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Francesco Caruso[*], Maria Carmela Ceparano[*] and Jacqueline Morgan[**]

Abstract

Regarding the approximation of Nash equilibria in games where the players have a continuum of strategies, there exist various algorithms based on best response dynamics and on its relaxed variants: from one step to the next, a player's strategy is updated by using explicitly a best response to the strategies of the other players that come from the previous steps. These iterative schemes generate sequences of strategy profiles which are constructed by using continuous optimization techniques and they have been shown to converge in the following situations: in zero-sum games or, in non zero-sum ones, under contraction assumptions or under linearity of best response functions. In this paper, we propose an algorithm which guarantees the convergence to a Nash equilibrium in two-player non zero-sum games when the best response functions are not necessarily linear, both their compositions are not contractions and the strategy sets are Hilbert spaces.

Firstly, we address the issue of uniqueness of the Nash equilibrium extending to a more general class the result obtained by Caruso, Ceparano, and Morgan [J. Math. Anal. Appl., 459 (2018), pp. 1208-1221] for weighted potential games. Then, we describe a theoretical approximation scheme based on a non-standard (non-convex) relaxation of best response iterations which converges to the unique Nash equilibrium of the game. Finally, we define a numerical approximation scheme relying on a derivative-free continuous optimization technique applied in a finite dimensional setting and we provide convergence results and error bounds.

Keywords: zero-sum game; saddle point; non-cooperative non zero-sum game; Nash equilibrium; uniqueness; theoretical and numerical approximation; fixed point; super monotone operator; best response algorithm; convex and non-convex relaxation; local variation method; error bound.

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1 Introduction

Algorithms for the approximation of a Nash equilibrium in non-cooperative deterministic games where players have a continuum of strategies have been widely investigated both in Game Theory and in Optimization literature.

One of the most explored iterative schemes involves *best response dynamics*: from one step to the next one, a player's strategy is obtained choosing a best response to the strategies of the other players that come from the previous steps. Hence, algorithms based on such schemes generate sequences of strategy profiles which are constructed by using continuous optimization techniques.

In particular, in a two-player zero-sum games framework, Cherruault and Loridan proposed in [17] two methods to approach a Nash equilibrium (i.e., a saddle point of the payoff function of any player) when the strategy sets are Euclidean spaces, the payoff function of each player is jointly twice continuously differentiable, strictly convex and coercive in his variable, and one of the two compositions of the best response functions is a contraction. When the strategy sets are Hilbert spaces and the two compositions of the best response functions are not necessarily a contraction, Morgan introduced in [42] a theoretical algorithm which converges to a saddle point. Such an algorithm relies on a relaxed variant of the best response dynamics, where a player's strategy is obtained through a convex combination of his previous step strategy and of the best response. Moreover, a scheme of discretization is presented, together with a numerical algorithm for the approximation of the discretized problem, error bounds computations, and applications to differential games.

Then, in a general non-cooperative N-player games framework, Gabay and Moulin defined in [29] two types of relaxed procedures (connected to the Jacobi and the Gauss-Seidel methods with relaxation) when the strategy set of each player is the interval $[0, +\infty[$ and the Jacobian matrix (that is the $N \times N$ matrix whose general ij element is given by the partial derivative of player i's payoff function with respect to player i's variable and player j's variable) is strictly diagonally dominant. Li and Başar proposed in [37] an inaccurate search algorithm when the strategy sets are Hilbert spaces, the payoff function of each player is strongly convex in his variable and one of the two compositions of the best response functions is a contraction. Başar investigated in [5] the convergence of some relaxation algorithms for the approximation of Nash equilibria when the strategy sets are $\mathbb{R}$ or $\mathbb{R}^2$ and the best response functions are linear, even when the two compositions of the best response functions are not a contraction. Attouch, Redont and Soubeyran presented in [2] an alternating proximal algorithm, also used to approach a Nash equilibrium for a special class of two-player weighted potential games: the players have the same strategy sets, assumed to be a Hilbert space, and the payoff functions are the sum of an individual component depending on their own strategy and of a quadratic component, the same for both players, depending on their joint strategies (hence, such payoff functions define a class of weighted potential games, in light of [12, Proposition 2]).

Table 1 summarizes the theoretical results previously mentioned; for a best response iterative method applied to an economic model see, for example, [36].

Hence, to the best of our knowledge, algorithms involving a best response-based approach which guarantee the convergence to a Nash equilibrium are not yet defined in the following situation for a two-player non zero-sum game: the best response functions are not assumed to be linear, the two compositions of the best response functions are not assumed to be contractions and the strategy sets are Hilbert spaces. Aim of this paper is to propose an iterative method which fills this lack. The iterative scheme we present involves a non-standard relaxation of the classical best response algorithm. In the usual relaxation techniques applied to a game theoretical setting (as in [42]), a current player's strategy

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Table 1: Some existing literature

	Game class	Strategy sets	Payoff functions assumptions	Composition of BR functions
Cherruault, Loridan [17]	two-player zero-sum	finite dimensional spaces	strictly convex and coercive in its argument, differentiable	contraction
Morgan [42, 40, 41]	two-player zero-sum	Hilbert spaces	strictly convex and coercive in its argument, differentiable	not necessarily
Gabay, Moulin [29]	N-player	$[0, +\infty[$	strict diagonally dominant	not necessarily
Li, Başar [37]	two-player	Hilbert spaces	strongly convex in its argument and differentiable	contraction
Başar [5]	two-player	$\mathbb{R}$ or $\mathbb{R}^2$	strongly convex in its argument and quadratic	not necessarily
Attouch, Redont, Soubeyran [2]	two-player weighted potential	Hilbert spaces	lower semicontinuous and stricly convex in its argument	contraction when the strategy sets are $\mathbb{R}$

is obtained via a convex combination of his previous step strategy and of the best response to the current strategy of the other player. Differently, in our approach the relaxation is obtained via an affine non-convex combination. Such non-standard combination is carried out through the so-called *inverse convex combinator* as defined in Definition 2.1. Motivations for its introduction will be illustrated at the beginning of Section 3, after the presentation of the suitable mathematical tools. The iterative method will be applied to a class of games for which the existence and uniqueness of the Nash equilibrium is ensured by extending a previous result obtained in [12] for weighted potential games.

Moreover, it will be designed a numerical approximation scheme (applicable offline with full knowledge of the game) for the unique Nash equilibrium which exploits a derivative-free optimization technique called *local variation method*, that was introduced in [15] for variational problems and used, in particular, in [17] for functional minimization problems and in [42, 19] for zero-sum games. The features of such a method will allow to prove the convergence of our numerical scheme and to compute error bounds and rates of convergence. However, apart from the local variation method, also first order and second order methods could be exploited in the implementation of the numerical approximation scheme; this investigation will be the subject of a future research.

We highlight that for constrained zero-sum games various numerical methods which make use of derivatives have been employed: for example, in [19] the local variation method, a conjugate gradient method and a quasi-Newton method have been used and compared, in [46, 14] *primal-dual methods* are involved, in [44, 31] *proximal-like methods* are entailed, and in [45] a comparison between two methods for stochastic optimization is illustrated.

For the sake of completeness we report that, beyond the best response dynamics approaches, various algorithms for finding Nash equilibria of general games make use of: the *Nikaido-Isoda function* (see, e.g., [53, 34, 18, 1]), the characterization of a Nash equilibrium by a variational inequality (see, e.g., the just cited [44, 46, 31] and [21, 23] for further discussions and references), the *ordinary differential equation (ODE) methods* (see, e.g., [51, 26, 13, 28, 50]) or sequences of Nash equilibria of "better-behaved" games (meaning that such approximating Nash equilibria are easier to compute, see e.g. [27] and [43, Section 4]). For algorithms on Generalized Nash equilibria (also called Social Nash equilibria) see, e.g, [22, Section 5].

Finally, we just mention that in situations where players have only partial knowledge of the strategy sets

and payoff functions, a broad literature concerns how a Nash equilibrium can be reached by means of adaptive or learning procedures (see, e.g., [49, 9, 11, 38] and references therein).

The paper is structured as follows. Section 2 concerns the issue of uniqueness of Nash equilibria: notation, assumptions and preliminary results are provided (Section 2.1), the existence of a unique Nash equilibrium is proved (Section 2.2) and a fitting class of games is defined together with a differential game example (Section 2.3). A theoretical iterative method for the approximation of the (unique) Nash equilibrium, called *Inverse-Adjusted Best Response Algorithm*, is presented in Section 3: the convergence is shown and error estimations are obtained. Section 4 is devoted to games with finite dimensional strategy sets. First, new assumptions are given in order to also handle situations where the best response functions are not analytically available and examples are presented. Then, a numerical approximation scheme, called *Numerical Inverse-Adjusted Best Response Algorithm*, is described by combining the theoretical iterative method with a continuous optimization technique (the local variation method, Section 4.1) in order to approximate the Nash equilibrium (Section 4.2). Finally, error bounds and rates of convergence for such an algorithm are proved in Section 4.3.

