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Ambiguous Games without a State Space and Full Rationality

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Abstract

This work aims to differentiate and to better understand the assumptions that must be imposed on the structure of ambiguity and on the attitudes towards ambiguity in order to have existence of equilibria in games under ambiguous belief correspondences. In the present paper, this class of games is studied under substantially weaker assumptions on agents' preferences, as they are not required to be rational and therefore do not have any functional representation. A new approach is required to deal with preferences that are not rational, in this particular framework; in fact, the present work shows that the attitudes of agents towards the imprecision of probabilistic beliefs play a key role in the issue of equilibrium existence, whenever they are combined with some property of convexity/concavity of the ambiguous belief correspondences. The paper also studies the role played by these assumptions in different specific models, (such as incomplete information games with multiple priors or games under strategic ambiguity), so as to illustrate the applicability of the results of equilibrium existence and connections with previous literature.

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1 Introduction

As shown in the seminal paper by Ellsberg (1961), in the problem of decision under uncertainty, beliefs cannot always be represented by conventional probabilities. There is substantial evidence, both from theory and applications, that *ambiguity* emerges even more clearly in game theory as the concept of equilibrium has a specific source of uncertainty in the expectations of players about their opponents's behavior (strategic ambiguity). Therefore, it seems to be noteworthy to study games and their equilibrium concepts in case of ambiguity. This is the issue that this paper addresses; more precisely, this work looks at games in which uncertainty is described by sets of probability distributions and studies sufficient conditions for the existence of equilibria in this framework. The paper provides a generalization of different results already presented in the literature. In particular, the assumption of rationality (completeness and transitivity) of preferences, that underlies the previous literature on ambiguous games[2], is here removed. The present work points out that this issue (lack of rationality of preferences) entails a new approach, as it is shown below that the attitudes of players towards the imprecision of probabilistic beliefs play a key role to obtain the minimal convexity assumption for preferences that is required in the equilibrium existence theorems, (whenever some additional assumption on the representation of ambiguous beliefs are imposed).

In order to clarify where the contribution of this work lies, it seems appropriate to relate the approach of this paper to two important issues which emerge more or less explicitly in the literature on decision theory under ambiguity. A first issue is the "structure of ambiguity", that is, the way ambiguous beliefs are represented, regardless of the preference of the agent. The *theory of imprecise probabilities* (see, for instance, Walley (2000) for a recent survey) studies in which way *exogenous* or *objective* ambiguity can be represented by probability judgments and it analyzes the relations between the different representations. The other issue deals with attitudes of the agents towards ambiguity; optimism and pessimism, in the broad sense, are recognized as the most significant behavioral traits related to ambiguity; essentially, they depend on whether or not the agent expects that ambiguity will be resolved in his favor. Gilboa and Marinacci (2015) surveys some of the main findings in the decision theory literature devoted to the axiomatization of different preference relations over the set of ambiguous alternatives. In this strand of literature, the source of uncertainty is described by an underlying state space and decision makers are endowed with preferences over a set of alternatives, called *acts*, i.e. functions from the state space to a space of *consequences*. Representation theorems for preferences axiomatize the fact that the decision maker is endowed with an utility function which gives a numerical outcome for every possible consequence, and with a probability judgment on the state space[3]. In this particular framework, the probabilistic belief is completely endogenous and depends only on the attitude of the decision maker towards ambiguity. However, there is another strand of research that is devoted to the axiomatization of preferences in case ambiguous alternatives have an exogenous or objective probabilistic representation (see for instance the survey by Walley (1991) or Coletti, Petturitti and Vantaggi (2015), Yager and Alajlan (2015) and references therein for recent developments). Gajdos, Hayashi, Tallon an Vergnaud (2008) merge the two

[2]Indeed, even the literature on Decisions under Uncertainty provides few models of preferences that are not rational; *Interval Dominance* is perhaps the most known example.

[3]In the *multiple prior* approach, firstly studied in Gilboa and Schmeidler (1989), the probability judgment is provided by a convex set of probability distributions. In the Choquet Expected Utility approach, firstly introduced by Schmeidler (1990), the probability judgment is given by a convex capacity.

approaches: in a classical decision making problem with a state space, they look at preferences over pairs (P, f) where P is a set of probability distributions over the state space (representing the objective information available) and f is an act. They provide a maxmin representation for complete preferences in this setting which has an important feature: a pessimistic attitude towards the imprecision of the probability judgments (i.e. imprecision aversion) is behind the Gilboa and Schmeidler's definition of *ambiguity aversion* and the maxmin representation.

Aim of this paper is to differentiate and to better understand the assumptions that must be imposed on the structure of ambiguity and on the attitudes towards ambiguity in order to have equilibrium existence in a game under ambiguity. In particular, it will be shown that a notion of imprecision aversion and its optimistic counterpart are key assumptions for the existence of equilibria. The approach considered in this work is different from the papers which investigate the effects of uncertainty aversion in incomplete information games with multiple priors[4]. In the present paper, ambiguous games are, instead, regarded from a different perspective as the approach here follows another strand of research in decision theory, (proposed by Ahn (2008), Olszewski (2007) and Stinchcombe (2003)), in which the decision-maker, facing ambiguity, is not able to understand what the relevant states are, so that the information available can be expressed entirely in the space of lotteries (probabilities) over consequences. This approach seems particularly useful to study ambiguity in games. In fact, on the one hand, the model *without the state space* and the classical *multiple prior* model can be reconciled as an ambiguous act can be evaluated by its induced set of distributions over consequences (see Ahn (2008) and Olszewski (2007)). On the other hand, game theory provides further evidence that ambiguity cannot always be reconducted to the classical approach with a state space and multiple priors. The literature on ambiguous games (see for instance Dow and Werlang (1994), Lo (1996), Klibanoff (1996), Eichberger and Kelsey (2000) and Marinacci (2000), Lehrer (2012), Riedel and Sass (2013), Beauchêne (2015)) has shown that the classical equilibrium notion embodies a specific source of ambiguity: In equilibrium, players choose their optimal strategies provided that they have correct expectations about the behavior of their opponents. However, agents may have ambiguous beliefs about opponents' strategy choices; in this case, rational agents take this issue into account and face a problem of decision making under ambiguity[5].

In previous papers, De Marco and Romaniello (2012, 2013, 2015a) introduced and studied the (so called) model of *game under ambiguous belief correspondences*[6] which provides a rather general tool to study ambiguity in games. The key point of these papers is that, for every player, ambiguity is directly represented by a belief correspondence which maps the set of strategy profiles into the set of all subsets of probability distributions over the outcomes of the game. For each player and for every given strategy profile, the belief correspondence gives the set of probability distributions over the possible outcomes of the game that the corresponding player perceives to be feasible and consistent with the actual strategy

[4]Kajii and Ui (2005) first consider this approach, Bade (2011) considers games à la Aumann (1997) under more general preferences. Azrieli and Teper (2011) characterize equilibrium existence in terms of the preferences of the players; the evidence from this paper is s that equilibria exist if and only if agents are ambiguity averse, as ambiguity aversion is deeply related to some form of convexity of preferences.

[5]There is no evidence in the literature showing that this kind of ambiguity can be properly reconducted to incomplete information games à la Harsanyi (that is, with a state space) under multiple priors which, in turn, must be generalized in order to encompass this specific game theoretical issue.

[6]De Marco and Romaniello (2012) presents the general model, an existence theorem and many motivating examples. Stability of the equilibria is studied in De Marco and Romaniello (2013,b). De Marco and Romaniello (2015a) extends the model to the case of variational preferences.

profile. It follows that belief correspondences might represent objective (exogenous) ambiguity as done in Ahn (2008) and Olszewski (2007); but, at the same time, it turns out (see the examples in De Marco and Romaniello (2013) and also Section 6 below) that many existing models of ambiguous game have an equivalent formulation in terms of belief correspondences. For example, a notion of equilibrium in incomplete information games with multiple priors and the concept of equilibrium under partially specified probabilities by Lehrer (2012) can both be regarded as particular cases of the notion of *equilibrium under ambiguous belief correspondences* (see De Marco and Romaniello (2013)).

In the present paper, games under ambiguous belief correspondences are studied under minimal assumptions on preferences. In particular agents are here endowed with preferences over the set of all possible subsets of probability distributions on the outcomes of the game and these preferences are not necessarily complete or transitive. The motivations for this approach are immediately clear. Firstly, it allows to study the existence of equilibria in ambiguous games regardless of the assumptions that must be imposed in the representation theorems for preferences, so as to understand what is truly required for the existence of equilibria. Secondly, it makes it possible to analyze models in which agents do not have rational preferences on *certain* alternatives, that is, models with general (although finite) outcome spaces. In this framework, (in which actions are evaluated by their induced sets of probability distributions on outcomes), a decision maker can be immediately recognized as pessimistic if he shows the following behavior: he does not prefer a subset of probability distributions to another one if the former is a subset of the latter. Similarly, an optimistic decision maker would not prefer a set of probability distributions to another one if the latter is a subset of the former. These behavioral traits can be reconducted to the idea of aversion (resp. inclination) towards imprecision presented in Gajdos, Hayashi, Tallon an Vergnaud (2008). The present paper builds upon such behavioral assumptions in order to obtain equilibrium existence theorems; roughly speaking, the main result is that equilibrium existence is guaranteed with pessimistic players provided that the ambiguous belief is a convex set-valued map and, similarly, with optimistic players provided that the ambiguous belief is a concave set-valued map[7]. In particular, the weakest form of disinclination (resp. inclination) towards imprecision is here called *imprecision aversion* (resp. *imprecision loving*) and it characterizes pessimism (resp. optimism) in the broad sense. Moreover, it is also shown below that imprecision aversion (resp. imprecision loving) can be refined in different ways. Some of these refinements allow to relax the assumption of convexity of preferences in the equilibrium existence theorem.

Besides the relevant issue of the attitudes towards ambiguity, the assumptions (continuity and convexity/concavity) imposed on the structure of ambiguity, i.e. the belief correspondences, are key for the existence of equilibria. Therefore, it seems important to investigate how much demanding these assumptions are in specific models so as to illustrate the existence results's applicability. Two specific models are considered in this paper. In the first model, the belief correspondences derive from a game model of incomplete information having multiple priors, while, in the second one, they arise from models of strategic ambiguity, that is, models in which ambiguity concerns agents' expectations about opponents' behavior. It turns out that in all these models the belief correspondences are convex but not concave set-valued maps. Hence, equilibrium existence is guaranteed with pessimistic agents and not with optimistic ones; these results reconcile the approach of this paper with the existing literature which shows

[7]There obviously are additional continuity assumptions on preferences and belief correspondences. Moreover, convexity of the preference relations over sets of probability distribution is also required

that equilibria exist provided that the agents are ambiguity averse. Indeed, it is also shown in this paper that, in case of rational preferences, imprecision aversion together with convexity of the belief correspondences guarantee the convexity of preferences with respect to their strategy choice. This latter convexity property, in turn, characterizes ambiguity aversion in the game model by Azrieli and Teper (2011).

The paper is organized as follows. Section 2 is a preliminary section which presents the model, the equilibrium notion and the general equilibrium existence theorem by Shafer and Sonnenschein (1975). Section 3 differentiate the assumptions on the structure of ambiguity from those on the attitudes towards ambiguity in the general equilibrium existence theorem. Section 4 introduces the notion of imprecision aversion/loving and studies its properties and its role in the equilibrium existence problem. Section 5 studies in deepen imprecision aversion/loving and all its possible refinements. Sections 6 studies the properties of the belief correspondences in the two specific models arising from incomplete information games and games under strategic ambiguity.

2 The model and equilibrium existence

2.1 The game

Consider a finite set on players $I = \{1, \dots n\}$; for every player i, $\Psi_i = \{\psi_i^1, \dots, \psi_i^{k(i)}\}$ is the (finite) pure strategy set of player i, $\Psi = \prod_{i \in I} \Psi_i$ and $\Psi_{-i} = \prod_{j \neq i} \Psi_j$. Denote with X_i the set of mixed strategies of player i and each strategy $x_i \in X_i$ is a vector $x_i = (x_i(\psi_i))_{\psi_i \in \Psi_i} \in \mathbb{R}_+^{k(i)}$ such that $\sum_{\psi_i \in \Psi_i} x_i(\psi_i) = 1$. Denote also with $X = \prod_{j=1}^n X_j$ and with $X_{-i} = \prod_{j \neq i} X_j$.

First, the primitives of the model are introduced. They are: 1) the structure of ambiguity, 2) the preferences over ambiguous alternatives.

The structure of ambiguity

The set of all the possible outcomes of the game is the finite set[8] Ω, with $|\Omega| = m$. Denote with $\mathcal{P}$ the set of all the probability distributions over Ω, i.e. $\mathcal{P} = \{\varrho \in \mathbb{R}^m \mid \sum_{\omega \in \Omega} \varrho(\omega) = 1, \varrho(\omega) \geqslant 0 \, \forall \omega \in \Omega\}$. so that beliefs will be represented by subsets of $\mathcal{P}$. Beliefs are unambiguous if they are singletons, they are ambiguous otherwise. The information (about the outcomes of the game) available to each player i is summarized by an exogenous set-valued map $\mathcal{B}_i : X \rightsquigarrow \mathcal{P}$, called *belief correspondence*, which gives to player i and for every strategy profile $x \in X$, the ambiguous belief over outcomes $\mathcal{B}_i(x) \subseteq \mathcal{P}$. The set $\mathcal{B}_i(x)$ represents the set of probability distributions over Ω which are feasible and consistent, in view of player i, with the actual strategy profile[9] x. Note that, the standard (unambiguous) normal form games (even under incomplete information) give rise to single valued beliefs correspondences. While complete ignorance would be represented by correspondences such that $\mathcal{B}_i(x) = \mathcal{P}$ for every x. As a final remark, previous literature shows that belief correspondences may come from different sources of ambiguity; Sections 6 provides two examples arising from well known game models under ambiguity.

[8] In previous papers or in some applications, the set of outcomes Ω is a subset of $\mathbb{R}^m$ and each ω_i represents the payoff of player i when outcome $\omega \in \Omega$ is realized. In this paper, Ω does not have necessarily any topological or algebraic structure.

[9] In this view, the strategy set X has a double use: first it represents the set of objects of choice of players but, at the same time, it stands for the set of variables that parameterize the beliefs of each player.

