1 - F_l(p)))(v(1) - p) \\ &\quad + (1 - q_\theta)(1 - \lambda_{0,h}(1 - F_h(p)) - \lambda_{0,l}(1 - F_l(p)))(v(0) - p). \end{aligned}$$

For any bid $p$ in the interior of the support of $F_\theta(\cdot)$, we can differentiate the value with respect to $p$ and require that the derivative be zero:

$$U'_\theta(p) = q_\theta (1 - \lambda_{1,h}(1 - F_h(p)) - \lambda_{1,l}(1 - F_l(p)))\left[(F'_h(p)\lambda_{1,h} + F'_l(p)\lambda_{1,l})(v(1) - p) - 1\right]$$

$$+ (1 - q_\theta)(1 - \lambda_{0,h}(1 - F_h(p)) - \lambda_{0,l}(1 - F_l(p)))\left[(F'_h(p)\lambda_{0,h} + F'_l(p)\lambda_{0,l})(v(0) - p) - 1\right].$$

From this point onwards, the proof follows exactly the same steps at the proof of Proposition 1 and we omit the details. ■

[9] We deal with the corner solutions separeately in the subsection on revenue comparisons.

## 6.3 Formal Auctions

By Theorem 1 in Wang, the unique symmetric equilibrium in the formal first-price auction with two bidders is one where the low type bidders submit the bid

$$b_l = \mathbb{E}\left(v \left| \theta_1 = \theta_2 = l \right.\right),$$

and high type bidders mix on an interval $[b_l, p^{\max}]$. Since $b_l$ is in the support of the high type bidders and at that bid they win if and only if the other bidder is of low type, we conclude that formal FPA and formal SPA result in the same payoff to both types. Furthermore we see tha the expected payoff is the same as in the informal SPA. Hence it is sufficient to compare the expected revenues between the informal FPA and the informal SPA.

## 6.4 Equilibrium Entry and Revenue Comparisons

We start with the symmetric planner's problem where the objective is to maximize the benefit from allocating the object net of the participation costs. Hence the objective is to

$$\begin{aligned} &\max_{\pi,,\pi_h} q[\left(1-(1-\alpha\pi_h-(1-\alpha)\pi_l)^2\right)v(1)-2c(\alpha\pi_h+(1-\alpha)\pi_l)] \\ &+(1-q)\left[\left(1-(1-\beta\pi_h-(1-\beta)\pi_l)^2\right)v(0)-2c(\beta\pi_h+(1-\beta)\pi_l)\right]. \end{aligned}$$

It is easy to verify that this objective is strictly concave in $(\pi_l, \pi_h)$. The two first-order conditions for an interior solution are given by:

$$\begin{aligned} & \alpha q(1-\alpha\pi_h-(1-\alpha)\pi_l)v(1)+(1-\beta)(1-q)(1-\beta\pi_h-(1-\beta)\pi_l)v(0) \\ = \;& c(q\alpha+(1-q)\beta), \\ & (1-\alpha)q(1-\alpha\pi_h-(1-\alpha)\pi_l)v(1)+(1-\beta)(1-q)(1-\beta\pi_h-(1-\beta)\pi_l)v(0) \\ = \;& c(q(1-\alpha)+(1-q)(1-\beta)). \end{aligned}$$

These two equations can be thought of as the planner's reaction curves for the two entry probabilities separately. The first determines the optimal $\pi_h^*(\pi_l)$ for any given level of $\pi_l$ and the second gives $\pi_l^*(\pi_h)$ .The socially optimal entry profile is found at the intersection $(\pi_l^*, \pi_h^*)$ of these reaction curves.

Notice that the second reaction curve also gives the equilibrium entry probability for the low type bidders in any interior equilibrium. The characterizations of the informal FPA and SPA imply immediately that low type bidders make a rent only if they are the sole entrants. Hence their expected payoff from entering is given by

$$\frac{(1-\alpha)\,q\,(1-\alpha\pi_h-(1-\alpha)\,\pi_l)\,v\,(1)+(1-\beta)\,(1-q)\,(1-\beta\pi_h-(1-\beta)\,\pi_l)\,v\,(0)}{(1-\alpha)\,q+(1-\beta)\,(1-q)}.$$

Equating this payoff to the cost of entry $c$ as required for an optimal $\pi_l \in (0,1)$ yields the planner's reaction curve $\pi_l^*(\pi_h)$.This observation allows us to conclude that entry of the low types is at the conditionally efficient level in both informal auctions. With this observation, we see that it is sufficient to compare the equilibrium entry rates of the high type bidders when comparing the social surplus generated in these auctions.

The private costs of a potential entrant coincide with the social cost. Since the private benefit is at least equal to the social benefit, we see that our auction formats generate excessive entry relative to social optimum.[10] Conditionally efficient entry by the low types is linear in $\pi_h$. This observation together with the concavity of the social objective function implies that the auction format that generates less entry by high types generates a higher social surplus. Whenever we have interior entry probabilities, both types of bidders must be indifferent between participating and not participating in the auction. This implies that the seller collects the entire expected social surplus in expected revenues in the auction. A comparison of the entry rates of the high types then also gives us a revenue ranking for the auction formats.

**Theorem 1** *With two potential bidders, the informal FPA generates a higher expected revenue than the Informal SPA and the formal auctions .*

**Proof.** Consider first the case where the equilibrium entry probabilities $\pi_\theta^{FPA} \in (0,1)$ for $\theta \in \{l,h\}$ in the informal FPA are interior. By Proposition 10, there are two cases to consider. If $0 \in \text{supp}F_l(\cdot) \cap \text{supp}F_h(\cdot)$, entry is at

[10] By bidding zero in either of the auction formats, both types of bidders can secure at least the social value (i.e. the value of the object in the event that there are no other bidders).

the socially optimal level in the informal FPA since the private gain from a bid of zero coincides with the social gain. Hence $\pi_\theta^{FPA} = \pi_\theta^*$ for $\theta \in \{l, h\}$ In the informal SPA, bidders with $\theta = h$ are indifferent between entering and not if

$$\begin{aligned} &\alpha q\left(1-\alpha \pi_h-(1-\alpha) \pi_l\right) v(1)+\beta(1-q)\left(1-\beta \pi_h-(1-\beta) \pi_l\right) v(0) \quad (12) \\ +&\alpha(1-\alpha) q \pi_l\left(v(1)-b_l\right)+\beta(1-\beta)(1-q) \pi_l\left(v(0)-b_l\right)=c(q \alpha+(1-q) \beta) . \end{aligned}$$

Since the sum of the first two terms on the second line is positive for $\pi_l > 0$, we see that the equilibrium entry curve $\pi_h^{SPA}(\pi_l)$ of the high types in the informal SPA lies above the social planner's reaction curve $\pi_h^*(\pi_l)$ . Hence we conclude that $\pi_h^{SPA} > \pi_h^*$, and the claim follows.

Consider next the case where $\text{supp}F_l(\cdot) = [0, p']$ and $\text{supp}F_h(\cdot) = [p', p^{\max}]$ for some $p' > 0$, and consider the equilibrium entry profile $\left(\pi_l^{FPA}, \pi_h^{FPA}\right)$. We want to compare the equilibrium expected payoff of the high type bidders in the informal FPA and SPA at these fixed entry probabilities. Notice first that by bidding 0 in the informal FPA, the low type bidder wins if and only if no other bidders enter. By bidding $p'$, she wins if and only if no high type bidder enters. Since both of these bids are in her bid support, they must yield the same expected payoff:

$$\begin{aligned} & \Pr\left(N^l = 0 \,\middle|\, \theta = l, N^h = 0\right) \mathbb{E}\left(v \,\middle|\, \theta = l, N^h = 0, N^l = 0\right) - 0 \\ = \; & \Pr\left(N^l = 0 \,\middle|\, \theta = l, N^h = 0\right) \mathbb{E}\left(v \,\middle|\, \theta = l, N^h = 0, N^l = 0\right) \\ & + \Pr\left(N^l = 1 \,\middle|\, \theta = l, N^h = 0\right) \mathbb{E}\left(v \,\middle|\, \theta = l, N^h = 0, N^l > 0\right) - p' \end{aligned}$$

so that

$$p' = \Pr\left(N^l = 1 \,\middle|\, \theta = l, N^h = 0\right) \mathbb{E}\left(v \,\middle|\, \theta = l, N^h = 0, N^l = 1\right).$$

The bid $p'$ is also in the support of the high type bidder.

Observe that by deviating to some bid between $b_l$ and $b_h$, the high type bidder does not change her equilibrium payoff in the informal SPA (as long as $\varepsilon$ is small enough). At this deviating bid, the allocation of the deviating bidder is exactly the same as allocation from the equilibrium bid of $p'$ in the informal FPA. To compare the expected payoff to the high type bidder across the two auction formats, we then need to compare only the expected payment in these formats.

The expected payment of the deviating high type bidder in the informal SPA is

$$\mathbb{E}(p) = \Pr\left(N^l = 0 \left| \theta = h, N^h = 0\right.\right) \cdot 0 + \Pr\left(N^l = 1 \left| \theta = h, N^h = 0\right.\right) b_l,$$

where

$$b_l = \mathbb{E}\left(v \left| \theta = l, N^h = 0, N^l = 1\right.\right)$$

is the bid of a low type in SPA

By comparing $\mathbb{E}(p)$ and $p'$, we see that the expected payment in the second price auction is smaller than in the FPA since

$$\Pr\left(N^l = 1 \left| \theta = h, N^h = 0\right.\right) < \Pr\left(N^l = 1 \left| \theta = l, N^h = 0\right.\right)$$

because the bidder with signal $\theta = h$ considers state $\omega = 1$ more likely than the bidder with signal $\theta = l$ and $(1 - \alpha) < (1 - \beta)$. This shows that at the equilibrium entry probabilities $\pi_\theta^{FPA}$, the expected payoff to the high types in the informal SPA is higher than the expected payoff $c$ that they get in the informal FPA.

In the bidding equilibrium, the expected payoff to the high types is decreasing in $\pi_h$ for a fixed level of $\pi_l$. This allows us to conclude that $\pi_h^{SPA} > \pi_h^{FPA}$ and the claim follows from the fact that the expected revenue coincides with the expected social surplus whenever the equilibrium entry probabilities are interior.

The same argument goes through unchanged for the case where $\pi_h^{FPA} = 1$. In this case, the equilibrium entry rates are the same across the two auction formats, but the rent going to the high type bidders is higher in the informal SPA.

$\pi_\theta^{FPA} = 0$ is possible only when the socially optimal entry rate is also zero for the low types. In this case, $\pi_\theta^{FPA} = \pi_\theta^{SPA} = \pi_\theta^*$ for $\theta \in \{l, h\}$. ■

## 6.5 Two Bidders and Ex Ante Entry

The characterizations in the bidding stage are exactly the same in the case of ex ante entry as well. The argument for revenue comparison is also unchanged. In this case, entry is always with an interior probability and the informal FPA strictly dominates the informal SPA in all cases.

# 7 Appendix B: Ex Ante Entry Decisions

## 7.1 Planner's problem

When entry decisions are taken at the ex ante stage, the equilibrium determination of equilibrium entry rates is straightforward. Since all players are ex ante symmetric, only a single entry rate $\pi$ needs to be determined.

