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# *An Adjustment Process-based Algorithm with Error Bounds for Approximating a Nash Equilibrium*

**Francesco Caruso[*], Maria Carmela Ceparano[*] and Jacqueline Morgan[**]**

**Abstract**
Regarding the approximation of Nash equilibria in games where the players have a continuum of strategies, there exist various algorithms based on adjustment processes: from one step to the next, one player updates his strategy solving an optimization problem where the strategies of the other players come from the previous steps. These iterative schemes generate sequences of strategy profiles which are constructed by using continuous optimization techniques and which converge to a Nash equilibrium of the game. In this paper, we propose an adjustment process based on continuous optimization which guarantee the convergence to a Nash equilibrium in two-player non zero-sum games when the best response functions are not linear, both compositions of the best response functions are not contractions, and the strategy sets are Hilbert spaces. Firstly, we address the issue of uniqueness of the Nash equilibrium extending to a more general class the result obtained in F. Caruso, M.C. Ceparano, and J. Morgan (J Math Anal Appl 2018) for weighted potential games. Secondly, we describe a theoretical adjustment process involving the best response functions which converges to the unique Nash equilibrium of the game. Finally, in order to ap- proximate the unique Nash equilibrium of the game, we present a discretization scheme and a numerical adjustment process based on continuous optimization, and we compute error bounds.

**Keywords**: Non-cooperative game; Approximation of Nash equilibrium; Uniqueness of Nash equilibrium; Fixed point; Adjustment process; Best response; Error bound..

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## Table of contents

## 1 Introduction

In Game Theory as well as in Optimization literature, there exist various algorithms for the numerical approximation of a Nash equilibrium based on *adjustment processes*: from one step to the next, one player updates his strategy solving an optimization problem where the strategies of the other players come from the previous steps (see e.g. [9, 21, 14, 18, 3, 17, 1]). In games where the players have a continuum of strategies, these iterative schemes generate sequences of strategy profiles which are constructed by using continuous optimization techniques and which converge to a Nash equilibrium of the game. In particular, Cherruault and Loridan [9] proposed two iterative schemes in order to approach a Nash equilibrium of a two-player zero-sum game when the strategy sets are Euclidean spaces, the payoff function of each player is jointly twice continuously differentiable, strictly convex and coercive in his variable, and one of the two compositions of the best response functions is a contraction. In Morgan [21] it is described a theoretical algorithm converging to a Nash equilibrium of a two-player zero-sum game when the strategy sets are Hilbert spaces and the two compositions of the best response functions are not necessarily a contraction. A scheme of discretization is presented, together with a numerical algorithm in order to approximate the discretized problem, error bounds computations, and applications to differential games. Gabay and Moulin [14] defined two types of adjustment processes when the strategy set of each of the $N$ players is the interval $[0, +\infty[$ and the $N \times N$ matrix whose general $ij$ element is given by the partial derivative of player $i$'s payoff function with respect to player $i$'s variable and player $j$'s variable is strictly diagonally dominant. Li and Başar [18], proposed an inaccurate search algorithm when the strategy sets are Hilbert spaces, the payoff function of each player is strongly convex in his variable and one of the two compositions of the best response functions is a contraction. In Başar [3] the convergence of some relaxation algorithms for the approximation of Nash equilibria is discussed when the strategy sets are $\mathbb{R}$ or $\mathbb{R}^2$ and the best response functions are linear, even when the two compositions of the best response functions are not a contraction. Attouch, Redont and Soubeyran [1] presented an alternating proximal algorithm to approach a Nash equilibrium for a special class of two-player weighted potential games: the players have the same strategy sets, assumed to be a Hilbert space, and the payoff functions are the sum of an individual component depending on their own strategy and a quadratic component depending on their joint strategies, which is the same for both players. Table 1 summarizes the results previously mentioned.

Hence, to the best of our knowledge, algorithms involving an adjustment process which guarantee the convergence to a Nash equilibrium of a two-player non zero-sum game are not yet defined in the following situation: the best response functions are not assumed to be linear, the two compositions of the best response functions are not assumed to be contractions and the strategy sets are Hilbert spaces. Hence, in this paper we propose

| | Game class | Strategy set | Payoff function assumptions | Composition of BR functions |
|---|---|---|---|---|
| Cherruault, Loridan [9] | two-player zero-sum | finite dimensional space | strictly convex and coercive in its argument, differentiable | contraction |
| Morgan [21] | two-player zero-sum | Hilbert | strictly convex and coercive in its argument, differentiable | not necessarily |
| Gabay, Moulin [14] | $N$-player | $[0, +\infty[$ | strict diagonally dominant | not necessarily |
| Li, Başar [18] | two-player | Hilbert | strongly convex in its argument and differentiable | contraction |
| Başar [3] | two-player | $\mathbb{R}$ or $\mathbb{R}^2$ | strongly convex in its argument and quadratic | not necessarily |
| Attouch, Redont, Soubeyran [1] | two-player weighted potential | Hilbert | lower semicontinuous and stricly convex in its argument | contraction when the strategy sets are $\mathbb{R}$ |

Table 1: Some existing literature

an iterative method based on continuous optimization that fills this lack and we apply such a method to a class of games for which the existence and uniqueness of the Nash equilibrium is ensured by extending a previous result obtained in [6] for weighted potential games.

The paper is structured as follows. In Section 2, after the class of games we deal with is defined and some properties of the best response functions are shown, the existence of a unique Nash equilibrium for games belonging to such a class is proved. A theoretical adjustment process for the approximation of the (unique) Nash equilibrium is presented in Section 3: the convergence is shown and error estimations are obtained. In Section 4, new assumptions are given in order to also handle situations where the best response functions are not analytically available and examples are presented. Then, we focus on the numerical approximation of the Nash equilibrium. Firstly, the equilibrium is approached through a sequence of Nash equilibria of games where the strategy sets are finite dimensional spaces (the discretization scheme of Subsection 4.1); secondly, a numerical adjustment process is described by combining the theoretical adjustment process with a continuous optimization technique (the *local variation method*, Subsection 4.2) in order to approximate each Nash equilibrium of the sequence obtained from the discretization (Subsection 4.3); finally, error bounds for such an approximation are proved (Subsection 4.4).

## 2 Uniqueness of the Nash equilibrium

Let $\Gamma := \{2, X, Y, F, G\}$ be a two-player normal form game. The first player's strategy set $X$ and the second player's strategy set $Y$ are real Hilbert spaces with inner products

$(\cdot\,,\cdot)_X$ and $(\cdot\,,\cdot)_Y$, respectively, and associated norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. The payoff functions $F$ and $G$ of the first and second player, respectively, are defined on $X \times Y$ with values in $\mathbb{R}$. We denote by $R_1$ the *best response correspondence of player 1*, i.e. $R_1$ is the set-valued map defined on $Y$ by

$$R_1(y) \coloneqq \operatorname*{Arg\,max}_{x\in X} F(x,y) = \{x' \in X \mid F(x',y) \geq F(x,y), \text{ for any } x \in X\} \subseteq X.$$

Analogously, we denote by $R_2$ the *best response correspondence of player 2*, that is the set-valued map defined on $X$ by $R_2(x) \coloneqq \operatorname{Arg\,max}_{y\in Y} G(x,y) \subseteq Y$. Recall that a *Nash equilibrium* of $\Gamma$ is a couple $(\bar{x},\bar{y}) \in X \times Y$ such that $(\bar{x},\bar{y}) \in R_1(\bar{y}) \times R_2(\bar{x})$. When $R_1$ (resp. $R_2$) is single-valued, the function $r_1$ (resp. $r_2$) defined by $\{r_1(y)\} \coloneqq R_1(y)$ for any $y \in Y$ (resp. $\{r_2(x)\} \coloneqq R_2(x)$ for any $x \in X$) is called *best response function of player 1* (resp. *player 2*) and we denote by $\rho$ the function from $X$ to itself defined by

$$\rho(x) \coloneqq (r_1 \circ r_2)(x) = r_1(r_2(x)). \tag{1}$$

Let us introduce the following hypothesis regarding the best response correspondences that will be used for the uniqueness result.

**(H$_1$)** The best response correspondences in $\Gamma$ are single-valued.

The next definition was introduced in [6, Section 3, p. 1213] to prove the existence of a unique Nash equilibrium in weighted potential games.

**Definition 2.1** Let $\Gamma = \{2, X, Y, F, G\}$ be a game satisfying (H$_1$) and let $\delta > 1$. The *$\delta$-inverse convex combinator of* $\Gamma$ is the function $g^\delta : X \to X$ defined by

$$g^\delta(x) \coloneqq \delta x - (\delta - 1)\rho(x) \tag{2}$$

where $\rho$ is defined in (1).

Such a function is called $\delta$-inverse convex combinator of $\Gamma$ since $x$ is a convex combination of $g^\delta(x)$ and $\rho(x)$, for any $x \in X$. The next lemma is a straightforward result regarding the connections between the fixed points of $g^\delta$ and the Nash equilibria of $\Gamma$, whose proof is omitted.

**Lemma 2.1.** *Let $\delta > 1$ and assume that $\Gamma = \{2, X, Y, F, G\}$ satisfies (H$_1$). Then, the following statements are equivalent:*

*(i) $\bar{x}$ is a fixed point of $g^\delta$.*
*(ii) $\bar{x}$ is a fixed point of $g^\tau$, for any $\tau > 1$.*
*(iii) $\bar{x}$ is a fixed point of $\rho$.*
*(iv) $(\bar{x}, r_2(\bar{x}))$ is a Nash equilibrium of $\Gamma$.*

Now, recall some usual notations. Let $S$ and $T$ be normed vector spaces equipped with the norms $\|\cdot\|_S$ and $\|\cdot\|_T$ respectively, and let $\mathcal{L}(S,T)$ be the normed vector space

of all continuous linear operators from $S$ to $T$, with the usual norm $\|\Lambda\|_{\mathcal{L}(S,T)} \coloneqq \sup\{\|\Lambda(s)\|_T \colon \|s\|_S = 1\}$. The space of continuous linear operators from $S$ to $\mathbb{R}$ is denoted by $S^*$, and the duality operation between $S^*$ and $S$ by $\langle \cdot, \cdot \rangle_{S^* \times S}$.
Let $f$ be a function from $S$ to $T$. If $f$ is twice differentiable on $S$, then $Df \colon S \to \mathcal{L}(S,T)$ and $D^2 f \colon S \to \mathcal{L}(S, \mathcal{L}(S,T))$ denote, respectively, the *Fréchet derivative of* $f$ and the *second Fréchet derivative of* $f$, and by $Df(s) \in \mathcal{L}(S,T)$ and $D^2 f(s) \in \mathcal{L}(S, \mathcal{L}(S,T))$ we mean, respectively, the *derivative of* $f$ *at* $s \in S$ and the *second derivative of* $f$ *at* $s \in S$. Moreover, $d_G f \colon S \to \mathcal{L}(S,T)$ and $d_G f(s) \in \mathcal{L}(S,T)$ denote, respectively, the *Gâteaux derivative of* $f$ and the *Gâteaux derivative of* $f$ *at* $s \in S$. When $S = S_1 \times \cdots \times S_n$, $D_{s_i} f \colon S \to \mathcal{L}(S_i, T)$ denotes the *partial derivative of* $f$ *with respect to* $s_i$, and $D_{s_j}(D_{s_i} f) \colon S \to \mathcal{L}(S_j, \mathcal{L}(S_i, T))$ and $D^2_{s_i} f \colon S \to \mathcal{L}(S_i, \mathcal{L}(S_i, T))$, respectively, the *second partial derivative of* $f$ *with respect to* $s_i$ *and* $s_j$ and the *second partial derivative of* $f$ *with respect to* $s_i$, for any $i, j \in \{1, \ldots, n\}$ (clearly, $D_{s_i}(D_{s_i} f) \equiv D^2_{s_i} f$ for any $i \in \{1, \ldots, n\}$).
Finally, let $\mathcal{GL}(S,T) \subseteq \mathcal{L}(S,T)$ be the set of all bijective continuous linear operators from $S$ to $T$ with continuous (and linear) inverse. If $f \in \mathcal{GL}(S,T)$, then $f^{-1} \colon T \to S$ denotes the inverse operator of $f$, where $f^{-1} \in \mathcal{L}(T,S)$.
Then, $D^2_x F(x,y) \in \mathcal{L}(X, X^*)$, $D_y(D_x F)(x,y) \in \mathcal{L}(Y, X^*)$, $D^2_y G(x,y) \in \mathcal{L}(Y, Y^*)$, $D_x(D_y G)(x,y) \in \mathcal{L}(X, Y^*)$, for any $(x,y) \in X \times Y$, and we can define

$$\lambda_1 \coloneqq \sup_{(x,y) \in X \times Y} \|[D^2_x F(x,y)]^{-1} \circ D_y(D_x F)(x,y)\|_{\mathcal{L}(Y,X)}, \tag{3}$$

$$\lambda_2 \coloneqq \sup_{(x,y) \in X \times Y} \|[D^2_y G(x,y)]^{-1} \circ D_x(D_y G)(x,y)\|_{\mathcal{L}(X,Y)}, \tag{4}$$

$$\lambda \coloneqq \lambda_1 \cdot \lambda_2. \tag{5}$$

Throughout the paper, we deal with the class of games described in the next definition.