Preferences, game and equilibria

The other primitive of the model is given by the preference relations $\succsim_{i,\mathcal{P}}$ over the subsets of $\mathcal{P}$, for every player i. Only closed subsets of $\mathcal{P}$ are taken into account[10] so that $\succsim_{i,\mathcal{P}}$ is a binary relation over the set $\mathbb{K}(\mathcal{P})$ of all the closed subsets of $\mathcal{P}$. Throughout the paper we assume that

Assumption 2.1: *The preference relation $\succsim_{i,\mathcal{P}}$ is reflexive, that is, $A \succsim_{i,\mathcal{P}} A$ for every $A \in \mathbb{K}(\mathcal{P})$.*

Denote with $A \not\succsim_{i,\mathcal{P}} B$ when it is not true that $A \succsim_{i,\mathcal{P}} B$; then, the strict preference $\succ_{i,\mathcal{P}}$ induced by $\succsim_{i,\mathcal{P}}$ is defined by

$$A \succ_{i,\mathcal{P}} B \iff A \succsim_{i,\mathcal{P}} B \text{ and } B \not\succsim_{i,\mathcal{P}} A \tag{1}$$

Moreover, the indifference relation $\sim_{i,\mathcal{P}}$ induced by $\succsim_{i,\mathcal{P}}$ is defined by

$$A \sim_{i,\mathcal{P}} B \iff A \succsim_{i,\mathcal{P}} B \text{ and } B \succsim_{i,\mathcal{P}} A. \tag{2}$$

Remark 2.2: Note that

$$A \succsim_{i,\mathcal{P}} B \implies B \not\succ_{i,\mathcal{P}} A \text{ and } A \succ_{i,\mathcal{P}} B \implies B \not\succsim_{i,\mathcal{P}} A.$$

Recall that

Definition 2.3: *The preference relation $\succsim_{i,\mathcal{P}}$ is said to be complete if for every pair of subsets A, B in $\mathbb{K}(\mathcal{P})$ it follows that $A \succsim_{i,\mathcal{P}} B$ or $B \succsim_{i,\mathcal{P}} A$ or both.*

Remark 2.4: Note that if $\succsim_{i,\mathcal{P}}$ is complete then

$$A \succsim_{i,\mathcal{P}} B \iff B \not\succ_{i,\mathcal{P}} A \text{ and } A \succ_{i,\mathcal{P}} B \iff B \not\succsim_{i,\mathcal{P}} A.$$

An important role will be played by the following set-valued map $U_{i,\mathcal{P}} : \mathbb{K}(\mathcal{P}) \rightsquigarrow \mathbb{K}(\mathcal{P})$ defined by

$$U_{i,\mathcal{P}}(A) = \{B \in \mathbb{K}(\mathcal{P}) \mid B \succ_{i,\mathcal{P}} A\} \quad \forall A \in \mathbb{K}(\mathcal{P})$$

which gives the *strict upper level sets* of the preference $\succsim_{i,\mathcal{P}}$ for every $A \in \mathbb{K}(\mathcal{P})$.

Therefore, the preference relation $\succsim_i$ of player i over X, is naturally defined as follows

$$x \succsim_i x' \iff \mathcal{B}_i(x) \succsim_{i,\mathcal{P}} \mathcal{B}_i(x')$$

so that the game model considered in this paper is the following

$$\Gamma = \{I; (X_i)_{i\in I}; (\succsim_i)_{i\in I}\}. \tag{3}$$

Denote with $\succ_i$ the strict preference induced by $\succsim_i$, that is

$$y \succ_i x \iff y \succsim_i x \text{ and } x \not\succsim_i y,$$

[10] Applications usually involve closed sets of probability distributions. Moreover, this assumption makes the problem more tractable from the mathematical point of view. Nevertheless, as pointed out by Walley (2000), reasonable probability judgments do not have to be necessarily closed sets.

then, it is possible to define the set-valued map $U_i : X \rightsquigarrow X_i$ as follows

$$U_i(x_i, x_{-i}) = \{x'_i \in X_i \,|\, (x'_i, x_{-i}) \succ_i (x_i, x_{-i})\} \quad \forall (x_i, x_{-i}) \in X$$

or equivalently

$$U_i(x_i, x_{-i}) = \{x'_i \in X_i \,|\, \mathcal{B}_i(x'_i, x_{-i}) \in U_{i,\mathcal{P}}(\mathcal{B}_i(x_i, x_{-i}))\} \quad \forall (x_i, x_{-i}) \in X$$

which gives the *strict upper level sets* in X_i of the preference $\succsim_i$ for every $x \in X$. Hence, the equilibrium notion is naturally the following one:

Definition 2.5: *A strategy profile $\overline{x} \in X$ is an equilibrium under beliefs correspondences $\mathcal{B}_i$ of the game Γ if $U_i(\overline{x}) = \emptyset$ for every $i \in I$.*

The notion of equilibrium under beliefs correspondences is the natural generalization of the classical concept of Nash equilibrium for the present model. In fact, the classical Nash equilibrium concept assumes that rational players will choose the most preferred strategy given their beliefs about what other players will do and it imposes the consistency condition that all players' beliefs are correct. Similarly, in an equilibrium under ambiguous beliefs correspondences x, the strategy x_i is *maximal* to player i with respect to his preference $\succsim_i$, given that his information on the consequences of each strategy choice $x'_i \in X_i$ is consistent with the actual strategy profile chosen, i.e. it is provided by the ambiguous belief over outcomes $\mathcal{B}_i(x'_i, x_{-i})$.

2.2 Equilibrium existence

Aim of this subsection is to provide a general existence results for equilibria under ambiguous belief correspondences as presented in Definition 2.5 building upon the existence result for equilibria in generalized games as presented in Shafer and Sonnenschein (1975).

Preliminaries on set-valued maps

Firstly, well known definitions and results on set-valued maps are recalled. Following Aubin and Frankowska (1990), if Z and Y are two metric spaces and $C : Z \rightsquigarrow Y$ a set-valued map, then

$$i)\ \operatorname*{Lim\,inf}_{z \to z'} C(z) = \left\{ y \in Y \mid \lim_{z \to z'} d(y, C(z)) = 0 \right\},$$

$$ii)\ \operatorname*{Lim\,sup}_{z \to z'} C(z) = \left\{ y \in Y \mid \liminf_{z \to z'} d(y, C(z)) = 0 \right\}$$

and $\operatorname*{Lim\,inf}_{z \to z'} C(z) \subseteq C(z') \subseteq \operatorname*{Lim\,sup}_{z \to z'} C(z)$. Moreover

Definition 2.6: *Given the set valued map $C : Z \rightsquigarrow Y$, then*

i) C is lower semicontinuous in z' if $C(z') \subseteq \operatorname*{Lim\,inf}_{z \to z'} C(z)$; that is, C is lower semicontinuous in z' if for every $y \in C(z')$ and every sequence $(z_\nu)_{\nu \in \mathbb{N}}$ converging to z' there exists a sequence $(y_\nu)_{\nu \in \mathbb{N}}$ converging to y such that $y_\nu \in C(z_\nu)$ for every $\nu \in \mathbb{N}$. Moreover, C is lower semicontinuous in Z if it is lower semicontinuous for all z' in Z.

ii) C is closed in z' if $\operatorname{Lim\,sup}_{z \to z'} C(z) \subseteq C(z')$*; that is, C is closed in z' if for every sequence $(z_\nu)_{\nu \in \mathbb{N}}$ converging to z' and every sequence $(y_\nu)_{\nu \in \mathbb{N}}$ converging to y such that $y_\nu \in C(z_\nu)$ for every $\nu \in \mathbb{N}$, it follows that $y \in C(z')$. Moreover, C is closed in Z if it is closed for all z' in Z;*

iii) C is upper semicontinuous in z' if for every open set U such that $C(z') \subseteq U$ there exists $\eta > 0$ such that $C(z) \subseteq U$ for all $z \in B_Z(z', \eta) = \{\zeta \in Z \mid \|\zeta - z'\| < \eta\}$;

iv) C is continuous (in the sense of Painlevé-Kuratowski) in z' if it is lower semicontinuous and upper semicontinuous in z'.

v) C has an open graph if $Graph(C) = \{(z, y) \in Z \times Y \mid y \in C(z)\}$ is an open set in $Z \times Y$ endowed with the product topology.

The following proposition (see for instance Aubin and Frankowska (1990)) is very useful in this work.

Proposition 2.7: *Assume that Z is closed, Y is compact and the set-valued map $C : Z \rightsquigarrow Y$ has closed values, i.e. $C(z)$ is closed for all $z \in Z$. Then, C is upper semicontinuous in $z \in Z$ if and only if C is closed in z.*

The general equilibrium existence theorem

Denote with $co(D)$ the convex hull of a set D. Then, existence of equilibria under ambiguous beliefs correspondences follows directly from Shafer and Sonnenschein (1975) existence theorem.

Theorem 2.8: *Assume that for every player $i \in I$:*

i) U_i has an open graph;

ii) $x_i \notin co(U_i(x_i, x_{-i}))$ for every $(x_i, x_{-i}) \in X$.

Then, the game Γ defined by (3) has a least an equilibrium.

Proof. The proof is a direct application of Shafer and Sonnenschein (1975) existence theorem. □

Remark 2.9: Shafer and Sonnenschein's theorem has been generalized to the case of discontinuous preferences in many different papers (see, for instance, Scalzo (2015) and references therein). However, the way those results can be applied to the model in this paper, in order to have more refined condition on the primitives of the model, remains an open problem.

3 Differentiating ambiguity from the attitudes towards ambiguity in the equilibrium existence problem

Theorem 2.8 gives a general existence result. This subsection looks more deeply at the primitives of the model and finds conditions for the set-valued maps $\mathcal{B}_i$ and the preferences $\succsim_{i,\mathcal{P}}$ which guarantee that the assumptions of Theorem 2.8 hold. In particular, two proposition are given below. The first states that U_i has an open graph provided that $\mathcal{B}_i$ is a continuous set-valued map and that the correspondence

$U_{i,\mathcal{P}}$ has an open graph. The second result gives conditions in order to obtain that $x_i \notin co(U_i(x_i, x_{-i}))$ for every $(x_i, x_{-i}) \in X$. This latter result is the most interesting. In fact, it is shown that sufficient conditions are obtained by combining a convexity/concavity property of the structure of ambiguity with a condition on the sets $co(U_{i,\mathcal{P}}(A))$ for every $A \in \mathbb{K}(\mathcal{P})$. More precisely, the property of convexity of the set-valued map $\mathcal{B}_i$ must be combined with a pessimistic attitude with respect to imprecision, while the property of concavity of the set-valued map $\mathcal{B}_i$ must be combined with an optimistic attitude with respect to imprecision.

3.1 Continuity

Recall that given the classical Euclidean distance d in $\mathbb{R}^m$ and two closed subsets $A, B \subseteq \mathbb{R}^m$ then the Hausdorff distance is defined by

$$d_H(A, B) = \max\left\{\max_{a\in A} \min_{b\in B} d(a, b), \max_{b\in B} \min_{a\in A} d(a, b)\right\}$$

It is well known that $d_H(\cdot, \cdot)$ is a metric in the set of all closed subset of $\mathbb{R}^n$. Then, the set $\mathbb{K}(\mathcal{P})$ of all the closed subsets of $\mathcal{P}$ is endowed with the topology induced by the Hausdorff distance d_H. The set $\mathcal{P}$ is compact; so, a sequence of sets $\{A_\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{K}(\mathcal{P})$ converges to $A \in \mathbb{K}(\mathcal{P})$ in the Hausdorff metric iff[11]

$$\operatorname*{Lim\,sup}_{\nu\to\infty} A_\nu = \operatorname*{Lim\,inf}_{\nu\to\infty} A_\nu = \operatorname*{Lim}_{\nu\to\infty} A_\nu = A$$

where

$$\operatorname*{Lim\,inf}_{\nu\to\infty} A_\nu = \left\{\varrho \in \mathcal{P} \mid \lim_{\nu\to\infty} d(\varrho, A_\nu) = 0\right\}$$

$$\operatorname*{Lim\,sup}_{\nu\to\infty} A_\nu = \left\{\varrho \in \mathcal{P} \mid \liminf_{\nu\to\infty} d(\varrho, A_\nu) = 0\right\}$$

Then, the proposition below gives explicit conditions on $\mathcal{B}_i$ and $U_{i,\mathcal{P}}$ which guarantee that the assumption *(i)* in Theorem 2.8 is satisfied:

Proposition 3.1: *Assume that the correspondence $U_{i,\mathcal{P}} : \mathbb{K}(\mathcal{P}) \rightsquigarrow \mathbb{K}(\mathcal{P})$ has an open graph and the correspondence $\mathcal{B}_i$ is a continuous set-valued map. Then, the correspondence $U_i : X \rightsquigarrow X_i$ has an open graph.*

Proof. Fix a point $(x, y_i) \in Graph(U_i)$. We must show that there exists a neighborhood $J(x, y_i) \subset X \times X_i$ of the point (x, y_i) such that $J(x, y_i) \subset Graph(U_i)$. Now, $(x, y_i) \in Graph(U_i)$ implies that $(\mathcal{B}_i(x_i, x_{-i}), \mathcal{B}_i(y_i, x_{-i})) \in Graph(U_{i,\mathcal{P}})$. Moreover, $U_{i,\mathcal{P}}$ has an open graph, so there exist ε_1 and ε_2 such that for every pair $(S_1, S_2) \in \mathbb{K}(\mathcal{P}) \times \mathbb{K}(\mathcal{P})$ with

$$d_H(S_1, \mathcal{B}_i(x_i, x_{-i})) < \varepsilon_1 \quad \text{and} \quad d_H(S_2, \mathcal{B}_i(y_i, x_{-i})) < \varepsilon_2$$

it follows that $S_2 \in U_{i,\mathcal{P}}(S_1)$.

[11] Theorem 3.79 in Aliprantis and Border (1999).