As in the main text, we start with the socially optimal choice of $\lambda$. Since the expected number of entrants is equal to the Poisson parameter $\pi$, the planner's problem is then to

$$\max_{\pi \geq 0} \overline{v} \left(1 - e^{-\pi}\right) - \pi c,$$

where $\overline{v} = qv(1) + (1-q)v(0)$. Note that the marginal benefit from increasing the entry intensity is the probability that there are no entrants times the value of the object. This problem is strictly concave and has an interior solution

$$\pi^* = \ln\left(\frac{\overline{v}}{c}\right).$$

since we are assuming that $v(0) > c$. It is also clear that the entry stage for any of our four auction formats will have a unique interior entry rate that balances the cost and benefit of entry and keeps the potential entrants indifferent between entering and not entering. This means that we can rank the expected revenue of our auction formats by computing the social surplus induced entry rates as before. By the concavity we know that if $\pi^{SPA} > \pi^{FPA} \geq \pi^*$, then the informal FPA dominates the informal SPA in terms of expected revenue.

## 7.2 Bidding equilibria

Since entry decisions are taken at a stage where all potential entrants are symmetrically informed about the value of the product, the realized number of entrants conveys no information. This distinguishes the current ex ante entry model from the interim entry model of the main text. Nevertheless if the two types of bidders use different strategies at the bidding stage, then winning the auction conveys information about the realized types of the other bidders and hence about the true value of the object.

In the proof of the unicity of the symmetric equilibrium in the FPA, we ruled out pooling equilibria by using the fact that both types of bidders are indifferent between entering and not entering. Unfortunately this condition is not available in the case of ex ante entry and we have not been able to rule out pooling equilibria. Nevertheless the steps in the proof of Proposition 1 demonstrating the uniqueness in symmetric atomless equilibria applies equally well for the special case where $\pi_h = \pi_l$. We concentrate on this equilibrium of the informal FPA in our revenue comparisons. Proposition 1 implies that the low type bidders' expected payoff at coincides with the value of the object conditional on being the only participating bidder in the auction.

The informal SPA has multiple symmetric pooling equilibria, and as in the main text, we concentrate on the bidder optimal equilibrium of the informal SPA. In the bidder optimal equilibrium of the informal SPA, low types pool on the bid $\underline{v}$ solving

$$\underline{v} = \frac{L^{lose}}{1 + L^{lose}} v(1) + \frac{1}{1 + L^{lose}} v(0),$$

where

$$L^{lose} = \frac{q_l}{1 - q_l} \frac{\left(1 - \frac{1 - e^{-\lambda_{1,l}}}{\lambda_{1,l}}\right)}{\left(1 - \frac{1 - e^{-\lambda_{0,l}}}{\lambda_{0,l}}\right)}.$$

The high type bidders bid according to an atomless bidding strategy with $\underline{v} < \min\{p \,|p \in \text{supp} F_h(\cdot)\}$.

## 7.3 Revenue comparisons

In this subsection, we compare the expected revenue across the informal auctions. We concentrate on the atoless symmetric equilibrium of the FPA and the bidder optimal bidding equilibrium of the informal SPA. The main result in this Appendix in the following theorem.

**Theorem 2** *The unique symmetric equilibrium in atomless strategies of the informal FPA raises strictly higher expected revenue than the bidder optimal equilibrium of the informal SPA.*

**Proof.** By Proposition 1, either $0 \in \text{supp}F_l(\cdot) \cap \text{supp}F_h(\cdot)$ or $\text{supp}F_l(\cdot) = [0, p']$ and $\text{supp}F_h(\cdot) = [p', p^{\max}]$ for some $p' > 0$ in the informal FPA. In the first case, both types receive as their equilibrium payoffs their contribution to the social surplus and as a result, entry is at the socially efficient level in the informal FPA. Since the pooling bid of the low type bidders in the informal SPA is below the low type bidder's expected value of object conditional on winning the rationing, a low type bidder has a strictly positive expected equilibrium payoff in the event where other low type bidders have entered the auction. This implies that the expected payoff of the low types is strictly higher in the informal SPA than in the informal FPA at a fixed entry rate $\pi$. Since the payoff to the high type is at least as high as the value of the object conditional on no entry by other bidders, the claim follows for the first case.

Consider next the second case. Since $0 \in \text{supp}F_l(\cdot)$, the conclusion for the low type bidders holds exactly as in the previous case. Compare next the payoff to the high type bidder in the informal SPA and in the informal FPA at bid $p'$. Since the allocation to the bidder is the same across the two cases, we get the payoff difference by comparing the expected payment. The high type bidder has a higher expected payoff in the informal SPA if

$$\mathbb{E}\left(p \,\middle|\, \theta = h, N^h = 0, \text{SPA}\right) < p'. \tag{13}$$

The left-hand side of this equation is

$$\begin{aligned}
& \mathbb{E}\left(p \,\middle|\, \theta = h, N^h = 0, \text{SPA}\right) \\
= \;& \Pr\left(N^l > 0 \,\middle|\, \theta = h, N^h = 0\right) \underline{v} \\
= \;& \frac{\left(1 - e^{-\lambda_{0,l}}\right) + L^h \left(1 - e^{-\lambda_{1,l}}\right)}{1 + L^h} \underline{v},
\end{aligned}$$

where

$$L^h = \frac{q}{1-q} \frac{\alpha}{\beta} e^{-\pi(\alpha - \beta)}.$$

To compute the right-hand side of the equation, we note that $p'$ and 0 are both in the support of the low type:

$$\begin{aligned}
& \Pr\left(N^l = 0 \left| \theta = l, N^h = 0\right.\right) \mathbb{E}\left(v \left| \theta = l, N^h = 0, N^l = 0\right.\right) - 0 \\
= \; & \Pr\left(N^l = 0 \left| \theta = l, N^h = 0\right.\right) \mathbb{E}\left(v \left| \theta = l, N^h = 0, N^l = 0\right.\right) \\
& + \Pr\left(N^l > 0 \left| \theta = l, N^h = 0\right.\right) \mathbb{E}\left(v \left| \theta = l, N^h = 0, N^l > 0\right.\right) - p'.
\end{aligned}$$

From this we see that:

$$p' = \Pr\left(N^l > 0 \left| \theta = l, N^h = 0\right.\right) \mathbb{E}\left(v \left| \theta = l, N^h = 0, N^l > 0\right.\right).$$

The first term on the right hand side in this expression can be written as:

$$\begin{aligned}
\Pr\left(N^l > 0 \left| \theta = l, N^h = 0\right.\right) &= \frac{L^l}{1+L^l}\left(1 - e^{-\lambda_{1,l}}\right) + \frac{1}{1+L^l}\left(1 - e^{-\lambda_{0,l}}\right) \\
&= \frac{\left(1 - e^{-\lambda_{0,l}}\right) + L^l\left(1 - e^{-\lambda_{1,l}}\right)}{1+L^l},
\end{aligned}$$

where $L^l$ is the likelihood ratio conditional on having a low signal and no high type entrants are present:

$$L^l = \frac{q}{1-q}\frac{1-\alpha}{1-\beta}\frac{e^{-\lambda_{1,h}}}{e^{-\lambda 0,h}}.$$

By the definition of $\underline{v}$, we get:

$$p' = \frac{\left(1 - e^{-\lambda_{0,l}}\right) + L^l\left(1 - e^{-\lambda_{1,l}}\right)}{1+L^l}\underline{v},$$

Since

$$\frac{\left(1 - e^{-\lambda_{0,l}}\right) + L^h\left(1 - e^{-\lambda_{1,l}}\right)}{1+L^h} < \frac{\left(1 - e^{-\lambda_{0,l}}\right) + L^h\left(1 - e^{-\lambda_{1,l}}\right)}{1+L^h},$$

(13) holds and hence also the high type bidder has a higher expected revenue in SPA. This means that the expected revenue is higher in FPA. Since the expected payoff of both types of bidders is decreasing in $\pi$, we concude that to achieve indifference for the potential bidders in the informal SPA, we must have $\pi^{SPA} > \pi^{FPA} \geq \pi^*$. ■

We can also compare the payoffs between informal and formal auctions with ex ante entry decisions. In the case where the bid distributions in the informal FPA are overlapping, the informal FPA dominates all other symmetric mechanisms. Hence we have the following proposition.

**Proposition 11** *Assume that*

$$\frac{v(1)}{v(0)} < \frac{1-\beta}{1-\alpha}.$$

*Then there exists a $c' < \overline{v}$ such that whenever $c \in (c', \overline{v})$, the informal FPA with ex ante entry raises a strictly higher revenue than any of the other auction formats.*

**Proof.** Immediate from Proposition 1. ■

# 8 Appendix C: Proofs

## 8.1 Proof of Proposition 1

We prove Proposition 1 by constructing the equilibrium bidding functions by requiring indifference over intervals of bids. Since the equations determining this indifference have a unique solution, we get the uniqueness of atomless bidding equilibria as a by-product of this procedure.

To begin, let us specify the range of bids where both types can potentially be indifferent simultaneously. Let us denote by $U_\theta(p)$ the payoff of type $\theta$ who bids $p$, when bidding distributions are given by $F_\theta(p)$:

$$U_\theta(p) = q_\theta e^{-\lambda_{1,h}(1-F_h(p))-\lambda_{1,l}(1-F_l(p))}(v(1)-p) \quad (14)$$
$$+(1-q_\theta)e^{-\lambda_{0,h}(1-F_h(p))-\lambda_{0,l}(1-F_l(p))}(v(0)-p). \quad (15)$$

Differentiating with respect to $p$, we have:

$$U'_\theta(p) = q_\theta e^{-\lambda_{1,h}(1-F_h(p))-\lambda_{1,l}(1-F_l(p))}\left[(F'_h(p)\lambda_{1,h}+F'_l(p)\lambda_{1,l})(v(1)-p)-1\right]$$
$$+(1-q_\theta)e^{-\lambda_{0,h}(1-F_h(p))-\lambda_{0,l}(1-F_l(p))}\left[(F'_h(p)\lambda_{0,h}+F'_l(p)\lambda_{0,l})(v(0)-p)-1\right]$$

In equilibrium, $F'_\theta(p) > 0$ requires $U'_\theta(p) = 0$ to maintain indifference within bidding support. To analyse when this can hold, we denote the two bracketed terms in this equation by $B(1)$ and $B(0)$:

$$B(1) = (F'_h(p)\lambda_{1,h}+F'_l(p)\lambda_{1,l})(v(1)-p)-1,$$
$$B(0) = (F'_h(p)\lambda_{0,h}+F'_l(p)\lambda_{0,l})(v(0)-p)-1.$$

These terms are weighted in (8.1) by positive terms that depend on $\theta$ only through $q_\theta$. Since $q_h > q_l$, we note that $U_h'(p)$ puts more weight on term $B(1)$ than $B(0)$, relative to $U_l'(p)$. It is immediate that for both $U_h'(p)$ and $U_l'(p)$ to be zero, it must be that $B(1) = B(0) = 0$. Since $\lambda_{1,h} > \lambda_{0,h}$ and $\lambda_{0,l} > \lambda_{1,l}$, simultaneous indifference of both types is only possible if

$$\lambda_{0,l}(v(0) - p) > \lambda_{1,l}(v(0) - p),$$

which can only hold for low values of $p$. Let us define $\widetilde{p}$ as the cutoff value such that the above inequality holds for $p < \widetilde{p}$:

$$\widetilde{p} = \max\left(0, \frac{v(0)\lambda_{0,l} - v(1)\lambda_{1,l}}{\lambda_{0,l} - \lambda_{1,l}}\right).$$

(Note that we define $\widetilde{p} = 0$ if indifference is never possible). We summarize the implications of this reasoning in the following Lemma. Part 1 says that the overlap of bidding supports is possible only for $p < \widetilde{p}$. Part 2 says that in a range where only low type randomizes, value of high type is U-shaped with minimum at $p = \widetilde{p}$. Part 3 says that in a range where only high type randomizes, value of low type is decreasing and hence the low type prefers lower bids.

