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Abstract

In this paper, we look at the classical problem of aggregating individual utilities and study social orderings which are based on the concept of Ordered Weighted Averaging (OWA) Aggregating Operator. In these social orderings, called OWA social welfare functions (swf), weights are assigned a priori to the positions in the social ranking and, for every possible alternative, the total welfare is calculated as a weighted sum in which the weight corresponding to the *k-th* position multiplies the utility in the *k-th* position. In the α–OWA swf, the utility in the *k-th* position is the *k-th* smallest value assumed by the utility functions, whereas, in the β–OWA swf, it is the utility of the *k-th* poorest individual. We emphasize the differences between the two concepts, analyze the continuity issue and provide maximum points existence results.

Keywords: Ordered weighted averaging, social welfare functions, maximum points, maxmin.

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1 Introduction

A *social welfare function (swf)* maps the set of individual preferences (represented by utility functions defined over a fixed set of alternatives) for every representative member of a society to a social ordering which takes naturally the form of a real-valued function and gives a rule for ranking alternative social states.

The two most known models are the maxmin or rawlsian (Rawls (1971,1974)) and the utilitarian (Harsanyi (1955)) social welfare functions. These two approaches express very different views about how a society of individuals maximizes welfare. The maxmin principle uses as measure of social welfare the utility of the worst-off member of society, for every possible alternative. Indeed this approach reflects an extreme form of uncertainty aversion on the part of society as a whole since it corresponds to the preferences of extremely risk-averse individuals who do not know in advance what position in society they will occupy and consequently assign probability one to the worst condition that a member of society could face, for every possible alternative. On the other hand, the (weighted) utilitarian principle asserts that the best social policy is the one which maximizes the (weighted) sum of utility functions of all representative individuals in the society. Weights represent the relative importance of each representative individual in the total welfare of the society. Moreover, each weight could also be interpreted as the probability that an individual, randomly drawn from the society, will be represented by the corresponding representative member of the society.

Therefore, previous arguments show that the utilitarian swf can be regarded as an expectation over the set of representative individuals of the society while the maxmin swf can be regarded as a degenerate expectation over the position in the social ranking which gives probability one to the worst-off position. The concept of *ordered weighted averaging aggregation operator* defined in the framework of multicriteria decision making by Yager (1988) allows to construct social welfare functions which merge those two different approaches. The idea is to assign a priori a system of weights to the positions in the social ranking - rather than to the representative individuals - and then, for every possible alternative, the total welfare is naturally calculated as a weighted sum in which the weight corresponding to the k-th position multiplies the utility in the k-th position in the society (where the order is from the worst to the best). Each weight represents the relative importance of the corresponding position in the total welfare of the society but it can also be regarded as the probability that an individual randomly drawn from the society will occupy the corresponding position in the social ranking independently from the alternative; in other words, the underlying idea is to consider (not necessarily degenerate) expectations over the positions in the social ranking. However, we show in this paper that there is an immediate problem in the definition of the corresponding swf: which utility should be considered in the k-th position is not unequivocal. In fact, the utility in the k-th position could be the k-th smallest value assumed by the utility functions or simply the utility of the k-th poorest individual. In this paper, we refer to the first case as the $\alpha-OWA$ *social welfare function* and to the second one as the $\beta-OWA$ *social welfare function*. As far as the graphs of the utility functions (of the representative individuals) never intersect, then the two OWA social welfare functions coincide and, in case of continuous utility functions, they must coincide with an utilitarian criterion. Instead, we show that if the graphs of two (or more) utility functions intersect for some alternative then these two criteria may differ and provide different optima. In this case they both differ from the utilitarian approach since it may happen that the utility function of a representative individual is weighted differently depending on the alternatives. Moreover, if a weight equal to one is assigned to the worst position (all the other weights being equal to zero), then both the two OWA social welfare functions always coincide with the maxmin swf.