**Definition 2.2** $\mathcal{H}$ is the set of games $\Gamma = \{2, X, Y, F, G\}$ which satisfies the following assumptions:

- $X$ and $Y$ are real Hilbert spaces;
- $F$ is twice continuously differentiable on $X \times Y$, $D^2_x F(x,y) \in \mathcal{GL}(X, X^*)$ for any $(x,y) \in X \times Y$, and $\lambda_1$ defined in (3) is a real number;
- $G$ is twice continuously differentiable on $X \times Y$, $D^2_y G(x,y) \in \mathcal{GL}(Y, Y^*)$ for any $(x,y) \in X \times Y$, and $\lambda_2$ defined in (4) is a real number.

The next lemma states some regularity properties of the best response functions $r_1$ and $r_2$, and of their composition $\rho$. The proof is obtained by extending to the class of games $\mathcal{H}$ the proofs of Propositions 3 and 4 in [6] given for weighted potential games.

**Lemma 2.2.** *Assume* $\Gamma \in \mathcal{H}$ *and satisfies* $(H_1)$*. Then* *(i)* $r_1$ *is continuously differentiable on* $Y$ *and Lipschitz continuous with Lipschitz constant no greater than* $\lambda_1$*;*

*(ii)* $r_2$ *is continuously differentiable on* $X$ *and Lipschitz continuous with Lipschitz constant no greater than* $\lambda_2$*;*

*(iii)* $\rho$ *is continuously differentiable on* $X$ *and Lipschitz continuous with Lipschitz constant no greater than* $\lambda$*;*

*Proof.* Let $y \in Y$. Since $F$ is differentiable on $X \times Y$, the pair $(r_1(y), y)$ satisfies the equation $D_x F(r_1(y), y) = 0$. Therefore, by applying the Implicit Function Theorem, $r_1$ is continuously differentiable on $Y$ and

$$Dr_1(y) = [D_x^2 F(r_1(y), y)]^{-1} \circ [D_y(D_x F)(r_1(y), y)] \in \mathcal{L}(Y, X). \tag{6}$$

Moreover, by the Mean Value Inequality and the definition of $\lambda_1$

$$\|r_1(y_1) - r_2(y_2)\|_X \le \sup_{t\in[0,1]} \|Dr_1(ty_1 + (1-t)y_2)\|_{\mathcal{L}(Y,X)} \|y_1 - y_2\|_Y \le \lambda_1 \|y_1 - y_2\|_Y$$

for any $y_1, y_2 \in Y$. Hence, $r_1$ is Lipschitz continuous with Lipschitz constant no greater than $\lambda_1$. Analogously, $r_2$ is continuously differentiable on $X$,

$$Dr_2(x) = [D_y^2 G(x, r_2(x))]^{-1} \circ [D_x(D_y G)(x, r_2(x))] \in \mathcal{L}(X, Y) \tag{7}$$

for any $x \in X$, and $r_2$ is Lipschitz continuous with Lipschitz constant no greater than $\lambda_2$. Finally, by the chain rule and (6)-(7), $\rho$ is continuously differentiable on $X$ and

$$\begin{aligned} D\rho(x) =& Dr_1(r_2(x)) \circ Dr_2(x) \\ =& [D_x^2 F(\rho(x), r_2(x))]^{-1} \circ [D_y(D_x F)(\rho(x), r_2(x))] \\ & \circ [D_y^2 G(x, r_2(x))]^{-1} \circ [D_x(D_y G)(x, r_2(x))] \in \mathcal{L}(X, X) \end{aligned} \tag{8}$$

for any $x \in X$. Furthermore, in light of (8) and the definition of $\lambda$,

$$\sup_{x\in X} \|D\rho(x)\|_{\mathcal{L}(X,X)} \le \lambda_1 \cdot \lambda_2 = \lambda. \tag{9}$$

Hence, $\rho$ is Lipschitz continuous with Lipschitz constant no greater than $\lambda$. □

Before proving the uniqueness of the Nash equilibrium for $\Gamma$ we need the following definition and preliminary results.

**Definition 2.3** ([23, 16]) An operator $\Lambda\colon X \to X$ is said to be *super monotone* iff $\Lambda$ is strongly monotone with constant $\gamma > 1$, i.e. there exists $\gamma > 1$ such that

$$(\Lambda x_1 - \Lambda x_2, x_1 - x_2)_X \ge \gamma \|x_1 - x_2\|_X^2 \quad \text{for any } x_1, x_2 \in X.$$

**Proposition 2.1.** *If* $\Lambda\colon X \to X$ *is differentiable on* $X$ *and there exists* $\gamma > 1$ *such that, for any* $x \in X$

$$(D\Lambda(x)\varphi, \varphi)_X \ge \gamma \|\varphi\|_X^2 \quad \textit{for any } \varphi \in X,$$

*then* $\Lambda$ *is super monotone with constant* $\gamma$*.*

*Proof.* Let $x_1, x_2 \in X$. By applying the Mean Value Theorem to the real-valued function $f$ defined by

$$f(s) := (\Lambda(sx_1 + (1 - sx_2)), x_1 - x_2)_X, \quad \text{for any } s \in [0, 1],$$

there exists $t \in ]0, 1[$ such that

$$(\Lambda(x_1) - \Lambda(x_2), x_1 - x_2)_X = (D\Lambda(tx_1 + (1 - t)x_2)(x_1 - x_2), x_1 - x_2)_X. \tag{10}$$

By hypothesis, in light of (10), the result is proved. □

**Proposition 2.2.** *If $\Lambda \colon X \to X$ is super monotone with constant $\gamma$ and differentiable on $X$, then*

$$\sup_{x \in X} \|D\Lambda(x)\|_{\mathcal{L}(X,X)} \geq \gamma > 1.$$

*Proof.* Let $x_1, x_2 \in X$ and $x_1 \neq x_2$. In light of the super monotonicity of $\Lambda$, the Cauchy-Schwarz inequality and the Mean Value Inequality, we have

$$\begin{aligned} \gamma \|x_1 - x_2\|_X^2 \leq & (\Lambda(x_1) - \Lambda(x_2), x_1 - x_2)_X \\ \leq & \|\Lambda(x_1) - \Lambda(x_2)\|_X \|x_1 - x_2\|_X \\ \leq & \sup_{t \in [0,1]} \|D\Lambda(tx_1 + (1 - t)x_2)\|_{\mathcal{L}(X,X)} \|x_1 - x_2\|_X^2. \end{aligned}$$

Hence, $\sup_{x \in X} \|D\Lambda(x)\|_{\mathcal{L}(X,X)} \geq \gamma$. □

Finally, let us introduce a further crucial hypothesis.

**(H$_2$)** $\rho$, as defined in (1), is super monotone with constant $\gamma$.

The next result is a straightforward consequence of Proposition 2.2 and inequality (9).

**Lemma 2.3.** *If $\Gamma \in \mathcal{H}$ and satisfies (H$_1$)–(H$_2$), then $\lambda \geq \gamma > 1$.*

**Remark 2.1** It is worth to note that when $\Gamma \in \mathcal{H}$ and (H$_1$)–(H$_2$) are satisfied, then $\lambda > 1$. Hence, the composition of the best response functions $\rho$ may be not a contraction.

Now we prove the existence of one and only one Nash equilibrium for a game belonging to the class $\mathcal{H}$ and which satisfies hypotheses (H$_1$)–(H$_2$). The proof is obtained arguing similarly to the proof of Theorem 1 in [6].

**Theorem 2.1.** *Assume $\Gamma \in \mathcal{H}$ and satisfies (H$_1$)–(H$_2$). Let $I_{\lambda,\gamma} := ]1, \frac{\lambda^2 - 1}{\lambda^2 - 2\gamma + 1}[$. Then, for any $\delta \in I_{\lambda,\gamma}$ the function $g^\delta$ as defined in (2) is a contraction and the unique Nash equilibrium of $\Gamma$ is $(\bar{x}, r_2(\bar{x}))$, where $\bar{x}$ is the unique fixed point of $g^\delta$.*

*Proof.* Firstly, let us note that $I_{\lambda,\gamma} \neq \emptyset$ since $\lambda^2 - 2\gamma + 1 > 0$ and $\frac{\lambda^2 - 1}{\lambda^2 - 2\gamma + 1} > 1$ by Lemma 2.3. Let $\delta > 1$ and $x_1, x_2 \in X$, then

$$\begin{aligned} \|g^\delta(x_1) - g^\delta(x_2)\|_X^2 = & \|\delta[x_1 - x_2] - (\delta - 1)[\rho(x_1) - \rho(x_2)]\|_X^2 \\ = & \delta^2 \|x_1 - x_2\|_X^2 + (\delta - 1)^2 \|\rho(x_1) - \rho(x_2)\|_X^2 \\ & - 2\delta(\delta - 1)(\rho(x_1) - \rho(x_2), x_1 - x_2)_X. \end{aligned} \tag{11}$$

In light of Lemma 2.2*(iii)* and hypothesis ($H_2$), from (11) it follows

$$\|g^\delta(x_1) - g^\delta(x_2)\|_X^2 \leq \left[\delta^2 + (\delta-1)^2\lambda^2 - 2\delta(\delta-1)\gamma\right] \|x_1 - x_2\|_X^2.$$

Let $\kappa : ]1, +\infty[ \to \mathbb{R}$ be the function defined by

$$\kappa(\delta) := \delta^2 + (\delta-1)^2\lambda^2 - 2\delta(\delta-1)\gamma. \tag{12}$$

Being $\lambda \geq \gamma > 1$ (by Lemma 2.3), $\kappa$ is a convex quadratic function with minimum at $\frac{\lambda^2-\gamma}{\lambda^2-2\gamma+1} > 1$ and, since $\kappa(\frac{\lambda^2-\gamma}{\lambda^2-2\gamma+1}) = \frac{\lambda^2-\gamma^2}{\lambda^2-2\gamma+1} \geq 0$, then $\kappa(\delta) \geq 0$ for any $\delta > 1$. Furthermore, $\kappa(\delta) < 1$ if and only if $\delta + 1 + (\delta-1)\lambda^2 - 2\delta\gamma < 0$, that is if and only if $\delta \in I_{\lambda,\gamma}$. Therefore $g^\delta$ is a contraction for any $\delta \in I_{\lambda,\gamma}$. Finally, let $\bar{x} \in X$ be the unique fixed point of $g^\delta$. Then, in light of Lemma 2.1, $(\bar{x}, r_2(\bar{x}))$ is the unique Nash equilibrium of $\Gamma$. □

**Remark 2.2** We highlight that if $\Gamma$ is a weighted potential game ([20] and also, for example, [13, 5]), then the assumptions of Theorem 2.1 guarantee the existence of a unique Nash equilibrium without necessarily requiring either the strict concavity of the potential function or the existence of a maximizer for such a function (see [6, Theorem 1 and Remark 6]). In particular, assumed $X = Y = \mathbb{R}$ in Theorem 2.1, the unique Nash equilibrium of $\Gamma$ is not a maximizer of the potential function, as proved in [6, Proposition 6]. Furthermore, note that such a function is not strictly concave and it does not admit maximizers.

## 3 Adjustment process for the approximation of the Nash equilibrium

Now, we illustrate an iterative method that allows to approach the unique Nash equilibrium when $\Gamma \in \mathcal{H}$ and satisfies ($H_1$)–($H_2$). Let

$$\nu := \frac{\lambda^2 - \gamma}{\lambda^2 - 2\gamma + 1}. \tag{13}$$

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***Adjustment process*** ($\mathcal{A}$)

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(*Step 0*) Choose an initial point $y_0 \in Y$ and compute $x_0 = r_1(y_0)$.

(*Step 1*) Compute $\begin{cases} y_1 = r_2(x_0) \\ x_1 = \nu x_0 - (\nu - 1) r_1(y_0) = g^\nu(x_0). \end{cases}$

⋮

(*Step n*) Compute $\begin{cases} y_n = r_2(x_{n-1}) \\ x_n = \nu x_{n-1} - (\nu - 1) r_1(y_n) = g^\nu(x_{n-1}). \end{cases}$

⋮

---

**Remark 3.1** In the special case where the strategy sets are $\mathbb{R}$ and the best response functions are assumed to be linear, the adjustment process $(\mathcal{A})$ corresponds to a relaxation algorithm described in [3, equations (3.4) p. 536].

In the next theorem the convergence of the algorithm is stated.

**Theorem 3.1.** *Assume $\Gamma \in \mathcal{H}$ and satisfies $(H_1)$–$(H_2)$. Let $(\bar{x}, \bar{y})$ be the unique Nash equilibrium of $\Gamma$.*
*Then, the sequence $(x_n, y_n)_n$ generated by the adjustment process $(\mathcal{A})$ is strongly convergent to $(\bar{x}, \bar{y})$ in $X \times Y$. Furthermore*

$$\lim_{n\to+\infty} F(x_n, y_n) = F(\bar{x}, \bar{y}), \qquad \lim_{n\to+\infty} G(x_n, y_n) = G(\bar{x}, \bar{y}). \tag{14}$$

*Proof.* The existence of a unique Nash equilibrium of $\Gamma$ is guaranteed by Theorem 2.1. Since $\bar{y} = r_2(\bar{x})$, then $\bar{x}$ is the unique fixed point of $g^\nu$ by Lemma 2.1. In light of Theorem 2.1, $g^\nu$ is a contraction since $\nu \in I_{\lambda,\gamma}$ and

$$\|x_n - \bar{x}\|_X = \|g^\nu(x_{n-1}) - g^\nu(\bar{x})\|_X \le \kappa(\nu)\|x_{n-1} - \bar{x}\|_X \le \cdots \le \kappa(\nu)^{n-1}\|x_1 - \bar{x}\|,$$

for any $n \in \mathbb{N}$, where $\kappa$ is defined in (12). As $\kappa(\nu) < 1$, then $\lim_{n\to+\infty}\|x_n - \bar{x}\|_X = 0$. Therefore, the sequence $(x_n)_n$ is strongly convergent to $\bar{x}$. Furthermore, by Lemma 2.2 *(ii)*

$$\|y_n - \bar{y}\|_Y = \|r_2(x_{n-1}) - r_2(\bar{x})\|_Y \le \lambda_2\|x_{n-1} - \bar{x}\|_X, \quad \text{for any } n \in \mathbb{N}.$$

Since $\lim_{n\to+\infty} \|x_{n-1} - \bar{x}\|_X = 0$, the sequence $(y_n)_n$ is strongly convergent to $\bar{y}$. Finally, equalities in (14) follow from the continuity of $F$ and $G$. □

**Remark 3.2** In light of Theorem 2.1, in the definition of the adjustment process $(\mathcal{A})$ any $\delta$-inverse convex combinator of $\Gamma$, with $\delta \in I_{\lambda,\gamma}$, could be used and the convergence results stated in Theorem 3.1 would be valid. We chose $g^\nu$, where $\nu$ is defined in (13), as it is the inverse convex combinator whose estimation of the contraction constant obtainable via (12) is minimal, i.e. $\kappa(\nu) = \min_{\delta\in]1,+\infty[} \kappa(\delta)$ (see the proof of Theorem 2.1).