**Lemma 2** *Let $\lambda_{1,h} > \lambda_{0,h}$ and $\lambda_{0,l} > \lambda_{1,l}$ be given, and let $F_\theta(p)$, $\theta = h, l$, be an atomless equilibrium bidding distribution. Then:*

1. If $F_\theta'(p) > 0$ for $\theta = h, l$, then $p < \widetilde{p}$.
2. If $F_l'(p) > 0$ and $F_h'(p) = 0$, then
$$U_h'(p)\begin{cases} < 0 \text{ for } p < \widetilde{p} \\ > 0 \text{ for } p > \widetilde{p} \end{cases}.$$
3. If $F_h'(p) > 0$ and $F_l'(p) = 0$, then $U_l'(p) < 0$.

**Proof.** Part 1: $F_\theta'(p) > 0$ for $\theta = h, l$ requires that $B(1) = B(0) = 0$, which is only possible if $p < \widetilde{p}$. Part 2: If $F_l'(p) > 0$, then $U_l'(p) = 0$. If $F_h'(p) = 0$, then $U_l'(p) = 0$ implies that $B(1) < 0 < B(0)$ for $p < \widetilde{p}$, and

$B(0) < 0 < B(1)$ for $p > \widetilde{p}$. Since $q_h > q_l$, the result follows. Part 3: If $F_h'(p) > 0$, then $U_h'(p) = 0$. If $F_l'(p) = 0$, then $U_h'(p) = 0$ implies that $B(0) < 0 < B(1)$. Since $q_l < q_h$, the result follows. ■

With this preliminary result in place, we can construct the equilibrium in the two cases separately. Assume first that

$$\left(1 - e^{-\lambda_{1,l}}\right) v(1) > \left(1 - e^{-\lambda_{0,l}}\right) v(0).$$

Starting from $p = 0$, assume that only low types bid for low $p$ and define the low type bidding distribution $F_l(p)$ over some interval $[0, p']$ in such a way that a low type bidder is indifferent throughout, and $F_l(p') = 1$. In particular, a low type bidder must be indifferent between bidding 0 and $p'$, which gives the following condition:

$$\begin{aligned} & q_l e^{-\lambda_{1,h}-\lambda_{1,l}} v(1) + (1 - q_l) e^{-\lambda_{0,h}-\lambda_{0,l}} v(0) \\ = \ & q_l e^{-\lambda_{1,h}} (v(1) - p') + (1 - q_l) e^{-\lambda_{0,h}} (v(0) - p'), \end{aligned}$$

which can be written as

$$q_l e^{-\lambda_{1,h}} \left[\left(1 - e^{-\lambda_{1,l}}\right) v(1) - p'\right] + (1 - q_l) e^{-\lambda_{0,h}} \left[\left(1 - e^{-\lambda_{0,l}}\right) v(0) - p'\right] = 0.$$

For this to hold, one of the terms in square-brackets must be positive and the other one must be negative. Since we assume $\left(1 - e^{-\lambda_{1,l}}\right) v(1) > \left(1 - e^{-\lambda_{0,l}}\right) v(0)$, it must be the term $\left[\left(1 - e^{-\lambda_{1,l}}\right) v(1) - p'\right]$ that is positive. But then, since $q_h > q_l$, this implies that

$$q_h e^{-\lambda_{1,h}} \left[\left(1 - e^{-\lambda_{1,l}}\right) v(1) - p'\right] + (1 - q_h) e^{-\lambda_{0,h}} \left[\left(1 - e^{-\lambda_{0,l}}\right) v(0) - p'\right] > 0$$

or

$$\begin{aligned} & q_h e^{-\lambda_{1,h}-\lambda_{1,l}} v(1) + (1 - q_h) e^{-\lambda_{0,h}-\lambda_{0,l}} v(0) \\ < \ & q_h e^{-\lambda_{1,h}} (v(1) - p') + (1 - q_h) e^{-\lambda_{0,h}} (v(0) - p'), \end{aligned}$$

so that high type prefers strictly to bid $p'$ rather than 0. Combining this with part 2 of Lemma 2, we note that

$$U_h(p') > U_h(p) \text{ for all } p \in [0, p')$$

and so the high type does not have an incentive to deviate to any $p < p'$. It is then easy to finish the construction of the equilibrium by defining $p''$ so that the high type is indiffent between winning for sure by bidding $p''$ and winning if and only there are no other high types by bidding $p'$:

$$\begin{aligned} & q_h \left(v\left(1\right) - p''\right) + \left(1 - q_h\right) \left(v\left(0\right) - p''\right) \\ = \; & q_h e^{-\lambda_{1,h}} \left(v\left(1\right) - p'\right) + \left(1 - q_h\right) e^{-\lambda_{0,h}} \left(v\left(0\right) - p'\right). \end{aligned}$$

It is then clear that there is a unique distribution $F_h\left(p\right)$ such that $F_h\left(p'\right) = 0$, $F_h\left(p''\right) = 1$, and for which high type is indifferent between any $p \in [p', p'']$:

$$U_h\left(p\right) = U_h\left(p'\right) \text{ for all } p \in \left(p', p''\right].$$

We have hence constructed an equilibrium where low type bidding support is $[0, p']$ and high type bidding support is $[p', p'']$.

Assume next that

$$\left(1 - e^{-\lambda_{1,l}}\right) v\left(1\right) < \left(1 - e^{-\lambda_{0,l}}\right) v\left(0\right).$$

If we now try to construct equilibrium as above, high type bidders have an incentive to deviate and bid zero. In particular, take any $p' > 0$ and assume that only low types bid on interval $[0, p']$ and $U_l\left(p\right) = U_l\left(0\right)$ for all $p \leq p'$. Then by part 1 of Lemma 2 we have $U_h\left(p\right) < U_h\left(0\right)$ for all $p \leq p'$, and high types will deviate to zero. It follows that in any equilibrium with atomless distributions, 0 must be contained in the bidding distributions of both types. We can now construct the equilibrium bidding distributions as follows. First, we define the $p''$ such that a high type bidder is indifferent between winning for sure by bidding $p''$ and winning if and only if there are no other bidders by bidding 0:

$$\begin{aligned} & q_h \left(v\left(1\right) - p''\right) + \left(1 - q_h\right) \left(v\left(0\right) - p''\right) \\ = \; & q_h e^{-\lambda_{1,h}} e^{-\lambda_{1,l}} v\left(1\right) + \left(1 - q_h\right) e^{-\lambda_{0,h}} e^{-\lambda_{0,l}} v\left(0\right), \end{aligned}$$

so that

$$p'' = q_h \left(1 - e^{-\lambda_{1,h} - \lambda_{1,l}}\right) v\left(1\right) + \left(1 - q_h\right) \left(1 - e^{-\lambda_{0,h} - \lambda_{0,l}}\right) v\left(0\right).$$

Then we can proceed downwards from $p''$ by defining $F_h(p)$ in such a way that

$$U_h(p) = U_h(p'') \text{ for } p < p'',$$

where $U_h(p)$ is given by (14) with $F_l(p) = 1$. At the same time, by part 3 of Lemma 2, we have $U_l'(p) < 0$ over this range, and at some point $p$ will reach a point $\overrightarrow{p}$ where low types want to become active. This point is pinned down by the condition that low type is indifferent between bidding $\overrightarrow{p}$ and zero:

$$\overrightarrow{p} = \left\{ p : U_l(p) = q_l e^{-\lambda_{1,h}} e^{-\lambda_{1,l}} v(1) + (1 - q_l) e^{-\lambda_{0,h}} e^{-\lambda_{0,l}} v(0) \right\},$$

where $U_l(p)$ is given by (14). We have then two different cases depending on whether $\overrightarrow{p}$ is below or above $\widetilde{p}$.

Case 1: $\overrightarrow{p} \leq \widetilde{p}$. We can define $F_l(p)$ and $F_h(p)$ below $\overrightarrow{p}$ so that both types are indifferent for all $p \leq \overrightarrow{p}$, that is, $U_l(p) = U_l(\overrightarrow{p})$ and $U_h(p) = U_h(\overrightarrow{p})$. As a result, we end up with an equilibrium, where the low type bidding support is $[0, \overrightarrow{p}]$ and high type bidding support is $[0, p'']$.

Case 2: $\overrightarrow{p} > \widetilde{p}$. By part 1 of Lemma 2, we cannot have indifference simultaneously for both types above $\widetilde{p}$, and hence the same structure as in Case 1 is not possible. Instead, we will construct an interval $[\overleftarrow{p}, \overrightarrow{p}]$ containing $\widetilde{p}$, where only low type is active: define $F_l(p)$ within $[\overleftarrow{p}, \overrightarrow{p}]$ such that $U_l(p) = U_l(\overrightarrow{p})$ for $p \in (\overleftarrow{p}, \overrightarrow{p})$, where $U_l(p)$ is given by (14) with $F_h(p) = F_h(\overrightarrow{p})$. By part 2 of Lemma 2, $U_h'(p) > 0$ for $p > \widetilde{p}$ and $U_h'(p) < 0$ for $p < \widetilde{p}$. We then define $\overleftarrow{p}$ as the point where $U_h(\overleftarrow{p}) = U_h(\overrightarrow{p})$. Since $\overleftarrow{p} < \widetilde{p}$, we can define $F_l(p)$ and $F_h(p)$ below $\overleftarrow{p}$ so that both types are indifferent, that is $U_l(p) = U_l(\overleftarrow{p})$ and $U_h(p) = U_h(\overleftarrow{p})$ for all $p \leq \overleftarrow{p}$. As a result we have an equilibrium, where the low type bidding support is $[0, \overrightarrow{p}]$ and high type bidding support is disconnected and given by $[0, \overleftarrow{p}] \cup [\overrightarrow{p}, p'']$.

## 8.2 Other proofs

**Proof of Proposition.** 3 i) Second Price Auction

We use the following notation for the total support of bids and the supports of the two different bid distributions:

$$\text{supp}F(\cdot) = \text{supp}F_h(\cdot) \cup \text{supp}F_l(\cdot).$$

A bid $p \in \text{supp} F(\cdot)$ is said to be non-pooling if $\Pr\{b_h = p\} = \Pr\{b_l = p\} = 0$. We denote the event where a bidder of type $\theta$ is tied for the highest bid at $p$ by $B_\theta(p)$, and the event where the bidder of type $\theta$ is tied and wins (loses) the auction at $p$ by $B_\theta^{win}(p)$ $(B_\theta^{lose}(p))$.