The $\beta-$OWA swf corresponds to the direct application of the Yager's concept of OWA operator to the social choice problem. Moreover, since the weights in the OWA operator give rise to a non additive measure (capacity) on the set of representative individuals in the society (see Grabisch, Orlovski and Yager (1998)), the $\beta-$OWA swf can be regarded as an expectation over representative individuals under ambiguity[1]. However, by definition, the $\beta-$OWA swf is affected by the following drawback: the weight (probability) of the $k-$th worst value attained by utility functions varies depending on how many representative individuals attain such value but also each other $i-$th worst value for $i \leq k-1$. In contrast, the $\alpha-$OWA swf is more suited for the situation in which individuals take into account only the possible outcomes that the whole society might face since weights (probabilities) of each $k-$th worst value are independent on whether or not utilities of representative individuals coincide for some alternative. In other words, different individuals, whose utilities coincide, are considered (weighted) as the same (unique) individual. In turn, we show that this approach implies that the $\alpha-$OWA swf is not consistent with any version of the Pareto principle[2].

This paper shows another crucial difference between the two OWA social welfare functions in relation with the continuity issue. In fact, the continuity of the OWA operator (see for instance Grabisch, Orlovski and Yager (1998)) immediately guarantees, on the one hand, that the $\beta-$OWA swf is continuous in X if the utility functions are continuous; on the other hand, optima are stable with respect to suitable perturbations of the utility functions. Conversely, an example shows that the $\alpha-$OWA swf is discontinuous even when utility functions are continuous, but we prove that the $\alpha-$OWA swf is upper semicontinuous so that optima exist in compact set of alternatives. Finally a counterexample shows that the set of optima is unstable even in the case in which the utility functions converge uniformly.

2 OWA social welfare functions

We consider a society whose set of representative individuals is $I = \{1, \dots n\}$; each individual $i \in I$ is endowed with a preference relation over a set of alternatives[3] $X \subseteq \mathbb{R}^l$; this preference relation is represented by an utility function $f_i : X \to \mathbb{R}$. Aim of this section is to *aggregate* the family of utility functions $(f_i)_{i \in I}$ into social orderings which give rules for ranking the alternatives in X and which are based on the concept of OWA aggregating operator. In particular we consider two social welfare functions that here we call $\alpha-$OWA swf and $\beta-$OWA swf. As already explained in the Introduction, in a OWA swf, weights are assigned a-priori to the positions in the social ranking and, for every possible alternative, the total welfare is calculated as a weighted sum in which the weight corresponding to the k-th position multiplies the utility in the k-th position in the society. In the $\alpha-$OWA swf, the k-th position is the k-th smallest value assumed by the utility functions, whereas, in the $\beta-$OWA swf, it is the k-th poorest

[1] See Schmeidler (1989) for relations between ambiguity and expected utility with respect to capacities.

[2] Consistency of the $\beta-$OWA swf with a strong version of Pareto optimality follows immediately from an analogous property of the OWA aggregating operator (see Yager 1988).

[3] All the results contained in this paper could be easily extended to the infinite dimensional case under suitable assumptions.

individual. In this section, we formalize these concepts and emphasize differences by means of an example.

$\alpha-$model

Denote with $x \rightsquigarrow I_1(x)$ the set-valued map such that $I_1(x) = I$ for every $x \in X$,

$$b_1(x) = \min_{h \in I_1(x)} f_h(x) \quad \text{and} \quad H_1(x) = \{h \in I \mid f_h(x) = b_1(x)\} \quad \text{for every } x \in X$$

Suppose defined $x \rightsquigarrow I_{k-1}(x)$, $x \to b_{k-1}(x)$ and $x \rightsquigarrow H_{k-1}(x)$, we define by induction $x \rightsquigarrow I_k(x)$, $x \to b_k(x)$ and $x \rightsquigarrow H_k(x)$ as follows:

- $I_k(x) = I_{k-1}(x) \setminus H_{k-1}(x)$ for every $x \in X$.
- $b_k(x) = \min_{h \in I_k(x)} f_h(x)$ for every $x \in X$
- $H_k(x) = \{h \in I \mid f_h(x) = b_k(x)$ for every $x \in X$

Note that if $I_k(x) \neq \emptyset$, then $b_k(x)$ is well-posed, i.e. $\min_{h \in I_k(x)} f_h(x)$ exists, and $H_k(x) \neq \emptyset$.
Then