In the next proposition the error estimations of the sequences generated by the adjustment process $(\mathcal{A})$ are computed. Let

$$k := \frac{\lambda^2 - \gamma^2}{\lambda^2 - 2\gamma + 1}. \tag{15}$$

It is worth to note that $k = \kappa(\nu)$, where $\kappa$ and $\nu$ are defined, respectively, in (12) and (13).

**Proposition 3.1.** *Assume $\Gamma \in \mathcal{H}$ and satisfies $(H_1)$–$(H_2)$. Let $(\bar{x}, \bar{y})$ be the unique Nash equilibrium of $\Gamma$. Then:*

$$\|x_n - \bar{x}\|_X \le \frac{k^n}{1-k}\|x_1 - x_0\|_X \quad \textit{for any } n \in \mathbb{N} \tag{16}$$

$$\|y_{n+1} - \bar{y}\|_Y \le \frac{k^n\lambda_2}{1-k}\|x_1 - x_0\|_X \quad \textit{for any } n \ge 0. \tag{17}$$

*where $k$ is defined in (15).*

*Proof.* In light of Theorem 2.1, $g^\nu$ is a contraction and the relative (estimated) contraction constant is $k$, so

$$\|x_{n+1} - x_n\|_X \le k\|x_n - x_{n-1}\|_X \le \ldots \le k^n\|x_1 - x_0\|_X \quad \text{for any } n \in \mathbb{N}.$$

Consequently, for any $p \in \mathbb{N}$ we get

$$\begin{aligned}\|x_{n+p} - x_n\|_X &\le \sum_{j=1}^{p} \|x_{n+j} - x_{n+j-1}\|_X \\ &\le \sum_{j=1}^{p} k^{n+j-1}\|x_1 - x_0\|_X = \frac{k^n(1-k^p)}{1-k}\|x_1 - x_0\|_X.\end{aligned} \tag{18}$$

Hence, inequality (16) follows from (18) taking the limit as $p \to +\infty$ since $k < 1$. Finally, by Lemma 2.2*(ii)* and (16) we get

$$\|y_{n+1} - \bar{y}\|_Y = \|r_2(x_n) - r_2(\bar{x})\|_Y \le \lambda_2\|x_n - \bar{x}\|_X \le \frac{k^n\lambda_2}{1-k}\|x_1 - x_0\|_X,$$

so inequality (17) is proved. □

## 4 Approximation of the Nash equilibrium via the local variation method

The adjustment process $(\mathcal{A})$ illustrated in Section 3 involves the best response functions of the game $\Gamma$. In this section we propose a numerical method which can be fruitfully used when the analytic expressions of the best response functions are not available. Firstly, we approximate the Nash equilibrium of $\Gamma$ through a sequence of Nash equilibria of games whose strategy sets are finite dimensional spaces. Then, we approximate each term of the sequence of Nash equilibria by combining the adjustment process $(\mathcal{A})$ with the *local variation method*, a derivative-free algorithm introduced in [7] for finding solutions of variational problems and used, in particular, in [8] for functional minimization problems and in [21, 11] for zero-sum games.

In order to achieve these goals, in this section we consider the following assumptions on $\Gamma = \{2, X, Y, F, G\}$.

**(A$_1$)** the function $F$ is strongly concave on $X$ uniformly on $Y$, i.e. there exists a constant $m_F > 0$ such that, for any $x', x'' \in X$, any $y \in Y$ and any $t \in [0, 1]$

$$F(tx' + (1-t)x'', y) \ge tF(x', y) + (1-t)F(x'', y) + m_F t(1-t)\|x' - x''\|_X^2;$$

and the function $G$ is strongly concave on $Y$ uniformly on $X$, i.e. there exists a constant $m_G > 0$ such that, for any $y', y'' \in Y$, any $x \in X$ and any $t \in [0, 1]$

$$G(x, ty' + (1-t)y'') \ge tG(x, y') + (1-t)G(x, y'') + m_G t(1-t)\|y' - y''\|_Y^2;$$

**(A$_2$)** there exists $\gamma > 1$ such that, for any $x_1, x_2 \in X$ and $y \in Y$

$$(H(x_1, x_2, y)\varphi, \varphi)_X \ge \gamma\|\varphi\|_X^2 \quad \text{for any } \varphi \in X,$$

where $H(x_1, x_2, y) : X \to X$ is the operator defined by

$$H(x_1, x_2, y) := [D_x^2 F(x_1, y)]^{-1} \circ D_y(D_x F)(x_1, y) \circ [D_y^2 G(x_2, y)]^{-1} \circ D_x(D_y G)(x_2, y). \tag{19}$$

**Remark 4.1** If $F$ is strongly concave on $X$ uniformly on $Y$, then the function $F(\cdot, y)$ is strongly concave for any $y \in Y$. The converse is not true in general (this is the case, for example, if $F$ is defined on $\mathbb{R}^2$ by $F(x, y) = -x^2 e^y$). Clearly, a function can be strongly concave on $X$ uniformly on $Y$ and can be not concave on $X \times Y$ (take, for example, $F$ defined on $\mathbb{R}^2$ by $F(x, y) = -x^2(e^y + 1)$).

**Remark 4.2** The new conditions $(A_1)$–$(A_2)$ are more restrictive than $(H_1)$–$(H_2)$. In fact, $(A_1)$ implies $(H_1)$ in light of, for example, [4, Corollary 11.16], whereas $(A_2)$ implies $(H_2)$ in light of Proposition 2.1 and (8) since $D\rho(x) = H(\rho(x), x, r_2(x))$ for any $x \in X$. Therefore, all the results obtained in Section 2 and 3 apply when we replace $(H_1)$–$(H_2)$ with $(A_1)$–$(A_2)$.

We emphasize that, although $(A_1)$–$(A_2)$ are more restrictive, they will allow to handle situations where the best response functions are not explicit. In fact, the following examples illustrate two games which belong to the class $\mathcal{H}$, satisfy assumptions $(A_1)$–$(A_2)$, and where the best response functions of both players cannot be computed explicitly.

**Example 4.1** Let $\Gamma = \{2, X, Y, F, G\}$ be the game where $X = Y = \mathbb{R}$ and the payoff functions are defined by

$$F(x, y) = -x^2 - \cos x \sin y - 5xy,$$
$$G(x, y) = \frac{1}{1 + y^2} - 4y^2 + y - 12xy.$$

The function $F$ is strongly concave in $x$ uniformly in $y$ since $D_x^2 F(x, y) = -2 + \cos x \sin y \leq -1$ for any $(x, y) \in \mathbb{R}^2$, and the function $G$ is strongly concave in $y$ uniformly in $x$ since $D_y^2 G(x, y) = [(6y^2 - 2)/(1 + y^2)^3] - 8 \leq -15/2$ for any $(x, y) \in \mathbb{R}^2$. So $(A_1)$ holds. Moreover

$$\frac{4}{3} \leq \lambda_1 = \sup_{(x,y)\in\mathbb{R}^2} \left| \frac{D_y(D_x F)(x, y)}{D_x^2 F(x, y)} \right| = \sup_{(x,y)\in\mathbb{R}^2} \frac{5 - \sin x \cos y}{2 - \cos x \sin y} \leq 6,$$
$$\lambda_2 = \sup_{(x,y)\in\mathbb{R}^2} \left| \frac{D_x(D_y G)(x, y)}{D_y^2 G(x, y)} \right| = \sup_{(x,y)\in\mathbb{R}^2} \frac{6(y^2 + 1)^3}{4y^6 + 12y^4 + 9y^2 + 5} = \frac{8}{5}.$$

Therefore $\Gamma \in \mathcal{H}$. Furthermore

$$H(x_1, x_2, y) = \frac{(5 - \sin x_1 \cos y)[6(y^2 + 1)^3]}{(2 - \cos x_1 \sin y)(4y^6 + 12y^4 + 9y^2 + 5)}$$
$$\geq \inf_{(x,y)\in\mathbb{R}^2} \frac{5 - \sin x \cos y}{2 - \cos x \sin y} \cdot \inf_{(x,y)\in\mathbb{R}^2} \frac{6(y^2 + 1)^3}{4y^6 + 12y^4 + 9y^2 + 5} \geq \frac{4}{3} \cdot \frac{6}{5} = \frac{8}{5} > 1,$$

for any $x_1, x_2 \in \mathbb{R}$ and $y \in \mathbb{R}$. Hence $(A_2)$ is satisfied by taking $\gamma = 8/5$. Note that $\lambda > 1$ since $\lambda = \lambda_1\lambda_2 \in [32/15, 48/5]$, and $\Gamma$ has a unique Nash equilibrium in light of Theorem 2.1.

**Example 4.2** Let $\Gamma = \{2, X, Y, F, G\}$ be the weighted potential game where $X = Y = \mathbb{R}$ and the potential function $P$ is defined on $\mathbb{R}^2$ by

$$P(x, y) = \frac{1}{1 + x^2} + \frac{1}{1 + y^2} - 4x^2 + x - 4y^2 + y - 12xy,$$

that is, the game considered in [6, Example 2]. Since $D_x^2 P(x, y) \leq -15/2$ and $D_y^2 P(x, y) \leq -15/2$ for any $(x, y) \in \mathbb{R}^2$, then $P$ is both strongly concave in $x$ uniformly in $y$ and strongly concave in $y$ uniformly in $x$, so $(A_1)$ holds. Moreover, $\lambda_1 = \lambda_2 = 8/5$ and $H(x_1, x_2, y) \geq 36/25$ for any $x_1, x_2 \in \mathbb{R}$ and $y \in \mathbb{R}$, hence $\Gamma \in \mathcal{H}$ and $(A_2)$ is satisfied by taking $\gamma = 36/25$ (in [6, Example 2] the main computations are provided). It is worth to note that, in light of Theorem 2.1 and Remark 2.2, $\Gamma$ has a unique Nash equilibrium which is not a maximizer of $P$ on $\mathbb{R}^2$.

### 4.1 Discretization scheme

Let $(X_p)_{p\in\mathbb{N}}$ be a sequence of spaces such that $X_p$ is a $p$-dimensional subspace of $X$ with inner product $(\cdot, \cdot)_{X_p}$ and related norm $\|\cdot\|_{X_p}$ (restrictions of the inner product and the norm of the space $X$, respectively), and such that $\cup_{p\in\mathbb{N}} X_p$ is dense in $X$. Let $(Y_q)_{q\in\mathbb{N}}$ be a sequence of spaces such that $Y_q$ is a $q$-dimensional subspace of $Y$ with inner product $(\cdot, \cdot)_{Y_q}$ and related norm $\|\cdot\|_{Y_q}$ (restrictions of the inner product and the norm of the space $Y$, respectively), and such that $\cup_{q\in\mathbb{N}} Y_q$ is dense in $Y$. For any $p$, $q \in \mathbb{N}$, let $\Gamma_{p,q} := \{2, X_p, Y_q, F, G\}$ be the two-player game obtained by replacing in $\Gamma$ the strategy sets $X$ and $Y$ with $X_p$ and $Y_q$, respectively, and the payoff functions $F$ and $G$ with their restrictions to $X_p \times Y_q$ (with an abuse of notation the payoff functions in $\Gamma_{p,q}$ are still denoted by $F$ and $G$).