Being $\mathcal{B}_i$ a continuous set-valued map, it follows that[12] there exist an open neighborhood $I_1(x_i, x_{-i})$ of (x_i, x_{-i}) and an open neighborhood $I_2(y_i, x_{-i})$ of (y_i, x_{-i}) such that $d_H(\mathcal{B}_i(x'_i, x'_{-i}), \mathcal{B}_i(x_i, x_{-i})) < \varepsilon_1$ for every $(x'_i, x'_{-i}) \in I_1(x_i, x_{-i})$ and $d_H(\mathcal{B}_i(y''_i, x''_{-i}), \mathcal{B}_i(y_i, x_{-i})) < \varepsilon_2$ for every $(y''_i, x''_{-i}) \in I_2(y_i, x_{-i})$. Then,

$$(\mathcal{B}_i(x'_i, x'_{-i}), \mathcal{B}_i(y''_i, x''_{-i})) \in Graph(U_{i,\mathcal{P}}) \quad \text{whenever} \quad (x'_i, x'_{-i}) \in I_1(x_i, x_{-i}) \text{ and } (y''_i, x''_{-i}) \in I_2(y_i, x_{-i}). \quad (4)$$

Now, let $J(x, y_i)$ be the subset of $X \times X_i$ defined by

$$(\widehat{x}, \widehat{y_i}) \in J(x, y_i) \iff (\widehat{x_i}, \widehat{x_{-i}}) \in I_1(x_i, x_{-i}) \quad \text{and} \quad (\widehat{y_i}, \widehat{x_{-i}}) \in I_2(y_i, x_{-i}), \quad (5)$$

and

$$J_1(x, y_i) = \{(\widehat{x}, \widehat{y_i}) \in X \times X_i \,|\, (\widehat{x_i}, \widehat{x_{-i}}) \in I_1(x_i, x_{-i})\}$$
$$J_2(x, y_i) = \{(\widehat{x}, \widehat{y_i}) \in X \times X_i \,|\, (\widehat{y_i}, \widehat{x_{-i}}) \in I_2(y_i, x_{-i})\}$$

It follows that $J_1(x, y_i)$ and $J_2(x, y_i)$ are open subsets of $X \times X_i$ so that $J(x, y_i) = J_1(x, y_i) \cap J_2(x, y_i)$ is a neighborhood of (x, y_i). Moreover, Equations 4 and 5 imply that $(\mathcal{B}_i(\widehat{x_i}, \widehat{x_{-i}}), \mathcal{B}_i(\widehat{y_i}, \widehat{x_{-i}})) \in Graph(U_{i,\mathcal{P}})$ for every $(\widehat{x}, \widehat{y_i}) \in J(x, y_i)$ which implies that $\widehat{y_i} \in U_i(\widehat{x_i}, \widehat{x_{-i}})$ for every $(\widehat{x}, \widehat{y_i}) \in J(x, y_i)$. So $J(x, y_i) \subset Graph(U_i)$ and we get the assertion. □

Remark 3.2: In Shafer and Sonnenschein (1975) the open graph property of the strict upper level set correspondence defines the continuity of the corresponding preference relation. It can be checked that in case of rational preferences, the open graph property is equivalent to the standard continuity notion: $\succsim_{i,\mathcal{P}}$ is continuous if given two sequences $\{A_\nu\}_{\nu\in\mathbb{N}}$ and $\{B_\nu\}_{\nu\in\mathbb{N}}$ in $\mathcal{P}$, such that $A_\nu \succsim_{i,\mathcal{P}} B_\nu$ for every ν and $\operatorname*{Lim}_{\nu\to\infty} A_\nu = A \subseteq \mathcal{P}$ and $\operatorname*{Lim}_{\nu\to\infty} B_\nu = B \subseteq \mathcal{P}$, then it follows that $A \succsim_{i,\mathcal{P}} B$.

3.2 Convexity

In this subsection, sufficient conditions on the primitives of the model are given. They guarantee that the relaxed convexity assumption *(ii)* in Theorem 2.8 is satisfied.

Let W and Y be two linear spaces. Recall that:

Definition 3.3: *Let Z be a convex subset of W, then, the set valued map $C : Z \rightsquigarrow Y$ is a said to be concave if*

$$tC(\overline{z}) + (1-t)C(\widehat{z}) \subseteq C(t\overline{z} + (1-t)\widehat{z}) \quad \forall\, \overline{z}, \widehat{z} \in Z,\ \forall\, t \in [0,1] \quad (6)$$

while it is convex[13] *if*

$$C(t\overline{z} + (1-t)\widehat{z}) \subseteq tC(\overline{z}) + (1-t)C(\widehat{z}) \quad \forall\, \overline{z}, \widehat{z} \in Z,\ \forall\, t \in [0,1]. \quad (7)$$

Finally, $C : Z \rightsquigarrow Y$ is a said to have convex images if $C(z)$ is a convex subset of Y for every $z \in Z$.

Then,

[12] Since the continuous set-valued map $\mathcal{B}_i$ has compact values contained in $\mathcal{P}$, then it can be regarded as a continuous function from X to $\mathbb{K}(\mathcal{P})$ endowed with the topology induced by the Hausdorff metric.

[13] Note that a set-valued map is concave if and only if its graph is a convex set. For this reason, some authors call convex set-valued maps those that here we call concave.

Proposition 3.4: *If one of the following conditions hold*

i) $\mathcal{B}_i(\cdot, x_{-i})$ *is a convex set-valued map in* X_i *for every* $x_{-i} \in X_{-i}$ *and*

$$\forall\, x \in X \quad \nexists A \in co(U_{i,\mathcal{P}}\,(\mathcal{B}_i(x))) \text{ such that } \mathcal{B}_i(x) \subseteq A \tag{8}$$

ii) $\mathcal{B}_i(\cdot, x_{-i})$ *is a concave set-valued map in* X_i *for every* $x_{-i} \in X_{-i}$ *and*

$$\forall\, x \in X \quad \nexists A \in co\,(U_{i,\mathcal{P}}(\mathcal{B}_i(x))) \text{ such that } A \subseteq \mathcal{B}_i(x) \tag{9}$$

then $x_i \notin co(U_i(x_i, x_{-i}))$ *for every* $(x_i, x_{-i}) \in X$.

Proof. Assume that *i)* holds. Suppose that there exists $(x_i, x_{-i}) \in X$ such that $x_i \in co(U_i(x_i, x_{-i}))$. This implies that there exist $y_i, z_i \in U_i(x_i, x_{-i})$ and $t \in [0, 1]$ such that $x_i = ty_i + (1 - t)z_i$. So

$$\mathcal{B}_i(x_i, x_{-i}) = \mathcal{B}_i(ty_i + (1 - t)z_i, x_{-i})$$

Being $\mathcal{B}_i$ a convex set-valued map, it follows that

$$\mathcal{B}_i(ty_i + (1 - t)z_i, x_{-i}) \subseteq t\mathcal{B}_i(y_i, x_{-i}) + (1 - t)\mathcal{B}_i(z_i, x_{-i}) \in co(U_{i,\mathcal{P}}(\mathcal{B}_i(x_i, x_{-i})))$$

which contradicts (8). So $x_i \notin co(U_i(x_i, x_{-i}))$ and we get the assertion. □

Theorem 3.5: *If the assumptions of Proposition 3.1 and one of the assumptions (i), (ii) in Proposition 3.4 hold, then the game has at least an equilibrium.*

Proof. Propositions 3.1, 3.4 imply that the assumptions of Theorem 2.8 are satisfied so that equilibria exist. □

4 Imprecision aversion, equilibria and convexity of preferences

Theorem 3.5 highlights the different role played by the correspondences $\mathcal{B}_i$ and $U_{i,\mathcal{P}}$, for $i = 1, \ldots, n$ to have the required relaxed convexity of preferences *(ii)* in Theorem 2.8. In other words, it allows to differentiate the conditions imposed on the structure of ambiguity (i.e. correspondences $\mathcal{B}_i$) from those imposed on the attitudes towards ambiguity (i.e. the preferences $\succsim_{i,\mathcal{P}}$). In particular, Proposition 3.4 tells that, in order to obtain that $x_i \notin co(U_i(x_i, x_{-i})$ for every $x \in X$, the convexity/concavity properties of the correspondences $\mathcal{B}_i(\cdot, x_{-i})$ must be combined with conditions (8)/(9). These latter conditions are rather general but lack an intuitive interpretation, at a first sight. However, it is shown below that such conditions involve the attitudes of the decision maker towards the imprecision of the probabilistic belief, (i.e. they involve the attitudes towards the inclusion relation between sets of probability distributions over outcomes). To illustrate this idea and keep the analysis simple, consider the case in which the set-valued map $U_{i,\mathcal{P}}$ has convex images so that $U_{i,\mathcal{P}}(S) = co(U_{i,\mathcal{P}}(S))$ for every set $S \in \mathbb{K}(\mathcal{P})$; it follows that condition (8) (resp. condition 9) will be satisfied when agent i does not prefer a set A to a set B if the latter is a subset of the former[14](resp. if the former is a subset of the latter). An agent showing such behavioral feature is called in this paper *imprecision averse*[15] (resp. *imprecision loving*).

In this section, it is firstly investigated the role played by imprecision averse (imprecision loving) preferences and their refinements in the problem of existence of equilibria. Then, imprecision aversion (resp. imprecision loving) is studied in the framework of rational preferences, highlighting the relation with the notion of convex preferences and with the classical models such as maximin, maximax and interval dominance preferences (see for example Troffaes 2007).

4.1 Imprecision aversion and equilibrium existence

Definition 4.1: *The preference relation $\succsim_{i,\mathcal{P}}$ is said to be*

i) imprecision averse if

$$A \subseteq B \implies B \not\succ_{i,\mathcal{P}} A$$

ii) imprecision loving if

$$A \subseteq B \implies A \not\succ_{i,\mathcal{P}} B$$

Section 5 below shows that imprecision averse (resp imprecision loving) preferences represent a minimally pessimistic (resp. optimistic) attitude with respect to imprecision; In fact, there exist other reasonable definitions of pessimistic (resp. optimistic) behavioral trait which indeed are refinements of imprecision aversion (loving), i.e. they all imply imprecision aversion (loving). Section 5 presents a detailed analysis of the interconnections between these concepts; in the present Section 4, we give just the definition of β-imprecision monotone (resp. δ-imprecision monotone) preferences which are a refinement of imprecision averse (loving) preferences. It will be shown below in propositions (4.3,4.4) that imprecision monotone preferences allow to relax (in the existence theorem) the assumption of convexity imposed on the images of the set-valued map $U_{i,\mathcal{P}}$.

More precisely,

Definition 4.2: *The preference relation $\succsim_{i,\mathcal{P}}$ is said to be*

i) β-imprecision monotone if

$$A \succ_{i,\mathcal{P}} B \implies A \subset B$$

ii) δ-imprecision monotone if

$$A \succ_{i,\mathcal{P}} B \implies B \subset A$$

Then,

Proposition 4.3: *Assume that $\mathcal{B}_i(\cdot, x_{-i})$ is a convex set-valued map. If one of the following two conditions hold*

[14]This idea will be better formalized by Propositions (4.3 4.4).

[15]The idea to look at the effects of preference situations which rank imprecise probabilistic uncertain alternatives according to the set inclusion order has already been investigated in Gajdos, Tallon and Vergnaud (2004). However, the approach is different. Firstly they deal with the issue of the representation of rational preferences by real functionals. Secondly, the ranking depends also on anchors which are elements that are external to the set of probability distributions.

1) $\succsim_{i,\mathcal{P}}$ is β imprecision monotone, the set-valued map $\mathcal{B}_i$ has convex images and $S \notin co\left(U_{i,\mathcal{P}}(S)\right)$ for every $S \in \mathbb{K}(\mathcal{P})$,

2) $\succsim_{i,\mathcal{P}}$ is imprecision averse and the set-valued map $U_{i,\mathcal{P}}$ has convex images,

then $x_i \notin co(U_i(x_i, x_{-i}))$ for every $(x_i, x_{-i}) \in X$.

Proof. 1). Let $I \in \mathbb{K}(\mathcal{P})$ be a convex set. Suppose that there exists $A \in co(U_{i,\mathcal{P}}(I))$ such that $I \subseteq A$. Then, there exist $B, C \in U_{i,\mathcal{P}}(I)$ and $t \in [0,1]$ such that $A = tB + (1-t)C$. Since $B, C \succ_{i,\mathcal{P}} I$ and $\succsim_{i,\mathcal{P}}$ is β imprecision monotone then $B, C \subset I$. Being I a convex set, it follows that $A = tB + (1-t)C \subseteq I$. Since $I \subseteq A$ then, it must be that $I = A$. Since $A \in co(U_{i,\mathcal{P}}(I))$ then $I \in co(U_{i,\mathcal{P}}(I))$ which contradicts the assumption. Hence, for every convex and closed subset I of $\mathcal{P}$ there does not exists any subset $A \in co(U_{i,\mathcal{P}}(I))$ such that $I \subseteq A$.

Now, $\mathcal{B}_i(x)$ is a convex set, then it follows that $\nexists A \in co\left(U_{i,\mathcal{P}}(\mathcal{B}_i(x))\right)$ such that $\mathcal{B}_i(x) \subseteq A$. Since this holds for every $x \in X$ and $\mathcal{B}_i$ is a convex set-valued map then, from Proposition 3.4, the assertion follows.

2). Fix $I \in \mathbb{K}(\mathcal{P})$ and suppose that there exists $A \in U_{i,\mathcal{P}}(I)$ such that $I \subseteq A$. Since $\succsim_{i,\mathcal{P}}$ is imprecision averse, it follows that $A \not\succ_{i,\mathcal{P}} I$. But $A \succ_{i,\mathcal{P}} I$ and hence we get a contradiction. Therefore, for every $I \in \mathbb{K}(\mathcal{P})$, there does not exist $A \in U_{i,\mathcal{P}}(I)$ such that $I \subseteq A$. From the assumption $co\left(U_{i,\mathcal{P}}(I)\right) = U_{i,\mathcal{P}}(I)$ for every $I \in \mathbb{K}(\mathcal{P})$ and then for every $I \in \mathbb{K}(\mathcal{P})$ it follows that

$$\nexists A \in co(U_{i,\mathcal{P}}(I)) \text{ such that } I \subseteq A.$$

Hence, it follows that $\nexists A \in co\left(U_{i,\mathcal{P}}(\mathcal{B}_i(x))\right)$ such that $\mathcal{B}_i(x) \subseteq A$ for every $x \in X$. Since $\mathcal{B}_i$ is a convex set-valued map then, from Proposition 3.4, the assertion follows. □

In case of optimistic preferences, the following result holds:

Proposition 4.4: *Assume that $\mathcal{B}_i(\cdot, x_{-i})$ is a concave set-valued map for every $x_{-i} \in X_{-i}$. If one of the following two conditions hold*

1) $\succsim_{i,\mathcal{P}}$ is δ imprecision monotone and $S \notin co\left(U_{i,\mathcal{P}}(S)\right)$ for every $S \in \mathbb{K}(\mathcal{P})$,

2) $\succsim_{i,\mathcal{P}}$ is imprecision loving and the set-valued map $U_{i,\mathcal{P}}$ has convex images,

then $x_i \notin co(U_i(x_i, x_{-i}))$ for every $(x_i, x_{-i}) \in X$.

Proof. The proof is substantially the same of Proposition 4.3 and it is omitted. □

Note that Proposition 4.4 is specular to Proposition 4.3 with the unique exception that it is not required to impose the additional assumption that the belief correspondences $\mathcal{B}_i$ has convex images; in fact, by definition it follows that if $\mathcal{B}_i(\cdot, x_{-i})$ is concave then $\mathcal{B}_i$ has convex images.