2 Uniqueness of the Nash equilibrium

Let $\Gamma := \{2, X, Y, F, G\}$ be a two-player normal form game. The first player's strategy set X and the second player's strategy set Y are real Hilbert spaces with inner products $(\cdot, \cdot)_X$ and $(\cdot, \cdot)_Y$, respectively, and associated norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. The payoff functions F and G of the first and second player, respectively, are defined on $X \times Y$ with values in $\mathbb{R}$. We denote by R_1 the *best response correspondence of player 1*, i.e. R_1 is the set-valued map defined on Y by

$$R_1(y) := \underset{x \in X}{\operatorname{Arg\,max}} F(x, y) = \{x' \in X \mid F(x', y) \geq F(x, y), \text{ for any } x \in X\} \subseteq X.$$

Analogously, we denote by R_2 the *best response correspondence of player 2*, that is the set-valued map defined on X by $R_2(x) := \operatorname{Arg\,max}_{y \in Y} G(x, y) \subseteq Y$. Recall that a *Nash equilibrium* of Γ is a couple $(\bar{x}, \bar{y}) \in X \times Y$ such that $(\bar{x}, \bar{y}) \in R_1(\bar{y}) \times R_2(\bar{x})$. When R_1 and R_2 are single-valued, the function r_1, defined by $\{r_1(y)\} := R_1(y)$ for any $y \in Y$, and the function r_2, defined by $\{r_2(x)\} := R_2(x)$ for any $x \in X$, are called *best response function of player 1* and *best response function of player 2*, respectively, and we denote by $\rho \colon X \to X$ the function defined by

$$\rho(x) := (r_1 \circ r_2)(x) = r_1(r_2(x)). \tag{1}$$

In the next subsections, firstly the assumptions we deal with are stated together with some preliminary results, secondly the existence of a unique Nash equilibrium is proved for games satisfying such assumptions, finally a class of games fitting the uniqueness theorem is described. Such a class involves games with infinite dimensional strategy spaces and it contains also an example of differential games.

2.1 Assumptions

Let us introduce the following hypothesis on the best response correspondences that will be used for the uniqueness result.

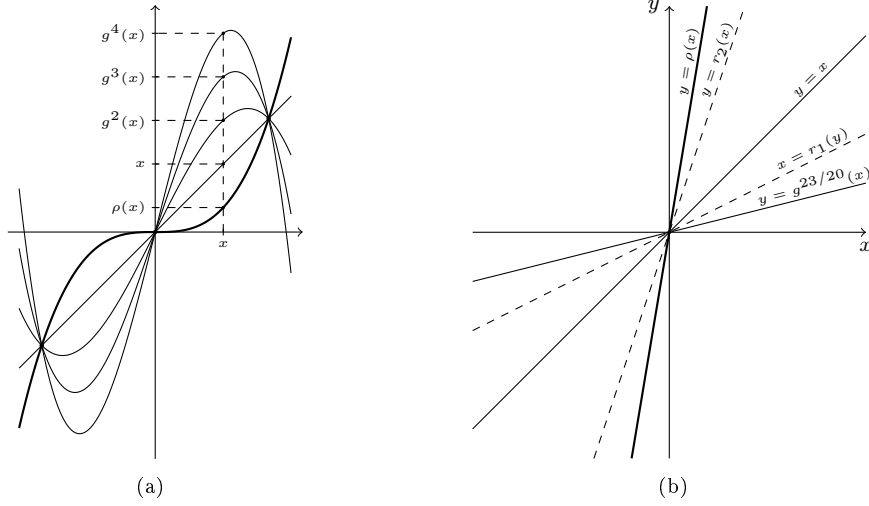($\mathscr{H}_1$) The best response correspondences R_1 and R_2 in Γ are single-valued.

Figure 1: Some graphical representations of g^δ.

Remark 2.1 Assumption $(\mathscr{H}_1)$ is satisfied if, for example, the function $F(\cdot, y)$ is strongly concave on X for any $y \in Y$ and the function $G(x, \cdot)$ is strongly concave on Y for any $x \in X$ (see, e.g., [7, Corollary 11.16]).

The next definition introduces an operator which will play a key role in all the paper.

Definition 2.1 Let $\Gamma = \{2, X, Y, F, G\}$ be a game satisfying $(\mathscr{H}_1)$ and let $\delta > 1$. The δ-*inverse convex combinator of* Γ is the function $g^\delta : X \to X$ defined by

$$g^\delta(x) := \delta x - (\delta - 1)\rho(x), \tag{2}$$

where ρ is defined in (1).

Such a function, employed in [12, Section 3, p. 1213] in order to prove the existence of a unique Nash equilibrium in weighted potential games, is called δ-inverse convex combinator of Γ since x is a convex combination of $g^\delta(x)$ and $\rho(x)$ for any $x \in X$. Indeed, rearranging (2) we get $x = \alpha g^\delta(x) + (1-\alpha)\rho(x)$, with $\alpha = 1/\delta \in]0, 1[$.

In particular, Figure 1 provides some graphical representations of g^δ. In Figure 1a, for a given function ρ we depict g^δ for some values of δ. In Figure 1b, we represent the composition ρ of the best response functions of the game $\Gamma = \{2, \mathbb{R}, \mathbb{R}, F, G\}$ where $F(x, y) = -x^2 + 4xy$, $G(x, y) = -y^2 + 6xy$ and we compute g^δ for $\delta = 23/20$.

Note that in Figure 1a g^δ and ρ have the same three fixed points and in Figure 1b they have the same unique fixed point. In fact, the next lemma, whose proof is straightforward and is omitted, summarizes the connections among the fixed points of g^δ, the fixed points of ρ and the Nash equilibria of Γ.

Lemma 2.1. *Let $\delta > 1$ and assume that $\Gamma = \{2, X, Y, F, G\}$ satisfies $(\mathscr{H}_1)$. Then, the following statements are equivalent:*

(i) *$\bar{x}$ is a fixed point of g^δ.*
(ii) *$\bar{x}$ is a fixed point of g^τ, for any $\tau > 1$.*
(iii) *$\bar{x}$ is a fixed point of ρ.*
(iv) *$(\bar{x}, r_2(\bar{x}))$ is a Nash equilibrium of Γ.*

The introduction of the δ-inverse convex combinator of Γ will allow to prove, in this section, the existence of a unique Nash equilibrium of Γ when ρ is not a contraction, by means of the equivalences stated in Lemma 2.1. Afterward, such a combinator will play a crucial role in the definition of our theoretical iterative method for the approximation of the unique Nash equilibrium: the leading idea will be to replace the best response function of one player with a function (the δ-inverse convex combinator g^δ) whose properties allow to transform a divergent procedure into a convergent one. More detailed intuition and motivations underlying the latter issue will be explained at the beginning of Section 3 and they will be put in evidence in Figure 2.

Now, recall some usual notations. Let S and T be normed vector spaces equipped with the norms $\|\cdot\|_S$ and $\|\cdot\|_T$ respectively, and let $\mathcal{L}(S,T)$ be the normed vector space of all continuous linear operators from S to T, with the usual norm $\|\Lambda\|_{\mathcal{L}(S,T)} := \sup\{\|\Lambda(s)\|_T : \|s\|_S = 1\}$. The space of all continuous linear operators from S to $\mathbb{R}$ is denoted by S^*, and the duality operation between S^* and S by $\langle \cdot, \cdot \rangle_{S^* \times S}$. Let f be a function from S to T. If f is twice differentiable on S, then $Df : S \to \mathcal{L}(S,T)$ and $D^2 f : S \to \mathcal{L}(S, \mathcal{L}(S,T))$ denote, respectively, the *Fréchet derivative of f* and the *second Fréchet derivative of f*, and by $Df(s) \in \mathcal{L}(S,T)$ and $D^2 f(s) \in \mathcal{L}(S, \mathcal{L}(S,T))$ we mean, respectively, the *derivative of f at $s \in S$* and the *second derivative of f at $s \in S$*. Moreover, $d_G f : S \to \mathcal{L}(S,T)$ and $d_G f(s) \in \mathcal{L}(S,T)$ denote, respectively, the *Gâteaux derivative of f* and the *Gâteaux derivative of f at $s \in S$*. When $S = S_1 \times \cdots \times S_n$, $D_{s_i} f : S \to \mathcal{L}(S_i, T)$ denotes the *partial derivative of f with respect to s_i*, and $D_{s_j}(D_{s_i} f) : S \to \mathcal{L}(S_j, \mathcal{L}(S_i, T))$ and $D^2_{s_i} f : S \to \mathcal{L}(S_i, \mathcal{L}(S_i, T))$, respectively, the *second partial derivative of f with respect to s_i and s_j* and the *second partial derivative of f with respect to s_i*, for any $i, j \in \{1, \ldots, n\}$ (clearly, $D_{s_i}(D_{s_i} f) \equiv D^2_{s_i} f$ for any $i \in \{1, \ldots, n\}$).

Finally, let $\mathcal{GL}(S,T) \subseteq \mathcal{L}(S,T)$ be the set of all bijective continuous linear operators from S to T with continuous (and linear) inverse. If $f \in \mathcal{GL}(S,T)$, then $f^{-1} : T \to S$ denotes the inverse operator of f and $f^{-1} \in \mathcal{L}(T,S)$.