We start by constructing an atomless distribution $F_h(p)$ of bids for the high-type bidders with the property that $p \in \text{supp} F_h(\cdot) \Rightarrow p > \underline{p}(n)$. Since $F_h(p)$ is by assumption an atomless distribution, the informational content in $B_h(p)$, $B_h^{win}(p)$, and $B_h^{lose}(p)$ is the same.[11] For any $p \in \text{supp} F_h(\cdot)$, we compute:

$$\begin{aligned}
\Pr\{\omega &= 1 \,|B_h(p)\} = \Pr\{\omega = 1 \,\big|B_h^{win}(p)\} = \Pr\{\omega = 1 \,\big|B_h^{win}(p)\} \\
: &= q^{SPA}(p) = \frac{\alpha^2(1-\alpha(1-F_h(p)))^{N-2}}{\alpha^2(1-\alpha(1-F_h(p)))^{N-2} + \beta^2(1-\beta(1-F_h(p)))^{N-2}}.
\end{aligned}$$

Whenever the value of the object conditional on $B_h^{win}(p)$ and $B_h^{lose}(p)$ is the same, the symmetric equilibrium bid must equal the expected value by the same reasoning as in Milgrom & Weber (1982). Hence we can solve for $F_h(p)$ from:

$$\begin{aligned}
p &= q^{SPA}(p)\, v(1) + (1 - q^{SPA}(p)) v(0) \\
&= \frac{\alpha^2(1-\alpha(1-F_h(p)))^{N-2} v(1) + \beta^2(1-\beta(1-F_h(p)))^{N-2} v(0)}{\alpha^2(1-\alpha(1-F_h(p)))^{N-2} + \beta^2(1-\beta(1-F_h(p)))^{N-2}}.
\end{aligned}$$

To show uniqueness, start by observing that if $p$ is non-pooling, then $p \notin \text{supp} F_h(\cdot) \cap \text{supp} F_l(\cdot)$. This follows from the simple observation that

$$\mathbb{E}[v \,|B_h(p)] \neq \mathbb{E}[v \,|B_l(p)],$$

and the fact that at non-pooling bids winning or losing the auction conveys no new information.

The next general observation is that it is not possible to have an equilibrium bid $p$ such that $\Pr\{b_h = p\} > 0$ and $\Pr\{b_l = p\} = 0$. To see this, consider the two cases: $F_l(p) = 0$ and $F_l(p) > 0$. In the first case, we must have

$$p = \mathbb{E}\left[v \,\middle|\, N^h = n\right] \tag{16}$$

[11] This follows from the fact that the probability of tying with more than a single other bidder is negligible in comparison to tying with exactly one other bidder.

contradicting the oprimality of $F_l(\cdot)$. In the second case,

$$\mathbb{E}[v\,|B_h(p)] > \mathbb{E}[v\,\big|B_h^{win}(p)]$$

by affiliation. In other words, we have a negative rationing effect. Since at any equilibrium bid, the expected payoff is non-negative, this implies that $\theta = h$ has a profitable deviation to a higher price similar to the informal auction.

Let $p_{\min} = \min\{p\,|p \in \text{supp}F(\cdot)\}$. It is not possible to have $F_h(p_{\min}) > 0$ and $F_l(p_{\min}) > 0$. To see this, assume to the contrary and notice that by tying for the highest bid at $p_{\min}$, each bidder knows that they have tied with $n-1$ other bidders. Hence there cannot be any rationing effects and

$$p_{\min} = \mathbb{E}[v\,|B_h(p)] = \mathbb{E}[v\,|B_l(p)].$$

By affiliation, the last two expected values are different and hence we have a contradiction.

The same observation that led to equation (16) shows that it is not possible to have nonpooling $p \in \text{supp}F_h(\cdot)$ such that $F_l(p) = 0$. Hence we conclude that $p_{\min} \in \text{supp}F_l(\cdot)$ and $p_{\min} \notin \text{supp}F_h(\cdot)$. But this impies again that for and $p < \min_p\{\text{supp}F_h(\cdot)\}$,

$$p = \mathbb{E}\left[v \,\middle|\, N^h = 0\right],$$

and $p_{\min}$ is thus a pooling bid for low type bidders.

We show next that high type bidders cannot pool. Conisder first pooling at bid $p_{h,\min} = \min_p\{\text{supp}F_h(\cdot)\}$. To see that this is not possible, notice that the rationing effect at $p_{h,\min}$ is negative and hence the bidders will have a profitable deviation to a higher price. Consider then the lowest pooling bid $p_{h,\text{pool}}$ by the high type bidders. This implies that $p_{h,\text{pool}}$ is a pooling bid for the low type bidders as well. If both types of bidders bid in equilibrium at $p < p_{h,\text{pool}}$, then

$$\mathbb{E}[v\,|B_\theta(p_{h,\text{pool}})] < \mathbb{E}[v\,\big|B_\theta^{win}(p_{h,\text{pool}})] \text{ for } \theta = h, l.$$

By the previous argument, there must be non-pooling bids $p \in \text{supp}F_h(\cdot)$ with $p < p_{h,\text{pool}}$. Consider a sequence (of non-pooling) $p_n \in \text{supp}F_h(\cdot)$ with

$$\lim_{n\to\infty} p_n = \sup\{p \in \text{supp}F_h(\cdot)\,|p < p_{h,\text{pool}}\}.$$

Since

$$\mathbb{E}[v \,|B_h \left(p_{h,\text{pool}}\right)] < \mathbb{E}[v \left|B_h^{win} \left(p_{h,\text{pool}}\right)\right.],$$

we have by construction

$$\lim_{n\to\infty} \mathbb{E}[v \,|B_h \left(p_n\right)] > \mathbb{E}[v \left|B_h^{win} \left(p_{h,\text{pool}}\right)\right.].$$

Since $p_n$ is non-pooling, we have

$$\begin{aligned}\lim_{n\to\infty} p_n &= \lim_{n\to\infty} \mathbb{E}[v \,|B_h \left(p_n\right)] > \mathbb{E}[v \left|B_h^{win} \left(p_{h,\text{pool}}\right)\right.] \\ &> \mathbb{E}[v \left|B_l^{win} \left(p_{h,\text{pool}}\right)\right.].\end{aligned}$$

But this contradicts the optimality of $F_l\left(\cdot\right)$.

Finally notice that we cannot have a pooling price for the low types strictly above $p_{\min}$. If such a bid $p'$ existed, then we have shown that $F_h\left(p'\right) > 0$. But then we can repeat the above argument to find a sequence of non-pooling $p_n \to p \leq p'$ with the property that $F_h\left(p\right) = F_h\left(p'\right)$. Along this sequence, the bids converge to a value strictly above $\mathbb{E}[v \,|B_l^{win}\left(p'\right)]$ contradicting the optimality of $F_l\left(\cdot\right)$.

The non-atomic bid distribution by the high types constructed in the first part of this proof is easily seen to be unique since for all non-pooling bids $p$, we must have

$$\mathbb{E}[v \,|B_h \left(p\right)].$$

ii) First-price auction. Wang (1991) proves that the strategy profile in the proposition constitutes a symmetric equilibrium. The uniqueness follows from the following steps. The equilibrium bid of $p_{\min}$ must result in a zero expected payoff. This is immediate if there is no atom at $p_{\min}$ and if an atom exists, zero expected payoff results from the fact that there are no profitable deviations. Hence if there is an atom at $p_{\min}$, it must be in the support of the low type bidders only.

If there are no atoms, then only the low type can have $p_{\min}$ in the support. To see this, note that by bidding the highest bid in supp$F_l\left(\cdot\right)$, the high type can secure a positive payoff. But $F_l\left(p_{\min}\right) = 0$ is not possible since nonnegative payoff for low types implies

$$p_{\min} \geq \mathbb{E}\left[v \,\middle|\, N^h = l\right],$$

and then at bid $p_{\min} + \varepsilon$, the low type makes a loss.

Hence we conclude that the low type must have an atom on $p_{\min}$. Since the high type does not have an atom at $p_{\min}$, there must be an interval $(p_{\min}, p_{\min} + \delta) \subset \text{supp} F_h(\cdot)$ and $(p_{\min}, p_{\min} + \delta) \cap \text{supp} F_l(\cdot) = \emptyset$. We can use the indifference condition of the high type to derife the bid distribution on this interval. By affiliation, we see that the low type's payoff is decreasing over this interval. Hence we conclude that the only symmetric equilibrium is that given in the proposition. ■

**Proof of Lemma 1.** If $\pi_l^*(\pi_h) = 0$, it is easy to check that (11) holds since $W(0, \pi_h)$ is concave in $\pi_h$. If $\pi_l^*(\pi_h) > 0$, the first order condition for $\pi_l$ must hold:

$$0 = (1-\alpha) q \left(v(1) e^{-\alpha\pi_h - (1-\alpha)\pi_l^*(\pi_h)} - c\right) \tag{17}$$
$$+ (1-\beta)(1-q)\left(v(0) e^{-\beta\pi_h - (1-\beta)\pi_l^*(\pi_h)} - c\right). \tag{18}$$

If $\pi_h = \pi_h^{opt}$, then $\pi_l^*(\pi_h) = \pi_l^{opt}$ and (18) is satisfied since

$$v(1) e^{-\alpha\pi_h^{opt} - (1-\alpha)\pi_l^{opt}(\pi_h)} - c = v(0) e^{-\beta\pi_h^{opt} - (1-\beta)\pi_l^{opt}} - c = 0.$$

If $\pi_h > (<) \pi_h^{opt}$, we note that since $\alpha > \beta$, (18) can only hold if

$$v(1) e^{-\alpha\pi_h - (1-\alpha)\pi_l^*(\pi_h)} - c < (>) 0 < (>) v(0) e^{-\beta\pi_h - (1-\beta)\pi_l^*(\pi_h)} - c.$$

But then, since $\alpha > (1-\alpha)$ and $\beta < (1-\beta)$, we have

$$\frac{\partial W(\pi_l^*(\pi_h), \pi_h)}{\pi_h} = \alpha q \left(v(1) e^{-\alpha\pi_h - (1-\alpha)\pi_l^*(\pi_h)} - c\right)$$
$$+ \beta(1-q)\left(v(0) e^{-\beta\pi_h - (1-\beta)\pi_l^*(\pi_h)} - c\right) < (>) 0.$$

The result then follows from the envelope theorem. ■

**Proof of Proposition 4.** Let $c \geq \bar{c}$ and assume that entry rates are given by $\pi_h = \pi_h^{opt} > 0$ and $\pi_l = \pi_l^{opt} = 0$. It is easy to check that in all the auction forms, the payoffs to the high and low type bidders are given by $V_h^0\left(\pi_h^{opt}, 0\right) = c$ and $V_l^0\left(\pi_h^{opt}, 0\right) < c$, respectively. Hence, socially optimal entry profile is an equilibrium. It is also straight forward to show that no other equilibria exist. (The only nontrivial case is the informal SPA, where equilibria entry profiles are not on reaction curve $\pi_l^*(\pi_h)$. Even there we can show using the same technique as in the proof of Proposition 6 that the indifference curves $V_l^{SPA}(\pi_h, \pi_l) = c$ and $V_h^{SPA}(\pi_h, \pi_l) = c$ cannot cross if $c > \overline{c}$.) ■

**Proof of Proposition 5.** Suppose that only high types enter and that there is an atom in the bidding distribution at some $b > 0$. Since entry rate is higher in state $\omega = 1$, winning with pooled bid is more likely in state $\omega = 0$ than in state $\omega = 1$ and hence winning is bad news about $v$. But then deviation slightly above $b$ is profitable since it increases chance of winning and avoids winner's curse. Pooling cannot exist.