Definition 2.1: Let $\lambda \in I$ be such that $I_i(x) \neq \emptyset$, for every $i \in \{1, \dots, \lambda\}$ and for every $x \in X$. Let $\omega = (\omega_1, \dots, \omega_\lambda) \in \mathbb{R}^\lambda$ be a vector such that $\sum_{i=1}^{\lambda} \omega_i = 1$ and $\omega_i \geq 0$ for all $i \in \{1, \dots, \lambda\}$, then the $\alpha-$*OWA social welfare function* with vector of weights ω is the function $F_\omega : X \to \mathbb{R}$ defined by

$$F_\omega(x) = \sum_{i=1}^{\lambda} \omega_i b_i(x) \quad \text{for all } x \in X. \tag{1}$$

$\beta-$model

For every $x \in X$, let $\pi_x : I \to I$ be a permutation in I such that, for every $i \in I$, $\pi_x(i)$ is the individual (criterion) who happens to be the ith-worst off at x, i.e.:

$$f_{\pi_x(1)}(x) \leq f_{\pi_x(2)}(x) \leq \cdots \leq f_{\pi_x(n)}(x). \tag{2}$$

Note that, given $x \in X$, there might be different permutations satisfying (2). Denote with $t_i : X \to \mathbb{R}$ the function defined by

$$t_i(x) = f_{\pi_x(i)}(x) \quad \text{for every } x \in X. \tag{3}$$

It can be easily seen that, for every $x \in X$ and every $i \in I$, $t_i(x)$ is the same whatever is the permutation π_x of I satisfying (2). Note also that $b_1(x) = t_1(x)$ for all $x \in X$ while an example below shows that (whenever $i \geq 2$) $b_i(x)$ and $t_i(x)$ might be different for some x in which the graphs of two or more utility functions intersect.

Definition 2.2: Let $\lambda \in I$ and $\omega = (\omega_1, \dots, \omega_\lambda) \in \mathbb{R}^\lambda$ be a vector such that $\sum_{i=1}^{\lambda} \omega_i = 1$ and $\omega_i \geq 0$ for all $i \in \{1, \dots, \lambda\}$, then the $\beta-$*OWA social welfare function* with vector of weights ω is the function $G_\omega : X \to \mathbb{R}$ defined by

$$G_\omega(x) = \sum_{i=1}^{\lambda} \omega_i t_i(x) \quad \text{for all } x \in X. \tag{4}$$

Note that, if the graphs of the utility functions never intersect then $F_\omega(x) = G_\omega(x)$ for all $x \in X$. In this case and whenever the utility functions are continuous, the function $x \to \pi_x(i)$ is constant for every $i \in I$; therefore, both the two swf coincide with a weighted utilitarian swf in which the weight of individual j is ω_i if $\pi_x(i) = j$ and $i \in \{1, \dots, \lambda\}$ or it is equal to 0 otherwise.

REMARK 2.3: In the previous definition, the $\beta-$OWA swf is expressed in terms of the functions $t_i(x)$, (defined by (3)), which make it simpler to compare the $\alpha-$OWA and $\beta-$OWA swf. However, the $\beta-$OWA swf can be equivalently be defined by taking explicitly into account the Yager's OWA aggregating operator. Indeed, if $g_\omega : \mathbb{R}^n \to \mathbb{R}$ is the function defined by $g_\omega(u) = \sum_{i=1}^{\lambda} \omega_i u_{(i)}$ where $(\cdot)$ is a permutation of I such that $u_{(1)} \le u_{(2)} \le \cdots \le u_{(n)}$ then $G_\omega(x) = g_\omega(f_1(x), \dots, f_n(x))$ for all $x \in X$. The function g_ω is exactly the *OWA aggregating operator* corresponding to $\omega = (\omega_1, \dots, \omega_\lambda)$. It will be useful to recall that g_ω is continuous in its domain (see for instance Grabisch, Orlovski and Yager (1998)).

The next example shows that the α-model and the β-model are different and may provide different optima.