**Theorem 4.1.** *Assume $\Gamma \in \mathcal{H}$ and satisfies $(A_1)$–$(A_2)$.*
*Then, the game $\Gamma_{p,q}$ has a unique Nash equilibrium $(\bar{x}_{p,q}, \bar{y}_{p,q}) \in X_p \times Y_q$ for any $p, q \in \mathbb{N}$. Moreover, if the sequence $(\bar{x}_{p,q}, \bar{y}_{p,q})_{p,q}$ is strongly convergent to $(\bar{x}, \bar{y}) \in X \times Y$, as $p, q \to +\infty$, then $(\bar{x}, \bar{y})$ is the unique Nash equilibrium of $\Gamma$ and*

$$\lim_{p,q\to+\infty} F(\bar{x}_{p,q}, \bar{y}_{p,q}) = F(\bar{x}, \bar{y}), \qquad \lim_{p,q\to+\infty} G(\bar{x}_{p,q}, \bar{y}_{p,q}) = G(\bar{x}, \bar{y}). \tag{20}$$

*Proof.* Firstly, let us note that the existence of a unique Nash equilibrium of $\Gamma$ is guaranteed by Theorem 2.1. Since $\Gamma \in \mathcal{H}$ and satisfies $(A_1)$–$(A_2)$, then even $\Gamma_{p,q}$ belongs to $\mathcal{H}$ and satisfies $(A_1)$–$(A_2)$, for any $p, q \in \mathbb{N}$. So, in light of Theorem 2.1, $\Gamma_{p,q}$ has a unique Nash equilibrium $(\bar{x}_{p,q}, \bar{y}_{p,q}) \in X_p \times Y_q$, for any $p, q \in \mathbb{N}$.
Let $(\bar{x}, \bar{y}) \in X \times Y$ be the limit of the sequence $(\bar{x}_{p,q}, \bar{y}_{p,q})_{p,q} \subseteq X_p \times Y_q$ and let $x \in X$. Since $\cup_{p\in\mathbb{N}} X_p$ is dense in $X$, there exists a sequence $(\hat{x}_p)_p$ strongly convergent to $x$ and

such that $\hat{x}_p \in X_p$ for any $p \in \mathbb{N}$. Being $(\bar{x}_{p,q}, \bar{y}_{p,q})$ the Nash equilibrium of $\Gamma_{p,q}$, for any $p, q \in \mathbb{N}$ we have $F(\bar{x}_{p,q}, \bar{y}_{p,q}) \geq F(\hat{x}_p, \bar{y}_{p,q})$. Given the above, by the continuity of $F$ we get

$$F(\bar{x}, \bar{y}) = \lim_{p,q \to +\infty} F(\bar{x}_{p,q}, \bar{y}_{p,q}) \geq \lim_{p,q \to +\infty} F(\hat{x}_p, \bar{y}_{p,q}) = F(x, \bar{y}).$$

Analogously, one can show that $G(\bar{x}, \bar{y}) \geq G(\bar{x}, y)$ for any $y \in Y$. Hence, $(\bar{x}, \bar{y})$ is the Nash equilibrium of $\Gamma$ and equalities in (20) hold. □

**Remark 4.3** It is worth to note that a more general discretization scheme could be used. Such a scheme was introduced in [19], and adapted in [10] for the approximation of zero-sum games. Let $(X_s)_{s>0}$ and $(Y_t)_{t>0}$ (where $s$ and $t$ will converge to 0) be two nets of $N(s)$-dimensional spaces and $M(t)$-dimensional spaces (with $N(s), M(t) \in \mathbb{N}$), respectively, and consider two nets $(P_s^X)_{s>0}$ and $(P_t^Y)_{t>0}$ of injective operators such that:

$$P_s^X \in \mathcal{L}(X_s, X) \text{ for any } s > 0, \text{ and } \cup_{s>0} P_s^X(X_s) \text{ is dense in } X;$$
$$P_t^Y \in \mathcal{L}(Y_t, Y) \text{ for any } t > 0, \text{ and } \cup_{t>0} P_t^Y(Y_t) \text{ is dense in } Y.$$

Let $F_{(s,t)}$ and $G_{(s,t)}$ be the real-valued functions defined on $X_s \times Y_t$ by

$$F_{(s,t)}(x_s, y_t) := F(P_s^X x_s, P_t^Y y_t) \quad \text{and} \quad G_{(s,t)}(x_s, y_t) := G(P_s^X x_s, P_t^Y y_t),$$

respectively. Therefore, considering the games $\Gamma_{s,t} := \{2, X_s, Y_t, F_{(s,t)}, G_{(s,t)}\}$ for any $s, t \in ]0, +\infty[$, an analogous result of Theorem 4.1 applies. For the sake of clarity and simplicity we used the discretization scheme described at the beginning of this subsection.

Once applied the discretization scheme presented above, we can move the analysis on the approximation of the Nash equilibrium from the infinite dimensional setting where $\Gamma$ lays towards the finite dimensional settings in which the games $\Gamma_{p,q}$ are defined for any $p, q \in \mathbb{N}$. Hence, from now on we focus on the issue of approximating the unique Nash equilibrium $(\bar{x}_{p,q}, \bar{y}_{p,q}) \in X_p \times Y_q$ of the game $\Gamma_{p,q} = \{2, X_p, Y_q, F, G\}$, where $X_p$ is a $p$-dimensional subspace of $X$ and $Y_q$ is a $q$-dimensional subspace of $Y$, with $p, q \in \mathbb{N}$ fixed.

### 4.2 Approximation of the finite dimensional problem: the local variation method

Before addressing the issue emphasized at the end of the previous subsection, we describe now the local variation method that allows, by using only the values of the function, both to find an approximation of the unique maximizer of a strongly concave real-valued function defined on a finite dimensional space and to obtain an estimation of the distance between the approximation calculated and the (exact) maximizer. Such a method will be used to define a numerical method to approximate the unique Nash equilibrium of $\Gamma_{p,q}$.

Let $V_N$ be an $N$-dimensional real vector space endowed with an inner product $(\cdot,\cdot)_{V_N}$ and related norm $\|\cdot\|_{V_N}$, and let $\mathcal{B} = \{b_1, b_2, \dots, b_N\} \subseteq V_N$ be the basis of $V_N$ such that the matrix of $(\cdot,\cdot)_{V_N}$ relative to $\mathcal{B}$ is the identity matrix of size $N$ [1]. In this way, the inner product $(\cdot,\cdot)_{V_N}$ of $V_N$ can be represented via the usual inner product of $\mathbb{R}^N$, denoted by $(\cdot,\cdot)_{\mathbb{R}^N}$. In fact, for any $u, v \in V_N$

$$(u, v)_{V_N} = x^T I_N y = x_1 y_1 + \cdots + x_N y_N = (x, y)_{\mathbb{R}^N}, \tag{21}$$

where $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ is the coordinate vector of $u$ relative to $\mathcal{B}$, defined by $\sum_{i=1}^N x_i b_i = u$ and denoted with $c_{\mathcal{B}}(u)$, and $y = (y_1, \dots, y_N) \in \mathbb{R}^N$ is the coordinate vector of $v$ relative to $\mathcal{B}$, defined by $\sum_{i=1}^N y_i b_i = v$ and denoted with $c_{\mathcal{B}}(v)$. Consequently, the norm $\|\cdot\|_{V_N}$ of $V_N$ can be represented via the the Euclidean norm of $\mathbb{R}^N$, denoted by $\|\cdot\|_{\mathbb{R}^N}$. In fact, in light of (21), for any $u \in V_N$

$$\|u\|_{V_N} = \sqrt{(u, u)_{V_N}} = \sqrt{x_1^2 + \cdots + x_N^2} = \|x\|_{\mathbb{R}^N}, \tag{22}$$

where $x = c_{\mathcal{B}}(u)$.

Let $J\colon V_N \to \mathbb{R}$ be a strongly concave function. Maximizing $J$ over $V_N$ is equivalent to maximize the function $f\colon \mathbb{R}^N \to \mathbb{R}$ defined by

$$f(z_1, \dots, z_N) := J(z_1 b_1 + \cdots + z_N b_N), \tag{23}$$

in the sense that even $f$ is strongly concave (on $\mathbb{R}^N$) and

$$\{\bar{z}_1 b_1 + \dots \bar{z}_N b_N\} = \operatorname*{Arg\,max}_{v \in V_N} J(v) \Leftrightarrow \{(\bar{z}_1, \dots, \bar{z}_N)\} = \operatorname*{Arg\,max}_{(z_1,\dots,z_N) \in \mathbb{R}^N} f(z_1, \dots, z_N)$$

$$\max_{v \in V_N} J(v) = \max_{(z_1,\dots,z_N) \in \mathbb{R}^N} f(z_1, \dots, z_N).$$

Following the scheme proposed in [8], we illustrate the local variation method (introduced in [7]) used to find an approximation of the unique maximizer of $f\colon \mathbb{R}^N \to \mathbb{R}$.

[1]In a $k$-dimensional real vector space $E$ with inner product $(\cdot,\cdot)_E$, denoted by $A \in \mathbb{R}^{k \times k}$ the matrix of $(\cdot,\cdot)_E$ relative to a basis $\mathcal{E} = \{e_1, e_2, \dots, e_N\}$, i.e. $a_{ij} = (e_i, e_j)_E$ for any $i, j \in \{1, \dots, k\}$, the following representation holds: for any $u, v \in E$

$$(u, v)_E = x^T A y,$$

where $x = (x_1, \dots, x_k) \in \mathbb{R}^k$ and $y = (y_1, \dots, y_k) \in \mathbb{R}^k$ are the coordinate vectors of $u$ and $v$ relative to $\mathcal{E}$, i.e. $\sum_{i=1}^k x_i e_i = u$ and $\sum_{i=1}^k y_i e_i = v$, and $x^T$ is the transpose of the vector x.

**_Local variation method_**

($Step\ 0$) Fix an initial point $(z_1^0, z_2^0, \dots, z_N^0) \in \mathbb{R}^N$ and a range $\epsilon > 0$.

($Step\ 1$) Define:

$$\begin{aligned}\Theta_1 &\coloneqq f(z_1^0, z_2^0, z_3^0, \dots, z_N^0),\\ \Theta_1^+ &\coloneqq f(z_1^0 + \epsilon, z_2^0, z_3^0, \dots, z_N^0),\\ \Theta_1^- &\coloneqq f(z_1^0 - \epsilon, z_2^0, z_3^0, \dots, z_N^0).\end{aligned}$$

Find the point in the set $\{z_1^0, z_1^0 + \epsilon, z_1^0 - \epsilon\}$ which corresponds to the maximum of the set $\{\Theta_1, \Theta_1^+, \Theta_1^-\}$ and denote it by $z_{1,1}^\epsilon$.

($Step\ 2$) Define:

$$\begin{aligned}\Theta_2 &\coloneqq f(z_{1,1}^\epsilon, z_2^0, z_3^0, \dots, z_N^0),\\ \Theta_2^+ &\coloneqq f(z_{1,1}^\epsilon, z_2^0 + \epsilon, z_3^0, \dots, z_N^0),\\ \Theta_2^- &\coloneqq f(z_{1,1}^\epsilon, z_2^0 - \epsilon, z_3^0, \dots, z_N^0).\end{aligned}$$

Find the point in the set $\{z_2^0, z_2^0 + \epsilon, z_2^0 - \epsilon\}$ which corresponds to the maximum of the set $\{\Theta_2, \Theta_2^+, \Theta_2^-\}$ and denote it by $z_{1,2}^\epsilon$.

$\vdots$

($Step\ i$) Define:

$$\begin{aligned}\Theta_i &\coloneqq f(z_{1,1}^\epsilon, z_{1,2}^\epsilon, \dots, z_{1,i-1}^\epsilon, z_i^0, z_{i+1}^0, \dots, z_N^0),\\ \Theta_i^+ &\coloneqq f(z_{1,1}^\epsilon, z_{1,2}^\epsilon, \dots, z_{1,i-1}^\epsilon, z_i^0 + \epsilon, z_{i+1}^0, \dots, z_N^0),\\ \Theta_i^- &\coloneqq f(z_{1,1}^\epsilon, z_{1,2}^\epsilon, \dots, z_{1,i-1}^\epsilon, z_i^0 - \epsilon, z_{i+1}^0, \dots, z_N^0).\end{aligned}$$

Find the point in the set $\{z_i^0, z_i^0 + \epsilon, z_i^0 - \epsilon\}$ which corresponds to the maximum of the set $\{\Theta_i, \Theta_i^+, \Theta_i^-\}$ and denote it by $z_{1,i}^\epsilon$.

$\vdots$

($Step\ N$) Define:

$$\begin{aligned}\Theta_N &\coloneqq f(z_{1,1}^\epsilon, z_{1,2}^\epsilon, \dots, z_{1,N-1}^\epsilon, z_N^0),\\ \Theta_N^+ &\coloneqq f(z_{1,1}^\epsilon, z_{1,2}^\epsilon, \dots, z_{1,N-1}^\epsilon, z_N^0 + \epsilon),\\ \Theta_N^- &\coloneqq f(z_{1,1}^\epsilon, z_{1,2}^\epsilon, \dots, z_{1,N-1}^\epsilon, z_N^0 - \epsilon).\end{aligned}$$

Find the point in the set $\{z_N^0, z_N^0+\epsilon, z_N^0-\epsilon\}$ which corresponds to the maximum of the set $\{\Theta_N, \Theta_N^+, \Theta_N^-\}$ and denote it by $z_{1,N}^\epsilon$. Hence, the vector $z_1^\epsilon \coloneqq (z_{1,1}^\epsilon, z_{1,2}^\epsilon, \dots, z_{1,N}^\epsilon)$ is constructed.

($Step\ R$) Repeat the steps from 1 to N choosing $z_1^\epsilon$ as initial point and the same range $\epsilon$, and get $z_2^\epsilon \coloneqq (z_{2,1}^\epsilon, z_{2,2}^\epsilon, \dots, z_{2,N}^\epsilon)$.