Suppose that both types enter. Consider the possibility of pooling by low type bidders. If the low types pool on a bid $b \geq v(0)$, then their expected payoff in state $\omega = 0$ from the pooling bid is non-positive. Since the expected payoff in equilibrium is $c$, the payoff in $\omega = 1$ is strictly positive. Bidders with high signals think that $\omega = 1$ is more likely than the low types and as a consequence a bid of $b$ gives the high types a payoff that is strictly larger than the payoff to the low types. This is a contradiction with interim entry decisions since in equilibrium, both types must get an expected payoff of $c$ in the bidding stage.

Pooling at $b < v(0)$ is not possible for either of the types since in this case the payoff from winning the auction is strictly positive in both states and hence it is optimal to deviate upwards from the assumed pooling bid. We conclude that there cannot be equilibria with an atom, and hence the atomless equilibrium of Proposition 1 is the only candidate for the bidding stage equilibrium. As stated in Proposition 1, zero is in the support of the low type. The upper bound $b_h$ follows from the break-even condition of a high type bidder who by bidding the highest bid in the support wins with probability 1. For an arbitrary entry profile $(\pi_h, \pi_l)$, denote the expected payoff of a player with signal $\theta$ in the atomless bidding equilibrium by $V_\theta^{FPA}(\pi_h, \pi_l)$, $\theta = h, l$.

Let us now consider the entry stage. We want to show that for each $c \in (0, \overline{c})$ there is a unique $(\pi_h, \pi_l)$ such that $V_h^{FPA}(\pi_l, \pi_h) = V_l^{FPA}(\pi_l, \pi_h) = c$. To do that, we will show that the following properties hold:

P1. $V_h^{FPA}(\pi_l, \pi_h)$ and $V_l^{FPA}(\pi_l, \pi_h)$ are continuous in $(\pi_l, \pi_h)$

P2. $V_h^{FPA}(\pi_l, \pi_h)$ and $V_l^{FPA}(\pi_l, \pi_h)$ are strictly decreasing in $\pi_l$

P3. There are unique points $\pi_l^{h*} > \pi_l^{l*} > 0$ and $\pi_h^{l*} > \pi_h^{h*} > 0$ such that

$$\begin{aligned} V_h^{FPA}\left(\pi_l^{h*}, 0\right) &= V_l^{FPA}\left(\pi_l^{l*}, 0\right) = c, \\ V_h^{FPA}\left(0, \pi_h^{h*}\right) &= V_l^{FPA}\left(0, \pi_h^{l*}\right) = c. \end{aligned}$$

P4. For every $(\pi_l, \pi_h)$ such that $V_h^{FPA}(\pi_l, \pi_h) = V_l^{FPA}(\pi_l, \pi_h) = c$, we have

$$\frac{\partial V_l^{FPA}(\pi_l, \pi_h)}{\partial \pi_l} \frac{\partial V_h^{FPA}(\pi_l, \pi_h)}{\partial \pi_h} > \frac{\partial V_h^{FPA}(\pi_l, \pi_h)}{\partial \pi_l} \frac{\partial V_l^{FPA}(\pi_l, \pi_h)}{\partial \pi_h}. \tag{19}$$

To see that these properties implie a unique $(\pi_h, \pi_l)$ such that $V_h^{FPA}(\pi_l, \pi_h) = V_l^{FPA}(\pi_l, \pi_h) = c$, note first that P1 - P3 imply that there are continuous and decreasing curves $\pi_l^{h*}(\pi_h)$ and $\pi_l^{l*}(\pi_h)$ such that

$$\pi_l^{h*}(0) = \pi_l^{h*},\ \pi_l^{l*}(0) = \pi_l^{l*},\ \pi_l^{h*}\left(\pi_h^{h*}\right) = 0,\ \pi_l^{l*}\left(\pi_h^{l*}\right) = 0$$

and

$$\begin{aligned} V_l^{FPA}\left(\pi_l^{l*}(\pi_h), \pi_h\right) &= c \text{ for all } \pi_h \in \left[0, \pi_h^{l*}\right], \\ V_h^{FPA}\left(\pi_l^{h*}(\pi_h), \pi_h\right) &= c \text{ for all } \pi_h \in \left[0, \pi_h^{h*}\right], \end{aligned}$$

and that these curves must cross each other at least once at some $(\pi_h, \pi_l)$ where $V_h^{FPA}(\pi_l, \pi_h) = V_l^{FPA}(\pi_l, \pi_h) = c$. We will now show that they cannot cross more than once. Suppose that the curves cross at $(\pi_h, \pi_l)$. Then totally differentiate $V_h^{FPA}(\pi_l, \pi_h)$, and consider an infinitesimal movement along $\pi_l^{h*}(\pi_h)$ in the direction where $d\pi_h > 0$. Along that curve

$$\frac{\partial V_h^{FPA}(\pi_l, \pi_h)}{\partial \pi_h} d\pi_h + \frac{\partial V_h^{FPA}(\pi_l, \pi_h)}{\partial \pi_l} d\pi_l = 0$$

so that

$$\frac{\partial V_h^{FPA}(\pi_l, \pi_h)}{\partial \pi_h} \Big/ \frac{\partial V_h^{FPA}(\pi_l, \pi_h)}{\partial \pi_l} = -\frac{d\pi_l}{d\pi_h}. \tag{20}$$

By Property P4, we have

$$\frac{\partial V_l^{FPA}(\pi_l, \pi_h)}{\partial \pi_l} \frac{\partial V_h^{FPA}(\pi_l, \pi_h)}{\partial \pi_h} > \frac{\partial V_h^{FPA}(\pi_l, \pi_h)}{\partial \pi_l} \frac{\partial V_l^{FPA}(\pi_l, \pi_h)}{\partial \pi_h}.$$

Since by Property P2 we have $\frac{\partial V_l^{FPA}(\pi_l,\pi_h)}{\partial \pi_l} < 0$ and $\frac{\partial V_h^{FPA}(\pi_l,\pi_h)}{\partial \pi_l} < 0$, this is equivalent to

$$\frac{\partial V_h^{FPA}(\pi_l,\pi_h)}{\partial \pi_h} / \frac{\partial V_h^{FPA}(\pi_l,\pi_h)}{\partial \pi_l} > \frac{\partial V_l^{FPA}(\pi_l,\pi_h)}{\partial \pi_h} / \frac{\partial V_l^{FPA}(\pi_l,\pi_h)}{\partial \pi_l}$$

so that combining with (20) we have

$$\frac{\partial V_l^{FPA}(\pi_l,\pi_h)}{\partial \pi_h} / \frac{\partial V_l^{FPA}(\pi_l,\pi_h)}{\partial \pi_l} < -\frac{d\pi_l}{d\pi_h}.$$

But since $\frac{\partial V_l^{FPA}(\pi_l,\pi_h)}{\partial \pi_l} < 0$ and $d\pi_h > 0$, this means that

$$\frac{\partial V_l}{\partial \pi_h} d\pi_h + \frac{\partial V_l}{\partial \pi_l} d\pi_l > 0,$$

and hence in any crossing point $\pi_l^{h*}(\pi_h)$ crosses $\pi_l^{l*}(\pi_h)$ from above when going in the direction of increasing $\pi_h$. Since $\pi_l^{h*}(\pi_h)$ and $\pi_l^{l*}(\pi_h)$ are continuous curves, this implies that there cannot be more than one crossing point. There is a unique $(\pi_h, \pi_l)$ where $V_h^{FPA}(\pi_l,\pi_h) = V_l^{FPA}(\pi_l,\pi_h) = c$.

The final step is to check that properties P1 - P4 hold. In both of the two cases in Proposition 1, the low type payoff can be written as:

$$V_l^{FPA}(\pi_l,\pi_h) = V_l^0(\pi_l,\pi_h).$$

This is clearly a continuous function and it is easy to show that

$$\frac{\partial V_l^{FPA}(\pi_l,\pi_h)}{\partial \pi_l} < 0, \; \frac{\partial V_l^{FPA}(\pi_l,\pi_h)}{\partial \pi_h} < 0.$$

To derive an expression for $V_h^{FPA}(\pi_l,\pi_h)$, we need to consider separately the two cases of Proposition 1. Consider first the case, where 0 is in the bidding supports of both types. In that case we have also

$$V_h^{FPA}(\pi_l,\pi_h) = V_h^0(\pi_l,\pi_h),$$

which is a continuous function and

$$\frac{\partial V_h^{FPA}(\pi_l,\pi_h)}{\partial \pi_l} < 0, \; \frac{\partial V_h^{FPA}(\pi_l,\pi_h)}{\partial \pi_h} < 0.$$

It is also straightforward to show that

$$\frac{\partial V_l^0(\pi_l,\pi_h)}{\partial \pi_l}\frac{\partial V_h^0(\pi_l,\pi_h)}{\partial \pi_h} > \frac{\partial V_h^0(\pi_l,\pi_h)}{\partial \pi_l}\frac{\partial V_l^0(\pi_l,\pi_h)}{\partial \pi_h}$$

so that also property P4 holds when $V_h^{FPA}(\pi_l,\pi_h) = V_h^0(\pi_l,\pi_h) = c$ and $V_l^{FPA}(\pi_l,\pi_h) = V_l^0(\pi_l,\pi_h) = c$.

Consider next the second case, where the bidding supports of the two types do not overlap. Let $p'(\pi_h,\pi_l)$ denote the only common point in the two supports, and note that this is a function of the entry rates. Since $p'(\pi_h,\pi_l)$ is in the support of both type of players' bidding strategy, we can write the payoffs of the two types as

$$V_\theta^{FPA}(\pi_h,\pi_l) = q_\theta w_1(\pi_h,\pi_l) + (1-q_\theta)\, w_0(\pi_h,\pi_l)\,,\ \theta = h,l,$$

where

$$q_h = \frac{\alpha q}{\alpha q + \beta(1-q)} \text{ and } q_l = \frac{\beta(1-q)}{\alpha q + \beta(1-q)}$$

are the posteriors of the two types, and $w_\omega(\pi_h,\pi_l)$, $\omega = 0,1$, is the payoff from bidding $p'(\pi_h,\pi_l)$ conditional on state. By bidding $p'(\pi_h,\pi_l)$, a player wins if and only if there are no high type entrants, and therefore we may write the state conditional payoffs from this strategy as:

$$\begin{aligned} w_1(\pi_l,\pi_h) &= e^{-\alpha\pi_h}\left(v(1) - p'(\pi_l,\pi_h)\right), \\ w_0(\pi_l,\pi_h) &= e^{-\beta\pi_h}\left(v(0) - p'(\pi_l,\pi_h)\right), \end{aligned} \tag{21}$$

where $p'(\pi_l,\pi_h)$ can be solved from the indifference condition for a low type bidder between bidding 0 and bidding $p'(\pi_l,\pi_h)$:

$$\begin{aligned} & q_l e^{-\alpha\pi_h-(1-\alpha)\pi_l} v(1) + (1-q_l)\, e^{-\beta\pi_h-(1-\beta)\pi_l} v(0) \\ = \ & q_l e^{-\alpha\pi_h}\left(v(1) - p'(\pi_l,\pi_h)\right) + (1-q_l)\, e^{-\beta\pi_h}\left(v(0) - p'(\pi_l,\pi_h)\right), \end{aligned}$$

which gives

$$p'(\pi_l,\pi_h) = \frac{q_l e^{-\alpha\pi_h}\left(1-e^{-(1-\alpha)\pi_l}\right) v(1) + (1-q_l)\, e^{-\beta\pi_h}\left(1-e^{-(1-\beta)\pi_l}\right) v(0)}{q_l e^{-\alpha\pi_h} v(1) + (1-q_l)\, e^{-\beta\pi_h} v(0)}.$$