4.2 Imprecision aversion and rational preferences

In this subsection, the particular case of rational (that is complete and transitive) preferences is studied. Firstly, it is shown that in case of rational preferences and under the assumption of propositions (4.3,4.4) imprecision aversion and imprecision loving imply the convexity of players' preferences which, in turn, is a standard and key assumption in the classical game models where preferences are defined by utility functions. In the second part of this subsection, some classical models of preference relation are studied and it is shown that they satisfy the assumptions of propositions (4.3,4.4).

Convexity

Firstly, recall that

Definition 4.5: *The preference relation $\succsim_{i,\mathcal{P}}$ is said to be convex if*

$$A \succsim_{i,\mathcal{P}} S \text{ and } B \succsim_{i,\mathcal{P}} S \implies tA + (1-t)B \succsim_{i,\mathcal{P}} S \quad \forall t \in [0,1],\ \forall A, B \subseteq \mathcal{P}.$$

and

Definition 4.6: *The preference $\succsim_i$ is convex in player i's strategy if, for every $x_{-i} \in X_{-i}$, it satisfies the following condition:*

$$(x_i', x_{-i}) \succsim_i (x_i, x_{-i}) \text{ and } (x_i'', x_{-i}) \succsim_i (x_i, x_{-i}) \implies (tx_i' + (1-t)x'', x_{-i}) \succsim_i (x_i, x_{-i}) \quad \forall t \in [0,1].$$

Then,

Proposition 4.7: *Assume that*

i) the correspondence $\mathcal{B}_i(\cdot, x_{-i})$ is convex (resp. concave) in X_i for every $x_{-i} \in X_{-i}$,

ii) the preference $\succsim_{i,\mathcal{P}}$ is

ii,a) complete and transitive,

ii,b) imprecision averse (resp. imprecision loving),

ii,c) convex.

Then, the preference $\succsim_i$ is convex in player i's strategy.

Proof. We give the proof in the case $\mathcal{B}_i(\cdot, x_{-i})$ is convex and $\succsim_{i,\mathcal{P}}$ is imprecision averse. The other case follows immediately.

Let

$$(x_i', x_{-i}) \succsim_i (x_i, x_{-i}) \text{ and } (x_i'', x_{-i}) \succsim_i (x_i, x_{-i}).$$

By definition it follows that

$$\mathcal{B}_i(x_i', x_{-i}) \succsim_{i,\mathcal{P}} \mathcal{B}_i(x_i, x_{-i}) \text{ and } \mathcal{B}_i(x_i'', x_{-i}) \succsim_{i,\mathcal{P}} \mathcal{B}_i(x_i, x_{-i})$$

Since $\succsim_{i,\mathcal{P}}$ is convex, it follows that

$$t\mathcal{B}_i(x_i', x_{-i}) + (1-t)\mathcal{B}_i(x_i'', x_{-i}) \succsim_{i,\mathcal{P}} \mathcal{B}_i(x_i, x_{-i}).$$

Then, being $\mathcal{B}_i(\cdot, x_{-i})$ a convex set-valued map, we get

$$\mathcal{B}_i(tx_i'(1-t)x_i'', x_{-i}) \subseteq t\mathcal{B}_i(x_i', x_{-i}) + (1-t)\mathcal{B}_i(x_i'', x_{-i})$$

The preference $\succsim_{i,\mathcal{P}}$ is imprecision averse so

$$t\mathcal{B}_i(x_i', x_{-i}) + (1-t)\mathcal{B}_i(x_i'', x_{-i}) \not\succ_{i,\mathcal{P}} \mathcal{B}_i(tx_i'(1-t)x_i'', x_{-i})$$

The completeness of $\succsim_{i,\mathcal{P}}$ implies that

$$\mathcal{B}_i(tx_i'(1-t)x_i'', x_{-i}) \succsim_{i,\mathcal{P}} t\mathcal{B}_i(x_i', x_{-i}) + (1-t)\mathcal{B}_i(x_i'', x_{-i}).$$

The transitivity of $\succsim_{i,\mathcal{P}}$ finally implies that $\mathcal{B}_i(tx_i'(1-t)x_i'', x_{-i}) \succsim_{i,\mathcal{P}} \mathcal{B}_i(x_i, x_{-i})$ and the assertion follows. □

Special cases

Here we look at classical models of rational preferences in case of imprecise probabilities, (see for instance Walley (1991) or Troffaes 2007)), such as maximin preferences, maximax preferences and interval dominance and show that they are imprecision averse/loving and convex preferences.

Let $f_i : \mathcal{P} \to \mathbb{R}$ be a continuous function which gives to player i the utility $f_i(\varrho)$ of every lottery $\varrho \in \mathcal{P}$[16]. Then

Definition 4.8: *For every $A \in \mathbb{K}(\mathcal{P})$, let $F_i(A) = \min_{\varrho\in A} f_i(\varrho)$. Then the preference $\succsim^m_{i,\mathcal{P}}$ defined, for every $(A, B) \in \mathcal{P}\times\mathcal{P}$, by*

$$A \succsim^m_{i,\mathcal{P}} B \iff F_i(A) \geqslant F_i(B)$$

is a maximin preference.

It immediately follows that:

Proposition 4.9: *Given $f_i : \mathcal{P} \to \mathbb{R}$, let $\succsim^m_{i,\mathcal{P}}$ be the corresponding maximin preference. Then,*

i) $\succsim^m_{i,\mathcal{P}}$ is imprecision averse.

ii) If f_i is a concave function then $\succsim^m_{i,\mathcal{P}}$ is a convex preference.

Proof. *i)* Obvious, in fact $A \subseteq B$ implies that $\min_{\varrho\in B} f_i(\varrho) \leqslant \min_{\varrho\in A} f_i(\varrho)$.

ii) Let f_i be a concave function. For every $t \in [0, 1]$, $F_i(tA + (1-t)B) = f_i(t\varrho_A + (1-t)\varrho_B)$ for some $\varrho_A \in A$ and $\varrho_B \in B$. By definition, it follows that $f_i(\varrho_A) \geqslant F_i(A)$ and $f_i(\varrho_B) \geqslant F_i(B)$. Since f_i a concave function, it follows that $f_i(t\varrho_A + (1-t)\varrho_B) \geqslant t f_i(\varrho_A) + (1-t) f_i(\varrho_B)$. Summarizing:

$$F_i(tA + (1-t)B) = f_i(t\varrho_A + (1-t)\varrho_B) \geqslant t f_i(\varrho_A) + (1-t) f_i(\varrho_B) \geqslant tF_i(A) + (1-t)F_i(B).$$

Since the previous inequality holds for every $A, B \subseteq \mathcal{P}$ and every $t \in [0, 1]$, it immediately follows that the preference $\succsim^m_{i,\mathcal{P}}$ deriving from F_i is convex. □

Definition 4.10: *For every $A \in \mathbb{K}(\mathcal{P})$, let $G_i(A) = \max_{\varrho\in A} f_i(\varrho)$. Then the preference $\succsim^M_{i,\mathcal{P}}$ defined, for every $(A, B) \in \mathcal{P}\times\mathcal{P}$, by*

$$A \succsim^M_{i,\mathcal{P}} B \iff G_i(A) \geqslant G_i(B)$$

is a maximax preference.

It immediately follows that:

Proposition 4.11: *Given $f_i : \mathcal{P} \to \mathbb{R}$, let $\succsim^M_{i,\mathcal{P}}$ be the corresponding maximax preference. Then,*

i) $\succsim^M_{i,\mathcal{P}}$ is imprecision loving.

ii) If f_i is a a quasi-concave function[17]*, then $\succsim^M_{i,\mathcal{P}}$ is a convex preference.*

[16] Usually, f_i is the classical expected utility, but it can be something different as in the variational preference model.

[17] That is $f_i(t\varrho' + (1-t)\varrho'') \geqslant \min\{f_i(\varrho'), f_i(\varrho'')\}$ for every $\varrho', \varrho'' \in \mathcal{P}$ and $t \in [0, 1]$. Recall also that if f_i is concave then it is quasi-concave but the converse statement does not hold.

Proof. *i)* Obvious, in fact $A \subseteq B$ implies that $\max_{\varrho\in B} f_i(\varrho) \geqslant \max_{\varrho\in A} f_i(\varrho)$.

ii) Let $\varrho_A \in A$ and $\varrho_B \in B$ be such that $G_i(A) = g_i(\varrho_A)$ and $G_i(B) = g_i(\varrho_B)$. Fix $t \in [0,1]$, then $t\varrho_A + (1-t)\varrho_B \in tA + (1-t)B$ which implies that $G_i(tA + (1-t)B) \geqslant f_i(t\varrho_A + (1-t)\varrho_B)$. Since f_i is quasi-concave, it follows that

$$G_i(tA + (1-t)B) \geqslant f_i(t\varrho_A + (1-t)\varrho_B) \geqslant \min\{f_i(\varrho_A), f_i(\varrho_B)\} = \min\{G_i(A), G_i(B)\}.$$

Since the previous inequality holds for every $A, B \subseteq \mathcal{P}$ and every $t \in [0,1]$, it immediately follows that the preference $\succsim^M_{i,\mathcal{P}}$ deriving from G_i is convex. □

Definition 4.12: *For every $A \in \mathbb{K}(\mathcal{P})$, let $F_i(A) = \min_{\varrho\in A} f_i(\varrho)$ and $G_i(A) = \max_{\varrho\in A} f_i(\varrho)$. Then the preference $\succsim^D_{i,\mathcal{P}}$ defined, for every $(A, B) \in \mathcal{P}\times\mathcal{P}$, by*

$$A \succsim^D_{i,\mathcal{P}} B \iff F_i(A) \geqslant G_i(B)$$

is an interval dominance preference.

It immediately follows that:

Proposition 4.13: *Given $f_i : \mathcal{P} \to \mathbb{R}$, let $\succsim^D_{i,\mathcal{P}}$ be the corresponding interval dominance preference. Then,*

i) $\succsim^D_{i,\mathcal{P}}$ is imprecision averse.

ii) If f_i is concave, then $\succsim^D_{i,\mathcal{P}}$ is a convex preference.

Proof. *i)* If $A \subseteq B$ then it immediately follows that

$$F_i(B) \leqslant F_i(A) \leqslant G_i(A) \leqslant G_i(B).$$

Therefore $A \not\succsim^D_{i,\mathcal{P}} B$ and $B \not\succsim^D_{i,\mathcal{P}} A$ and hence if $A \subseteq B$ then $B \not\succ^D_{i,\mathcal{P}} A$ and $\succsim^D_{i,\mathcal{P}}$ is imprecision averse.

ii) Suppose that $A \succsim^D_{i,\mathcal{P}} S$ and $B \succsim^D_{i,\mathcal{P}} S$, meaning that $F_i(A), F_i(B) \geqslant G_i(S)$. For every $t \in [0,1]$, $F_i(tA + (1-t)B) = f_i(t\varrho_A + (1-t)\varrho_B)$ for some $\varrho_A \in A$ and $\varrho_B \in B$. By definition, it follows that $f_i(\varrho_A) \geqslant F_i(A)$ and $f_i(\varrho_B) \geqslant F_i(B)$. Since f_i a concave function, it follows that $f_i(t\varrho_A + (1-t)\varrho_B) \geqslant t f_i(\varrho_A) + (1-t) f_i(\varrho_B)$. Summarizing:

$$F_i(tA + (1-t)B) = f_i(t\varrho_A + (1-t)\varrho_B) \geqslant t f_i(\varrho_A) + (1-t) f_i(\varrho_B) \geqslant tF_i(A) + (1-t)F_i(B) \geqslant G_i(S).$$

Since the previous inequality holds for every $A, B \subseteq \mathcal{P}$ and every $t \in [0,1]$, it immediately follows that the preference $\succsim^D_{i,\mathcal{P}}$ is convex. □

Remark 4.14: It is immediate to check that $\succsim^m_{i,\mathcal{P}}$ is not β imprecision monotone and $\succsim^M_{i,\mathcal{P}}$ is not δ imprecision monotone. Moreover, the proof of Proposition 4.13 shows that $\succsim^D_{i,\mathcal{P}}$ is not β imprecision monotone as well.

5 Attitudes towards ambiguity

This section is devoted to imprecision averse and imprecision loving preferences and their refinements. The previous section highlights the role of imprecision aversion/loving and β/δ imprecision monotonicity in the equilibrium existence problem. Aim of this section is to provide a more comprehensive analysis of the possible attitudes towards ambiguity in the particular framework of this paper in which uncertainty is described by sets of probability distributions. Therefore, it is shown below that there exist other reasonable definitions of pessimistic (resp. optimistic) behavioral trait which turn to be refinements of imprecision aversion (loving), i.e. they all imply imprecision aversion (loving). Particular attention is given to the way all those concepts are mutually related in the general case but also in case of rational preferences.

The definition of imprecision averse and imprecision loving preferences suggests that the characterizations of agent i's attitude towards imprecision can be expressed in terms of implications of the following form

$$A\mathcal{R}_1 B \implies A\mathcal{R}_2 B \tag{10}$$

where

$$\mathcal{R}_1 \in \{\succsim_{i,\mathcal{P}},\ \succ_{i,\mathcal{P}}, \not\succsim_{i,\mathcal{P}},\ \not\succ_{i,\mathcal{P}},\} \quad \text{and} \quad \mathcal{R}_2 \in \{\subseteq,\ \subset,\ \not\supseteq,\ \not\supset,\ \not\subseteq,\ \not\subset,\ \supseteq,\ \supset\}$$

or

$$\mathcal{R}_1 \in \{\subseteq,\ \subset,\ \not\subseteq,\ \not\subset\} \quad \text{and} \quad \mathcal{R}_2 \in \{\succsim_{i,\mathcal{P}},\ \succ_{i,\mathcal{P}},\ \not\precsim_{i,\mathcal{P}},\ \not\prec_{i,\mathcal{P}},\ \not\succsim_{i,\mathcal{P}},\ \not\succ_{i,\mathcal{P}},\ \precsim_{i,\mathcal{P}},\ \prec_{i,\mathcal{P}}\}$$

where $A \precsim_{i,\mathcal{P}} B$ means that $B \succsim_{i,\mathcal{P}} A$.