Hence, if F and G are twice differentiable we have $D^2_x F(x,y) \in \mathcal{L}(X, X^*)$, $D^2_y G(x,y) \in \mathcal{L}(Y, Y^*)$, $D_y(D_x F)(x,y) \in \mathcal{L}(Y, X^*)$, $D_x(D_y G)(x,y) \in \mathcal{L}(X, Y^*)$, for any $(x,y) \in X \times Y$, and we can define

$$\lambda_1 := \sup_{(x,y) \in X \times Y} \|[D^2_x F(x,y)]^{-1} \circ D_y(D_x F)(x,y)\|_{\mathcal{L}(Y,X)}, \tag{3a}$$

$$\lambda_2 := \sup_{(x,y) \in X \times Y} \|[D^2_y G(x,y)]^{-1} \circ D_x(D_y G)(x,y)\|_{\mathcal{L}(X,Y)}, \tag{3b}$$

$$\lambda := \lambda_1 \cdot \lambda_2, \tag{3c}$$

provided that $D^2_x F(x,y) \in \mathcal{GL}(X, X^*)$ and $D^2_y G(x,y) \in \mathcal{GL}(Y, Y^*)$ for any $(x,y) \in X \times Y$. Throughout the paper, we deal with the class of games described in the next definition.

Definition 2.2 $\mathcal{H}$ is the set of games $\Gamma = \{2, X, Y, F, G\}$ which satisfy the following assumptions:

- X and Y are real Hilbert spaces;

- F is twice continuously differentiable on $X \times Y$, $D^2_x F(x,y) \in \mathcal{GL}(X, X^*)$ for any $(x,y) \in X \times Y$, and λ_1 defined in (3a) is a real number;

- G is twice continuously differentiable on $X \times Y$, $D^2_y G(x,y) \in \mathcal{GL}(Y, Y^*)$ for any $(x,y) \in X \times Y$, and λ_2 defined in (3b) is a real number.

The next lemma states some regularity properties of the best response functions r_1 and r_2, and of their composition ρ. The proof is obtained by extending to the class of games $\mathcal{H}$ the proofs of Propositions 3

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and 4 in [12] given for weighted potential games.

Lemma 2.2. *Assume* $\Gamma \in \boldsymbol{\mathcal{H}}$ *and satisfies* *($\mathscr{H}_1$)*. *Then*

  (i) *r_1 is continuously differentiable on Y and Lipschitz continuous with Lipschitz constant no greater than λ_1;*

  (ii) *r_2 is continuously differentiable on X and Lipschitz continuous with Lipschitz constant no greater than λ_2;*

  (iii) *ρ is continuously differentiable on X and Lipschitz continuous with Lipschitz constant no greater than $\lambda = \lambda_1 \cdot \lambda_2$.*

Proof. Let $y \in Y$. Since F is differentiable on $X \times Y$, the pair $(r_1(y), y)$ satisfies the equation $D_x F(r_1(y), y) = 0$. Therefore, by applying the Implicit Function Theorem, r_1 is continuously differentiable on Y and

$$Dr_1(y) = [D_x^2 F(r_1(y), y)]^{-1} \circ [D_y(D_x F)(r_1(y), y)] \in \mathcal{L}(Y, X). \tag{4}$$

Moreover, by the Mean Value Inequality and the definition of λ_1

$$\|r_1(y_1) - r_1(y_2)\|_X \leq \sup_{t \in [0,1]} \|Dr_1(ty_1 + (1-t)y_2)\|_{\mathcal{L}(Y,X)} \|y_1 - y_2\|_Y \leq \lambda_1 \|y_1 - y_2\|_Y$$

for any $y_1, y_2 \in Y$. Hence, r_1 is Lipschitz continuous with Lipschitz constant no greater than λ_1. Analogously, r_2 is continuously differentiable on X,

$$Dr_2(x) = [D_y^2 G(x, r_2(x))]^{-1} \circ [D_x(D_y G)(x, r_2(x))] \in \mathcal{L}(X, Y) \tag{5}$$

for any $x \in X$, and r_2 is Lipschitz continuous with Lipschitz constant no greater than λ_2. Finally, by the chain rule and (4) and (5), ρ is continuously differentiable on X and

$$
\begin{aligned}
D\rho(x) =& Dr_1(r_2(x)) \circ Dr_2(x) \\
=& [D_x^2 F(\rho(x), r_2(x))]^{-1} \circ [D_y(D_x F)(\rho(x), r_2(x))] \\
& \circ [D_y^2 G(x, r_2(x))]^{-1} \circ [D_x(D_y G)(x, r_2(x))] \in \mathcal{L}(X, X)
\end{aligned} \tag{6}
$$

for any $x \in X$. Furthermore, in light of (6) and the definition of λ,

$$\sup_{x \in X} \|D\rho(x)\|_{\mathcal{L}(X,X)} \leq \lambda_1 \cdot \lambda_2 = \lambda. \tag{7}$$

Hence, ρ is Lipschitz continuous with Lipschitz constant no greater than λ. $\qquad\square$

In order to introduce a further assumption, we state the following notion of monotonicity, used in [55] for solving functional equations (see also [33]), together with preliminary results.

Definition 2.3 An operator $\Lambda \colon X \to X$ is said to be *super monotone with constant γ* iff Λ is strongly monotone with constant γ, that is

$$(\Lambda x_1 - \Lambda x_2, x_1 - x_2)_X \geq \gamma \|x_1 - x_2\|_X^2 \quad \text{for any } x_1, x_2 \in X,$$

and, moreover, $\gamma > 1$.

Proposition 2.1. *Let $\Lambda \colon X \to X$. Then*

  (i) *if Λ is super monotone with constant γ, then Λ is strictly monotone (and, hence, monotone);*

(ii) if Λ is super monotone with constant γ, then Λ is expansive, i.e. there exists $\sigma > 1$ such that

$$\|\Lambda(x_1) - \Lambda(x_2)\|_X \geq \sigma\|x_1 - x_2\|_X \quad \text{for any } x_1, x_2 \in X,$$

and, moreover, the expansive constant σ is equal to γ;

(iii) if Λ is differentiable and super monotone with constant γ, then

$$\sup_{x \in X}\|D\Lambda(x)\|_{\mathcal{L}(X,X)} \geq \gamma > 1;$$

(iv) if Λ is differentiable and there exists $\gamma > 1$ such that, for any $x \in X$

$$(D\Lambda(x)\varphi, \varphi)_X \geq \gamma\|\varphi\|_X^2 \quad \text{for any } \varphi \in X, \tag{8}$$

then Λ is super monotone with constant γ.

Proof. (i) It follows immediately from Definition 2.3 and the definitions of monotone and strictly monotone operators.

(ii) Let Λ be super monotone with constant γ and let $x_1, x_2 \in X$. The Cauchy-Schwarz inequality implies that

$$\|\Lambda(x_1) - \Lambda(x_2)\|_X \|x_1 - x_2\|_X \geq (\Lambda(x_1) - \Lambda(x_2), x_1 - x_2)_X \geq \gamma\|x_1 - x_2\|_X^2$$

with $\gamma > 1$. Then

$$\|\Lambda(x_1) - \Lambda(x_2)\|_X \geq \gamma\|x_1 - x_2\|_X.$$

As $\gamma > 1$ we have that Λ is expansive with expansive constant γ.

(iii) Let Λ be differentiable and super monotone with constant γ and let $x_1, x_2 \in X$. In light of Proposition 2.1(ii) and the Mean Value Inequality, we have

$$\gamma\|x_1 - x_2\|_X \leq \|\Lambda(x_1) - \Lambda(x_2)\|_X$$
$$\leq \sup_{t \in [0,1]} \|D\Lambda(tx_1 + (1-t)x_2)\|_{\mathcal{L}(X,X)}\|x_1 - x_2\|_X,$$

with $\gamma > 1$. Hence, $\sup_{x \in X}\|D\Lambda(x)\|_{\mathcal{L}(X,X)} \geq \gamma > 1$.

(iv) Let $\gamma > 1$ such that (8) holds and let $x_1, x_2 \in X$. Thus, by applying the Mean Value Theorem to the real-valued function f defined by

$$f(s) := (\Lambda(sx_1 + (1-s)x_2), x_1 - x_2)_X, \quad \text{for any } s \in [0,1],$$

there exists $t \in]0,1[$ such that

$$(\Lambda(x_1) - \Lambda(x_2), x_1 - x_2)_X = (D\Lambda(tx_1 + (1-t)x_2)(x_1 - x_2), x_1 - x_2)_X$$
$$\geq \gamma\|x_1 - x_2\|_X^2.$$

Therefore, Λ is super monotone with constant γ. $\qquad\square$

For additional discussion on notions stronger than monotonicity see, for example, [7, Chapter 22]. For connections between expansiveness and existence of fixed points see, for example, [54]. We are now ready to introduce a further crucial hypothesis on games in the class $\mathcal{H}$ satisfying $(\mathscr{H}_1)$.

$(\mathscr{H}_2)$ The function $\rho = r_1 \circ r_2$ is super monotone with constant γ.

In the following remarks a sufficient condition for $(\mathscr{H}_2)$ and a straightforward consideration on ρ are provided.

Remark 2.2 Let $H(x_1, x_2, y) : X \to X$ be the operator defined by

$$H(x_1, x_2, y) := [D_x^2 F(x_1, y)]^{-1} \circ D_y(D_x F)(x_1, y)$$
$$\circ [D_y^2 G(x_2, y)]^{-1} \circ D_x(D_y G)(x_2, y),$$

where $x_1, x_2 \in X$ and $y \in Y$. Equality (6) implies that $D\rho(x) = H(\rho(x), x, r_2(x))$ for any $x \in X$. Hence, in light of Proposition 2.1(iv) assumption $(\mathscr{H}_2)$ is satisfied if there exists $\gamma > 1$ such that, for any $x_1, x_2 \in X$ and $y \in Y$

$$(H(x_1, x_2, y)\varphi, \varphi)_X \geq \gamma \|\varphi\|_X^2 \quad \text{for any } \varphi \in X.$$

Remark 2.3 When $\Gamma \in \mathcal{H}$ and $(\mathscr{H}_1)$–$(\mathscr{H}_2)$ are satisfied, then the composition ρ of the best response functions cannot be a contraction and $\lambda \geq \gamma > 1$, as a straightforward consequence of Proposition 2.1 and inequality (7).