This is continuous in $(\pi_h, \pi_l)$, and so are $V_\theta^{FPA}(\pi_h, \pi_l)$, $\theta = h, l$. From this we can also show that

$$\begin{aligned} \frac{\partial p'(\pi_l, \pi_h)}{\partial \pi_l} &> 0, \\ \frac{\partial p'(\pi_l, \pi_h)}{\partial \pi_h} &< 0. \end{aligned}$$

Differentiating $w_\omega(\pi_l, \pi_h)$, we have

$$\begin{aligned} \frac{\partial w_1(\pi_l, \pi_h)}{\partial \pi_h} &= -e^{-\alpha \pi_h}\left(\alpha\left(v(1) - p'(\pi_l, \pi_h)\right) + \frac{\partial p'(\pi_l, \pi_h)}{\partial \pi_h}\right), \\ \frac{\partial w_0(\pi_l, \pi_h)}{\partial \pi_h} &= -e^{-\beta \pi_h}\left(\beta\left(v(0) - p'(\pi_l, \pi_h)\right) + \frac{\partial p'(\pi_l, \pi_h)}{\partial \pi_h}\right), \\ \frac{\partial w_1(\pi_l, \pi_h)}{\partial \pi_l} &= -e^{-\alpha \pi_h}\frac{\partial p'(\pi_l, \pi_h)}{\partial \pi_l}, \\ \frac{\partial w_0(\pi_l, \pi_h)}{\partial \pi_l} &= -e^{-\beta \pi_h}\frac{\partial p'(\pi_l, \pi_h)}{\partial \pi_l}, \end{aligned}$$

and we see immediately that

$$\frac{\partial w_1(\pi_l, \pi_h)}{\partial \pi_l} < 0, \ \frac{\partial w_0(\pi_l, \pi_h)}{\partial \pi_l} < 0, \tag{22}$$

which implies that also in this case we have

$$\frac{\partial V_h^{FPA}(\pi_l, \pi_h)}{\partial \pi_l} < 0.$$

To check property P3, note first that since $V_l^{FPA}(\pi_h, \pi_l) = V_l^0(\pi_h, \pi_l)$ for all $(\pi_h, \pi_l)$, we have

$$\begin{aligned} V_l^{FPA}(\pi_h, 0) &= V_l^0(\pi_h, 0) \text{ and} \\ V_l^{FPA}(0, \pi_l) &= V_l^0(0, \pi_l). \end{aligned}$$

For the high type, it is easy to show that if there are no low types, the payoff of high type is equal to the social contribution and

$$V_h^{FPA}(\pi_h, 0) = V_h^0(\pi_h, 0).$$

Moreover, a high type always gets at least the social contribution so that

$$V_h^{FPA}(0,\pi_l) \geq V_h^0(0,\pi_l).$$

It is easy to check using these that P3 holds whenever $c < \overline{c}$.

Finally, we consider property P4 in the case where bidding supports do not overlap. First, we can write (19) as:

$$\begin{aligned} & \left(q_l \frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_l} + (1-q_l)\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_l}\right)\left(q_h \frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_h} + (1-q_h)\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_h}\right) \\ > & \left(q_h \frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_l} + (1-q_h)\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_l}\right)\left(q_l \frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_h} + (1-q_l)\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_h}\right). \end{aligned}$$

By straightforward algebra, this can be written as

$$\begin{aligned} & q_l\left(\frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_l}\right)\left(\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_h}\right) + q_h\left(\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_l}\right)\left(\frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_h}\right) \\ > & q_h\left(\frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_l}\right)\left(\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_h}\right) + q_l\left(\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_l}\right)\left(\frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_h}\right). \end{aligned}$$

Since $q_h > q_l$, we see that (19) is equivalent to

$$\frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_h}\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_l} > \frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_l}\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_h}. \tag{23}$$

Differentiating (21), we have:

$$\begin{aligned} & \frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_h}\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_l} \\ = & e^{-(\alpha-\beta)\pi_h}\left(\alpha\left(v(1) - p'(\pi_l,\pi_h)\right) + \frac{\partial p'(\pi_l,\pi_h)}{\partial \pi_h}\right)\frac{\partial p'(\pi_l,\pi_h)}{\partial \pi_l}, \\ & \frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_h}\frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_l} \\ = & e^{-(\alpha-\beta)\pi_h}\left(\beta\left(v(0) - p'(\pi_l,\pi_h)\right) + \frac{\partial p'(\pi_l,\pi_h)}{\partial \pi_h}\right)\frac{\partial p'(\pi_l,\pi_h)}{\partial \pi_l}, \end{aligned}$$

from which we can check that (23) holds, which is equivalent to (19). ■

**Proof of Proposition 6.** We need to show that for each $c \in (0,\overline{c})$ there is a unique $(\pi_h,\pi_l)$ such that $V_h^{SPA}(\pi_l,\pi_h) = V_l^{SPA}(\pi_l,\pi_h) = c$, where

$V_{\theta}^{SPA}(\pi_l,\pi_h)$ is the payoff of an entrant type $\theta$ in the bidder optimal equilibrium given entry profile $(\pi_h,\pi_l)$. Following the proof of Proposition 5, we need to show that the four properties P1 - P4 hold for $V_h^{SPA}(\pi_l,\pi_h)$ and $V_l^{SPA}(\pi_l,\pi_h)$.

First, since $\underline{v}(\pi_l,\pi_h)$ is lower than expected value of the object conditonal on winning rationing, both types of bidders get a strictly positive payoff in the auction even if other low type bidders are present. Therefore $V_{\theta}^{SPA}(\pi_l,\pi_h) > V_{\theta}^{0}(\pi_l,\pi_h)$ for $\theta = h,l$, and therefore $\pi_h^{SPA} > \pi_l^{*}\left(\pi_l^{SPA}\right)$ and $\pi_l^{SPA} > \pi_l^{*}\left(\pi_h^{SPA}\right)$.

We can derive the exact expressions for $V_{\theta}^{SPA}(\pi_l,\pi_h)$ by considering the payoff from bidding $p' \in (\underline{v}(\pi_l,\pi_h), b')$, where $b'$ is defined in (6). With this strategy, a bidder wins if and only if there are no (other) high type bidders, and pays either $\underline{v}(\pi_l,\pi_h)$ or 0 depending on whether there are (other) low type bidders or not. Since $\underline{v}(\pi_l,\pi_h)$ is defined so that a low type bidder is indifferent between bidding $\underline{v}(\pi_l,\pi_h)$ and overbidding, bidding $p'$ is clearly a weak best response to both type of players (for high type it results in exacly the same allocation and payment as bidding $b'$). Therefore, we may use that strategy to write the payoffs as follows:

$$\begin{aligned} V_h^{SPA}(\pi_l,\pi_h) &= q_h w_1(\pi_l,\pi_h) + (1-q_h)\, w_0(\pi_l,\pi_h), \\ V_l^{SPA}(\pi_l,\pi_h) &= q_l w_1(\pi_l,\pi_h) + (1-q_l)\, w_0(\pi_l,\pi_h), \end{aligned} \tag{24}$$

where $w_{\omega}(\pi_l,\pi_h)$, $\omega = 0,1$, is the payoff from bidding $p'$ conditional on state $\omega$:

$$\begin{aligned} w_1(\pi_l,\pi_h) &= e^{-\alpha\pi_h}\left[v(1) - \left(1 - e^{-(1-\alpha)\pi_l}\right)\underline{v}(\pi_l,\pi_h)\right], \\ w_0(\pi_l,\pi_h) &= e^{-\beta\pi_h}\left[v(0) - \left(1 - e^{-(1-\beta)\pi_l}\right)\underline{v}(\pi_l,\pi_h)\right], \end{aligned}$$

where

$$\underline{v}(\pi_l,\pi_h) = q^{lose}(\pi_l,\pi_h)\, v(1) + \left(1 - q^{lose}(\pi_l,\pi_h)\right) v(0).$$

Clearly $V_{\theta}^{SPA}(\pi_l,\pi_h)$ are continuous so that P1 holds. We have

$$\frac{\partial V_{\theta}^{SPA}(\pi_l,\pi_h)}{\partial \pi_l} = q_{\theta}\frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_l} + (1-q_{\theta})\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_l},\ \theta = h,l,$$

where

$$\frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_l} = -e^{-\alpha\pi_h}\left[e^{-(1-\alpha)\pi_l}(1-\alpha)\underline{v}(\pi_l,\pi_h)+\left(1-e^{-(1-\alpha)\pi_l}\right)\frac{\partial \underline{v}(\pi_l,\pi_h)}{\partial \pi_l}\right],$$
$$\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_l} = -e^{-\beta\pi_h}\left[e^{-(1-\beta)\pi_l}(1-\beta)\underline{v}(\pi_l,\pi_h)+\left(1-e^{-(1-\beta)\pi_l}\right)\frac{\partial \underline{v}(\pi_l,\pi_h)}{\partial \pi_l}\right],$$

We can show that

$$\frac{\partial q^{lose}(\pi_h,\pi_l)}{\partial \pi_l} > 0,$$

so that

$$\frac{\partial \underline{v}(\pi_l,\pi_h)}{\partial \pi_l} > 0.$$

From these expressions, we see immediately that:

$$\frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_l} < 0, \ \frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_l} < 0$$

and hence

$$\frac{\partial V_h^{SPA}(\pi_l,\pi_h)}{\partial \pi_l} < 0, \ \frac{\partial V_l^{SPA}(\pi_l,\pi_h)}{\partial \pi_l} < 0$$

and P2 holds.

Consider now property P3. If $\pi_h = 0$, we have clearly $w_1(0,\pi_l) > w_0(0,\pi_l)$, so that $V_h^{SPA}(0,\pi_l) > V_l^{SPA}(0,\pi_l)$. We also have for $V_\theta^{SPA}(0,0) > c$, $V_\theta^{SPA}(0,\pi_l) < c$ for $\pi_l$ large enough, and $V_\theta^{SPA}(0,\pi_l)$ are decreasing in $\pi_l$ for $\theta = h, l$. It follows that there are $\pi_l^{h*} > \pi_l^{l*} > 0$ such that

$$V_h^{SPA}\left(\pi_l^{h*},0\right) = V_l^{SPA}\left(\pi_l^{l*},0\right) = c.$$

If $\pi_l = 0$, we have $V_\theta^{SPA}(0,\pi_h) = V_\theta^0(0,\pi_h)$ for $\theta = h, l$, so there are $\pi_h^{l*} > \pi_h^{h*} > 0$ such that

$$V_h^{SPA}\left(0,\pi_h^{h*}\right) = V_l^{SPA}\left(0,\pi_h^{l*}\right) = c,$$

so that P3 holds.