EXAMPLE 2.4: Consider the set $X = [0, 4]$ and $f_i : X \to \mathbb{R}$ for $i = 1, 2, 3$, defined by

$$f_1(x) = \begin{cases} x+3 & \text{if } x \in [0,1] \\ -\frac{4}{3}x + \frac{16}{3} & \text{if } x \in]1,4] \end{cases}, \quad f_2(x) = \begin{cases} \frac{4}{3}x & \text{if } x \in [0,3] \\ -x+7 & \text{if } x \in]3,4] \end{cases}, \quad f_3(x) = 6 \quad \forall x \in X.$$

It can be checked that

$$t_1(x) = \begin{cases} \frac{4}{3}x & \text{if } x \in [0,2] \\ -\frac{4}{3}x + \frac{16}{3} & \text{if } x \in]2,4] \end{cases} \qquad t_2(x) = \begin{cases} x+3 & \text{if } x \in [0,1] \\ -\frac{4}{3}x + \frac{16}{3} & \text{if } x \in]1,2] \\ \frac{4}{3}x & \text{if } x \in]2,3] \\ -x+7 & \text{if } x \in]3,4] \end{cases}$$

Fix $\omega = (\frac{1}{3}, \frac{2}{3})$, then

$$G_\omega(x) = \begin{cases} \frac{10}{9}x + 2 & \text{if } x \in [0,1] \\ -\frac{4}{9}x + \frac{32}{9} & \text{if } x \in]1,2] \\ \frac{4}{9}x + \frac{16}{9} & \text{if } x \in]2,3] \\ -\frac{10}{9}x + \frac{58}{9} & \text{if } x \in]3,4] \end{cases}$$

The function G_ω has two maximum points in X, $\overline{x}_1 = 1$ and $\overline{x}_2 = 3$ and $G_\omega(\overline{x}_i) = \frac{28}{9}$. On the other hand

$$b_1(x) = t_1(x) \ \forall x \in X, \quad b_2(x) = \begin{cases} t_2(x) & \text{if } x \neq 2 \\ 6 & \text{if } x = 2 \end{cases}$$

Hence

$$F_\omega(x) = \begin{cases} G_\omega(x) & \text{if } x \neq 2 \\ \frac{44}{9} & \text{if } x = 2 \end{cases}$$

Therefore, F_ω has a unique maximum point in X, that is $\overline{x}_3 = 2$. Finally, note that F_ω is not continuous in $\overline{x}_3$ while G_ω is continuous in its domain.

REMARK 2.5: In Yager (1988) it is shown that the OWA operator is consistent with a (strong) version of the Pareto principle (therein called monotonicity). This consistency condition immediately translates to the $\beta-$OWA swf as follows: given x_1 and x_2 in X, if $f_i(x_1) \ge f_i(x_2)$ for every $i \in I$, then, for every vector of weights ω, it follows that $G_\omega(x_1) \ge G_\omega(x_2)$.
On the other hand the α-model is not consistent with any version of the Pareto principle. In

fact, consider the following example of a society with 3 representative individual whose utility functions are f_1, f_2, f_3. Suppose that there exist two alternatives x_1 and x_2 in X such that $(f_1(x_1), f_2(x_1), f_3(x_1)) = (0, 0, 4)$ and $(f_1(x_2), f_2(x_2), f_3(x_2)) = (1, 2, 5)$. Obviously x_2 strongly (Pareto) dominates x_1. However $b_1(x_1) = 0$ and $b_2(x_1) = 4$ while $b_1(x_2) = 1$ and $b_2(x_2) = 2$ so that if $\omega = (\omega_1, \omega_2) = (1/2, 1/2)$ it follows that $F_\omega(x_1) > F_\omega(x_2)$.

3 OWA social optima

In this section we focus on the continuity issue and its implications to the problem of existence and stability of the maximum points. This analysis emphasizes the difference between the two social welfare functions as the β-model is much more *regular* than the α-model. Nevertheless we show that the $\alpha-$OWA swf is sufficiently regular to sustain a maximum points existence result under classical assumptions.

3.1 Existence of optimal points

As previously recalled, the OWA aggregation operator g_ω is continuous in its domain. This obviously implies that the $\beta-$OWA swf G_ω is continuous in X if, for instance, each utility function f_i is continuous in X. In this case and if X is compact, we immediately obtain the existence of maximum points for G_ω in X that we call *$\beta-$OWA social optima*.