2.2 Uniqueness theorem

Before proving the existence of one and only one Nash equilibrium, let us associate to any game $\Gamma \in \mathcal{H}$ satisfying $(\mathscr{H}_1)$–$(\mathscr{H}_2)$ the following interval

$$I_{\lambda, \gamma} := \left] 1, \frac{\lambda^2 - 1}{\lambda^2 - 2\gamma + 1} \right[. \tag{9}$$

It is worth to note that $I_{\lambda, \gamma} \neq \emptyset$ since $\lambda^2 - 2\gamma + 1 > 0$ and $\frac{\lambda^2 - 1}{\lambda^2 - 2\gamma + 1} > 1$ by Remark 2.3.
The proof of the uniqueness result is obtained arguing similarly to the proof of Theorem 1 in [12].

Theorem 2.1. *Assume $\Gamma \in \mathcal{H}$ and satisfies $(\mathscr{H}_1)$–$(\mathscr{H}_2)$. Then*

(i) the function g^δ as defined in (2) is a contraction, for any $\delta \in I_{\lambda, \gamma}$;
(ii) the contraction constant of g^δ is minimal for $\delta = \frac{\lambda^2 - \gamma}{\lambda^2 - 2\gamma + 1}$;
(iii) the game Γ has a unique Nash equilibrium $(\bar{x}, r_2(\bar{x}))$, where $\bar{x}$ is the unique fixed point of g^δ.

Proof. Let $\delta > 1$ and $x_1, x_2 \in X$, then

$$\begin{aligned}
\|g^\delta(x_1) - g^\delta(x_2)\|_X^2 &= \|\delta[x_1 - x_2] - (\delta - 1)[\rho(x_1) - \rho(x_2)]\|_X^2 \\
&= \delta^2 \|x_1 - x_2\|_X^2 + (\delta - 1)^2 \|\rho(x_1) - \rho(x_2)\|_X^2 \\
&\quad - 2\delta(\delta - 1)(\rho(x_1) - \rho(x_2), x_1 - x_2)_X.
\end{aligned} \tag{10}$$

In light of Lemma 2.2(iii) and hypothesis $(\mathscr{H}_2)$, from (10) it follows

$$\|g^\delta(x_1) - g^\delta(x_2)\|_X^2 \leq \left[\delta^2 + (\delta - 1)^2 \lambda^2 - 2\delta(\delta - 1)\gamma \right] \|x_1 - x_2\|_X^2.$$

Let $K :]1, +\infty[\to \mathbb{R}$ be the function defined by

$$K(\delta) := \delta^2 + (\delta - 1)^2 \lambda^2 - 2\delta(\delta - 1)\gamma. \tag{11}$$

Being $\lambda \geq \gamma > 1$ (by Remark 2.3), K is a convex quadratic function of δ with minimum at $\frac{\lambda^2 - \gamma}{\lambda^2 - 2\gamma + 1} > 1$ and, since $K(\frac{\lambda^2 - \gamma}{\lambda^2 - 2\gamma + 1}) = \frac{\lambda^2 - \gamma^2}{\lambda^2 - 2\gamma + 1} \geq 0$, then $K(\delta) \geq 0$ for any $\delta > 1$. Furthermore, $K(\delta) < 1$ if and only if $\delta + 1 + (\delta - 1)\lambda^2 - 2\delta\gamma < 0$, that is if and only if $\delta \in I_{\lambda, \gamma}$. Therefore g^δ is a contraction for any $\delta \in I_{\lambda, \gamma}$ and the contraction constant of g^δ is minimal for $\delta = \frac{\lambda^2 - \gamma}{\lambda^2 - 2\gamma + 1}$. Finally, let $\bar{x} \in X$ be the unique fixed point of g^δ. Then, in light of Lemma 2.1, $(\bar{x}, r_2(\bar{x}))$ is the unique Nash equilibrium of Γ. $\quad\square$

In the following remark we investigate how the assumptions of Theorem 2.1 can be reformulated in a more compact way in the case where the game belongs to the class of weighted potential games and, consequently, how Theorem 2.1 extends the uniqueness result proved in [12, Theorem 1].

Remark 2.4 For the sake of completeness, we recall that Γ is said to be a *weighted potential game* ([39] and also, for example, [25, 10]) if there exist $w_1 > 0, w_2 > 0$, called *weights*, and a real-valued function P defined on $X \times Y$, called *weighted potential of* Γ, such that

$$F(x,y) - F(x',y) = w_1(P(x,y) - P(x',y)),$$
$$G(x,y) - G(x,y') = w_2(P(x,y) - P(x,y')),$$

for any $x, x' \in X$ and any $y, y' \in Y$. When $w_1 = w_2 = 1$, Γ is said to be a *potential game* and we refer to P as *potential of* Γ.

In light of the characterization of weighted potential games given in [25, Theorem 2.1], the set of Nash equilibria of a weighted potential game $\Gamma = \{2, X, Y, F, G\}$ is equal to the set of Nash equilibria of the game $\Gamma_P = \{2, X, Y, P, P\}$. Hence, in this framework, we can require that assumptions of Theorem 2.1 hold for the "simplified" game Γ_P, and this can be employed by making assumptions directly on function P.

In fact, Γ_P belongs to $\mathcal{H}$ if P is twice continuously differentiable on $X \times Y$, $D_x^2 P(x,y) \in \mathcal{GL}(X, X^*)$ and $D_y^2 P(x,y) \in \mathcal{GL}(Y, Y^*)$ for any $(x,y) \in X \times Y$, and

$$\sup_{(x,y) \in X \times Y} \|D_x^2 P(x,y)^{-1} \circ D_y(D_x P)(x,y)\|_{\mathcal{L}(Y,X)} \in \mathbb{R},$$
$$\sup_{(x,y) \in X \times Y} \|D_y^2 P(x,y)^{-1} \circ D_x(D_y P)(x,y)\|_{\mathcal{L}(X,Y)} \in \mathbb{R}.$$

Moreover, $(\mathcal{H}_1)$ holds if P is strongly concave in each argument (in light of Remark 2.1) whereas $(\mathcal{H}_2)$ holds if there exists $\gamma > 1$ such that, for any $x_1, x_2 \in X$ and $y \in Y$

$$([D_x^2 P(x_1,y)^{-1} \circ D_y(D_x P)(x_1,y) \circ D_y^2 P(x_2,y)^{-1} \circ D_x(D_y P)(x_2,y)]\varphi, \varphi)_X \geq \gamma\|\varphi\|_X^2,$$

for any $\varphi \in X$ (in light of Remark 2.2).

In [12, Theorem 1 and Proposition 6] we proved that the conditions stated above guarantee the existence of a unique Nash equilibrium regardless of either the strict concavity of P over $X \times Y$ or the existence of a maximizer of P, whereas the literature on uniqueness of Nash equilibrium in weighted potential games is essentially based on the strict concavity of P over $X \times Y$ and on the existence of a maximizer of P (see [47, Corollary of Theorem 1]). Moreover, recall that, when Γ is a potential game and the potential P is a differentiable function, the operator $-(D_x F, D_y G)$ equals the operator $-(D_x P, D_y P)$, so the strict (resp. strong) monotonicity of $-(D_x F, D_y G)$ is equivalent to the strict (resp. strong) concavity of P over $X \times Y$ (see, e.g., [32, Theorems 2.1 and 2.4]). Therefore, given all the above, the uniqueness result in Theorem 2.1 neither implies nor it is implied by the results on uniqueness of Nash equilibria based on the properties of monotonicity of $-(D_x F, D_y G)$, like for example [51, Theorem 2] or [32, Theorem 5.1].

In the next example, we illustrate a class of weighted potential games whose weighted potential function P satisfies the conditions stated in Remark 2.4. Such a class is similar to the one considered in [12, Subsection 4.2] and is exhibited for the sake of completeness.

Example 2.1 Let $\Gamma = \{2, X, Y, F, G\}$ be the game with $X = Y = \mathbb{R}$ and

$$F(x,y) = h_F(x) + k_F(y) \pm \beta_F xy$$
$$G(x,y) = h_G(y) + k_G(x) \pm \beta_G xy,$$

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where $h_F : \mathbb{R} \to \mathbb{R}$ and $h_G : \mathbb{R} \to \mathbb{R}$ are twice continuously differentiable functions,[1] $k_F : \mathbb{R} \to \mathbb{R}$, $k_G : \mathbb{R} \to \mathbb{R}$, $\beta_F > 0$ and $\beta_G > 0$.