$\vdots$

(*Step S*) Continue until obtaining a *stationary* vector $\bar{z}^\epsilon := (\bar{z}_1^\epsilon, \bar{z}_2^\epsilon, \ldots, \bar{z}_N^\epsilon)$, i.e. a vector which satisfies the following inequalities

$$\begin{cases} \begin{cases} \Theta_1^+ \leq \Theta_1 \\ \Theta_1^- \leq \Theta_1 \end{cases} \\ \vdots \\ \begin{cases} \Theta_N^+ \leq \Theta_N \\ \Theta_N^- \leq \Theta_N. \end{cases} \end{cases} \iff \begin{cases} f(\bar{z}_1^\epsilon \pm \epsilon, \bar{z}_2^\epsilon, \ldots, \bar{z}_N^\epsilon) \leq f(\bar{z}_1^\epsilon, \bar{z}_2^\epsilon, \ldots, \bar{z}_N^\epsilon) \\ \vdots \\ f(\bar{z}_1^\epsilon, \bar{z}_2^\epsilon, \ldots, \bar{z}_N^\epsilon \pm \epsilon) \leq f(\bar{z}_1^\epsilon, \bar{z}_2^\epsilon, \ldots, \bar{z}_N^\epsilon). \end{cases} \tag{24}$$

---

The existence of a vector verifying (24) is shown in [8]. For the sake of completeness we give below the statement and the proof of such a result.

**Lemma 4.1** (Lemme 1.1 in [8])**.** *Let $f\colon \mathbb{R}^N \to \mathbb{R}$ be a strongly concave function and let $\epsilon > 0$. Then, there exists a vector $\bar{z}^\epsilon$ satisfying (24), which is obtained by repeating a finite number of times the steps from 1 to $N$ of the local variation method.*

*Proof.* Firstly, it is worth to note that the strong concavity of $f$ implies

$$\lim_{\|z\|_{\mathbb{R}^N} \to +\infty} f(z) = -\infty, \tag{25}$$

Let $(z_k^\epsilon)_k$ be the sequence where $z_k^\epsilon := (z_{k,1}^\epsilon, z_{k,2}^\epsilon, \ldots, z_{k,N}^\epsilon)$ is the vector obtained after repeating $k$ times the steps from 1 to $N$. Then, the sequence $(z_k^\epsilon)_k$ is necessarily bounded. In fact, by contradiction, if $\lim_{k\to+\infty}\|z_k^\epsilon\|_{\mathbb{R}^N} = +\infty$ then $\lim_{k\to+\infty} f(z_k^\epsilon) = -\infty$, in light of (25). This is not possible since $(f(z_k^\epsilon))_k$ is an increasing sequence, by construction. Therefore $(z_k^\epsilon)_k$ is bounded and, consequently, there exists a constant $C > 0$ such that $|z_{k,i}^\epsilon| \leq C$ for any $i \in \{1, \ldots, N\}$ and $k \in \mathbb{N}$. Given the above and since $z_{k,i}^\epsilon = z_i^0 + m_{k,i}\epsilon$ for some $m_{k,i} \in \mathbb{Z}$, there exists $\bar{k} \in \mathbb{N}$ such that $z_k^\epsilon = z_{\bar{k}}^\epsilon$ for any $k > \bar{k}$ and $z_{\bar{k}}^\epsilon$ necessarily satisfies (24), so the result is proved. □

We emphasize that the convergence of the local variation method has been shown in [8] for functions defined on a finite dimensional space, whereas, in [21] an error estimation in the case of functions defined on $\mathbb{R}^N$ has been claimed in order to obtain error bounds in zero-sum games. In this paper, having in mind to obtain error estimations for non zero-sum games defined in finite dimensional spaces, we need to prove, preliminarily, error bounds and convergence of the local variation method for functions defined on $\mathbb{R}^N$. So, in the next lemma we address this issue, by exploiting the proof of Théorème 3.1 in [8]. Before proving the result, we recall that, when $f\colon \mathbb{R}^N \to \mathbb{R}$ is a differentiable function and given $x \in \mathbb{R}^N$, the Taylor's theorem guarantees

$$\exists \mathcal{I}_x \subseteq \mathbb{R}^N \text{ s.t. } \quad f(x+h) - f(x) = (\nabla f(x), h)_{\mathbb{R}^N} + r(x,h) \quad \forall h \in \mathcal{I}_x, \tag{26}$$

where $\mathcal{I}_x$ is a neighbourhood of 0 depending on $x$, $\nabla f(x) \in \mathbb{R}^N$ is the gradient of $f$ at $x$, and the remainder $r(x,h)$ satisfies $\lim_{h\to 0} r(x,h)/\|h\|_{\mathbb{R}^N} = 0$. Moreover, it is worth to remind the following characterization of the strong concavity: a differentiable function $f\colon \mathbb{R}^N \to \mathbb{R}$ is strongly concave if and only if there exists $m > 0$ such that

$$f(x'') - f(x') \le (\nabla f(x'), x'' - x')_{\mathbb{R}^N} - m\|x'' - x'\|^2_{\mathbb{R}^N}; \tag{27}$$

for any $x', x'' \in \mathbb{R}^N$.

**Lemma 4.2.** *Let $f\colon \mathbb{R}^N \to \mathbb{R}$ be a differentiable and strongly concave function on $\mathbb{R}^N$. Assume that there exist $C_1 > 0$, $C_0 \ge 0$ and $\tau > 1$ such that*

$$|r(x,h)| \le C_1\|h\|^{\tau}_{\mathbb{R}^N} + C_0\|h\|^{\tau+1}_{\mathbb{R}^N} \quad \text{for any } x \in \mathbb{R}^N \text{ and } h \in \mathcal{I}_x, \tag{28}$$

*where $r$ and $\mathcal{I}_x$ are defined in (26).*
*Let $\epsilon > 0$ and let $z^\epsilon \in \mathbb{R}^N$ be the stationary vector obtained at step $S$ of the local variation method applied to $f$, i.e. the vector satisfying the inequalities (24). Then*

$$\|z^\epsilon - z^{max}\|_{\mathbb{R}^N} \le \frac{\sqrt{N}(C_1 + \epsilon C_0)}{m}\epsilon^{\tau-1}, \tag{29}$$

*where $z^{max}$ is the unique maximizer of $f$ over $\mathbb{R}^N$ and $m$ is the constant related to the strong concavity of $f$, defined in (27). Moreover, if $(\epsilon_n)_{n\ge 0} \subseteq ]0,+\infty[$ is a sequence decreasing to zero, the sequence $(z^{\epsilon_n})_{n\ge 0}$ converges to $z^{max}$.*

*Proof.* Let us note that $z^\epsilon$ is well-defined in light of Lemma 4.1, and that the last part of the statement follows immediately from (29), as $\lim_{n\to+\infty} \epsilon_n = 0$ and $\tau > 1$. Therefore, we prove only inequality (29). Let $\{e_1, \dots, e_N\}$ be the standard basis of $\mathbb{R}^N$ and let us fix $i \in \{1, \dots, N\}$. Since $z^\epsilon$ verifies (24), by (26) we have

$$\begin{aligned} 0 &\ge f(z^\epsilon + \epsilon e_i) - f(z^\epsilon) = \epsilon(\nabla f(z^\epsilon), e_i)_{\mathbb{R}^N} + r(z^\epsilon, \epsilon e_i), \\ 0 &\ge f(z^\epsilon - \epsilon e_i) - f(z^\epsilon) = -\epsilon(\nabla f(z^\epsilon), e_i)_{\mathbb{R}^N} + r(z^\epsilon, -\epsilon e_i). \end{aligned} \tag{30}$$

So, in light of (30) and (28)

$$\begin{aligned} (\nabla f(z^\epsilon), e_i)_{\mathbb{R}^N} &\ge \frac{r(z^\epsilon, -\epsilon e_i)}{\epsilon} \ge -\frac{C_1\|-\epsilon e_i\|^{\tau}_{\mathbb{R}^N} + C_0\|-\epsilon e_i\|^{\tau+1}_{\mathbb{R}^N}}{\epsilon}, \\ (\nabla f(z^\epsilon), e_i)_{\mathbb{R}^N} &\le -\frac{r(z^\epsilon, \epsilon e_i)}{\epsilon} \le \frac{C_1\|\epsilon e_i\|^{\tau}_{\mathbb{R}^N} + C_0\|\epsilon e_i\|^{\tau+1}_{\mathbb{R}^N}}{\epsilon}. \end{aligned} \tag{31}$$

Hence, by (31) and since $\|e_i\|_{\mathbb{R}^N} = 1$ we get

$$|(\nabla f(z^\epsilon), e_i)_{\mathbb{R}^N}| \le \epsilon^{\tau-1}(C_1 + \epsilon C_0). \tag{32}$$

As $z^{max}$ is the maximizer of $f$ and in light of (27) and (32), then

$$\begin{aligned}
m\|z^\epsilon - z^{max}\|^2_{\mathbb{R}^N} &\leq -[f(z^{max}) - f(z^\epsilon)] + (\nabla f(z^\epsilon), z^{max} - z^\epsilon)_{\mathbb{R}^N} \\
&\leq |(\nabla f(z^\epsilon), z^{max} - z^\epsilon)_{\mathbb{R}^N}| \\
&= \left|(\nabla f(z^\epsilon), \sum_{i=1}^N (z^{max} - z^\epsilon)_i e_i)_{\mathbb{R}^N}\right| \\
&\leq \sum_{i=1}^N |(z^{max} - z^\epsilon)_i| \cdot |(\nabla f(z^\epsilon), e_i)_{\mathbb{R}^N}| \\
&\leq \epsilon^{\tau-1}(C_1 + \epsilon C_0)\|z^{max} - z^\epsilon\|_1 \\
&\leq \sqrt{N}(C_1 + \epsilon C_0)\epsilon^{\tau-1}\|z^{max} - z^\epsilon\|_{\mathbb{R}^N},
\end{aligned}$$

where $\|\cdot\|_1$ is the 1-norm of $\mathbb{R}^N$ and the last inequality follows from the equivalence of norms in $\mathbb{R}^N$, more precisely from the inequality $\|z\|_p \leq N^{(1/p-1/q)}\|z\|_q$ holding for any $z \in \mathbb{R}^N$ and $p, q \in [1, +\infty[$. Therefore (29) is proved and the proof is complete. □

**Remark 4.4** Let us note that the convergence of the local variation method is guaranteed by assuming only (25), i.e. the coercivity of $-f$, as shown in [8, Théorème 2.1]. In Lemma 4.2 we added further assumptions related to the differentiability of $f$ to obtain also an error estimation result (inequality (29) in the statement of Lemma 4.2).

Let us come back to the main goal of this subsection, namely the application of the local variation method to the function $J\colon V_N \to \mathbb{R}$ defined on a general finite dimensional space. Before showing the error estimation and the convergence results, it is worth to recall that, when $J$ is differentiable on $V_N$ and given $u \in V_N$, the Taylor's theorem ensures

$$\exists\, \mathcal{V}_u \subseteq V_N \text{ s.t. } \quad J(u+v) - J(u) = \langle DJ(u), v\rangle_{V_N^* \times V_N} + R(u,v) \quad \forall v \in \mathcal{V}_u, \tag{33}$$

where $\mathcal{V}_u$ is a neighbourhood of 0 depending on $u$ and $R(u,v)$ is the remainder, and furthermore that a differentiable function $J\colon V_N \to \mathbb{R}$ is strongly concave if and only if there exists $m > 0$ such that

$$J(u'') - J(u') \leq \langle DJ(u'), u'' - u'\rangle_{V_N^* \times V_N} - m\|u'' - u'\|^2_{V_N} \tag{34}$$

for any $u', u'' \in V_N$. Finally it is worth to remind that, given $u \in V_N$, we denoted by $c_{\mathcal{B}}(u) \in \mathbb{R}^N$ the coordinate vector of $u$ relative to $\mathcal{B}$.

**Theorem 4.2.** *Let $J\colon V_N \to \mathbb{R}$ be a differentiable and strongly concave function on $V_N$. Assume that there exist $C_1 > 0$, $C_0 \geq 0$ and $\tau > 1$ such that*

$$|R(u,v)| \leq C_1\|v\|^\tau_{\mathbb{R}^N} + C_0\|v\|^{\tau+1}_{V_N} \quad \textit{for any } u \in V_N \textit{ and } v \in \mathcal{V}_u, \tag{35}$$

*where $R$ and $\mathcal{V}_u$ are defined in (33).*
*Let $\epsilon > 0$ and let $w^\epsilon \in V_N$ be the point generated by applying the local variation method*

*to $J$, i.e. the point such that $c_{\mathcal{B}}(w^\epsilon)$ is the stationary vector obtained at step $S$ of the local variation method applied to the function $f$ defined in (23). Then*

$$\|w^\epsilon - w^{max}\|_{V_N} \leq \frac{\sqrt{N}(C_1 + \epsilon C_0)}{m}\epsilon^{\tau-1}, \tag{36}$$

*where $w^{max}$ is the unique maximizer of $J$ over $V_N$, and $m$ is the constant related to the strong concavity of $J$, defined in (34). Moreover, if $(\epsilon_n)_{n\geq 0} \subseteq ]0, +\infty[$ is a sequence decreasing to zero, the sequence $(w^{\epsilon_n})_{n\geq 0}$ converges to $w^{max}$.*

*Proof.* Let $u, v \in V_N$ be two points verifying (33), whose coordinate vectors relative to $\mathcal{B}$ are, respectively, $x := c_{\mathcal{B}}(u) \in \mathbb{R}^N$ and $h := c_{\mathcal{B}}(v) \in \mathbb{R}^N$, and let $f : \mathbb{R}^N \to \mathbb{R}$ be the function defined in (23). Since $J$ is differentiable and strongly concave over $V_N$, then $f$ is differentiable and strongly concave over $R^N$. Moreover, by definition of Gâteaux derivative and (23) we have

$$\begin{aligned}
\langle DJ(u), v\rangle_{V_N^* \times V_N} &= \langle d_G J(u), v\rangle_{V_N^* \times V_N} \\
&= \lim_{t\to 0} \frac{J(u + tv) - J(u)}{t} \\
&= \lim_{t\to 0} \frac{f(x + th) - J(x)}{t} \\
&= \langle d_G f(x), h\rangle_{\mathbb{R}^{N*} \times \mathbb{R}^N} \\
&= \langle Df(x), h\rangle_{\mathbb{R}^{N*} \times \mathbb{R}^N} = (\nabla f(x), h)_{\mathbb{R}^N}.
\end{aligned} \tag{37}$$