The complete list of combinations can be summarized by the following table:

	Pessimism	Optimism
$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_2$
$\subseteq$	$\succsim_{i,\mathcal{P}},\ \succ_{i,\mathcal{P}},\ \not\precsim_{i,\mathcal{P}},\ \not\prec_{i,\mathcal{P}}$	$\not\succsim_{i,\mathcal{P}},\ \not\succ_{i,\mathcal{P}},\ \precsim_{i,\mathcal{P}},\ \prec_{i,\mathcal{P}}$
$\subset$	$\succsim_{i,\mathcal{P}},\ \succ_{i,\mathcal{P}},\ \not\precsim_{i,\mathcal{P}},\ \not\prec_{i,\mathcal{P}}$	$\not\succsim_{i,\mathcal{P}},\ \not\succ_{i,\mathcal{P}},\ \precsim_{i,\mathcal{P}},\ \prec_{i,\mathcal{P}}$
$\succsim_{i,\mathcal{P}}$	$\subseteq,\ \subset,\ \not\supseteq,\ \not\supset$	$\not\subseteq,\ \not\subset,\ \supseteq,\ \supset$
$\succ_{i,\mathcal{P}}$	$\subseteq,\ \subset,\ \not\supseteq,\ \not\supset$	$\not\subseteq,\ \not\subset,\ \supseteq,\ \supset$
$\not\subseteq$	$\not\succsim_{i,\mathcal{P}},\ \not\succ_{i,\mathcal{P}},\ \precsim_{i,\mathcal{P}},\ \prec_{i,\mathcal{P}}$	$\succsim_{i,\mathcal{P}},\ \succ_{i,\mathcal{P}},\ \not\precsim_{i,\mathcal{P}},\ \not\prec_{i,\mathcal{P}}$
$\not\subset$	$\not\succsim_{i,\mathcal{P}},\ \not\succ_{i,\mathcal{P}},\ \precsim_{i,\mathcal{P}},\ \prec_{i,\mathcal{P}}$	$\succsim_{i,\mathcal{P}},\ \succ_{i,\mathcal{P}},\ \not\precsim_{i,\mathcal{P}},\ \not\prec_{i,\mathcal{P}}$
$\not\succsim_{i,\mathcal{P}}$	$\not\subseteq,\ \not\subset,\ \supseteq,\ \supset$	$\subseteq,\ \subset,\ \not\supseteq,\ \not\supset$
$\not\succ_{i,\mathcal{P}}$	$\not\subseteq,\ \not\subset,\ \supseteq,\ \supset$	$\subseteq,\ \subset,\ \not\supseteq,\ \not\supset$

(11)

Notice first that the 64 combinations have been divided in two groups which represent respectively pessimism and optimism. Indeed, it can be easily checked that imprecision aversion and its six refinements belong to the left hand side of the table, while the seven specular definitions involving imprecision loving agents belong to the right hand side of the table. More precisely, it turns out that in the left hand side of the table, 22 out of the 32 combinations are equivalent to one of the behavioral attitudes that are given in Definition 5.2 below. Moreover, it is shown below in Remark 5.10 that the remaining 10 combinations are not compatible with a reflexive preference relation, so that we are able to identify the role played by each of the 32 combinations. Identical arguments hold for the right hand side of the table.

Remark 5.1: The reason why, in formula (10), it is sufficient to consider the relation $\mathcal{R}_1$ only in the set $\{\succsim_{i,\mathcal{P}}, \succ_{i,\mathcal{P}}, \not\succsim_{i,\mathcal{P}}, \nsucc_{i,\mathcal{P}},\}$ and not in the larger set $\{\succsim_{i,\mathcal{P}}, \succ_{i,\mathcal{P}}, \not\precsim_{i,\mathcal{P}}, \nprec_{i,\mathcal{P}}, \not\succsim_{i,\mathcal{P}}, \nsucc_{i,\mathcal{P}}, \precsim_{i,\mathcal{P}}, \prec_{i,\mathcal{P}}\}$ is because both sets give precisely the same sets of implications. For instance, $A \succsim_{i,\mathcal{P}} B \implies A \subseteq B$ is equivalent to $A \precsim_{i,\mathcal{P}} B \implies A \supseteq B$. The same arguments hold in the second case where $\mathcal{R}_1$ belongs to $\{\subseteq, \subset, \nsubseteq, \not\subset\}$ instead of $\{\subseteq, \subset, \nsupseteq, \not\supset, \nsubseteq, \not\subset, \supseteq, \supset\}$.

5.1 Pessimism

Definition 5.2: *The preference relation $\succsim_{i,\mathcal{P}}$ is said to be*

P_1) *imprecision averse if the following equivalent conditions are satisfied:*

- *i)* $A \subseteq B \implies B \nsucc_{i,\mathcal{P}} A$
- *ii)* $A \succ_{i,\mathcal{P}} B \implies B \nsubseteq A$
- *iii)* $A \succ_{i,\mathcal{P}} B \implies B \not\subset A$
- *iv)* $A \subset B \implies B \nsucc_{i,\mathcal{P}} A$

P_2) *strongly imprecision averse if the following equivalent conditions are satisfied:*

- *i)* $A \subset B \implies B \not\succsim_{i,\mathcal{P}} A$
- *ii)* $A \succsim_{i,\mathcal{P}} B \implies B \not\subset A$

P_3) *α imprecision monotone if the following equivalent conditions are satisfied:*

- *i)* $A \subseteq B \implies A \succsim_{i,\mathcal{P}} B$
- *ii)* $A \not\succsim_{i,\mathcal{P}} B \implies A \nsubseteq B$
- *iii)* $A \subset B \implies A \succsim_{i,\mathcal{P}} B$
- *iv)* $A \not\succsim_{i,\mathcal{P}} B \implies A \not\subset B$

P_4) *β imprecision monotone if the following equivalent conditions are satisfied:*

- *i)* $A \succ_{i,\mathcal{P}} B \implies A \subset B$
- *ii)* $A \not\subset B \implies A \nsucc_{i,\mathcal{P}} B$
- *iii)* $A \succ_{i,\mathcal{P}} B \implies A \subseteq B$
- *iv)* $A \nsubseteq B \implies A \nsucc_{i,\mathcal{P}} B$

P_5) *strongly α imprecision monotone if the following equivalent conditions are satisfied:*

- *i)* $A \subset B \implies A \succ_{i,\mathcal{P}} B$
- *ii)* $A \nsucc_{i,\mathcal{P}} B \implies A \not\subset B$

P_6) *strongly β imprecision monotone if the following equivalent conditions are satisfied:*

i) $A \succsim_{i,\mathcal{P}} B \implies A \subseteq B$

ii) $A \nsubseteq B \implies A \not\succsim_{i,\mathcal{P}} B$

P_7*) noninclusion averse if the following equivalent conditions are satisfied:*

i) $A \nsubseteq B \implies B \succsim_{i,\mathcal{P}} A$

ii) $A \not\succsim_{i,\mathcal{P}} B \implies B \subseteq A$

iii) $A \not\subset B \implies B \succsim_{i,\mathcal{P}} A$

iv) $A \not\succsim_{i,\mathcal{P}} B \implies B \subset A$

REMARK 5.3: The reason for the equivalence of the conditions in each of the previous definitions comes from two observations:

1) Given a binary relation $\mathcal{R}$, denote with $A\mathcal{R}^{not}B$ if it is not true that $A\mathcal{R}B$. Now it is well known that the condition $A\mathcal{R}_1B \implies A\mathcal{R}_2B$ is equivalent to $A\mathcal{R}_2^{not}B \implies A\mathcal{R}_1^{not}B$. For instance, $A \subseteq B \implies B \not\succ_{i,\mathcal{P}} A$ is equivalent to $A \succ_{i,\mathcal{P}} B \implies B \nsubseteq A$.

2) $A \subset B \implies A \subseteq B$; $A \nsubseteq B \implies A \not\subset B$.

It is also useful to give the following

DEFINITION 5.4: *The preference relation $\succsim_{i,\mathcal{P}}$ is said to be noninclusion indifferent if*

$$\begin{cases} A \nsubseteq B \\ B \nsubseteq A \end{cases} \implies A \sim_{i,\mathcal{P}} B \tag{12}$$

We have the following implications

PROPOSITION 5.5: *Given the preference relation $\succsim_{i,\mathcal{P}}$ then:*

i) If one of the following conditions holds

a) $\succsim_{i,\mathcal{P}}$ *is α imprecision monotone*

b) $\succsim_{i,\mathcal{P}}$ *is β imprecision monotone*

then $\succsim_{i,\mathcal{P}}$ is imprecision averse.

ii) If one of the following conditions holds

a′) $\succsim_{i,\mathcal{P}}$ *is strongly α imprecision monotone*

b′) $\succsim_{i,\mathcal{P}}$ *is strongly β imprecision monotone*

then $\succsim_{i,\mathcal{P}}$ is strongly imprecision averse.

iii) If $\succsim_{i,\mathcal{P}}$ is strongly imprecision averse then $\succsim_{i,\mathcal{P}}$ is imprecision averse.

iv) If $\succsim_{i,\mathcal{P}}$ is strongly α imprecision monotone then $\succsim_{i,\mathcal{P}}$ is α imprecision monotone.

v) If $\succsim_{i,\mathcal{P}}$ is strongly β imprecision monotone then $\succsim_{i,\mathcal{P}}$ is β imprecision monotone.

vi) $\succsim_{i,\mathcal{P}}$ is noninclusion averse if and only if $\succsim_{i,\mathcal{P}}$ is noninclusion indifferent and α imprecision monotone.

vii) $\succsim_{i,\mathcal{P}}$ is noninclusion averse if and only if $\succsim_{i,\mathcal{P}}$ is complete and β imprecision monotone.

Proof. i) Suppose that (a) holds, let A, B be such that $A \subseteq B$. Then (a) implies that $A \succsim_{i,\mathcal{P}} B$ which immediately implies that $B \not\succ_{i,\mathcal{P}} A$. So, $\succsim_{i,\mathcal{P}}$ is imprecision averse.

Suppose that (b) holds, let A, B be such that $A \subseteq B$. If $B \succ_{i,\mathcal{P}} A$ then (b) implies that $B \subset A$, but this is a contradiction since $A \subseteq B$. It follows that $B \not\succ_{i,\mathcal{P}} A$ and $\succsim_{i,\mathcal{P}}$ is imprecision averse.

ii) Suppose that (a′) holds, let A, B be such that $A \subset B$. Then (a′) implies that $A \succ_{i,\mathcal{P}} B$ which immediately implies that $B \not\succsim_{i,\mathcal{P}} A$. So, $\succsim_{i,\mathcal{P}}$ is strongly imprecision averse.

Suppose that (b′) holds, let A, B be such that $A \subset B$. If $B \succsim_{i,\mathcal{P}} A$ then (b′) implies that $B \subseteq A$, but this is a contradiction since $A \subset B$. It follows that $B \not\succsim_{i,\mathcal{P}} A$ and $\succsim_{i,\mathcal{P}}$ is strongly imprecision averse.

iii) Suppose that $A \subseteq B$. Since $\succsim_{i,\mathcal{P}}$ is reflexive, if $A = B$ then it follows that $A \succsim_{i,\mathcal{P}} B$ and $B \succsim_{i,\mathcal{P}} A$, which imply that $B \not\succ_{i,\mathcal{P}} A$. Since $\succsim_{i,\mathcal{P}}$ is strongly imprecision averse, $A \subset B$ implies $B \not\succsim_{i,\mathcal{P}} A$ and then it immediately follows that $B \not\succ_{i,\mathcal{P}} A$. Hence, $\succsim_{i,\mathcal{P}}$ is imprecision averse.

iv) Suppose that $A \subseteq B$. Since $\succsim_{i,\mathcal{P}}$ is reflexive, $A = B$ implies that $A \succsim_{i,\mathcal{P}} B$. Since $\succsim_{i,\mathcal{P}}$ is strongly α imprecision monotone, $A \subset B$ implies $A \succ_{i,\mathcal{P}} B$. Hence, $A \subseteq B$ implies $A \succsim_{i,\mathcal{P}} B$ and $\succsim_{i,\mathcal{P}}$ is α imprecision monotone.

v) Suppose that $A \succ_{i,\mathcal{P}} B$. Since $\succsim_{i,\mathcal{P}}$ is strongly β imprecision monotone, then it follows that $A \subseteq B$. Since $\succsim_{i,\mathcal{P}}$ is reflexive, $A = B$ implies that $B \succsim_{i,\mathcal{P}} A$ which is not possible since $A \succ_{i,\mathcal{P}} B$. Hence, it must be that $A \subset B$. Therefore, $\succsim_{i,\mathcal{P}}$ is β imprecision monotone.

vi) If $\succsim_{i,\mathcal{P}}$ is noninclusion averse then it is noninclusion indifferent, by definition. Moreover, let $A \subset B$ then $B \nsubseteq A$ which implies that $A \succsim_{i,\mathcal{P}} B$. Since $\succsim_{i,\mathcal{P}}$ is reflexive then it follows that $A \subseteq B \implies A \succsim_{i,\mathcal{P}} B$ and $\succsim_{i,\mathcal{P}}$ is α monotone. Conversely, let $\succsim_{i,\mathcal{P}}$ be noninclusion indifferent and α monotone and let A, B be such that $A \nsubseteq B$. If $B \nsubseteq A$ then $A \sim_{i,\mathcal{P}} B$ since $\succsim_{i,\mathcal{P}}$ is noninclusion indifferent and then $B \succsim_{i,\mathcal{P}} A$. If $B \subseteq A$ then $B \succsim_{i,\mathcal{P}} A$ since $\succsim_{i,\mathcal{P}}$ is α monotone. Hence $\succsim_{i,\mathcal{P}}$ is noninclusion averse.

vii) Suppose that $\succsim_{i,\mathcal{P}}$ is noninclusion averse. If $A \nsubseteq B$ then $B \succsim_{i,\mathcal{P}} A$. If $A \subset B$ then $B \nsubseteq A$ and $A \succsim_{i,\mathcal{P}} B$. Since $\succsim_{i,\mathcal{P}}$ is reflexive then it follows that $\succsim_{i,\mathcal{P}}$ is complete. Now we show that $\succsim_{i,\mathcal{P}}$ is β monotone. Let $A \succ_{i,\mathcal{P}} B$ and suppose that $A \not\subset B$, then by definition it follows that $B \succsim_{i,\mathcal{P}} A$ which is a contradiction. So, it must be that $A \subset B$ and $\succsim_{i,\mathcal{P}}$ is β monotone. Conversely, suppose that $\succsim_{i,\mathcal{P}}$ is β monotone and complete. Let A, B be such that $A \nsubseteq B$. Suppose that $B \not\succsim_{i,\mathcal{P}} A$ then $A \succ_{i,\mathcal{P}} B$ since $\succsim_{i,\mathcal{P}}$ is reflexive and complete; on the other hand $\succsim_{i,\mathcal{P}}$ is β monotone so $A \succ_{i,\mathcal{P}} B$ implies that $A \subset B$ which is a contradiction. Therefore $B \succsim_{i,\mathcal{P}} A$ and $\succsim_{i,\mathcal{P}}$ is noninclusion averse.