Assume that

$$M_F := -\inf_{x \in \mathbb{R}} D^2 h_F(x) \in \mathbb{R}, \qquad m_F := -\sup_{x \in \mathbb{R}} D^2 h_F(x) > 0,$$

$$M_G := -\inf_{y \in \mathbb{R}} D^2 h_G(y) \in \mathbb{R}, \qquad m_G := -\sup_{y \in \mathbb{R}} D^2 h_G(y) > 0, \tag{12}$$

$$M_F M_G < \beta_F \beta_G.$$

Then, Γ is a weighted potential game with weights β_F, β_G and weighted potential

$$P(x,y) = \frac{h_F(x)}{\beta_F} + \frac{h_G(y)}{\beta_G} \pm xy,$$

and, for any $(x,y) \in \mathbb{R}^2$

$$D_x^2 P(x,y) = \frac{D^2 h_F(x)}{\beta_F} \leq -\frac{m_F}{\beta_F} < 0, \quad D_y^2 P(x,y) = \frac{D^2 h_G(y)}{\beta_G} \leq -\frac{m_G}{\beta_G} < 0. \tag{13}$$

Moreover,

$$\sup_{(x,y) \in \mathbb{R}^2} \left| \frac{D_y(D_x P)(x,y)}{D_x^2 P(x,y)} \right| = \frac{\beta_F}{m_F}, \quad \sup_{(x,y) \in \mathbb{R}^2} \left| \frac{D_x(D_y P)(x,y)}{D_y^2 P(x,y)} \right| = \frac{\beta_G}{m_G}, \tag{14}$$

and for any $x_1, x_2, y \in \mathbb{R}$

$$\frac{D_y(D_x P)(x_1,y) \cdot D_x(D_y P)(x_2,y)}{D_x^2 P(x_1,y) \cdot D_y^2 P(x_2,y)} \geq \frac{\beta_F \beta_G}{M_F M_G} > 1. \tag{15}$$

Then, by (13) to (15), the weighted potential P fulfills the conditions of Remark 2.4.

Assumptions in (12) hold when, for example

$$h_F(x) = \frac{1}{1+x^2} - 4x^2 + x, \quad h_G(y) = \frac{1}{1+y^2} - 4y^2 + y, \quad \beta_F = \beta_G = 12, \tag{16}$$

since, in this case, $M_F = M_G = 10$ and $m_F = m_G = 15/2$.

2.3 A fitting class of games

In this subsection, we propose a class of games satisfying the hypotheses of Theorem 2.1 and which involves games with infinite dimensional strategy spaces and quadratic payoff functions. Such a class includes the class of weighted potential games considered in [12, Subsection 4.1].

Let $\Gamma = \{2, X, Y, F, G\}$ be the game with

$$F(x,y) = -a_F(x,x) + L_F(x) + c_F + f(y) + b_F(y,x)$$
$$G(x,y) = -a_G(y,y) + L_G(y) + c_G + g(x) + b_G(x,y), \tag{17}$$

where $a_F : X \times X \to \mathbb{R}$, $b_F : Y \times X \to \mathbb{R}$, $a_G : Y \times Y \to \mathbb{R}$ and $b_G : X \times Y \to \mathbb{R}$ are bilinear continuous operators, $f : Y \to \mathbb{R}$, $g : X \to \mathbb{R}$ are twice continuously differentiable, $L_F \in X^*$, $L_G \in Y^*$, and $c_F, c_G \in \mathbb{R}$.

Assume that there exist $\alpha_F > 0, \alpha_G > 0$ such that for any $x \in X$ and any $y \in Y$

$$a_F(x,x) \geq \alpha_F \|x\|_X^2, \qquad a_G(y,y) \geq \alpha_G \|y\|_Y^2, \tag{18}$$

---

[1] It is worth to remind that in this case the derivatives $D_x^2 F$ and $D_y(D_x F)$ can be identified with the usual derivatives of real-valued functions defined on $\mathbb{R}^2$, $\left[D_x^2 F(x,y)\right]^{-1}$ exists provided that $D_x^2 F(x,y) \neq 0$, and $\left[D_x^2 F(x,y)\right]^{-1} = 1/D_x^2 F(x,y)$. Analogous considerations apply to $D_y^2 G$ and $D_x(D_y G)$.

and, moreover, let $A_F \in \mathcal{L}(X, X^*)$, $A_G \in \mathcal{L}(Y, Y^*)$, $B_F \in \mathcal{L}(Y, X^*)$ and $B_G \in \mathcal{L}(X, Y^*)$ such that for any $x, x_1, x_2 \in X$ and any $y, y_1, y_2 \in Y$

$$a_F(x_1, x_2) = \langle A_F x_1, x_2 \rangle_{X^* \times X}, \quad b_F(y, x) = \langle B_F y, x \rangle_{X^* \times X}, \tag{19a}$$

$$a_G(y_1, y_2) = \langle A_G y_1, y_2 \rangle_{Y^* \times Y}, \quad b_G(x, y) = \langle B_G x, y \rangle_{Y^* \times Y}. \tag{19b}$$

Then, F and G are twice continuously differentiable on $X \times Y$ and for any $(x, y) \in X \times Y$

$$\begin{aligned} D_x^2 F(x, y) &= -2A_F, \quad D_y(D_x F)(x, y) = B_F, \\ D_y^2 G(x, y) &= -2A_G, \quad D_x(D_y G)(x, y) = B_G. \end{aligned} \tag{20}$$

In light of (18), (19a) and (19b), the Lax-Milgram Theorem (see, e.g., [35]) guarantees that $D_x^2 F(x, y) \in \mathcal{GL}(X, X^*)$ and $D_y^2 G(x, y) \in \mathcal{GL}(Y, Y^*)$ for any $(x, y) \in X \times Y$ and, by definition of λ_1 and λ_2 in (3a)–(3b) and by (20), we get

$$\lambda_1 = \frac{1}{2} \| A_F^{-1} \circ B_F \|_{\mathcal{L}(Y, X)} \in \mathbb{R}, \quad \lambda_2 = \frac{1}{2} \| A_G^{-1} \circ B_G \|_{\mathcal{L}(X, Y)} \in \mathbb{R}.$$

Therefore, Γ belongs to $\boldsymbol{\mathcal{H}}$. Furthermore, Γ satisfies $(\mathscr{H}_1)$ in light of (20) and Remark 2.1, and $(\mathscr{H}_2)$ holds if there exists $\gamma > 1$ such that

$$([A_F^{-1} \circ B_F \circ A_G^{-1} \circ B_G] \varphi, \varphi)_X \geq 4\gamma \|\varphi\|_X^2 \quad \text{for any } \varphi \in X, \tag{21}$$

in light of Remark 2.2.

In the following proposition, sufficient conditions for inequality (21) are provided.

Proposition 2.2. *Let Γ be a game whose payoff functions are defined as in* (17) *and satisfy* (18), (19a) *and* (19b). *Assume that X and Y coincide with the same Hilbert space $Z := X = Y$, and let $(\cdot, \cdot)_Z$ and $((\cdot, \cdot))_Z$ be two inner products on Z. If there exist $\beta_F, \beta_G \in \mathbb{R}$ such that for any $z_1, z_2 \in Z$*

$$a_F(z_1, z_2) = \alpha_F \cdot (z_1, z_2)_Z, \quad b_F(z_2, z_1) = \beta_F \cdot (z_2, z_1)_Z, \tag{22a}$$

$$a_G(z_1, z_2) = \alpha_G \cdot ((z_1, z_2))_Z, \quad b_G(z_1, z_2) = \beta_G \cdot ((z_1, z_2))_Z, \tag{22b}$$

and $\beta_F \beta_G > 4\alpha_F \alpha_G$, then there exists $\gamma > 1$ whereby inequality (21) *holds.*

Proof. Let $\varphi \in Z$; then in light of (19b) and (22b)

$$\langle B_G \varphi, z_2 \rangle_{Z^* \times Z} = \beta_G \cdot ((\varphi, z_2))_Z, \quad \text{for any } z_2 \in Z. \tag{23}$$

Since $A_G \in \mathcal{GL}(Z, Z^*)$, then $A_G^{-1}(B_G \varphi)$ is the unique $z_2 \in Z$ such that $A_G z_2 = B_G \varphi$, that is $\langle A_G z_2, k \rangle_{Z^* \times Z} = \langle B_G \varphi, k \rangle_{Z^* \times Z}$ for any $k \in Z$. By (19b), (22b) and (23)

$$\alpha_G \cdot ((z_2, k))_Z = \beta_G \cdot ((\varphi, k))_Z, \quad \text{for any } k \in Z;$$

so $A_G^{-1}(B_G \varphi) = z_2 = \frac{\beta_G}{\alpha_G} \varphi$.

Hence, in light of (19a) and (22a) the operator $B_F(A_G^{-1}(B_G \varphi)) \in Z^*$ is defined by

$$\langle B_F(A_G^{-1}(B_G \varphi)), h \rangle_{Z^* \times Z} = \frac{\beta_F \beta_G}{\alpha_G} \cdot (\varphi, h)_Z, \quad \text{for any } h \in Z. \tag{24}$$

Since $A_F \in \mathcal{GL}(Z, Z^*)$, then $A_F^{-1}(B_F(A_G^{-1}(B_G \varphi)))$ is the unique $z_1 \in Z$ such that $A_F z_1 = B_F(A_G^{-1}(B_G \varphi))$, that is $\langle A_F z_1, h \rangle_{Z^* \times Z} = \langle B_F(A_G^{-1}(B_G \varphi)), h \rangle_{Z^* \times Z}$ for any $h \in Z$. Therefore, by (19a), (22a) and (24)

$$\alpha_F \cdot (z_1, h)_Z = \frac{\beta_F \beta_G}{\alpha_G} (\varphi, h)_Z, \quad \text{for any } h \in Z;$$

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so

$$A_F^{-1}(B_F(A_G^{-1}(B_G\varphi))) = z_1 = \frac{\beta_F\beta_G}{\alpha_F\alpha_G}\varphi.$$

Being φ arbitrary in Z, it follows that

$$([A_F^{-1} \circ B_F \circ A_G^{-1} \circ B_G]\varphi, \varphi)_Z = \frac{\beta_F\beta_G}{\alpha_F\alpha_G}\|\varphi\|_Z^2, \quad \text{for any } \varphi \in Z,$$

where $\|\cdot\|_Z$ is the norm associated with the inner product $(\cdot, \cdot)_Z$. Hence, since $\beta_F\beta_G > 4\alpha_F\alpha_G$, inequality (21) holds for $\gamma = \frac{\beta_F\beta_G}{4\alpha_F\alpha_G}$. $\qquad\square$

We highlight that also some differential games (for definitions see, e.g., [6, 20]) can be included in the class of games just presented. We illustrate an example below.