The example in the previous section has shown that the $\alpha-$OWA swf F_ω might be discontinuous even if each utility function f_i is continuous in X. Now we show that in this case F_ω is upper semicontinuous. This result immediately implies that F_ω has maximum points on a compact set[4] X that we call *$\alpha-$OWA social optima*. To this purpose, we recall (see, for instance, Aubin and Frankowska (1990)) that

Definition 3.1: A set-valued map $\mathcal{I} : X \rightsquigarrow \mathbb{N}$ is said to be *sequentially lower semicontinuous* in $x \in X$ if for every $h \in I(x)$ and every sequence $(x_\nu)_{\nu \in \mathbb{N}}$ converging to x $(x_\nu \to x)$, there exists $\overline{\nu} \in \mathbb{N}$ such that $h \in I(x_\nu)$ for all $\nu \geq \overline{\nu}$. The set-valued map $\mathcal{I}$ is said to be sequentially lower semicontinuous in X if it is sequentially lower semicontinuous in every $x \in X$.

Definition 3.2: A function $g : X \to \mathbb{R}$ is said to be *sequentially upper semicontinuous* in $x \in X$ if for every sequence $x_\nu \to x$ it follows that

$$\limsup_{\nu \to \infty} g(x_\nu) \leq g(x).$$

Since $X \subseteq \mathbb{R}^l$, this definition coincides with the classical (topological) definition of upper semicontinuous function; therefore, in remaining part of the paper, we simply refer to upper semicontinuous functions[5]. Finally, the function g is said to be sequentially upper semicontinuous in X if it is upper semicontinuous in every $x \in X$.

Therefore, we give the following:

Lemma 3.3: *Assume that f_i is continuous in X, for every $i \in I$, and $\mathcal{I} : X \rightsquigarrow I \subset \mathbb{N}$ is sequentially lower semicontinuous in X, with nonempty images for every $x \in X$.*

[4] This is the classical generalized Weirstrass theorem.

[5] Things change in the infinite dimensional case in which only the sequential upper semicontinuity property would be required.

Then, the function $g : X \to \mathbb{R}$, defined by

$$g(x) = \min_{h \in \mathcal{I}(x)} f_h(x) \quad \text{for all } x \in X,$$

is upper semicontinuous in X. Moreover, denote with $\mathcal{H}(x) = \{h \in I \mid f_h(x) = g(x)\}$, then the set-valued map $\mathcal{G} : X \rightsquigarrow I$, defined by

$$\mathcal{G}(x) = \mathcal{I}(x) \setminus \mathcal{H}(x) \quad \text{for all } x \in X,$$

is sequentially lower semicontinuous in X.

Proof. Fix $x \in X$ and let $h \in \mathcal{I}(x)$ be such that $f_h(x) = g(x)$. Consider a sequence $x_\nu \to x$. Being $\mathcal{I}$ sequentially lower semicontinuous, there exists $\overline{\nu} \in \mathbb{N}$ such that $h \in \mathcal{I}(x_\nu)$ for every $\nu \in \mathbb{N}$ such that $\nu \geq \overline{\nu}$; therefore, $g(x_\nu) \leq f_h(x_\nu)$ for every $\nu \geq \overline{\nu}$. Hence

$$\limsup_{\nu \to \infty} g(x_\nu) \leq \limsup_{\nu \to \infty} f_h(x_\nu) = f_h(x) = g(x)$$

and g is upper semicontinuous.

Suppose now that $\mathcal{G}$ is not sequentially lower semicontinuous in x, then there exists $\widehat{h} \in \mathcal{G}(x)$ and a sequence $x_\nu \to x$ such that for every $\overline{\nu} \in \mathbb{N}$ it results that $\widehat{h} \notin \mathcal{G}(x_\nu)$ for a $\nu \in \mathbb{N}$ such that $\nu \geq \overline{\nu}$. Hence there exists a subsequence $x_{\nu_k} \to x$ such that $\widehat{h} \notin \mathcal{G}(x_{\nu_k})$ for every $k \in \mathbb{N}$. Since $\widehat{h} \in \mathcal{I}(x)$ and $\mathcal{I}$ is sequentially lower semicontinuous in x, there exists $\overline{k} \in \mathbb{N}$ such that $\widehat{h} \in \mathcal{I}(x_{\nu_k})$ for every $k \geq \overline{k}$ which implies that $\widehat{h} \in \mathcal{H}(x_{\nu_k})$ for every $k \geq \overline{k}$. Hence $f_{\widehat{h}}(x_{\nu_k}) = g(x_{\nu_k})$ for every $k \geq \overline{k}$ and