Hence, in light of (23), (33) and (35) we get

$$\begin{aligned}
|f(x + h) - f(x) - (\nabla f(x), h)_{\mathbb{R}^N}| &= |J(u + v) - J(u) - \langle DJ(u), v\rangle_{V_N^* \times V_N}| \\
&= |R(u, v)| \\
&\leq C_1 \|v\|^\tau_{\mathbb{R}^N} + C_0 \|v\|^{\tau+1}_{V_N} \\
&= C_1 \|h\|^\tau_{\mathbb{R}^N} + C_0 \|h\|^{\tau+1}_{\mathbb{R}^N}.
\end{aligned}$$

Furthermore, the constant related to the strong concavity of $f$ is equal to the constant related to strong concavity of $J$. In fact, for any $x', x'' \in \mathbb{R}^N$, from (23), (34) and (37) it follows

$$\begin{aligned}
f(x'') - f(x') &= J(u'') - J(u') \\
&\leq \langle DJ(u'), u'' - u'\rangle_{V_N^* \times V_N} - m\|u'' - u'\|^2_{V_N} \\
&= (\nabla f(x'), x'' - x')_{\mathbb{R}^N} - m\|x'' - x'\|^2_{\mathbb{R}^N},
\end{aligned}$$

where $u' \in V_N$ and $u'' \in V_N$ are defined, respectively, by $x' = c_{\mathcal{B}}(u')$ and $x'' = c_{\mathcal{B}}(u'')$. Therefore $f$ satisfies the assumptions of Lemma 4.2, and so

$$\|z^\epsilon - z^{max}\|_{\mathbb{R}^N} \leq \frac{\sqrt{N}(C_1 + \epsilon C_0)}{m}\epsilon^{\tau-1}, \tag{38}$$

where $z^\epsilon \in \mathbb{R}^N$ is the stationary vector obtained at step $S$ of the local variation method applied to $f$, that verifies (24), and $z^{max} \in \mathbb{R}^N$ is the unique maximizer of $f$ over $\mathbb{R}^N$.

Thus, $z^\epsilon = c_{\mathcal{B}}(w^\epsilon)$ and $z^{max} = c_{\mathcal{B}}(w^{max})$. In light of (22) and (38),

$$\|w^\epsilon - w^{max}\|_{V_N} = \|z^\epsilon - z^{max}\|_{\mathbb{R}^N} \leq \frac{\sqrt{N}(C_1 + \epsilon C_0)}{m}\epsilon^{\tau-1},$$

so (36) holds. The last part of the result follows from (36), since $\lim_{n\to+\infty} \epsilon_n = 0$ and $\tau > 1$. □

### 4.3 Approximation of the finite dimensional problem: the numerical adjustment process via local variation method

Let $\Gamma_{p,q} = \{2, X_p, Y_q, F, G\}$ where $X_p$ is a $p$-dimensional subspace of $X$, $Y_q$ is a $q$-dimensional subspace of $Y$ (with $p, q \in \mathbb{N}$), and $F$ and $G$ are the restrictions to $X_p \times Y_q$ of the payoff functions of $\Gamma$. Now, we introduce a numerical method to approximate the unique Nash equilibrium $(\bar{x}_{p,q}, \bar{y}_{p,q})$ of $\Gamma_{p,q}$, referred as *numerical adjustment process via local variation method* and denoted by $(\mathcal{AL})$, which combines the local variation method described in the previous subsection and the adjustment process $(\mathcal{A})$ described in Section 3.

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***Numerical adjustment process*** $(\mathcal{AL})$

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(*Step 0*) Let $y_0$ be an arbitrary point in $Y_q$. Chosen an initial point $\tilde{x}_0 \in X_p$ and a range $\epsilon_0 > 0$, apply the local variation method to the function $F(\cdot, y_0)$ and get the stationary vector $x_0^* \in X_p$.

(*Step 1*) Taking $y_0$ as initial point and choosing the range $\epsilon_1 = \epsilon_0/2$, apply the local variation method to the function $G(x_0^*, \cdot)$ and get the stationary vector $y_1^* \in Y_q$. Then, starting from $x_0^*$ and using the range $\epsilon_1$, apply the local variation method to the function $F(\cdot, y_1^*)$ and get the stationary vector $\tilde{x}_1^* \in X_p$.
Hence, compute $x_1^* := \nu x_0^* - (\nu - 1)\tilde{x}_1^* \in X_p$, where $\nu$ is defined in (13).

(*Step 2*) Taking $y_1^*$ as initial point and choosing the range $\epsilon_2 = \epsilon_1/2$, apply the local variation method to the function $G(x_1^*, \cdot)$ and get the stationary vector $y_2^*$.
Then, let $\tilde{x}_2^*$ be the stationary vector obtained by applying the local variation method to the function $F(\cdot, y_2^*)$ with $x_1^*$ as initial point and $\epsilon_2$ as range.
Hence, compute $x_2^* := \nu x_1^* - (\nu - 1)\tilde{x}_2^*$.
⋮

At the generic step $n$, with $n > 2$, given $x_{n-1}^* \in X_p$, $y_{n-1}^* \in Y_q$ and $\epsilon_{n-1}$, we come to

(*Step n*) Apply the local variation method to the function $G(x_{n-1}^*, \cdot)$ with initial point $y_{n-1}^*$ and range $\epsilon_n = \epsilon_{n-1}/2$, and get $y_n^* \in Y_q$.
Apply the local variation method to the function $F(\cdot, y_n^*)$ with initial point $x_{n-1}^*$ and range $\epsilon_n$, and get $\tilde{x}_n^* \in X_p$.

Compute $x_n^* \coloneqq \nu x_{n-1}^* - (\nu - 1)\tilde{x}_n^* \in X_p$.

$\vdots$

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Figure 1 and Figure 2 provide, respectively, a schematization of the steps from 0 to 2 and a schematization of (*Step n*).

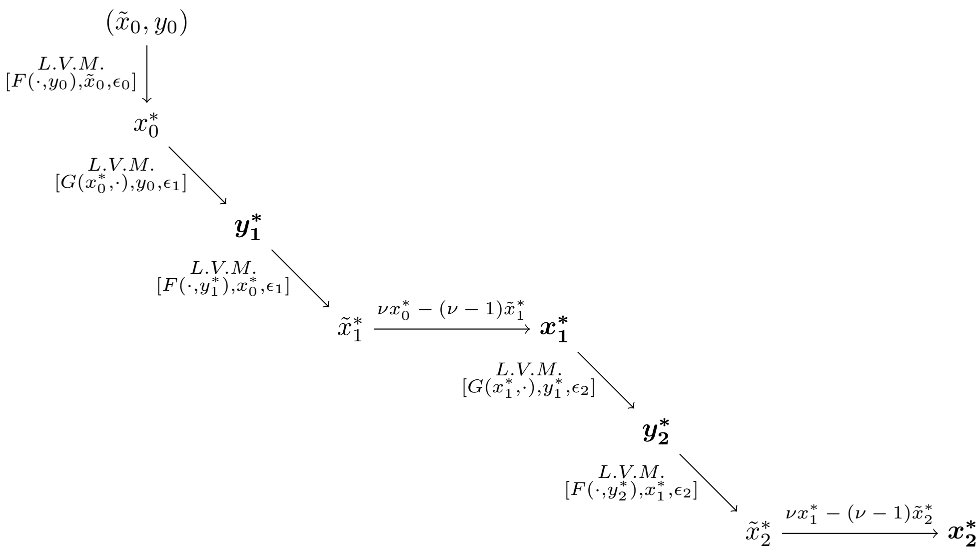


Figure 1: Starting of the numerical adjustment process

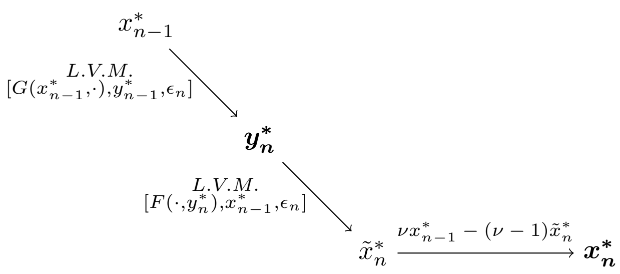


Figure 2: Generic step of the numerical adjustment process

In the next theorem the convergence of the numerical adjustment process $(\mathcal{AL})$ is shown. Let us remind, preliminarily, the Taylor's theorem applied to $F(\cdot, y)$ and fixed the point $x \in X_p$, that is

$$\begin{cases} \exists \mathcal{I}_{x,y} \subseteq X_p \text{ neighborhood of 0 depending on } x \text{ and } y \text{ such that} \\ F(x+h, y) - F(x, y) = \langle D_x F(x, y), h \rangle_{X_p^* \times X_p} + R_F(x, h, y) \quad \forall h \in \mathcal{I}_{x,y}, \end{cases} \tag{39}$$

where $R_F(x, h, y)$ is the remainder; and the Taylor's theorem applied to $G(x, \cdot)$ and fixed the point $y \in Y_q$, that is

$$\begin{cases} \exists\, \mathcal{J}_{y,x} \subseteq Y_q \text{ neighborhood of } 0 \text{ depending on } y \text{ and } x \text{ such that} \\ G(x, y+k) - G(x,y) = \langle D_y G(x,y), k\rangle_{Y_q^* \times Y_q} + R_G(y,k,x) \quad \forall k \in \mathcal{J}_{y,x}, \end{cases} \tag{40}$$

where $R_G(y,k,x)$ is the remainder.

**Theorem 4.3.** *Assume $\Gamma_{p,q} = \{2, X_p, Y_q, F, G\} \in \mathcal{H}$ satisfies $(A_1)$–$(A_2)$ and*

*(i) there exist $A_1 > 0$, $A_0 \geq 0$ and $\alpha > 1$ such that*

$$|R_F(x,h,y)| \leq A_1 \|h\|_{X_p}^{\alpha} + A_0 \|h\|_{X_p}^{\alpha+1}, \tag{41}$$

*for any $x \in X_p$, $y \in Y_q$ and $h \in \mathcal{I}_{x,y}$, where $R_F$ and $\mathcal{I}_{x,y}$ are defined in (39);*

*(ii) there exist $B_1 > 0$, $B_0 \geq 0$ and $\beta > 1$ such that*

$$|R_G(y,k,x)| \leq B_1 \|h\|_{Y_q}^{\beta} + B_0 \|h\|_{Y_q}^{\beta+1}, \tag{42}$$

*for any $y \in Y_q$, $x \in X_p$ and $k \in \mathcal{J}_{y,x}$, where $R_G$ and $\mathcal{J}_{y,x}$ are defined in (40).*

*Let $\epsilon_0 > 0$ and $\epsilon_n = \epsilon_0/2^n$ for any $n \in \mathbb{N}$, and let $(\tilde{x}_0, y_0) \in X_p \times Y_q$. Then, the sequence $(x_n^*, y_n^*)_n \subseteq X_p \times Y_q$ generated by the numerical adjustment process $(\mathcal{AL})$ is convergent to the unique Nash equilibrium $(\bar{x}_{p,q}, \bar{y}_{p,q})$ of $\Gamma_{p,q}$.*

*Proof.* The uniqueness of the Nash equilibrium of $\Gamma_{p,q}$ is guaranteed by the assumptions, Theorem 2.1 and Remark 4.2 (applied to $\Gamma_{p,q} = \{2, X_p, Y_q, F, G\}$). Moreover, the sequence $(x_n^*, y_n^*)_n \subseteq X_p \times Y_q$ is well-defined.
In order to show the result, let us define the following points[2], associated to $(x_n^*, y_n^*)_n$

$$z_n \coloneqq r_2(x_{n-1}^*) \in Y_q \tag{43}$$

$$\tilde{s}_n \coloneqq r_1(z_n) \in X_p \tag{44}$$

$$s_n \coloneqq \nu x_{n-1}^* - (\nu - 1)\tilde{s}_n = g^{\nu}(x_{n-1}^*) \in X_p \tag{45}$$

$$\tilde{t}_n \coloneqq r_1(y_n^*) \in X_p \tag{46}$$

$$t_n \coloneqq \nu x_{n-1}^* - (\nu - 1)\tilde{t}_n \in X_p, \tag{47}$$

for any $n \in \mathbb{N}$, where $x_0^*$ is defined in (*Step 0*) of $(\mathcal{AL})$. We represented below the connections among the sequence $(x_n^*, y_n^*)_n$, the points defined in (43)–(47) and the sequence generated by the adjustment process $(\mathcal{A})$ applied to $\Gamma_{p,q}$. In particular, Figure 3 schematizes such connections for (*Step 0*), (*Step 1*) and (*Step 2*) of the numerical adjustment process $(\mathcal{AL})$, whereas, Figure 4 schematizes such connections for (*Step n*), with $n > 2$.