Example 2.2 Let us consider a two-player infinite horizon differential game where the control variables u_F, u_G belong to $U := L^2([0, +\infty[)$, the state variable $x : [0, +\infty[\to \mathbb{R}$ evolves according to the equation

$$\dot{x}(t) = u_F(t) + u_G(t) - mx(t), \tag{25}$$

with $x(0) = x_0 > 0$ and $m > 0$, and the instantaneous profits of players at time t are

$$\pi_F(x(t), u_F(t), u_G(t)) := x(t) - \alpha_F[u_F(t)]^2 + \beta_F u_F(t)u_G(t),$$
$$\pi_G(x(t), u_F(t), u_G(t)) := x(t) - \alpha_G[u_G(t)]^2 + \beta_G u_F(t)u_G(t),$$

with $\alpha_F > 0, \alpha_G > 0, \beta_F \in \mathbb{R}$ and $\beta_G \in \mathbb{R}$. So, players' objective functional are

$$J_F(x, u_F, u_G) = \int_0^\infty e^{-i_F t}\pi_F(x(t), u_F(t), u_G(t))\, dt,$$
$$J_G(x, u_F, u_G) = \int_0^\infty e^{-i_G t}\pi_G(x(t), u_F(t), u_G(t))\, dt, \tag{26}$$

where $i_F \geq 0$ and $i_G \geq 0$ are the discount rates of the first and second player, respectively. The differential game described above has a structure similar to the one often employed in knowledge accumulation models (see, for example, [20, Example 7.1 and Section 9.5]) and also in advertising models (see, for example, [20, Section 11.3]).

Substituting the solution of the first-order differential equation (25) in (26), we can rewrite the players' objective functionals as functions of the control variables only.[2] Denoted such functions by F and G, we obtain

$$F(u_F, u_G) = \int_0^\infty e^{-i_F t}\left\{x_0 e^{-mt} - \alpha_F[u_F(t)]^2 + \beta_F u_F(t)u_G(t) + e^{-mt}\int_0^t [u_F(s) + u_G(s)]e^{ms}\, ds\right\} dt$$

$$G(u_F, u_G) = \int_0^\infty e^{-i_G t}\left\{x_0 e^{-mt} - \alpha_G[u_G(t)]^2 + \beta_G u_F(t)u_G(t) + e^{-mt}\int_0^t [u_F(s) + u_G(s)]e^{ms}\, ds\right\} dt$$

for any $(u_F, u_G) \in U \times U$. Then, the game $\Gamma = \{2, U, U, F, G\}$ belongs to the class of games considered

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[2]The solution of differential equation (25) is $x(t) = x_0 e^{-mt} + e^{-mt}\int_0^t [u_F(s) + u_G(s)]e^{ms}\, ds$.

in this subsection and characterized by (17), where

$$a_F(u_F', u_F'') = \alpha_F \int_0^\infty e^{-i_F t} u_F'(t) u_F''(t)\, dt, \quad \text{for any } u_F', u_F'' \in U; \tag{27a}$$

$$a_G(u_G', u_G'') = \alpha_G \int_0^\infty e^{-i_G t} u_G'(t) u_G''(t)\, dt, \quad \text{for any } u_G', u_G'' \in U; \tag{27b}$$

$$b_F(u_G, u_F) = \beta_F \int_0^\infty e^{-i_F t} u_F(t) u_G(t)\, dt, \quad \text{for any } u_F, u_G \in U; \tag{27c}$$

$$b_G(u_F, u_G) = \beta_G \int_0^\infty e^{-i_G t} u_F(t) u_G(t)\, dt, \quad \text{for any } u_F, u_G \in U; \tag{27d}$$

$$L_F(u_F) = \int_0^\infty e^{-(i_F + m)t} \left[\int_0^t e^{ms} u_F(s)\, ds \right] dt, \quad \text{for any } u_F \in U;$$

$$L_G(u_G) = \int_0^\infty e^{-(i_G + m)t} \left[\int_0^t e^{ms} u_G(s)\, ds \right] dt, \quad \text{for any } u_G \in U;$$

$$f(u_G) = \int_0^\infty e^{-(i_F + m)t} \left[\int_0^t e^{ms} u_G(s)\, ds \right] dt, \quad \text{for any } u_G \in U;$$

$$g(u_F) = \int_0^\infty e^{-(i_G + m)t} \left[\int_0^t e^{ms} u_F(s)\, ds \right] dt, \quad \text{for any } u_F \in U;$$

$$c_F = \int_0^\infty x_0 e^{-(i_F + m)t}\, dt, \quad c_G = \int_0^\infty x_0 e^{-(i_G + m)t}\, dt.$$

In particular, the operators a_F, a_G, b_F, b_G in (27a)–(27d) are of the same type of the operators described in (22a) and (22b) where $Z = U$ and the two inner products on U are defined by

$$(u_F, u_G)_U := \int_0^\infty e^{-i_F t} u_F(t) u_G(t)\, dt, \quad \text{for any } u_F, u_G \in U,$$

$$((u_F, u_G))_U := \int_0^\infty e^{-i_G t} u_F(t) u_G(t)\, dt, \quad \text{for any } u_F, u_G \in U.$$

Finally, we point out that $\Gamma = \{2, U, U, F, G\}$ is not, in general, a weighted potential game.

3 Theoretical approximation of the Nash equilibrium

The classical best response algorithm (where the player's strategy at the current step is the best response to the previous strategy of the other player) is well-known to converge both in zero-sum and in non zero-sum games if the composition ρ of the best response functions is a contraction, as shown in [17] and in [37]. Moreover, in zero-sum games, convex relaxations of such an algorithm (where the player's strategy at the current step is a convex combination of his previous strategy and of the best response to the current strategy of the other player) have been proved to converge for special choices of the convex combinations' coefficients even when ρ is not a contraction, as shown in [42]. Nevertheless, the classical best response algorithm as well as each of its convex relaxations may fail to converge in non zero-sum games when ρ is not a contraction. So, our motivating reason concerns how modifying a best response algorithm in order to ensure the convergence in non zero-sum games when ρ is not a contraction. The idea is to adjust "reversely" the best response of one player: we define an affine non-convex combination (instead of a convex combination) whereby the previous strategy is in-between the current strategy and the best response to the current strategy of the other player.

Relying on the just mentioned considerations, we formalize now an iterative method that allows to approach the Nash equilibrium of Γ (whose uniqueness is ensured by Theorem 2.1).

Hence, assume Γ belongs to $\boldsymbol{\mathcal{H}}$ and satisfies $(\mathscr{H}_1)$–$(\mathscr{H}_2)$; we remind that under such assumptions ρ is not a contraction (see Remark 2.3). Let $\delta \in I_{\lambda, \gamma}$, where $I_{\lambda, \gamma}$ is defined in (9).

Inverse-Adjusted Best Response Algorithm $(\mathcal{A}^\delta)$

(Step 0) Choose an initial point $y_0 \in Y$ and compute $x_0 = r_1(y_0)$.

(Step 1) Compute $\begin{cases} y_1 = r_2(x_0) \\ x_1 = \delta x_0 - (\delta - 1)r_1(y_1) = g^\delta(x_0). \end{cases}$

$\vdots$

(Step n) Compute $\begin{cases} y_n = r_2(x_{n-1}) \\ x_n = \delta x_{n-1} - (\delta - 1)r_1(y_n) = g^\delta(x_{n-1}). \end{cases}$

$\vdots$

Remark 3.1 In the special case where the strategy sets are $\mathbb{R}$ and the best response functions are assumed to be linear, $(\mathcal{A}^\delta)$ corresponds to a relaxation algorithm described in [5, equations (3.4) p. 536].

At step n, the algorithm $(\mathcal{A}^\delta)$ firstly selects the best response of the second player, i.e., $y_n = r_2(x_{n-1})$; then, it selects a non-convex combination of the strategy of the first player coming from step $n-1$ and of his best response to y_n, i.e. $x_n = \delta x_{n-1} - (\delta - 1)r_1(y_n)$ with $\delta \in I_{\lambda,\gamma} \subseteq]1, +\infty[$ (note that $(\mathcal{A}^\delta)$ would coincide with the classical best response algorithm when $\delta = 0$ and with its convex relaxations when varying $\delta \in]0, 1[$). Intuitively, it is as if the algorithm computes $r_1(y_n)$ in an imaginary intermediate step and then it adjusts such an $r_1(y_n)$ inversely with respect to x_{n-1}. Such an inversion is carried out by the δ-inverse convex combinator of Γ, that is the function g^δ defined in (2).