$$f_{\widehat{h}}(x) = \limsup_{k \to \infty} f_{\widehat{h}}(x_{\nu_k}) = \limsup_{k \to \infty} g(x_{\nu_k}) \leq g(x).$$

On the other hand, $\widehat{h} \in \mathcal{G}(x)$ implies that $\widehat{h} \in \mathcal{I}(x)$ but $\widehat{h} \notin \mathcal{H}(x)$ which means that $f_{\widehat{h}}(x) > g(x)$. But this is a contradiction and we deduce that $\mathcal{G}$ is sequentially lower semicontinuous. □

Now we deduce the upper semicontinuity result for the function F_ω from the previous Lemma. This result implies that $\alpha-$OWA social optima exist if the set of alternatives is compact.

PROPOSITION 3.4: *Assume that f_i is continuous in X, for every $i \in I$, and let $\lambda \in I$ be such that $I_i(x) \neq \emptyset$ for every $i \in \{1, \dots, \lambda\}$ and for every $x \in X$.*

Then, the $\alpha-$OWA social welfare function $F_\omega : X \to \mathbb{R}$ is upper semicontinuous in X for every vector of weights $\omega = (\omega_1, \dots, \omega_\lambda)$.

Proof. The proof works by induction. Obviously the set valued map I_1 is sequentially lower semicontinuous in X; moreover, from Lemma 3.3, b_1 is an upper semicontinuous function. Suppose I_{i-1} is a sequentially lower semicontinuous set-valued map and b_{i-1} is an upper semicontinuous function. From Lemma 3.3, also I_i is a sequentially lower semicontinuous set-valued map and b_i an upper semicontinuous function. Hence F_ω is an upper semicontinuous function since it is the sum of upper semicontinuous functions and the assertion follows. □

3.2 Stability of optimal points and values

The problem we address in this section is the following: given a sequence of utility functions $(f_i^\nu)_{\nu\in\mathbb{N}}$ for every representative individual $i \in I$, we look for conditions of convergence for such sequences which guarantee that converging sequences of corresponding OWA social optima have their limits in the set of OWA social optima corresponding to the limit utility functions. The stability property in the β-model follows easily (under the classical continuous convergence assumption) from the continuity of the OWA operator g_ω. An example shows instead that in the α-model sequences of optimal points may not converge to an optimal point even when the utility functions converge uniformly.

First we formalize the result for the β-model.

PROPOSITION 3.5: *Assume that X is closed and, for every player i, $(f_i^\nu)_{\nu\in\mathbb{N}}$ is a sequence of functions ($f_i^\nu : X \to \mathbb{R}$ for every $\nu \in \mathbb{N}$) which continuously converges (Poppe (1974)) to f_i, that is, for every $x \in X$ and for every sequence $(x_\nu)_{\nu\in\mathbb{N}} \subset X$ converging to x, it follows that*

$$\lim_{\nu\to\infty} f_i^\nu(x_\nu) = f_i(x).$$

If $(\overline{x}_\nu)_{\nu\in\mathbb{N}}$ is a sequence converging to $\overline{x}$ and each $\overline{x}_\nu$ is a $\beta-$OWA social optimum corresponding to the utility functions $(f_i^\nu)_{i\in I}$ and the vector of weights ω, then $\overline{x}$ is a $\beta-$OWA social optimum corresponding to the utility functions $(f_i)_{i\in I}$ and the vector of weights ω.