[2]In this setting, since $F$ and $G$ are the restrictions of the payoff functions of $\Gamma$ to $X_p \times Y_q$, then the best response $r_2$ is a function from $X_p$ to $Y_q$, the best response $r_1$ is a function from $Y_q$ to $X_p$, and $g^{\nu}$ is a function from $X_p$ to $X_p$ (the $\nu$-inverse convex combinator of $\Gamma_{p,q}$).

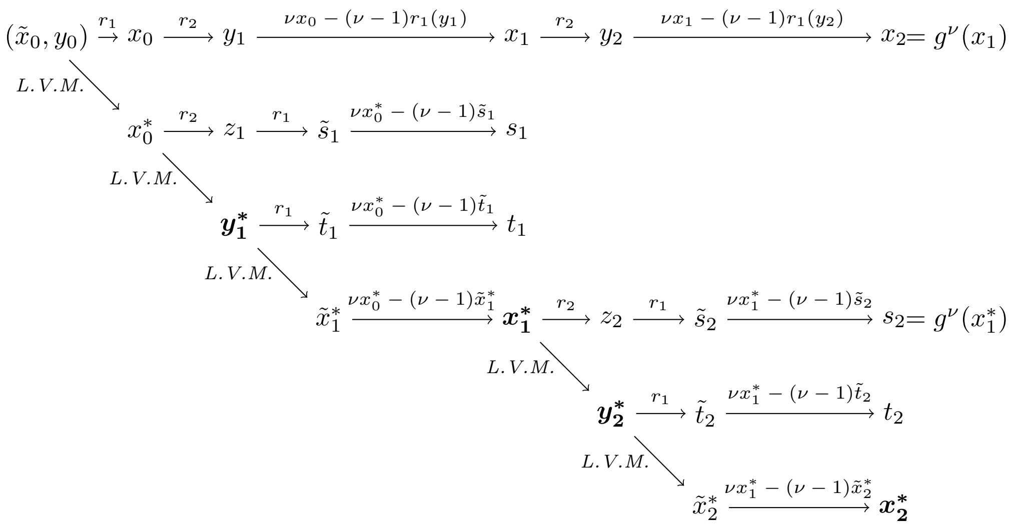


Figure 3: Representation of $z_n, \tilde{s}_n, s_n, \tilde{t}_n, t_n$, for $n = 1, 2$

We start by proving that $\lim_{n\to+\infty} \|x_n^* - \bar{x}_{p,q}\|_{X_p} = 0$, where $\bar{x}_{p,q}$ is the Nash equilibrium strategy of the first player in $\Gamma_{p,q}$. For any $n \in \mathbb{N}$

$$\|x_n^* - \bar{x}_{p,q}\|_{X_p} \le \|x_n^* - x_n\|_{X_p} + \|x_n - \bar{x}_{p,q}\|_{X_p}, \tag{48}$$

where $x_n$ is the point generated by the adjustment process $(\mathcal{A})$ applied to $\Gamma_{p,q}$. Since $\Gamma_{p,q}$ satisfies the assumptions of Theorem 3.1 (in light of Remark 4.2), then

$$\lim_{n\to+\infty} \|x_n - \bar{x}_{p,q}\|_{X_p} = 0. \tag{49}$$

So, focusing only on the first term in the right-hand side of (48), we have

$$\|x_n^* - x_n\|_{X_p} \le \|x_n^* - t_n\|_{X_p} + \|t_n - s_n\|_{X_p} + \|s_n - x_n\|_{X_p}. \tag{50}$$

Let us analyze the three terms in the right-hand side of (50).

1. By definition of $x_n^*$ in (*Step* $n$) and by (47), we get

$$\|x_n^* - t_n\|_{X_p} = (\nu - 1)\|\tilde{x}_n^* - \tilde{t}_n\|_{X_p}. \tag{51}$$

Let us note that $\tilde{x}_n^*$ is the approximation of the maximizer of $F(\cdot, y_n^*)$ over $X_p$ generated by applying the local variation method to $F(\cdot, y_n^*)$ with initial point $x_{n-1}^*$ and range $\epsilon_n$ (as represented in Figure 4), whereas $\tilde{t}_n$ is actually such a maximizer, by (46). In light of assumption $(i)$, from Theorem 4.2 we get

$$\|\tilde{x}_n^* - \tilde{t}_n\|_{X_p} \le \frac{\sqrt{p}(A_1 + \epsilon_n A_0)}{m_F} \epsilon_n^{\alpha-1}, \tag{52}$$

where $m_F$ is the constant related to the concavity[3] of $F$ on $X$.

[3] Since $F(\cdot, y)$ is differentiable for any $y \in Y_q$, then $F$ is strongly concave on $X$ uniformly on $Y$ if

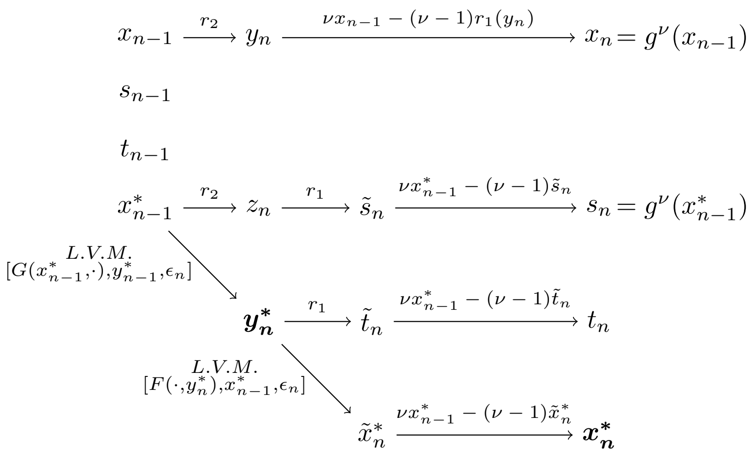


Figure 4: Representation of $z_n, \tilde{s}_n, s_n, \tilde{t}_n, t_n$, for $n > 2$

2. In light of (44)–(47) and Lemma 2.2($i$), we have

$$\begin{aligned} \|t_n - s_n\|_{X_p} =& (\nu-1)\|\tilde{t}_n - \tilde{s}_n\|_{X_p} \\ =& (\nu-1)\|r_1(y^*_n) - r_1(z_n)\|_{X_p} \le \lambda_1(\nu-1)\|y^*_n - z_n\|_{Y_q}. \end{aligned} \tag{53}$$

Similarly to the previous case, $y^*_n$ is the approximation of the maximizer of $G(x^*_{n-1},\cdot)$ over $Y_q$ come up by applying the local variation method to $G(x^*_{n-1},\cdot)$ with initial point $y^*_{n-1}$ and range $\epsilon_n$ (as represented in Figure 4), whereas $z_n$ is effectively such a maximizer, by (43). In light of assumption ($ii$), from Theorem 4.2 it follows

$$\|y^*_n - z_n\|_{Y_q} \le \frac{\sqrt{q}(B_1 + \epsilon_n B_0)}{m_G}\epsilon_n^{\beta-1}, \tag{54}$$

where $m_G$ is the constant related to the concavity[4] of $G$ on $Y$.

3. By definition of $x_n$ in ($\mathcal{A}$), by (45), and by Theorem 2.1 applied to $\Gamma_{p,q}$ we get

$$\|s_n - x_n\|_{X_p} = \|g^\nu(x^*_{n-1}) - g^\nu(x_{n-1})\|_{X_p} \le k\|x^*_{n-1} - x_{n-1}\|_{X_p}, \tag{55}$$

where $k$ is defined in (15).

Hence, using (51)-(55) into (50) and since $\epsilon_n \le \epsilon_0$, we have

$$\|x^*_n - x_n\|_{X_p} \le k\|x^*_{n-1} - x_{n-1}\|_{X_p} + \lambda_1(\nu-1)D\epsilon_n^{\beta-1} + (\nu-1)C\epsilon_n^{\alpha-1}, \tag{56}$$

and only if there exists a constant $m_F > 0$ such that

$$F(x'', y) - F(x', y) \le \langle D_x F(x', y), x'' - x'\rangle_{X_p^* \times X_p} - m_F\|x'' - x'\|^2_{X_p}, \quad \text{for any } x', x'' \in X_p,\ y \in Y_q.$$

[4]Since $G(x,\cdot)$ is differentiable for any $x \in X_p$, then $G$ is strongly concave on $Y$ uniformly on $X$ if and only if there exists a constant $m_G > 0$ such that

$$G(x, y'') - G(x, y') \le \langle D_y G(x, y'), y'' - y'\rangle_{Y_q^* \times Y_q} - m_G\|y'' - y'\|^2_{Y_q}, \quad \text{for any } y', y'' \in Y_q,\ x \in X_p.$$

where $D = \sqrt{q}(B_1 + \epsilon_0 B_0)/m_G$ and $C = \sqrt{p}(A_1 + \epsilon_0 A_0)/m_F$. Let $d_n = \|x_n^* - x_n\|_{X_p}$ for any $n \in \mathbb{N}$, and $d_0 = \|x_0^* - x_0\|_{X_p}$. Then by (56) it follows

$$
\begin{aligned}
d_n &\leq k d_{n-1} + \lambda_1(\nu - 1) D \epsilon_n^{\beta-1} + (\nu - 1) C \epsilon_n^{\alpha-1} \\
&\leq k[k d_{n-2} + \lambda_1(\nu - 1) D \epsilon_{n-1}^{\beta-1} + (\nu - 1) C \epsilon_{n-1}^{\alpha-1}] \\
&\quad + \lambda_1(\nu - 1) D \epsilon_n^{\beta-1} + (\nu - 1) C \epsilon_n^{\alpha-1} \\
&\vdots \\
&\leq \lambda_1(\nu - 1) D[\epsilon_n^{\beta-1} + k\epsilon_{n-1}^{\beta-1} + \cdots + k^{n-1}\epsilon_1^{\beta-1}] \\
&\quad + (\nu - 1) C[\epsilon_n^{\alpha-1} + k\epsilon_{n-1}^{\alpha-1} + \cdots + k^{n-1}\epsilon_1^{\alpha-1}] + d_0 k^n \\
&= \lambda_1(\nu - 1) D \sum_{m=0}^{n-1} \epsilon_{n-m}^{\beta-1} k^m + (\nu - 1) C \sum_{m=0}^{n-1} \epsilon_{n-m}^{\alpha-1} k^m + d_0 k^n.
\end{aligned}
\tag{57}
$$

The summation $\sum_{m=0}^{n} \epsilon_{n-m}^{\beta-1} k^m$ is the $n$-th term of the Cauchy product of the two series $\sum_{i=0}^{+\infty} k^i$ and $\sum_{j=0}^{+\infty} \epsilon_j^{\beta-1}$, that is

$$
\left( \sum_{i=0}^{+\infty} k^i \right) \cdot_c \left( \sum_{j=0}^{+\infty} \epsilon_j^{\beta-1} \right) = \sum_{n=0}^{+\infty} \sum_{m=0}^{n} \epsilon_{n-m}^{\beta-1} k^m, \tag{58}
$$

where $\cdot_c$ denotes the Cauchy product. The two series in the left-hand side of (58) are convergent: in fact, $\sum_{i=0}^{+\infty} k^i < +\infty$ since it is a geometric series with ratio $k < 1$, and $\sum_{j=0}^{+\infty} \epsilon_j^{\beta-1} < +\infty$ as $\epsilon_j = \epsilon_0/2^j$ with $\epsilon_0 > 0$, and $\beta > 1$. Therefore, in light of the Cauchy theorem (see, for example, [15, Theorem 160]), the series in the right-hand side of (58) is convergent, so $\lim_{n \to +\infty} \sum_{m=0}^{n} \epsilon_{n-m}^{\beta-1} k^m = 0$. Analogously, $\lim_{n \to +\infty} \sum_{m=0}^{n} \epsilon_{n-m}^{\alpha-1} k^m = 0$. Given the above and since $\lim_{n \to +\infty} k^n = 0$, by (57) we have

$$
\lim_{n \to +\infty} \|x_n^* - x_n\|_{X_p} = \lim_{n \to +\infty} d_n = 0, \tag{59}
$$

hence, in light of (48) and (49), the sequence $(x_n^*)_n$ is convergent to $\bar{x}_{p,q}$.

Now, let us prove that $\lim_{n \to +\infty} \|y_n^* - \bar{y}_{p,q}\|_{Y_q} = 0$, where $\bar{y}_{p,q}$ is the Nash equilibrium strategy of the second player in $\Gamma_{p,q}$. For any $n \in \mathbb{N}$

$$
\|y_n^* - \bar{y}_{p,q}\|_{Y_q} \leq \|y_n^* - z_n\|_{Y_q} + \|z_n - y_n\|_{Y_q} + \|y_n - \bar{y}_{p,q}\|_{Y_q}, \tag{60}
$$

where $y_n$ is the point generated by the adjustment process $(\mathcal{A})$ applied to $\Gamma_{p,q}$. Since $\Gamma_{p,q}$ satisfies the assumptions of Theorem 3.1 (in light of Remark 4.2), then

$$
\lim_{n \to +\infty} \|y_n - \bar{y}_{p,q}\|_{Y_q} = 0. \tag{61}
$$

Hence, let us consider the first and the second term in the right-hand side of (60).