Figure 2 provides some graphical insights related to the algorithm $(\mathcal{A}^\delta)$ applied to the game $\Gamma = \{2, \mathbb{R}, \mathbb{R}, F, G\}$, where $F(x, y) = -x^2 + 4xy$, $G(x, y) = -y^2 + 6xy$, choosing $\delta = 23/20$ (such a game belongs to $\mathcal{H}$ and satisfies $(\mathscr{H}_1)$–$(\mathscr{H}_2)$, the unique Nash equilibrium is $(0, 0)$). In particular, Figures 2a and 2b display the first two iterations of $(\mathcal{A}^\delta)$: we note that x_0 is in-between x_1 and $r_1(y_1)$, x_1 is in-between x_2 and $r_1(y_2)$, and that the approximations x_1, x_2 and y_1, y_2 approach the Nash equilibrium strategies. In Figure 2c we mainly focus on the restriction to g^δ of the first two iterations of $(\mathcal{A}^\delta)$: x_1, x_2 approach the unique fixed point of g^δ, which coincides with the unique fixed point of ρ (according to Lemma 2.1). Figure 2d depicts the first two iterations of the classical best response algorithm applied to Γ: such an algorithm clearly diverges (as ρ is not a contraction), as well as any convex relaxed variant.

In the next theorem the convergence of the algorithm is stated.

Theorem 3.1. *Assume $\Gamma \in \mathcal{H}$ and satisfies $(\mathscr{H}_1)$–$(\mathscr{H}_2)$. Let $(\bar{x}, \bar{y})$ be the unique Nash equilibrium of Γ. Equipped the product space $X \times Y$ with the norm defined by*

$$\|(x, y)\|_{X \times Y} := \|x\|_X + \|y\|_Y, \quad \text{for any } (x, y) \in X \times Y, \tag{28}$$

then the sequence $(x_n, y_n)_n$ generated by the algorithm $(\mathcal{A}^\delta)$ is strongly convergent to $(\bar{x}, \bar{y})$ in $X \times Y$, for any $\delta \in I_{\lambda,\gamma}$. Furthermore

$$\lim_{n \to +\infty} F(x_n, y_n) = F(\bar{x}, \bar{y}), \quad \lim_{n \to +\infty} G(x_n, y_n) = G(\bar{x}, \bar{y}). \tag{29}$$

Proof. The existence of a unique Nash equilibrium of Γ is guaranteed by Theorem 2.1(iii). Let $\delta \in I_{\lambda,\gamma}$. Since $\bar{y} = r_2(\bar{x})$, then $\bar{x}$ is the unique fixed point of g^δ by Lemma 2.1. In light of Theorem 2.1(i), g^δ is a contraction with related (estimated) contraction constant $\kappa(\delta)$ where

$$\kappa(\delta) := [K(\delta)]^{1/2} \tag{30}$$

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(a) *(Step 0)-(Step 1)* of $(\mathcal{A}^\delta)$ (b) *(Step 2)* of $(\mathcal{A}^\delta)$

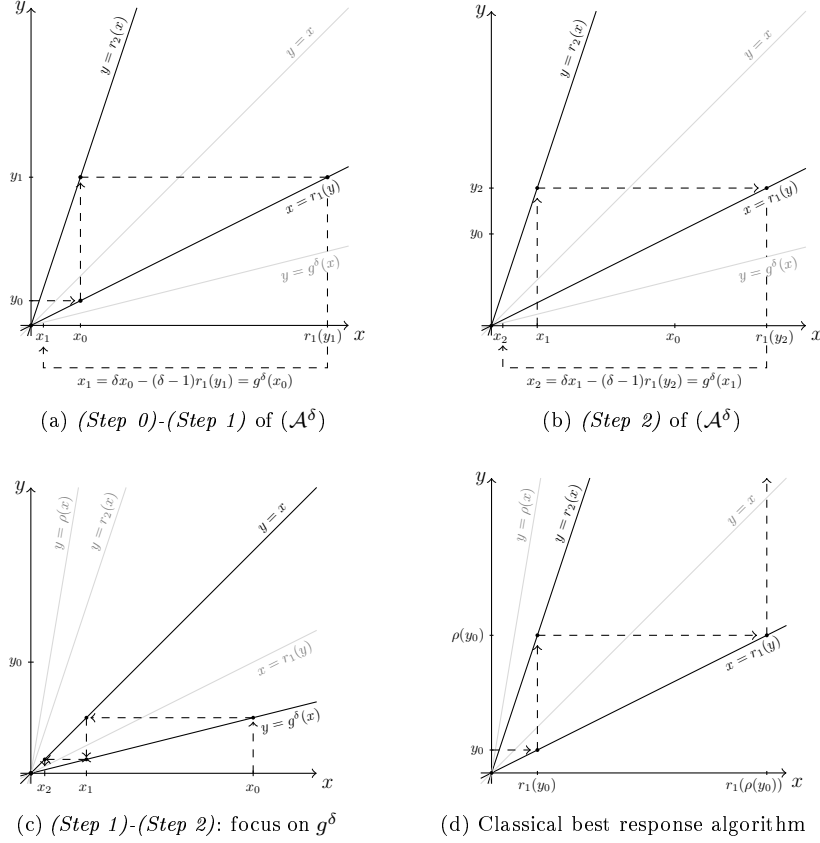(c) *(Step 1)-(Step 2)*: focus on g^δ (d) Classical best response algorithm

Figure 2: Some graphical representations related to the algorithm $(\mathcal{A}^\delta)$.

with $K(\delta)$ defined in (11). Moreover,

$$\|x_n - \bar{x}\|_X = \|g^\delta(x_{n-1}) - g^\delta(\bar{x})\|_X \leq \kappa(\delta)\|x_{n-1} - \bar{x}\|_X \leq \cdots \leq \kappa(\delta)^{n-1}\|x_1 - \bar{x}\|,$$

for any $n \in \mathbb{N}$. As $\kappa(\delta) < 1$, then $\lim_{n \to +\infty}\|x_n - \bar{x}\|_X = 0$. Therefore, the sequence $(x_n)_n$ is strongly convergent to $\bar{x}$. Furthermore, by Lemma 2.2(ii)

$$\|y_n - \bar{y}\|_Y = \|r_2(x_{n-1}) - r_2(\bar{x})\|_Y \leq \lambda_2\|x_{n-1} - \bar{x}\|_X, \quad \text{for any } n \in \mathbb{N}.$$

Since $\lim_{n \to +\infty}\|x_{n-1} - \bar{x}\|_X = 0$, the sequence $(y_n)_n$ is strongly convergent to $\bar{y}$. So, the sequence $(x_n, y_n)_n$ strongly converges to $(\bar{x}, \bar{y})$ in light of (28).

Finally, equalities in (29) follow from the continuity of F and G. □

In the next proposition the error estimations of the sequences generated by the algorithm $(\mathcal{A}^\delta)$ are computed.

Proposition 3.1. *Assume* $\Gamma \in \boldsymbol{\mathcal{H}}$ *and satisfies* $(\mathscr{H}_1)$–$(\mathscr{H}_2)$. *Let* $(\bar{x}, \bar{y})$ *be the unique Nash equilibrium of* Γ. *Then, for any* $\delta \in I_{\lambda,\gamma}$, *the following estimations hold*

$$\|x_n - \bar{x}\|_X \leq \frac{\kappa(\delta)^n}{1 - \kappa(\delta)}\|x_1 - x_0\|_X \quad \text{for any } n \in \mathbb{N}, \tag{31a}$$

$$\|y_{n+1} - \bar{y}\|_Y \leq \frac{\kappa(\delta)^n \lambda_2}{1 - \kappa(\delta)}\|x_1 - x_0\|_X \quad \text{for any } n \in \mathbb{N}, \tag{31b}$$

where κ *is defined in* (30).

Proof. Let $\delta \in I_{\lambda,\gamma}$. In light of Theorem 2.1(i), g^δ is a contraction and the relative (estimated) contraction constant is $\kappa(\delta)$, so

$$\|x_{n+1} - x_n\|_X \leq \kappa(\delta)\|x_n - x_{n-1}\|_X \leq \ldots \leq \kappa(\delta)^n\|x_1 - x_0\|_X \quad \text{for any } n \in \mathbb{N}.$$

Consequently, for any $p \in \mathbb{N}$ we get

$$\begin{aligned}
\|x_{n+p} - x_n\|_X &\leq \sum_{j=1}^{p} \|x_{n+j} - x_{n+j-1}\|_X \\
&\leq \sum_{j=1}^{p} \kappa(\delta)^{n+j-1}\|x_1 - x_0\|_X = \frac{\kappa(\delta)^n(1 - \kappa(\delta)^p)}{1 - \kappa(\delta)}\|x_1 - x_0\|_X.
\end{aligned} \tag{32}$$

Hence, inequality (31a) follows from (32) taking the limit as $p \to +\infty$ since $\kappa(\delta) < 1$.