□

Proposition 5.6: *Assume that the preference relation $\succsim_{i,\mathcal{P}}$ is complete. Then:*

i) If $\succsim_{i,\mathcal{P}}$ is imprecision averse then $\succsim_{i,\mathcal{P}}$ is α imprecision monotone.

ii) If $\succsim_{i,\mathcal{P}}$ is strongly imprecision averse then $\succsim_{i,\mathcal{P}}$ is strongly α imprecision monotone.

iii) $\succsim_{i,\mathcal{P}}$ cannot be strongly β imprecision monotone.

Proof. *i)* Let $A \subseteq B$, then $B \not\succ_{i,\mathcal{P}} A$ since $\succsim_{i,\mathcal{P}}$ is imprecision averse; $\succsim_{i,\mathcal{P}}$ is complete and Remark 2.4 implies that $A \succsim_{i,\mathcal{P}} B$. Therefore $\succsim_{i,\mathcal{P}}$ is α imprecision monotone.
ii) Let $A \subset B$, then $B \not\succsim_{i,\mathcal{P}} A$ since $\succsim_{i,\mathcal{P}}$ is strongly imprecision averse; $\succsim_{i,\mathcal{P}}$ is complete and Remark 2.4 implies that $A \succ_{i,\mathcal{P}} B$. Therefore $\succsim_{i,\mathcal{P}}$ is strongly α imprecision monotone.
iii) Since $\succsim_{i,\mathcal{P}}$ is complete, then, for every pair of subsets (A, B), it follows that $A \succsim_{i,\mathcal{P}} B$ or $B \succsim_{i,\mathcal{P}} A$ or both. Since $\succsim_{i,\mathcal{P}}$ is strongly β imprecision monotone then it follows that $A \subseteq B$ or $B \subseteq A$ or both. This is an obvious contradiction since it is possible to consider a pair of subsets A, B such that $A \nsubseteq B$ and $B \nsubseteq A$. □

Now examples are given showing which implications do not hold.

Example 5.7: Let $\succsim_{i,\mathcal{P}}$ be the total preference relation over $\mathbb{K}(\mathcal{P})$; that is, for every pair of set $(A, B) \in \mathbb{K}(\mathcal{P}) \times \mathbb{K}(\mathcal{P})$ it follows that $A \succsim_{i,\mathcal{P}} B$ (hence $B \succsim_{i,\mathcal{P}} A$). It can be immediately checked that $\succsim_{i,\mathcal{P}}$ is imprecision averse, α imprecision monotone and noninclusion averse. However, by definition, it is not strongly imprecision averse and strongly α imprecision monotone.

Example 5.8: Fix $\overline{\varrho} \in \mathcal{P}$ and $f : \mathbb{K}(\mathcal{P}) \to \{0, 1\}$ be the function defined, for every $S \in \mathbb{K}(\mathcal{P})$, by

$$f(S) = 1 \quad \text{if} \quad \overline{\varrho} \notin S \quad \text{and} \quad f(S) = 0 \quad \text{if} \quad \overline{\varrho} \in S.$$

Let $\succsim_{i,\mathcal{P}}$ be the complete preference relation defined by $A \succsim_{i,\mathcal{P}} B \iff f(A) \geqslant f(B)$. It clearly follows that $A \succ_{i,\mathcal{P}} B$ if and only if $\overline{\varrho} \in B \setminus A$ which implies that $B \nsubseteq A$. So $\succsim_{i,\mathcal{P}}$ is imprecision averse. However, it is possible to find two sets A, B such that $\overline{\varrho} \in B \setminus A$ and $A \neq A \cap B$, then $f(A) = 1 > f(B) = 0$. So, $A \succ_{i,\mathcal{P}} B$ and $A \not\subset B$ which implies that $\succsim_{i,\mathcal{P}}$ is not β monotone. Summarizing, we found a complete preference relation which is α monotone (imprecision averse) but it is not β monotone and noninclusion averse.

Remark 5.9: Finally, the previous results guarantee that β imprecision monotonicity implies imprecision aversion which, under completeness, implies α imprecision monotonicity. Hence a complete β imprecision monotone preference relation is a α imprecision monotone preference relation. However, these implications are not true when the preference relation is not complete. The argument shown below is simple.

Suppose that $\succsim_{i,\mathcal{P}}$ is β imprecision monotone and A and B are two subsets of $\mathbb{K}(\mathcal{P})$ such that $A \subset B$. Now, construct a new preference relation $\succsim'_{i,\mathcal{P}}$ such that $S \succsim'_{i,\mathcal{P}} T \iff S \succsim_{i,\mathcal{P}} T$ for every $(S, T) \in \mathbb{K}(\mathcal{P}) \times \mathbb{K}(\mathcal{P}) \setminus (A, B)$ and $A \not\succsim'_{i,\mathcal{P}} B$. It immediately follows that $\succsim'_{i,\mathcal{P}}$ is β imprecision monotone but violates α monotonicity as $A \subset B \not\Rightarrow A \succsim'_{i,\mathcal{P}} B$.

The previous propositions and counterexamples about reflexive preference relations can be summarized by the following scheme:

$$
\begin{array}{ccccc}
 & & \boxed{\text{nonincl.}\quad\text{averse}} & & \\
 & \Swarrow \; \not\Nearrow & & \Searrow \; \not\Nwarrow & \\
\boxed{\alpha\ \text{impr.}\quad\text{monotone}} & & \begin{array}{c} \nLeftarrow \\ \nRightarrow \end{array} & & \boxed{\beta\ \text{impr.}\quad\text{monotone}} \\
 & \Searrow \; \not\Nwarrow & & \Swarrow \; \not\Nearrow & \\
 & & \boxed{\text{impr.}\quad\text{averse}} & & \\
\Uparrow \not\Downarrow & & \Uparrow \not\Downarrow & & \Uparrow \not\Downarrow \\
 & & \boxed{\text{str. impr.}\quad\text{averse}} & & \\
 & \Nearrow \; \not\Swarrow & & \Nwarrow \; \not\Searrow & \\
\boxed{\text{str. } \alpha\ \text{impr.}\quad\text{monotone}} & & \begin{array}{c} \nLeftarrow \\ \nRightarrow \end{array} & & \boxed{\text{str. } \beta\ \text{impr.}\quad\text{monotone}}
\end{array}
$$

If the preference relation is also complete, the scheme is the following

$$
\begin{array}{c}
\boxed{\boxed{\beta\ \text{impr.}\quad\text{monotone}} \iff \boxed{\text{nonincl. averse}}} \\
\Downarrow \not\Uparrow \\
\boxed{\boxed{\alpha\ \text{impr.}\quad\text{monotone}} \iff \boxed{\text{impr.}\quad\text{averse}}} \\
\not\Downarrow \Uparrow \\
\boxed{\boxed{\text{str. } \alpha\ \text{impr.}\quad\text{monotone}} \iff \boxed{\text{str. impr.}\quad\text{averse}}}
\end{array}
$$

Remark 5.10: It can be checked that the following (pessimistic) combinations (grouped by two equivalent combinations) cannot be satisfied by any preference relation:

1. $A \nsubseteq B \implies B \succ_{i,\mathcal{P}} A$ or equivalently $A \not\succ_{i,\mathcal{P}} B \implies B \subseteq A$

2. $A \not\subset B \implies B \succ_{i,\mathcal{P}} A$ or equivalently $A \not\succ_{i,\mathcal{P}} B \implies B \subset A$

In fact, if one of the previous conditions hold, then, for every pair of subsets (A, B) such that $A \not\subset B$ and $B \not\subset A$, it follows that $A \succ_{i,\mathcal{P}} B$ and $B \succ_{i,\mathcal{P}} A$, which is impossible.

Moreover, in case the preference $\succsim_{i,\mathcal{P}}$ is reflexive (which is the basic assumption of this work), then $\succsim_{i,\mathcal{P}}$ cannot satisfy any of the following properties:

1. $A \subseteq B \implies B \not\succsim_{i,\mathcal{P}} A$ or equivalently $A \succsim_{i,\mathcal{P}} B \implies B \nsubseteq A$

2. $A \subseteq B \implies A \succ_{i,\mathcal{P}} B$ or equivalently $A \not\succ_{i,\mathcal{P}} B \implies A \nsubseteq B$

3. $A \succsim_{i,\mathcal{P}} B \implies A \subset B$ or equivalently $A \not\subset B \implies A \not\succsim_{i,\mathcal{P}} B$

It is also worth noting that (strongly) pessimistic behavioral traits provides sufficient (but not necessary) conditions for the the open graph property of the correspondence $U_{i,\mathcal{P}}$. More precisely,

Proposition 5.11: *If $\succsim_{i,\mathcal{P}}$ is strongly α imprecision monotone and β imprecision monotone, i.e., $A \subset B \iff A \succ_{i,\mathcal{P}} B$, then the corresponding set-valued map $U_{i,\mathcal{P}}$ has an open graph.*

Proof. Let $(A, B) \in Graph(U_{i,\mathcal{P}})$, then we must show that there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that for every pair $(S_1, S_2) \in \mathbb{K}(\mathcal{P}) \times \mathbb{K}(\mathcal{P})$ with $d_H(S_1, A) < \varepsilon_1$ and $d_H(S_2, B) < \varepsilon_2$ it follows that $S_2 \in U_{i,\mathcal{P}}(S_1)$ or, equivalently, $S_2 \succ_{i,\mathcal{P}} S_1$. Now, since $\succsim_{i,\mathcal{P}}$ is β imprecision monotone then $B \succ_{i,\mathcal{P}} A$ implies that $B \subset A$. Then it is possible to find $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $S_2 \subset S_1$ if $d_H(A, S_1) < \varepsilon_1$ and $d_H(B, S_2) < \varepsilon_2$. Therefore, since $\succsim_{i,\mathcal{P}}$ is strongly α imprecision averse and $S_2 \subset S_1$, then $S_2 \succ_{i,\mathcal{P}} S_1$. Therefore, there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $(S_1, S_2) \in Graph(U_{i,\mathcal{P}})$ if $d_H(S_1, A) < \varepsilon_1$ and $d_H(S_2, B) < \varepsilon_2$ so that the $Graph(U_{i,\mathcal{P}})$ is open. $\square$

5.2 Optimism

The counterpart of the concepts provided in Definition 5.2 is given below. The definition of optimistic behavioral traits can be easily obtained from Definition 5.2 by simply replacing $\succsim_{i,\mathcal{P}}$, $\succ_{i,\mathcal{P}}$, $\not\succsim_{i,\mathcal{P}}$ and $\not\succ_{i,\mathcal{P}}$ respectively with $\precsim_{i,\mathcal{P}}$, $\prec_{i,\mathcal{P}}$, $\not\precsim_{i,\mathcal{P}}$ and $\not\prec_{i,\mathcal{P}}$. More precisely, we have

Definition 5.12: *The preference relation $\succsim_{i,\mathcal{P}}$ is said to be*

O_1) *imprecision loving if the following equivalent conditions are satisfied:*

- *i)* $A \subseteq B \implies A \not\succ_{i,\mathcal{P}} B$
- *ii)* $A \succ_{i,\mathcal{P}} B \implies A \nsubseteq B$
- *iii)* $A \succ_{i,\mathcal{P}} B \implies A \not\subset B$
- *iv)* $A \subset B \implies A \not\succ_{i,\mathcal{P}} B$

O_2) *strongly imprecision loving if the following equivalent conditions are satisfied:*

- *i)* $A \subset B \implies A \not\succsim_{i,\mathcal{P}} B$
- *ii)* $A \succsim_{i,\mathcal{P}} B \implies A \not\subset B$

O_3) *γ imprecision monotone if the following equivalent conditions are satisfied:*

- *i)* $A \subseteq B \implies B \succsim_{i,\mathcal{P}} A$
- *ii)* $A \not\succsim_{i,\mathcal{P}} B \implies B \nsubseteq A$
- *iii)* $A \subset B \implies B \succsim_{i,\mathcal{P}} A$
- *iv)* $A \not\succsim_{i,\mathcal{P}} B \implies B \not\subset A$

O_4) *δ imprecision monotone if the following equivalent conditions are satisfied:*

- *i)* $A \succ_{i,\mathcal{P}} B \implies B \subset A$
- *ii)* $A \not\subset B \implies B \not\succ_{i,\mathcal{P}} A$
- *iii)* $A \succ_{i,\mathcal{P}} B \implies B \subseteq A$
- *iv)* $A \nsubseteq B \implies B \not\succ_{i,\mathcal{P}} A$

O_5) *strongly γ imprecision monotone if the following equivalent conditions are satisfied:*

i) $A \subset B \implies B \succ_{i,\mathcal{P}} A$

ii) $A \not\succ_{i,\mathcal{P}} B \implies B \not\subset A$

O_6) *strongly δ imprecision monotone if the following equivalent conditions are satisfied:*

i) $A \succsim_{i,\mathcal{P}} B \implies B \subseteq A$

ii) $A \nsubseteq B \implies B \not\succsim_{i,\mathcal{P}} A$

O_7) *noninclusion loving if the following equivalent conditions are satisfied:*

i) $A \nsubseteq B \implies A \succsim_{i,\mathcal{P}} B$

ii) $A \not\succsim_{i,\mathcal{P}} B \implies A \subseteq B$

iii) $A \not\subset B \implies A \succsim_{i,\mathcal{P}} B$

iv) $A \not\succsim_{i,\mathcal{P}} B \implies A \subset B.$

It is clear that trivial arguments allow to have specular results of those provided in Propositions 5.5, 5.6, Examples 5.7, 5.8 and Remark 5.9. Summarizing we have

$$
\begin{array}{ccccc}
 & & \boxed{\text{nonincl. loving}} & & \\
 & \Swarrow\ \not\Nearrow & & \Searrow\ \not\Nwarrow & \\
\boxed{\gamma\text{ impr. monotone}} & & \begin{array}{c}\not\Leftarrow\\ \not\Rightarrow\end{array} & & \boxed{\delta\text{ impr. monotone}} \\
 & \Searrow\ \not\Nwarrow & & \Swarrow\ \not\Nearrow & \\
 & & \boxed{\text{impr. loving}} & & \\
\Uparrow\not\Downarrow & & \Uparrow\not\Downarrow & & \Uparrow\not\Downarrow \\
 & & \boxed{\text{str. impr. loving}} & & \\
 & \Nearrow\ \not\Swarrow & & \Nwarrow\ \not\Searrow & \\
\boxed{\text{str. }\gamma\text{ impr. monotone}} & & \begin{array}{c}\not\Leftarrow\\ \not\Rightarrow\end{array} & & \boxed{\text{str. }\delta\text{ impr. monotone}}
\end{array}
$$