Finally, consider property P4. As a first step, consider the partial of $w_1(\pi_l, \pi_h)$ with respect to $\pi_h$:

$$
\begin{aligned}
&\frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_h} \\
=\;& -e^{-\alpha\pi_h}\left[\alpha\left(v(1)-\left(1-e^{-(1-\alpha)\pi_l}\right)\underline{v}(\pi_l,\pi_h)\right)+\left(1-e^{-(1-\alpha)\pi_l}\right)\frac{\partial \underline{v}(\pi_l,\pi_h)}{\partial \pi_h}\right] \\
=\;& -\alpha e^{-\alpha\pi_h}[v(1)-\left(1-e^{-(1-\alpha)\pi_l}\right)\underline{v}(\pi_l,\pi_h) \\
-\;& \left(1-e^{-(1-\alpha)\pi_l}\right)\frac{\alpha-\beta}{\alpha}q^{lose}(\pi_l,\pi_h)(v(1)-\underline{v}(\pi_l,\pi_h))] \\
=\;& -\alpha e^{-\alpha\pi_h}\left[v(1)-\left(1-e^{-(1-\alpha)\pi_l}\right)\left(\underline{v}(\pi_l,\pi_h)+\frac{\alpha-\beta}{\alpha}q^{lose}(\pi_l,\pi_h)(v(1)-\underline{v}(\pi_l,\pi_h))\right)\right].
\end{aligned}
$$

Noting that

$$
\underline{v}(\pi_l,\pi_h)+\frac{\alpha-\beta}{\alpha}q^{lose}(\pi_l,\pi_h)(v(1)-\underline{v}(\pi_l,\pi_h))<v(1),
$$

it follows that

$$
\frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_h}<0 \text{ for any } (\pi_h,\pi_l).
$$

When $V_h^{SPA}(\pi_l,\pi_h) = V_l^{SPA}(\pi_l,\pi_h) = c$, we must have $w_1(\pi_l,\pi_h) = w_0(\pi_l,\pi_h) = c$ by (24). Let us evaluate the following term containing the partial derivatives of $w_\omega(\pi_h,\pi_l)$ at some $(\pi_l,\pi_h)$, where $w_1(\pi_l,\pi_h) = w_0(\pi_l,\pi_h) = c$:

$$
\begin{aligned}
&\frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_h}\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_l}_{w_1=w_0=c} \\
=\;& \left[\alpha c+e^{-\alpha\pi_h}\left(1-e^{-(1-\alpha)\pi_l}\right)\frac{\partial \underline{v}(\pi_l,\pi_h)}{\partial \pi_h}\right] \\
&\cdot e^{-\beta\pi_h}\left[e^{-(1-\beta)\pi_l}(1-\beta)\underline{v}(\pi_l,\pi_h)+\left(1-e^{-(1-\beta)\pi_l}\right)\frac{\partial \underline{v}(\pi_l,\pi_h)}{\partial \pi_l}\right].
\end{aligned}
$$

Noting that $\underline{v}(\pi_l,\pi_h) \in (v(0), v(1))$ is decreasing in $\pi_h$ and increasing in $\pi_l$, the only negative term in this expression is $\frac{\partial \underline{v}(\pi_l,\pi_h)}{\partial \pi_h}$. But since we have already shown that $\frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_h} < 0$ and $\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_l} < 0$ for any $(\pi_h,\pi_l)$, we must have

$$
\alpha c > e^{-\alpha\pi_h}\left(1-e^{-(1-\alpha)\pi_l}\right)\frac{\partial \underline{v}(\pi_l,\pi_h)}{\partial \pi_h}.
$$

Let us next consider the term

$$\begin{aligned}
& \left.\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_h}\frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_l}\right|_{w_1=w_0=c} \\
= & \left[\beta c + e^{-\beta\pi_h}\left(1-e^{-(1-\beta)\pi_l}\right)\frac{\partial \underline{v}(\pi_l,\pi_h)}{\partial \pi_h}\right] \\
& \cdot e^{-\alpha\pi_h}\left[e^{-(1-\alpha)\pi_l}(1-\alpha)\,\underline{v}(\pi_l,\pi_h) + \left(1-e^{-(1-\alpha)\pi_l}\right)\frac{\partial \underline{v}(\pi_l,\pi_h)}{\partial \pi_l}\right].
\end{aligned}$$

A term by term comparison between the previous expression now clearly shows that

$$\frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_h}\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_l} > \frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_h}\frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_l}$$

when evaluated at $(\pi_h,\pi_h)$, where $w_1(\pi_l,\pi_h) = w_1(\pi_l,\pi_h) = c$, i.e. where $V_h^{FPA}(\pi_l,\pi_h) = V_l^{FPA}(\pi_l,\pi_h) = c$. By exactly the same argument as in the proof of Proposition 5, this is equivalent to

$$\frac{\partial V_l^{SPA}(\pi_l,\pi_h)}{\partial \pi_l}\frac{\partial V_h^{SPA}(\pi_l,\pi_h)}{\partial \pi_h} > \frac{\partial V_h^{SPA}(\pi_l,\pi_h)}{\partial \pi_l}\frac{\partial V_l^{SPA}(\pi_l,\pi_h)}{\partial \pi_h}$$

for any $(\pi_l,\pi_h)$ such that $V_h^{FPA}(\pi_l,\pi_h) = V_l^{FPA}(\pi_l,\pi_h) = c$, and property P4 holds. ■

**Proof of Proposition 7.** Write the entry rates of low type bidder conditional on state as $\lambda_{0,l} := (1-\beta)\pi_l$ and $\lambda_{1,l} := (1-\alpha)\pi_l$. Note that

$$\frac{1-e^{-(1-\beta)\pi_l}}{1-e^{-(1-\alpha)\pi_l}}$$

is decreasing in $\pi_l$ with $\lim_{\pi_l\to\infty}\frac{1-e^{-(1-\beta)\pi_l}}{1-e^{-(1-\alpha)\pi_l}} = 1$ and $\lim_{\pi_l\to 0}\frac{1-e^{-(1-\beta)\pi_l}}{1-e^{-(1-\alpha)\pi_l}} = \frac{1-\beta}{1-\alpha}$. Hence, if

$$\frac{1-\beta}{1-\alpha} > \frac{v(1)}{v(0)}$$

and $\pi_l$ is small enough, we have

$$\frac{1-e^{-\lambda_{0,l}}}{1-e^{-\lambda_{1,l}}} > \frac{v(1)}{v(0)},$$

so by Proposition 1, zero is in the support of both types of bidders. In this case, the payoff in the auction is $V_\theta^0(\pi_h, \pi_l)$ and equilibrium must be socially optimal:

$$\begin{aligned}\pi_h &= \pi_h^{opt} = \frac{1}{\alpha-\beta}\left((1-\beta)\log\left(\frac{v(1)}{c}\right) - (1-\alpha)\log\left(\frac{v(0)}{c}\right)\right),\\ \pi_l &= \pi_l^{opt} = \frac{1}{\alpha-\beta}\left(-\beta\log\left(\frac{v(1)}{c}\right) + \alpha\log\left(\frac{v(0)}{c}\right)\right).\end{aligned}$$

Clearly this must be the case when $c$ is sufficiently close to $\overline{c}$ (when $c = \overline{c}$, we have $\pi_l^{opt} = 0$). In all the other auction formats, at least the high type bidder gets more than $V_h^0(\pi_h, \pi_l)$, and hence $\pi_h > \pi_h^*(\pi_l)$, so that equilibrium is inefficient and generates a strictly lower total surplus than informal FPA. ■

**Proof of Proposition 8.** Let $\left\{\left(\pi_h^{FPA}, \pi_l^{FPA}\right), \left(F_h^{FPA}, F_l^{FPA}\right)\right\}$ denote the unique equilibrium entry and bidding strategies in FPA and let

$$\left\{\left(\pi_h^{SPA}, \pi_l^{SPA}\right), \left(F_h^{SPA}, F_l^{SPA}\right)\right\}$$

denote the unique bidder equilibrium entry and bidding strategies in SPA.

Consider first the first-price auction equilibrium. If zero is in the support of $F_h^{FPA}$, the result is immediate, therefore assume that bidding supports do not overlap and let $p'$ denote the common point in the two supports.

Fix entry rates at $\left(\pi_h^{FPA}, \pi_l^{FPA}\right)$, but consider buyer optimal bidding equilibrium of SPA. We know that $V_l^{SPA}\left(\pi_h^{FPA}, \pi_l^{FPA}\right) > V_l^{FPA}\left(\pi_h^{FPA}, \pi_l^{FPA}\right) = c$ so that a low type gets a excessive payoff relative to entry cost. Since $V_l^{SPA}(\pi_h, \pi_l)$ is decreasing in $\pi_l$, we can find some $\pi_l' > \pi_l^{FPA}$ such that $V_l^{SPA}\left(\pi_h^{FPA}, \pi_l'\right) = c$, i.e. adjust low type entry rate upwards while maintaining high type entry rate at $\pi_h^{FPA}$ so that low type is indifferent between the two cases. Then compare FPA with entry rates $\left(\pi_h^{FPA}, \pi_l^{FPA}\right)$ to SPA with entry rates $\left(\pi_h^{FPA}, \pi_l'\right)$. In FPA, bidding $p'$ the low type bidder wins if and only if there are no high type bidders. In SPA, similarly, by bidding some $p'' \in \left(\underline{v}\left(\pi_h^{FPA}, \pi_l'\right), b'\right)$, where $b'$ is defined in (6), the low type bidder wins under exactly the same condition. Note that we have not changed the high type entry rate so the probability of winning is the same in the two cases. Moreover, since we focus on the bidder optimal equilibirum, the low type is indifferent between $\underline{v}(\pi_l, \pi_h)$ and overbidding to $p''$.

Therefore, since low type is indifferent between the two cases as indicated by $V_l^{FPA}\left(\pi_h^{FPA}, \pi_l^{FPA}\right) = V_l^{SPA}\left(\pi_h^{FPA}, \pi_l'\right)$, her expected payment conditional on winning must be the same in the two cases, i.e. we have

$$\begin{aligned} p' &= \Pr\left(\text{at least one low type entrant} \middle| \theta = l, \pi = \left(\pi_h^{FPA}, \pi_l'\right), \text{ no high type entry}\right) \\ &\quad \cdot \underline{v}\left(\pi_h^{FPA}, \pi_l'\right) \\ &= \left[\widetilde{q}_l\left(1 - e^{-(1-\alpha)\pi_l'}\right) + (1-\widetilde{q}_l)\left(1 - e^{-(1-\beta)\pi_l'}\right)\right] \cdot \underline{v}\left(\pi_h^{FPA}, \pi_l'\right), \end{aligned}$$

where $\widetilde{q}_l$ is the belief of low type conditional on no high type entering:

$$\frac{\widetilde{q}_l}{1-\widetilde{q}_l} = \frac{q}{1-q}\frac{1-\alpha}{1-\beta}\frac{e^{-\alpha\pi_h^{FPA}}}{e^{-\beta\pi_h^{FPA}}}.$$

Consider next the payoffs of a high type. Bidding $p'$ in FPA and $b'$ (the lowest bid in the high type equilibrium support) in SPA, she also gets the object in both auctions if and only if there are no (other) high type bidders. Therefore, she prefers SPA if and only if her expected payment in FPA is higher than in SPA, that is, if

$$\begin{aligned} p' &> \Pr\left(\text{at least one low type entrant} \middle| \theta = h, \pi = \left(\pi_l', \pi_h^{FPA}\right), \text{ no high type entry}\right) \\ &\quad \cdot \underline{v}\left(\pi_h^{FPA}, \pi_l'\right) \\ &= \left[\widetilde{q}_h\left(1 - e^{-(1-\alpha)\pi_l'}\right) + (1-\widetilde{q}_h)\left(1 - e^{-(1-\beta)\pi_l'}\right)\right] \cdot \underline{v}\left(\pi_h^{FPA}, \pi_l'\right), \end{aligned}$$

where $\widetilde{q}_h$ is the belief of a high type conditional on no high type entering:

$$\frac{\widetilde{q}_h}{1-\widetilde{q}_h} = \frac{q}{1-q}\frac{\alpha}{\beta}\frac{e^{-\alpha\pi_h^{FPA}}}{e^{-\beta\pi_h^{FPA}}}.$$

Combining the above equations this is equivalent to

$$\widetilde{q}_l\left(1 - e^{-(1-\alpha)\pi_l'}\right) + (1-\widetilde{q}_l)\left(1 - e^{-(1-\beta)\pi_l'}\right) > \widetilde{q}_h\left(1 - e^{-(1-\alpha)\pi_l'}\right) + (1-\widetilde{q}_h)\left(1 - e^{-(1-\beta)\pi_l'}\right)$$

which holds since $\alpha > \beta$ and $q_h > q_l$.