1. Since $\lim_{n \to +\infty} \epsilon_n = 0$ and $\beta > 1$, then, by (54), we get

$$
\lim_{n \to +\infty} \|y_n^* - z_n\|_{Y_q} \leq \lim_{n \to +\infty} \frac{\sqrt{q}(B_1 + \epsilon_n B_0)}{m_G} \epsilon_n^{\beta-1} = 0. \tag{62}
$$

2. In light of (43), the definition of $y_n$, Lemma 2.2$(ii)$ and (59), we have

$$\begin{aligned}\lim_{n\to+\infty}\|z_n - y_n\|_{Y_q} &= \lim_{n\to+\infty}\|r_2(x^*_{n-1}) - r_2(x_n)\|_{Y_q} \\ &\leq \lambda_2 \lim_{n\to+\infty}\|x^*_{n-1} - x_n\|_{X_p} = 0.\end{aligned} \tag{63}$$

So, $\lim_{n\to+\infty}\|y^*_n - \bar{y}_{p,q}\|_{Y_q} = 0$ by (60)-(63). Therefore, the sequence $(y^*_n)_n$ is convergent to $\bar{y}_{p,q}$ and the proof is complete. □

**Remark 4.5** In the statement of Theorem 4.3, instead of setting $\epsilon_n = \epsilon_0/2^n$ for any $n \in \mathbb{N}$, we could choose any sequence $(\epsilon_n)_n$ such that the series $\sum_{j=0}^{+\infty}\epsilon_j^{\beta-1}$ and $\sum_{j=0}^{+\infty}\epsilon_j^{\alpha-1}$ are convergent.

**Remark 4.6** Let us provide a sufficient condition for hypotheses $(i)$ and $(ii)$ in Theorem 4.3. Assumption $(i)$ is satisfied if there exists a constant $A > 0$ such that $\|D^2_x F(x,y)\|_{\mathcal{L}(X_p,X_p^*)} \leq A$ for any $(x,y) \in X_p \times Y_q$. In fact, since $F$ is twice continuously differentiable, by applying the Taylor's theorem with Lagrange's form of remainder (see, e.g., [2, Formule de Taylor 3.5]), we get

$$|R_F(x,h,y)| = |F(x+h,y) - F(x,y) - \langle D_x F(x,y), h\rangle_{X_p^*\times X_p}| \leq \frac{A}{2}\|h\|^2_{X_p},$$

for any $x \in X_p$, $y \in Y_q$ and $h \in \mathcal{I}_{x,y}$, where $R_F$ and $\mathcal{I}_{x,y}$ are defined in (39). Hence, assumption $(i)$ holds setting $A_1 = A/2$, $A_0 = 0$, and $\alpha = 2$. Analogously, assumption $(ii)$ is satisfied if there exists a constant $B > 0$ such that $\|D^2_y G(x,y)\|_{\mathcal{L}(Y_q,Y_q^*)} \leq B$ for any $(x,y) \in X_p \times Y_q$.

**Remark 4.7** The games illustrated in Example 4.1 and Example 4.2 satisfy, in light of Remark 4.6, the assumptions of Theorem 4.3. Moreover, it is worth to note that the unique Nash equilibrium of the potential game in Example 4.2 cannot be approximated through the usual methods based on the potential function (which exploit the property that any maximizer of the potential function is a Nash equilibrium of the potential game), since such equilibrium is not a maximizer of the potential function (in light of Remark 2.2). See, for example, [12, 22] and reference therein for further discussion regarding methods based on the potential function.

### 4.4 Approximation of the finite dimensional problem: error bounds

In this subsection we provide the error estimations for the sequences $(x^*_n)_n$ and $(y^*_n)_n$ generated by the numerical adjustment process $(\mathcal{AL})$ represented in Subsection 4.3. Let us remind that $(\bar{x}_{p,q}, \bar{y}_{p,q})$ is the Nash equilibrium of $\Gamma_{p,q} = \{2, X_p, Y_q, F, G\}$ where $p, q \in \mathbb{N}$.

**Proposition 4.1.** *Under the assumptions of Theorem 4.3, there exist $L, M \in \mathbb{R}$ such that*

$$\|x^*_n - \bar{x}_{p,q}\|_{X_p} \leq Lk^n + \frac{M}{(2^{\alpha-1})^n} \quad \textit{for any } n \in \mathbb{N},$$

*where $k$ and $\alpha$ are defined, respectively, in (15) and (41).*

*Proof.* Let $n \in \mathbb{N}$, then

$$\|x_n^* - \bar{x}_{p,q}\|_{X_p} \leq \|x_n^* - x_n\|_{X_p} + \|x_n - \bar{x}_{p,q}\|_{X_p}, \tag{64}$$

where $x_n$ is the point generated by the adjustment process $(\mathcal{A})$ applied to $\Gamma_{p,q}$. Let us analyze the two terms in the right-hand side of (64).

1. In light of (57) we know that

$$\begin{aligned} \|x_n^* - x_n\|_{X_p} \leq \lambda_1(\nu-1)D \sum_{m=0}^{n-1} \epsilon_{n-m}^{\beta-1} k^m \\ + (\nu-1)C \sum_{m=0}^{n-1} \epsilon_{n-m}^{\alpha-1} k^m + k^n \|x_0^* - x_0\|_{X_p}, \end{aligned} \tag{65}$$

where $D = \sqrt{q}(B_1 + \epsilon_0 B_0)/m_G$ and $C = \sqrt{p}(A_1 + \epsilon_0 A_0)/m_F$. Since $x_0^*$ is the approximation of the maximizer of $F(\cdot, y_0)$ over $X_p$ obtained by applying the local variation method to $F(\cdot, y_0)$ with initial point $\tilde{x}_0$ and range $\epsilon_0$ (as represented in Figure 1), and $x_0$ is effectively such a maximizer (as defined at the beginning of the adjustment process $(\mathcal{A})$), then $\|x_0^* - x_0\|_{X_p} \leq C\epsilon_0^{\alpha-1}$ by Theorem 4.2. Given the above and since $\epsilon_n = \epsilon_0/2^n$, by (65) we have

$$\begin{aligned} \|x_n^* - x_n\|_{X_p} &\leq \lambda_1(\nu-1)D\epsilon_0^{\beta-1} \sum_{m=0}^{n-1} \frac{k^m}{(2^{\beta-1})^{n-m}} \\ &\quad + (\nu-1)C\epsilon_0^{\alpha-1} \sum_{m=0}^{n-1} \frac{k^m}{(2^{\alpha-1})^{n-m}} + C\epsilon_0^{\alpha-1}k^n \\ &= \frac{\lambda_1(\nu-1)D\epsilon_0^{\beta-1}}{(2^{\beta-1})^n} \left[\frac{1-(k2^{\beta-1})^n}{1-k2^{\beta-1}}\right] \\ &\quad + \frac{(\nu-1)C\epsilon_0^{\alpha-1}}{(2^{\alpha-1})^n} \left[\frac{1-(k2^{\alpha-1})^n}{1-k2^{\alpha-1}}\right] + C\epsilon_0^{\alpha-1}k^n, \end{aligned} \tag{66}$$

where the equality is obtained by exploiting the sum of the first $n$ terms of geometric series of ratio $k2^{\beta-1}$ and $k2^{\alpha-1}$.

2. In light of Proposition 3.1 (applied to $\Gamma_{p,q}$), recalling that $x_1 = \nu x_0 - (\nu - 1)r_1(y_1)$ and $x_0 = r_1(y_0)$ as defined in $(\mathcal{A})$, and that $r_1$ is Lipschitz continuous by Lemma 2.2$(i)$, we get

$$\begin{aligned} \|x_n - \bar{x}_{p,q}\|_{X_p} &\leq \frac{k^n}{1-k}\|x_1 - x_0\|_{X_p} \\ &= \frac{k^n}{1-k}\|(\nu-1)(x_0 - r_1(y_1))\|_{X_p} \\ &= \frac{(\nu-1)k^n}{1-k}\|r_1(y_0) - r_1(y_1)\|_{X_p} \\ &\leq \frac{\lambda_1(\nu-1)k^n}{1-k}\|y_1 - y_0\|_{Y_q} \\ &\leq \frac{\lambda_1(\nu-1)k^n}{1-k}[\|y_1 - z_1\|_{Y_q} + \|z_1 - y_1^*\|_{Y_q} + \|y_1^* - y_0\|_{Y_q}]. \end{aligned} \tag{67}$$

Since $y_1^*$ is the approximation of the maximizer of $G(x_0^*, \cdot)$ over $Y_q$ generated by applying the local variation method to $G(x_0^*, \cdot)$ with initial point $y_0$ and range $\epsilon_1$ (as represented in Figure 1), and $z_1$ is actually such a maximizer (in light of (43)), then $\|z_1 - y_1^*\|_{Y_q} \leq D\epsilon_1^{\beta-1}$ by Theorem 4.2. Given the above, by definition of $y_1$ in $(\mathcal{A})$, and the definition of $z_1$ in (43), it follows

$$\begin{aligned} \|y_1 - z_1\|_{Y_q} + \|z_1 - y_1^*\|_{Y_q} &\leq \|r_2(x_0) - r_2(x_0^*)\|_{Y_q} + D\epsilon_1^{\beta-1} \\ &\leq \lambda_2 C\epsilon_0^{\alpha-1} + D\frac{\epsilon_0^{\beta-1}}{2^{\beta-1}}, \end{aligned} \tag{68}$$

where the last inequality holds in light of Lemma 2.2$(ii)$, the inequality $\|x_0^* - x_0\|_{X_p} \leq C\epsilon_0^{\alpha-1}$ proved in the previous point, and the definition of $\epsilon_1$. Hence, (67)-(68) imply

$$\|x_n - \bar{x}_{p,q}\|_{X_p} \leq \frac{\lambda_1(\nu-1)k^n}{1-k}\left[\lambda_2 C\epsilon_0^{\alpha-1} + D\frac{\epsilon_0^{\beta-1}}{2^{\beta-1}} + \|y_1^* - y_0\|_{Y_q}\right]. \tag{69}$$

Finally, putting (66) and (69) into (64), we get

$$\begin{aligned} \|x_n^* - \bar{x}_{p,q}\|_{X_p} &\leq \frac{(\nu-1)C\epsilon_0^{\alpha-1}}{(2^{\alpha-1})^n}\left[\frac{1-(k2^{\alpha-1})^n}{1-k2^{\alpha-1}}\right] + C\epsilon_0^{\alpha-1}k^n \\ &\quad + \frac{\lambda_1(\nu-1)k^n}{1-k}\left[\lambda_2 C\epsilon_0^{\alpha-1} + D\frac{\epsilon_0^{\beta-1}}{2^{\beta-1}} + \|y_1^* - y_0\|_{Y_q}\right] \\ &= Lk^n + \frac{M}{(2^{\alpha-1})^n}, \end{aligned}$$

where

$$\begin{aligned} L &= C\epsilon_0^{\alpha-1}\left[\frac{2-\nu-k2^{\alpha-1}}{1-k2^{\alpha-1}}\right] \\ &\quad + \frac{\lambda_1(\nu-1)}{1-k}\left[\lambda_2 C\epsilon_0^{\alpha-1} + D\frac{\epsilon_0^{\beta-1}}{2^{\beta-1}} + \|y_1^* - y_0\|_{Y_q}\right], \\ M &= \frac{(\nu-1)C\epsilon_0^{\alpha-1}}{1-k2^{\alpha-1}}, \end{aligned} \tag{70}$$

therefore the result is proved. □

**Proposition 4.2.** *Under the assumptions of Theorem 4.3, there exist* $L', M', W \in \mathbb{R}$ *such that*

$$\|y_n^* - \bar{y}_{p,q}\|_{Y_q} \leq L'k^{n-1} + \frac{M'}{(2^{\alpha-1})^{n-1}} + \frac{W}{(2^{\beta-1})^n} \quad \textit{for any } n \in \mathbb{N},$$

*where* $k$*,* $\alpha$ *and* $\beta$ *are defined, respectively, in* (15)*,* (41) *and* (42)*.*

*Proof.* Let $n \in \mathbb{N}$, then

$$\|y_n^* - \bar{y}_{p,q}\|_{Y_q} \leq \|y_n^* - z_n\|_{Y_q} + \|z_n - \bar{y}_{p,q}\|_{Y_q}, \tag{71}$$

where $z_n$ is defined in (43). Noting that $\bar{y}_{p,q} = r_2(\bar{x}_{p,q})$ by definition of Nash equilibrium, then, in light of (54), (43) and Lemma 2.2(*ii*) we have

$$\begin{aligned} \|y_n^* - z_n\|_{Y_q} + \|z_n - \bar{y}_{p,q}\|_{Y_q} &\leq D\epsilon_n^{\beta-1} + \|r_2(x_{n-1}^*) - r_2(\bar{x}_{p,q})\|_{Y_q} \\ &\leq D\epsilon_n^{\beta-1} + \lambda_2\|x_{n-1}^* - \bar{x}_{p,q}\|_{X_p}, \end{aligned} \tag{72}$$

where $D = \sqrt{q}(B_1 + \epsilon_0 B_0)/m_G$ and $C = \sqrt{p}(A_1 + \epsilon_0 A_0)/m_F$. Recalling that $\epsilon_n = \epsilon_0/2^n$ and by applying Proposition 4.1, from (71) and (72) we get

$$\begin{aligned} \|y_n^* - \bar{y}_{p,q}\|_{Y_q} &\leq D\frac{\epsilon_0^{\beta-1}}{(2^{\beta-1})^n} + \lambda_2\left[Lk^{n-1} + \frac{M}{(2^{\alpha-1})^{n-1}}\right] \\ &= L'k^{n-1} + \frac{M'}{(2^{\alpha-1})^{n-1}} + \frac{W}{(2^{\beta-1})^n}, \end{aligned}$$

where $L' = \lambda_2 L$, $M' = \lambda_2 M$, $W = D\epsilon_0^{\beta-1}$, and $L$ and $M$ are defined in (70). □

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