If the preference relation is also complete, the scheme is the following

$$
\begin{array}{c}
\boxed{\boxed{\delta\text{ impr. monotone}} \iff \boxed{\text{nonincl. loving}}} \\
\Downarrow\not\Uparrow \\
\boxed{\boxed{\gamma\text{ impr. monotone}} \iff \boxed{\text{impr. loving}}} \\
\not\Downarrow\Uparrow \\
\boxed{\boxed{\text{str. }\gamma\text{ impr. monotone}} \iff \boxed{\text{str. impr. loving}}}
\end{array}
$$

Remark 5.13: In order to classify all the 32 cases in the right hand side of Table 11, it is worth noting that the 10 cases not involved in Definition 5.12 are not compatible with reflexive preference relations. The arguments are specular to those contained in Remark 5.10. More precisely, it can be checked that the following combinations cannot be satisfied by any preference relation:

1. $A \nsubseteq B \implies A \succ_{i,\mathcal{P}} B$ or equivalently $A \not\succ_{i,\mathcal{P}} B \implies A \subseteq B$

2. $A \not\subset B \implies A \succ_{i,\mathcal{P}} B$ or equivalently $A \not\succ_{i,\mathcal{P}} B \implies A \subset B$

Moreover, in case the preference $\succsim_{i,\mathcal{P}}$ is reflexive, then, it cannot satisfy any of the following properties:

1. $A \subseteq B \implies A \not\succsim_{i,\mathcal{P}} B$ or equivalently $A \succsim_{i,\mathcal{P}} B \implies A \nsubseteq B$

2. $A \subseteq B \implies B \succ_{i,\mathcal{P}} A$ or equivalently $A \not\succ_{i,\mathcal{P}} B \implies B \nsubseteq A$

3. $A \succsim_{i,\mathcal{P}} B \implies B \subset A$ or equivalently $A \not\subset B \implies B \not\succsim_{i,\mathcal{P}} A$

6 On the structure of ambiguity

In the previous sections, it is studied the role played by the attitudes towards ambiguity in the equilibrium existence problem. Another important question is to understand how much restrictive are the assumptions imposed to the correspondences $\mathcal{B}_i$ in Propositions 3.1 and 3.4. The present section looks at this particular problem and presents an analysis of the properties of continuity and concavity/convexity of the correspondences $\mathcal{B}_i$ in two specific models. In particular, already existing models of ambiguous games are here regarded from the perspective of the model with ambiguous belief correspondences. The first model looks at incomplete information games with multiple priors on the state space, while, the second one tackles the issue of strategic ambiguity.The main result that emerges in this section is that, in both models, the correspondences $\mathcal{B}_i(\cdot, x_{-i})$ are convex but not concave set-valued maps in general. As a consequence, equilibrium existence and convexity of players' preferences are guaranteed only in case of imprecision averse players.

6.1 Incomplete information games under multiple priors

There is a standard formulation of ambiguous games conforming to Harsanyi's classic work. Let Θ be a finite set of payoff relevant states. Denote again with Ψ_i and with X_i respectively the finite set of pure strategies and the set of mixed strategies of player i. Then, player i has a payoff function $f_i : \Psi \times \Theta \to \mathbb{R}$ and a set of priors $\mathscr{P}_i$ over Θ. For every probability $P_i \in \mathscr{P}_i$ and every mixed strategy profile $x \in X$ let

$$u_i(x_i, x_{-i}|P_i) = \sum_{\theta\in\Theta} \sum_{\psi_i\in\Psi_i} \sum_{\psi_{-i}\in\Psi_{-i}} x_i(\psi_i)x_{-i}(\psi_{-i})P_i(\theta)f_i(\psi_i, \psi_{-i}|\theta).$$

Hence the maxmin preference of player i is given by the following utility function

$$v_i(x) = \min_{P_i\in\mathscr{P}_i} [u_i(x_i, x_{-i}|P_i)] \quad \forall x \in X. \tag{13}$$

Then, it is possible to consider the game

$$\widetilde{\Gamma} = \{I; \Theta; (\mathscr{P}_i)_{i\in I}; (\Psi_i)_{i\in I}; (v_i)_{i\in I}\}. \tag{14}$$

The equilibrium notion given by Kajii and Ui (2005) is adapted for the present model:

Definition 6.1: *A strategy profile $x^* \in X$ is a mixed strategy equilibrium for the game with multiple priors $\widetilde{\Gamma}$ if*

$$v_i(x_i^*, x_{-i}^*) = \max_{x_i \in X_i} v_i(x_i, x_{-i}^*) \quad \forall i \in P.$$

Denote with $\widetilde{E}$ the set of equilibria of $\widetilde{\Gamma}$.

The game $\widetilde{\Gamma}$ has an equivalent formulation in terms of beliefs correspondences. In fact, denote with $f(\psi|\theta) = (f_1(\psi|\theta), \dots f_n(\psi|\theta))$ and $\Omega = \{f(\psi|\theta) \,|\, \psi \in \Psi,\ \theta \in \Theta\}$. Denote again with $\mathcal{P}$ the set of probability distributions over Ω, then $\mathscr{P}_i$ induces a beliefs correspondence $\mathcal{B}_i : X \rightsquigarrow \mathcal{P}$ in the obvious way:

$$\varrho \in \mathcal{B}_i(x) \iff \exists P_i \in \mathscr{P}_i \ s.t.\ \varrho(f(\psi|\theta)) = x_i(\psi_i)x_{-i}(\psi_{-i})P_i(\theta) \quad \forall (\psi, \theta) \in \Psi \times \Theta. \tag{15}$$

It follows immediately that

$$\min_{P_i \in \mathscr{P}_i} [u_i(x_i, x_{-i}|P_i)] = \min_{\varrho \in \mathcal{B}_i(x)} \left[\sum_{\omega \in \Omega} \varrho(\omega)\omega_i \right] \quad \forall x \in X. \tag{16}$$

So, for every $A \subseteq \mathcal{P}$, denote with $F_i(A) = \min_{\varrho \in A} \left[\sum_{\omega \in \Omega} \varrho(\omega)\omega_i\right]$ and consider the preference relation $\succsim_{i,\mathcal{P}}$ defined by $A \succsim_{i,\mathcal{P}} B \iff F_i(A) \geqslant F_i(B)$ where $A, B \subseteq \mathcal{P}$. Consider the corresponding preference relation $\succsim_i$, over strategy profiles, and the corresponding game $\Gamma = \{I; (X_i)_{i \in I}; (\succsim_i)_{i \in I}\}$. Equation (16) guarantees that the set E of equilibria under ambiguous belief correspondences of the game Γ coincides with the set of equilibria $\widetilde{E}$ of $\widetilde{\Gamma}$.

Below, the properties of the belief correspondences $\mathcal{B}_i$ defined by (15) are investigated. It turns out that $\mathcal{B}_i$ is a continuous and a convex set-valued map but it is not a concave set-valued map. Hence, the assumptions of the existence theorem in this case of imprecision averse agents are satisfied; while, the lack of concavity makes it impossible to have the existence result in case ambiguity loving agents.

Lemma 6.2: *If the set $\mathscr{P}_i$ is closed then the belief correspondence $\mathcal{B}_i : X \rightsquigarrow \mathcal{P}$ defined by (15) is continuous in X.*

Proof. Let $\{x_\nu\}_{\nu \in \mathbb{N}} \subset X$ be a sequence converging to $x \in X$ and $\{\varrho_\nu\}_{\nu \in \mathbb{N}}$ be a sequence of probability distributions converging to ϱ such that $\varrho_\nu \in \mathcal{B}_i(x_\nu)$ for every $\nu \in \mathbb{N}$. For every $\nu \in \mathbb{N}$, let $P_{i,\nu}$ be the marginal of ϱ_ν over Θ, that is $\varrho_\nu(f(\psi|\theta)) = x_{i,\nu}(\psi_i)x_{-i,\nu}(\psi_{-i})P_{i,\nu}(\theta)$ for every $(\psi, \theta) \in \Psi \times \Theta$. Since the sequence $\{\varrho_\nu\}_{\nu \in \mathbb{N}}$ converges to ϱ, then the sequence $\{P_{i,\nu}(\theta)\}_{\nu \in \mathbb{N}}$ converges for every $\theta \in \Theta$; so, let $P_i = \lim_{\nu \to \infty} P_{i,\nu}$, that is $P_i(\theta) = \lim_{\nu \to \infty} P_{i,\nu}(\theta)$ for every $\theta \in \Theta$. Then

$$\varrho(f(\psi|\theta)) = \lim_{\nu \to \infty} \varrho_\nu(f(\psi|\theta)) = \lim_{\nu \to \infty} x_{i,\nu}(\psi_i)x_{-i,\nu}(\psi_{-i})P_{i,\nu}(\theta) = x_i(\psi_i)x_{-i}(\psi_{-i})P_i(\theta) \quad \forall (\psi, \theta) \in \Psi \times \Theta$$

Since $\mathscr{P}_i$ is closed, then $P_i \in \mathscr{P}_i$. It follows that $\varrho \in \mathcal{B}_i(x)$ and $\mathcal{B}_i$ is a closed set-valued map. So $\mathcal{B}_i$ is also upper semicontinuous in X.

Now it is shown that $\mathcal{B}_i$ is lower semicontinuous. In fact, let $\varrho \in \mathcal{B}_i(x)$, then there exist $P_i \in \mathscr{P}_i$ such that $\varrho(f(\psi|\theta)) = x_i(\psi_i)x_{-i}(\psi_{-i})P_i(\theta)$ for all $(\psi, \theta) \in \Psi \times \Theta$. Now, for every sequence $\{x_\nu\}_{\nu \in \mathbb{N}} \subset X$ converging to $x \in X$, let $\{\varrho_\nu\}_{\nu \in \mathbb{N}}$ be a sequence defined, for every $\nu \in \mathbb{N}$, by $\varrho_\nu(f(\psi|\theta)) = x_{i,\nu}(\psi_i)x_{-i,\nu}(\psi_{-i})P_i(\theta)$ for every $(\psi, \theta) \in \Psi \times \Theta$. It immediately follows that $\varrho_\nu \in \mathcal{B}_i(x_\nu)$ and $\lim_{\nu \to \infty} \varrho_\nu = \varrho$. So $\mathcal{B}_i$ is lower semicontinuous and therefore continuous in X. □

LEMMA 6.3: *Let $\mathcal{B}_i : X \rightsquigarrow \mathcal{P}$ be defined by (15). Then, the set-valued map $\mathcal{B}_i(\cdot, x_{-i}) : X_i \rightsquigarrow \mathcal{P}$ is convex in X_i, for every $x_{-i} \in X_{-i}$.*

Proof. Let $\varrho \in \mathcal{B}_i(t\overline{x}_i + (1-t)\widehat{x}_i, x_{-i})$, then there exists $P_i \in \mathscr{P}_i$ such that $\varrho(f(\psi|\theta)) = (t\overline{x}_i(\psi_i) + (1-t)\widehat{x}_i(\psi_i))x_{-i}(\psi_{-i})P_i(\theta)$ for every $(\psi,\theta) \in \Psi \times \Theta$. It immediately follows that $\varrho = t\overline{\varrho} + (1-t)\widehat{\varrho}$ where $\overline{\varrho} \in \mathcal{B}_i(\overline{x}_i, x_{-i})$ and $\widehat{\varrho} \in \mathcal{B}_i(\widehat{x}_i, x_{-i})$ are defined by

$$\overline{\varrho}(f(\psi|\theta)) = \overline{x}_i(\psi_i)x_{-i}(\psi_{-i})P_i(\theta),\ \widehat{\varrho}(f(\psi|\theta)) = \widehat{x}_i(\psi_i)x_{-i}(\psi_{-i})P_i(\theta) \quad \forall(\psi,\theta) \in \Psi \times \Theta$$

and the assertion follows. □

LEMMA 6.4: *Assume that $\mathscr{P}_i$ is not a singleton and let $\mathcal{B}_i : X \rightsquigarrow \mathcal{P}$ be defined by (15). Then the set-valued map $\mathcal{B}_i(\cdot, x_{-i}) : X_i \rightsquigarrow \mathcal{P}$ is not concave in X_i, for every $x_{-i} \in X_{-i}$.*

Proof. Let $P'_i, P''_i \in \mathscr{P}_i$ be such that $P'_i \neq P''_i$. Let $\overline{x}_i, \widehat{x}_i$ be two mixed strategies such that there exist $\overline{\psi}_i, \widehat{\psi}_i$ with $\overline{\psi}_i \neq \widehat{\psi}_i$, $\overline{x}_i(\overline{\psi}_i) = \widehat{x}_i(\widehat{\psi}_i) = 0$ and $\overline{x}_i(\widehat{\psi}_i) \neq 0$, $\widehat{x}_i(\overline{\psi}_i) \neq 0$. Let $\overline{\varrho} \in \mathcal{B}_i(\overline{x}_i, x_{-i})$ and $\widehat{\varrho} \in \mathcal{B}_i(\widehat{x}_i, x_{-i})$ be defined for every $(\psi,\theta) \in \Psi \times \Theta$ by

$$\overline{\varrho}(f(\psi|\theta)) = \overline{x}_i(\psi_i)x_{-i}(\psi_{-i})P'_i(\theta) \quad \text{and} \quad \widehat{\varrho}(f(\psi|\theta)) = \widehat{x}_i(\psi_i)x_{-i}(\psi_{-i})P''_i(\theta).$$

Given $t \in]0,1[$, suppose that $t\mathcal{B}_i(\overline{x}_i, x_{-i}) + (1-t)\mathcal{B}_i(\widehat{x}_i, x_{-i}) \subseteq \mathcal{B}_i(t\overline{x}_i + (1-t)\widehat{x}_i, x_{-i})$; this latter inclusion implies that there exists $P^*_i \in \mathscr{P}_i$ such that

$$t\overline{x}_i(\psi_i)x_{-i}(\psi_{-i})P'_i(\theta) + (1-t)\widehat{x}_i(\psi_i)x_{-i}(\psi_{-i})P''_i(\theta) = (t\overline{x}_i(\psi_i) + (1-t)\widehat{x}_i(\psi_i))x_{-i}(\psi_{-i})P^*_i(\theta). \tag{17}$$

for every $(\psi,\theta) \in \Psi \times \Theta$. Let ψ_{-i} be such that $x_{-i}(\psi_{-i}) \neq 0$; then, from (17), $\overline{x}_i(\overline{\psi}_i) = 0$ and $\widehat{x}_i(\overline{\psi}_i) \neq 0$, it follows that

$$(1-t)\widehat{x}_i(\overline{\psi}_i)x_{-i}(\psi_{-i})P''_i(\theta) = (1-t)\widehat{x}_i(\overline{\psi}_i)x_{-i}(\psi_{-i})P^*_i(\theta) \quad \forall\theta \in \Theta$$

then $P''_i(\theta) = P^*_i(\theta)$ for every $\theta \in \Theta$. Similarly, from (17), $\widehat{x}_i(\widehat{\psi}_i) = 0$ and $\overline{x}_i(\widehat{\psi}_i) \neq 0$, it follows that

$$t\overline{x}_i(\widehat{\psi}_i)x_{-i}(\psi_{-i})P'_i(\theta) = t\overline{x}_i(\widehat{\psi}_i)x_{-i}(\psi_{-i})P^*_i(\theta) \quad \forall\theta \in \Theta$$

then $P'_i(\theta) = P^*_i(\theta)$ for every $\theta \in \Theta$. Then $P'_i(\theta) = P''_i(\theta)$ for every $\theta \in \Theta$. This is a contradiction. Therefore

$$t\mathcal{B}_i(\overline{x}_i, x_{-i}) + (1-t)\mathcal{B}_i(\widehat{x}_i, x_{-i}) \nsubseteq \mathcal{B}_i(t\overline{x}_i + (1-t)\widehat{x}_i, x_{-i})$$

and $\mathcal{B}_i(\cdot, x_{-i})$ is not concave. □

REMARK 6.5: Azrieli and Teper (2011) look at a more general version of incomplete information games under ambiguity but in the case in which ambiguous alternatives are acts and preferences are rational and represented by real functionals. They obtain existence of equilibria if and only if agents are *uncertainty averse*, which represent a pessimistic attitude towards ambiguity. The underlying idea is that players are uncertainty averse if and only if they have convex preferences with respect their own strategies. Convex preferences allow to have convex valued best replies which, in turn, lead to equilibrium existence.