We conclude that high type prefers SPA with entry rate $\left(\pi_h^{FPA}, \pi_l'\right)$ to FPA with entry rate $\left(\pi_h^{FPA}, \pi_l^{FPA}\right)$, i.e. $V_h^{SPA}\left(\pi_h^{FPA}, \pi_l'\right) > V_h^{FPA}\left(\pi_l^{FPA}, \pi_h^{FPA}\right) = c$. Moreover, since $V_l^{SPA}\left(\pi_h^{FPA}, \pi_l'\right) = c$, we have

$$V_h^{SPA}\left(\pi_h^{FPA}, \pi_l'\right) > V_l^{SPA}\left(\pi_h^{FPA}, \pi_l'\right) = c.$$

Since $V_h^{SPA}(\pi_h,\pi_l)$ is decreasing in $\pi_l$, there is some $\pi_l'' > \pi_l'$ such that $V_h^{SPA}\left(\pi_h^{FPA},\pi_l''\right) = c$. Following the proof of Proposition 6, the curves $\pi_h^{SPA}(\pi_l) := \left\{\pi_h : V_h^{SPA}(\pi_h,\pi_l) = c\right\}$ and $\pi_l^{SPA}(\pi_l) := \left\{\pi_h : V_l^{SPA}(\pi_h,\pi_l) = c\right\}$ are continuous functions of $\pi_l$ and cross exactly once. Also, from that proof we know that $V_h^{SPA}(0,\pi_l) > V_l^{SPA}(0,\pi_l)$ for all $\pi_l$, and hence the unique crossing point must be some $\left(\pi_h^{SPA},\pi_l^{SPA}\right)$, where $\pi_h^{SPA} > \pi_h^{FPA}$. Since $\pi_h^{opt} < \pi_h^{FPA} < \pi_h^{SPA}$, Lemma 1 implies that $W\left(\pi_h^{FPA},\pi_l^*\left(\pi_h^{FPA}\right)\right) > W\left(\pi_h^{SPA},\pi_l^*\left(\pi_h^{SPA}\right)\right)$. Furthermore, by definition $\pi_l^*\left(\pi_h^{SPA}\right)$ maximizes $W\left(\pi_h^{FPA},\pi_l\right)$ as a function of $\pi_l$ and therefore $W\left(\pi_h^{FPA},\pi_l^*\left(\pi_h^{SPA}\right)\right) > W\left(\pi_h^{FPA},\pi_l^{FPA}\right)$. Since $W\left(\pi_h^{FPA},\pi_l^{FPA}\right) = W\left(\pi_h^{FPA},\pi_l^*\left(\pi_h^{FPA}\right)\right)$, the above inequalities together impliy $W\left(\pi_h^{FPA},\pi_l^{FPA}\right) > W\left(\pi_h^{SPA},\pi_l^{SPA}\right)$, i.e. informal first price auction generates a higher social surplus which means that it generates a higher expected revenue. ■

**Proof of Proposition 9.** As a first step, we show that with an exogenous entry profile $(\pi_h,\pi_l)$ a high type gets a higher payoff in formal auctions than in informal FPA.

If zero is in the bidding support of both types for informal FPA, the result is immediate. Therefore, assume there is no overlap in the bidding supports in informal FPA and denote by $p'$ the common point in the two supports.

Let us contrast informal FPA to formal SPA, and consider the following bidding strategy. In informal FPA, let both types bid $p'$. Since this is in the support of both types, it generates the equilibrium payoff to both types. In formal SPA, let both types bid $\underline{p}(n)+\varepsilon$, where

$$\underline{p}(n) = E\left[v \left|\theta = l,\ N^l = n-1, N^h = 0\right.\right]$$

is the equilibrium pooling bid for the low type and $\varepsilon > 0$ is some number small enough so that $\underline{p}(n)+\varepsilon$ is strictly below the lowest point in the high type bidding support. With this strategy, a bidder wins if and only if there are no (other) high type bidders, and price in such a case is either zero or $\underline{p}(n)$. Clearly, this strategy is a weakly best-response for both types. Since using these strategies, a player gets the same allocation in both auction formats (get the object if and only if there are no high types present), we may compare the players' preference over the auction formats by contrasting their expected payment conditional on winning across the auction formats.

Start with the low type. Since the low type gets payoff zero for all $n \geq 2$ in both auction formats, she is indifferent between bidding $p'$ in informal FPA and bidding $\underline{p}(n) + \varepsilon$ in formal SPA. Therefore, the expected payment conditional on winning must be the same in the two cases, leading to:

$$\begin{aligned} p' &= E\left(p \,\middle|\, \theta = l, \text{ "win by bidding } \underline{p}(n) + \varepsilon\text{"}\right) \\ &= \Pr\left(N^l = 0 \,\middle|\, \theta = l, N^h = 0\right) \cdot 0 \\ &\quad + \sum_{k=1}^{\infty} \Pr\left(N^l = k \,\middle|\, \theta = l, N^h = 0\right) \underline{p}(k+1). \end{aligned}$$

Consider next the high type. Her expected payment when bidding $\underline{p}(n) + \varepsilon$ in formal SPA is

$$\begin{aligned} & E\left(p \,\middle|\, \theta = h, \text{ "win by bidding } \underline{p}(n) + \varepsilon\text{"}\right) \\ = & \sum_{k=1}^{\infty} \Pr\left(N^l = k \,\middle|\, \theta = h, N^h = 0\right) \underline{p}(k+1) \\ = & \sum_{k=1}^{\infty} \Pr\left(N^l = k \,\middle|\, \theta = h, N^h = 0\right) E\left(v \,\middle|\, \theta = l, N^h = 0, N^l = k\right) \end{aligned}$$

and hence she prefers the formal SPA if

$$\begin{aligned} & \sum_{k=1}^{\infty} \Pr\left(N^l = k \,\middle|\, \theta = h, N^h = 0\right) \underline{p}(k+1) \\ < \; & p' = \sum_{k=1}^{\infty} \Pr\left(N^l = k \,\middle|\, \theta = l, N^h = 0\right) \underline{p}(k+1), \end{aligned} \tag{25}$$

where

$$\underline{p}(k+1) := E\left(v \,\middle|\, \theta = l, N^h = 0, N^l = k\right)$$

is a decreasing function of $k$ (for $k \geq 1$). Note that the only difference in these formulas is that the probability

$$\Pr\left(N^l = k \,\middle|\, \theta, N^h = 0\right)$$

is conditioned on $\theta = h$ in the former and $\theta = l$ in the latter. Since a high signal makes state $\omega = 1$ more likely, a simple sufficient condition for (25) to hold is that

$$\Pr\left(N^l = k \,\middle|\, \omega = 1, N^h = 0\right) < \Pr\left(N^l = k \,\middle|\, \omega = 0, N^h = 0\right)$$

for all $k \geq 1$. Since

$$N^l \left| \omega = 1, N^h = 0 \sim Poisson\left(\lambda_{\omega,l}\right), \right.$$

we know that (25) holds if

$$\lambda_{0,l} e^{-\lambda_{0,l}} > \lambda_{1,l} e^{-\lambda_{1,l}},$$

that is

$$\left(1-\beta\right) \pi_l \, e^{-(1-\beta)\pi_l} > \left(1-\alpha\right) \pi_l \, e^{-(1-\alpha)\pi_l}$$

or

$$\pi_l < \frac{\log\left(1-\beta\right) - \log\left(1-\alpha\right)}{\alpha - \beta}.$$

Therefore, a high-type bidder has a lower expected payment in the formal SPA for $\pi_l$ small enough, which means that her expected payoff is higher in that auction formal. Since low type is always indifferent, this means that that expected revenue for the seller is higher in the informal FPA.

The second step is just to conclude that a change of auction format from informal FPA to a formal auction increases $\pi_h$, which is already too high in FPA. Since in both auction formats, $\pi_l = \pi_l^* \left(\pi_h\right)$, this further distortion will move the equilibrium entry point along $\pi_l^* \left(\pi_h\right)$ further away from social optimum, which by Lemma 1 will decrease surplus and hence revenue. ■

# References


Atakan, A. & M. Ekmekci. 2014. "Auctions, Actions, and the Failure of Information Aggregation." *American Economic Review (forthcoming)* .

French, K. & R. McCormick. 1984. "Sealed Bids, Sunk Costs, and the Process of Competition." *Journal of Business* 57:417–442.

Harstad, R. 1990. "Alternative Common Value Procedures: Revenue Comparisons with Free Entry." *Journal of Political Economy* 98:421–429.

Jehiel, Philippe & Laurent Lamy. 2015. "On Discrimination in Auctions with Endogenous Entry." *American Economic Review* 105(8):2595–2643.

Kim, Kyungmin & Philipp Kircher. 2015. "Efficient Competition Through Cheap Talk: The Case of Competing Auctions." *Econometrica* 83(5).

Lauermann, S. & A. Wolinsky. 2015. "A Common Value Auction with Bidder Solicitation.".

Levin, Dan & James Smith. 1994. "Equilibrium in Auctions with Entry." *American Economic Review* 84:585–599.

Matthews, S. 1977. Information Acqusition in Competitive Bidding Process. Technical report California Institute of Technology.

Matthews, S. 1984. Information Acquisition in Discriminatory Auctions. In *Bayesian Models in Economic Theory*, ed. M. Boyer & R.E. Kihlstrom. Amsterdam: North-Holland pp. 181–207.

Milgrom, P.R. & R.J. Weber. 1982. "A Theory of Auctions and Competitive Bidding." *Econometrica* 50:1089–1122.

Pekec, Aleksandar & Ilia Tsetlin. 2008. "Revenue Ranking of Discriminatory and Uniform Auctions with an Unknown Number of Bidders." *Management Science* 54(9):1610–1623.

Persico, N. 2000. "Information Acquisition in Auctions." *Econometrica* 68:135–148. mimeo, University of Pennsylvania.

Samuleson, W. 1985. "Competitive Bidding with Entry Costs." *Economics Letters* pp. 53–57.

Stegeman, M. 1996. "Participation Costs and Efficient Auctions." *Journal of Economic Theory* 71:228–259.

Wang, Ruqu. 1991. "Common-value auctions with discrete private information." *Journal of Economic Theory* 54(2):429 – 447.