Proof. Denote with G_ω^ν the $\beta-$OWA swf corresponding to the family of utility functions $(f_i^\nu)_{i\in I}$ and vector of weights ω, that is $G_\omega^\nu(x) = g_\omega(f_1^\nu(x), \ldots, f_n^\nu(x))$ for every $x \in X$. Given $x \in X$, let $(x_\nu)_{\nu\in\mathbb{N}} \subset X$ be a sequence converging to x. Since g_ω is continuous and $f_i^\nu(x_\nu) \to f_i(x)$ for every i, we get

$$\lim_{\nu\to\infty} G_\omega^\nu(x_\nu) = \lim_{\nu\to\infty} g_\omega(f_1^\nu(x_\nu), \ldots, f_n^\nu(x_\nu)) = g_\omega(f_1(x), \ldots, f_n(x)) = G_\omega(x).$$

Let $(\overline{x}_\nu)_{\nu\in\mathbb{N}}$ be a sequence converging to $\overline{x}$ where each $\overline{x}_\nu$ is a $\beta-$OWA social optimum corresponding to the family of utility functions $(f_i^\nu)_{i\in I}$ and vector of weights ω; then $G_\omega^\nu(\overline{x}_\nu) \geq G_\omega^\nu(x)$ for every $x \in X$; therefore it follows that

$$G_\omega(\overline{x}) = \lim_{\nu\to\infty} G_\omega^\nu(\overline{x}_\nu) \geq \lim_{\nu\to\infty} G_\omega^\nu(x) = G_\omega(x) \quad \text{for all } x \in X.$$

Moreover, X being closed, it follows that $\overline{x} \in X$; so, $\overline{x}$ is a $\beta-$OWA social optimum corresponding to the family of utility functions $(f_i)_{i\in I}$ and vector of weights ω and the assertion follows. □

The next example shows that the $\alpha-$OWA model is not stable with respect to perturbations.

EXAMPLE 3.6: Consider a society with 3 representative individuals and set of alternatives $X = [0, 2]$. Consider the sequences of utility functions $(f_i^\nu)_{\nu\in\mathbb{N}}$ with $i = 1, 2, 3$. For every $\nu \in \mathbb{N}$, the functions are defined by

$$f_1^\nu(x) = 0,\ f_2^\nu(x) = 10,\ f_3^\nu(x) = |x-1| + \frac{1}{\nu} \quad \forall x \in [0, 2]$$

Then we get

$$b_1^\nu(x) = f_1^\nu(x) = 0 \quad \text{and} \quad b_2^\nu(x) = f_3^\nu(x) = |x-1| + \frac{1}{\nu} \quad \forall x \in [0, 2].$$

Fix $\omega = (\frac{1}{2}, \frac{1}{2})$. Hence

$$F_\omega^\nu(x) = \frac{|x-1|}{2} + \frac{1}{2\nu} \quad \forall x \in [0,2]$$

For every $\nu \in \mathbb{N}$, the function F_ω^ν has two maximum points in $[0,2]$, $\overline{x} = 0$ and $\widehat{x} = 2$.

If we take the limit as $\nu \to \infty$, we get that the sequences $(f_i^\nu)_{\nu \in \mathbb{N}}$ (with $i = 1,2,3$) converge uniformly respectively to the function f_1, f_2, f_3 defined by

$$f_1(x) = 0,\ f_2(x) = 10,\ f_3(x) = |x-1| \quad \forall x \in [0,2]$$

Hence

$$b_1(x) = 0 \quad \forall x \in [0,2] \quad \text{and } b_2(x) = \begin{cases} |x-1| & \text{if } x \in [0,1[\cup]1,2] \\ 10 & \text{if } x = 1 \end{cases}$$

So

$$F_\omega(x) = \begin{cases} \frac{|x-1|}{2} & \text{if } x \in [0,1[\cup]1,2] \\ 5 & \text{if } x = 1 \end{cases}$$

which implies that F_ω has a unique maximum point in $[0,2]$, $\widetilde{x} = 1$, and therefore there is no convergence of the maximum points of the perturbed problems to the maximum point of the unperturbed one.

As a final remark, notice that the maximum values of the perturbed problems, (which are equal to $F_\omega^\nu(\overline{x}) = F_\omega^\nu(\widehat{x}) = 1/2 + 1/2\nu$ for every $\nu \in \mathbb{N}$), converge, as $\nu \to \infty$, to $1/2$ which is different from the maximum value of the unperturbed problem $F_\omega(\widetilde{x}) = 5$.

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