The previous Lemma 6.3 and 6.4 in this section, reconcile the existence results in this paper with the existence result in Azrieli and Teper (2011). In fact, in the present paper, equilibrium existence is guaranteed only if players are pessimistic (imprecision averse). The existence of equilibria in case of optimistic (imprecision loving) players would require the concavity of $\mathcal{B}_i(\cdot, x_{-i})$, but this is not the case of belief correspondences deriving from incomplete information games under multiple priors.

6.2 Strategic Ambiguity

Strategic ambiguity is a recent but increasingly relevant issue in theory of strategic games. Roughly speaking, it concerns the situations in which players have ambiguous expectations about opponents' behavior. This subsection looks at belief correspondences deriving from a model of strategic ambiguity; in particular, here ambiguous beliefs of a player depend on the actual play of his opponents[18]. This is the case when the agent does not know the strategy profile of his opponents in its entirety but can only assess some imprecise probability judgment; the probabilistic belief, in turn, might depend on the strategy profile and possibly reveal a part of the actual strategy profile.

The model is constructed as follows: each player i is endowed with a payoff function $f_i : \Psi \to \mathbb{R}$ and a belief correspondence about opponents' behavior which is given by a set-valued map $\mathcal{K}_i : X_{-i} \rightsquigarrow \Delta_{-i}$ which maps strategy profiles to correlated strategies of his opponents; in fact, Ψ_i and X_i are respectively the set of pure and mixed strategies of player i, $\Psi = \prod_{j=1}^{n} \Psi_j$ and Δ_{-i} is the set of all probability distributions[19] over the set of pure strategy profiles $\Psi_{-i} = \prod_{j\neq i} \Psi_j$. More precisely, when the strategy profile $x_{-i} \in \prod_{j\neq i} X_j$ is chosen by the agents $j \neq i$, player i does not know x_{-i} but he has only the ambiguous belief $\mathcal{K}_i(x_{-i}) \subseteq \Delta_{-i}$ over the set of pure strategy profiles chosen by his opponents. Examples can be easily constructed from previous literature as $\mathcal{K}_i(x_{-i})$ can be a set of distributions deriving from coherent lower expectations as in De Marco and Romaniello (2015,b) or from partially specified probabilities as in Lehrer (2012) (see De Marco and Romaniello (2013)).

In this model, the set of outcomes of the game is given by $\Omega = \{(f_1(\psi), \ldots, f_n(\psi)) \,|\, \psi \in \Psi\}$; then, each $\mathcal{K}_i$ induces the belief correspondence over outcomes $\mathcal{B}_i : X \rightsquigarrow \mathcal{P}$ in the obvious way:

$$\varrho \in \mathcal{B}_i(x) \iff \exists\, \mu \in \mathcal{K}_i(x_{-i}) \quad \text{such that}\, \varrho(f(\psi)) = x_i(\psi_i)\mu(\psi_{-i}) \quad \forall \psi \in \Psi. \tag{18}$$

Firstly, continuity of $\mathcal{B}_i$ is investigated.

Proposition 6.6: *Assume that $\mathcal{B}_i$ is defined by (18) and that $\mathcal{K}_i : X_{-i} \to \Delta_{-i}$ is a continuous set valued map with not empty and closed values. Then, $\mathcal{B}_i$ is a continuous set-valued map in X with closed values.*

Proof. Let $\varrho \in \mathcal{B}_i(x)$ be the probability distribution defined as in (18) by

$$\varrho(f(\psi)) = x_i(\psi_i)\mu(\psi_{-i}) \;\; \forall \psi \in \Psi, \quad \text{with}\, \mu \in \mathcal{K}_i(x_{-i})$$

and $(x_\nu)_{\nu\in\mathbb{N}}$, with $x_\nu = (x_{i,\nu}, x_{-i,\nu})$ for every $\nu \in \mathbb{N}$, be a sequence converging to $x = (x_i, x_{-i})$. Since for every i, $\mathcal{K}_i$ is lower semicontinuous in x_{-i} then there exists a sequence $(\mu_\nu)_{\nu\in\mathbb{N}}$ converging to μ with $\mu_\nu \in \mathcal{K}_i(x_{-i,\nu})$ for every $\nu \in \mathbb{N}$. Define ϱ_ν as follows:

$$\varrho_\nu(f(\psi)) = x_{i,\nu}(\psi_i)\mu_\nu(\psi_{-i}) \;\; \forall \psi \in \Psi.$$

[18] In De Marco and Romaniello (2015,a) it is studied a different model of strategic stability in which beliefs of a player i over his opponents' strategy profiles are given by the set of Nash equilibria of the game among his opponents once they have observed player i's action. The underlying idea in this model is that player i believes that his opponents will observe his action before choosing their strategies and then they will react optimally. Therefore, player i's beliefs about opponents' behavior depend only on his action choice and they are given by the equilibria of the game (between his opponents) which is induced by player i's strategy choice.

[19] Hence correlated strategies are possible.

If follows that $\varrho_\nu \to \varrho$ and $\varrho_\nu \in \mathcal{B}_i(x_\nu)$ for every $\nu \in \mathbb{N}$. So $\mathcal{B}_i$ is lower semicontinuous in x. Since x is arbitrary, $\mathcal{B}_i$ is lower semicontinuous in X.

Now we show that $\mathcal{B}_i$ is upper semicontinuous in X with closed images. Let $(x_\nu)_{\nu\in\mathbb{N}}$ be a sequence converging to x and $(\varrho_\nu)_{\nu\in\mathbb{N}}$ be a sequence converging to ϱ, with $\varrho_\nu \in \mathcal{B}_i(x_\nu)$ for every $\nu \in \mathbb{N}$. We show that $\varrho \in \mathcal{B}_i(x)$. By definition (18), $\varrho_\nu(f(\psi)) = x_{i,\nu}(\psi_i)\mu_\nu(\psi_{-i})$ for all $\psi \in \Psi$, where $\mu_\nu \in \mathcal{K}_i(x_{-i,\nu})$. It immediately follows that $(\mu_\nu)_{\nu\in\mathbb{N}}$ converges to a probability distribution $\mu \in \Delta_{-i}$; being $\mathcal{K}_i$ a closed set-valued map, then $\mu \in \mathcal{K}_i(x)$. So $\varrho_\nu \to \varrho$ where $\varrho(\psi) = x_i(\psi_i)\mu(\psi_{-i})$ for all $\psi \in \Psi$. Therefore, (18) implies that $\varrho \in \mathcal{B}_i(x)$ and $\mathcal{B}_i$ is closed in x. Since x is arbitrary then $\mathcal{B}_i$ is closed in X. Moreover, if $(x_\nu)_{\nu\in\mathbb{N}}$ is the constant sequence, with $x_\nu = x$ for every $\nu \in \mathbb{N}$, then the closedness of $\mathcal{B}_i$ in x implies that the set $\mathcal{B}_i(x)$ is closed. Therefore, $\mathcal{B}_i$ is upper semicontinuous in X. □

Now, it is shown that the model defined by (18) leads to convex but not concave belief correspondences. More precisely:

Proposition 6.7: *Given $\mathcal{K}_i : X_{-i} \rightsquigarrow \Delta_{-i}$, let $\mathcal{B}_i : X \rightsquigarrow \mathcal{P}$ be the belief correspondence defined by (18). Then,*

i) $\mathcal{B}_i(\cdot, x_{-i})$ is a convex set-valued map, for every $x_{-i} \in X_{-i}$.

ii) If $\mathcal{K}_i(x_{-i})$ is not a singleton then $\mathcal{B}_i(\cdot, x_{-i})$ is not a concave set-valued map.

Proof. i) Fix $x_{-i} \in X_{-i}$. Let $\varrho \in \mathcal{B}_i(tx_i' + (1-t)x_i'', x_{-i})$, then there exists $\mu \in \mathcal{K}_i(x_{-i})$ such that

$$\varrho(f(\psi)) = [tx_i'(\psi_i) + (1-t)x_i''(\psi_i)]\mu(\psi_{-i}) = tx_i'(\psi_i)\mu(\psi_{-i}) + (1-t)x_i''(\psi_i)\mu(\psi_{-i}) \quad \forall \psi \in \Psi \tag{19}$$

Let ϱ' and ϱ'' be defined respectively by

$$\varrho'(f(\psi)) = x_i'(\psi_i)\mu(\psi_{-i}) \quad \text{and} \quad \varrho''(f(\psi)) = x_i''(\psi_i)\mu(\psi_{-i}) \quad \forall \psi \in \Psi.$$

By definition, it follows that $\varrho' \in \mathcal{B}_i(x_i', x_{-i})$ and $\varrho'' \in \mathcal{B}_i(x_i'', x_{-i})$. So, from (19) we get that

$$\varrho = t\varrho' + (1-t)\varrho'' \in t\mathcal{B}_i(x_i', x_{-i}) + (1-t)\mathcal{B}_i(x_i'', x_{-i})$$

Being ϱ an arbitrary point in $\mathcal{B}_i(tx_i' + (1-t)x_i'', x_{-i})$, it follows that

$$\mathcal{B}_i(tx_i' + (1-t)x_i'', x_{-i}) \subseteq t\mathcal{B}_i(x_i', x_{-i}) + (1-t)\mathcal{B}_i(x_i'', x_{-i}).$$

Since x_i', x_i'' and t are arbitrary, it follows that $\mathcal{B}_i(\cdot, x_{-i})$ is a convex set-valued map.

ii) Let μ' and μ'' in $\mathcal{K}_i(x_{-i})$, with $\mu' \neq \mu''$, $t \in]0,1[$. Let x_i' and x_i'' in X_i be such that $x_i'(\psi_i') = 1$ and $x_i''(\psi_i'') = 1$ for some $\psi_i', \psi_i'' \in \Psi_i$ with $\psi_i' \neq \psi_i''$. Suppose that

$$t\mathcal{B}_i(x_i', x_{-i}) + (1-t)\mathcal{B}_i(x_i'', x_{-i}) \subseteq \mathcal{B}_i(tx_i' + (1-t)x_i'', x_{-i}) \tag{20}$$

Denote with ϱ' and ϱ'' the probability distributions over Ω defined respectively by

$$\varrho'(f(\psi)) = x_i'(\psi_i)\mu'(\psi_{-i}) \quad \text{and} \quad \varrho''(f(\psi)) = x_i''(\psi_i)\mu''(\psi_{-i}) \quad \forall \psi \in \Psi.$$

It immediately follows that $\varrho' \in \mathcal{B}_i(x_i', x_{-i})$ and $\varrho'' \in \mathcal{B}_i(x_i'', x_{-i})$. (20) implies that there exists $\varrho \in \mathcal{B}_i(tx_i' + (1-t)x_i'', x_{-i})$ such that $\varrho = t\varrho' + (1-t)\varrho''$. Since

$$\varrho(f(\psi)) = t\varrho'(f(\psi)) + (1-t)\varrho''(f(\psi)) = tx_i'(\psi_i)\mu'(\psi_{-i}) + (1-t)x_i''(\psi_i)\mu''(\psi_{-i}) \quad \forall \psi \in \Psi,$$

then, for every $\psi_{-i} \in \Psi_{-i}$ it follows that

$$\varrho(f(\psi_i', \psi_{-i})) = t\mu'(\psi_{-i}),\ \varrho(f(\psi_i'', \psi_{-i})) = (1-t)\mu''(\psi_{-i}),\ \varrho(f(\overline{\psi}_i, \psi_{-i})) = 0 \quad \forall \overline{\psi}_i \in \Psi_i \setminus \{\psi_i', \psi_i''\} \tag{21}$$

On the other hand, $\varrho \in \mathcal{B}_i(tx_i' + (1-t)x_i'', x_{-i})$ implies that there exists $\mu \in \mathcal{K}_i(x_{-i})$ such that

$$\varrho(f(\psi)) = [tx_i'(\psi_i) + (1-t)x_i''(\psi_i)]\mu(\psi_{-i}) \quad \forall \psi \in \Psi,$$

So, for every $\psi_{-i} \in \Psi_{-i}$ it follows that

$$\varrho(f(\psi_i', \psi_{-i})) = t\mu(\psi_{-i}),\ \varrho(f(\psi_i'', \psi_{-i})) = (1-t)\mu(\psi_{-i}),\ \varrho(f(\overline{\psi}_i, \psi_{-i})) = 0 \quad \forall \overline{\psi}_i \in \Psi_i \setminus \{\psi_i', \psi_i''\} \tag{22}$$

It follows from (21,22) that

$$\mu'(\psi_{-i}) = \mu(\psi_{-i}) = \mu''(\psi_{-i}) \quad \forall \psi \in \Psi$$

This latter condition implies that $\mu' = \mu''$ which is a contradiction. So

$$t\mathcal{B}_i(x_i', x_{-i}) + (1-t)\mathcal{B}_i(x_i'', x_{-i}) \nsubseteq \mathcal{B}_i(tx_i' + (1-t)x_i'', x_{-i})$$

and $\mathcal{B}_i(\cdot, x_{-i})$ is not a concave set-valued map. □

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