Finally, by Lemma 2.2(ii) and (31a) we get

$$\|y_{n+1} - \bar{y}\|_Y = \|r_2(x_n) - r_2(\bar{x})\|_Y \leq \lambda_2\|x_n - \bar{x}\|_X \leq \frac{\kappa(\delta)^n \lambda_2}{1 - \kappa(\delta)}\|x_1 - x_0\|_X,$$

so inequality (31b) is proved. $\qquad\square$

It is worth to note that the convergence of the theoretical algorithm $(\mathcal{A}^\delta)$ grounds on the fact that g^δ is a contraction, for any $\delta \in I_{\lambda,\gamma}$. As the estimation of the contraction constant of g^δ is given by $\kappa(\delta)$ defined in (30), a natural choice of the parameter in $I_{\lambda,\gamma}$ could be the one associated with the inverse convex combinator whose contraction constant is minimal. We call ν such a value, that is

$$\nu := \frac{\lambda^2 - \gamma}{\lambda^2 - 2\gamma + 1} \tag{33}$$

in light of Theorem 2.1(ii), and we denote by k the associated contraction constant, that is

$$k := \kappa(\nu) = \left(\frac{\lambda^2 - \gamma^2}{\lambda^2 - 2\gamma + 1}\right)^{1/2}. \tag{34}$$

Therefore, in the next section we deal only with the algorithm $(\mathcal{A}^\delta)$ when δ takes the value ν. For notational convenience we will refer to it as the algorithm $(\mathcal{A})$, illustrated below.

Inverse-Adjusted Best Response Algorithm $(\mathcal{A})$

(Step 0)   Choose an initial point $y_0 \in Y$ and compute $x_0 = r_1(y_0)$.

(Step 1)   Compute $\begin{cases} y_1 = r_2(x_0) \\ x_1 = \nu x_0 - (\nu - 1)r_1(y_1) = g^\nu(x_0). \end{cases}$

$\quad\vdots$

(Step n)   Compute $\begin{cases} y_n = r_2(x_{n-1}) \\ x_n = \nu x_{n-1} - (\nu - 1)r_1(y_n) = g^\nu(x_{n-1}). \end{cases}$

$\quad\vdots$

---

4 Numerical approximation of the Nash equilibrium

Inverse-Adjusted Best Response Algorithm $(\mathcal{A})$ illustrated in Section 3 involves the best response functions of the game Γ. In this section we propose a numerical method which can be fruitfully used when the analytic expressions of the best response functions are not available. For the sake of brevity, we consider from now on only games whose strategy sets are finite dimensional spaces and we defer to future research the case of infinite dimensional strategy spaces together with the application of the numerical approximation to classes of differential games.

So, let $\Gamma_{p,q} = \{2, X_p, Y_q, F, G\}$ where X_p is a p-dimensional space and Y_q is a q-dimensional space, and F and G now are real-valued functions defined on $X_p \times Y_q$. Firstly, we recall the *local variation method*, a derivative-free continuous optimization technique introduced in [15] for finding solutions of variational problems and used, in particular, in [16] for functional minimization problems and in [42, 19] for zero-sum games. Then, we approximate the Nash equilibrium of the game by combining the algorithm $(\mathcal{A})$ with the local variation method.

In order to achieve this goal, in this section we consider the following assumptions on $\Gamma_{p,q}$.

$(\mathscr{A}_1)$ the function F is strongly concave on X_p uniformly on Y_q and the function G is strongly concave on Y_q uniformly on X_p, i.e. there exist two constants $m_F > 0$ and $m_G > 0$ such that, for any $x, x', x'' \in X_p$, any $y, y', y'' \in Y_q$ and any $t \in [0, 1]$:

$$F(tx' + (1-t)x'', y) \geq tF(x', y) + (1-t)F(x'', y) + m_F t(1-t)\|x' - x''\|_{X_p}^2;$$
$$G(x, ty' + (1-t)y'') \geq tG(x, y') + (1-t)G(x, y'') + m_G t(1-t)\|y' - y''\|_{Y_q}^2;$$

$(\mathscr{A}_2)$ there exists $\gamma > 1$ such that, for any $x_1, x_2 \in X_p$ and $y \in Y_q$

$$(H(x_1, x_2, y)\varphi, \varphi)_{X_p} \geq \gamma\|\varphi\|_{X_p}^2 \quad \text{for any } \varphi \in X_p,$$

where $H(x_1, x_2, y) : X_p \to X_p$ is the operator defined by

$$H(x_1, x_2, y) := [D_x^2 F(x_1, y)]^{-1} \circ D_y(D_x F)(x_1, y)$$
$$\circ [D_y^2 G(x_2, y)]^{-1} \circ D_x(D_y G)(x_2, y).$$

Remark 4.1 If F is strongly concave on X_p uniformly on Y_q, then the function $F(\cdot, y)$ is strongly concave for any $y \in Y_q$. The converse is not true in general (this is the case, for example, if F is defined on $\mathbb{R}^2$ by $F(x, y) = -x^2 e^y$). Clearly, a function can be strongly concave on X_p uniformly on Y_q and can be not concave on $X_p \times Y_q$ (take, for example, F defined on $\mathbb{R}^2$ by $F(x, y) = -x^2(e^y + 1)$). Similar arguments hold also when G is strongly concave on Y_q uniformly on X_p.

Remark 4.2 Conditions $(\mathscr{A}_1)$–$(\mathscr{A}_2)$ are more restrictive than $(\mathscr{H}_1)$–$(\mathscr{H}_2)$. In fact, $(\mathscr{A}_1)$ implies $(\mathscr{H}_1)$ in light of Remarks 2.1 and 4.1, whereas $(\mathscr{A}_2)$ implies $(\mathscr{H}_2)$ in light of Remark 2.2. Therefore, all the results obtained in Sections 2 and 3 for $\Gamma = \{2, X, Y, F, G\}$ apply when we replace Γ with $\Gamma_{p,q} = \{2, X_p, Y_q, F, G\}$ and $(\mathscr{H}_1)$–$(\mathscr{H}_2)$ with $(\mathscr{A}_1)$–$(\mathscr{A}_2)$.

We emphasize that, although $(\mathscr{A}_1)$–$(\mathscr{A}_2)$ are more restrictive, they will allow to handle situations where the best response functions are not explicit. In fact, the following examples illustrate two games which belong to the class $\mathcal{H}$, satisfy assumptions $(\mathscr{A}_1)$–$(\mathscr{A}_2)$, and where the best response functions of both players cannot be computed explicitly.

Example 4.1 Let $\Gamma_{p,q} = \{2, X_p, Y_q, F, G\}$ be the game where $X_p = Y_q = \mathbb{R}$ and the payoff functions are defined by

$$F(x, y) = -x^2 - \cos x \sin y - 5xy,$$

$$G(x, y) = \frac{1}{1 + y^2} - 4y^2 + y - 12xy.$$

The function F is strongly concave on $X_p = \mathbb{R}$ uniformly on $Y_q = \mathbb{R}$ since $D_x^2 F(x, y) = -2 + \cos x \sin y \leq -1$ for any $(x, y) \in \mathbb{R}^2$, and the function G is strongly concave on $Y_q = \mathbb{R}$ uniformly on $X_p = \mathbb{R}$ since $D_y^2 G(x, y) = [(6y^2 - 2)/(1 + y^2)^3] - 8 \leq -15/2$ for any $(x, y) \in \mathbb{R}^2$. So $(\mathscr{A}_1)$ holds. Moreover

$$\frac{4}{3} \leq \lambda_1 = \sup_{(x,y) \in \mathbb{R}^2} \left| \frac{D_y(D_x F)(x, y)}{D_x^2 F(x, y)} \right| = \sup_{(x,y) \in \mathbb{R}^2} \frac{5 - \sin x \cos y}{2 - \cos x \sin y} \leq 6,$$

$$\lambda_2 = \sup_{(x,y) \in \mathbb{R}^2} \left| \frac{D_x(D_y G)(x, y)}{D_y^2 G(x, y)} \right| = \sup_{(x,y) \in \mathbb{R}^2} \frac{6(y^2 + 1)^3}{4y^6 + 12y^4 + 9y^2 + 5} = \frac{8}{5}.$$

Therefore $\Gamma_{p,q} \in \mathcal{H}$. Furthermore

$$H(x_1, x_2, y) = \frac{(5 - \sin x_1 \cos y)[6(y^2 + 1)^3]}{(2 - \cos x_1 \sin y)(4y^6 + 12y^4 + 9y^2 + 5)}$$

$$\geq \inf_{(x,y) \in \mathbb{R}^2} \frac{5 - \sin x \cos y}{2 - \cos x \sin y} \cdot \inf_{(x,y) \in \mathbb{R}^2} \frac{6(y^2 + 1)^3}{4y^6 + 12y^4 + 9y^2 + 5} \geq \frac{4}{3} \cdot \frac{6}{5} = \frac{8}{5} > 1,$$

for any $x_1, x_2 \in \mathbb{R}$ and $y \in \mathbb{R}$. Hence $(\mathscr{A}_2)$ is satisfied by taking $\gamma = 8/5$.

Note that $\Gamma_{p,q}$ does not belong either to the class of weighted potential games illustrated in Remark 2.4 or to the class of games described in Section 2.3.

Example 4.2 Let $\Gamma_{p,q} = \{2, X_p, Y_q, F, G\}$ be a weighted potential game belonging to the class considered in Example 2.1 with $h_F, h_G, \beta_F, \beta_G$ specified in (16) and $k_F \equiv k_G \equiv 0$. Since $D_x^2 F(x, y) = D_y^2 G(x, y) \leq -15/2$ for any $(x, y) \in \mathbb{R}^2$, then F is strongly concave on $X_p = \mathbb{R}$ uniformly on $Y_q = \mathbb{R}$ and G is strongly concave on $Y_q = \mathbb{R}$ uniformly on $X_p = \mathbb{R}$, so $(\mathscr{A}_1)$ holds. Moreover, in light of Remark 2.4, $\Gamma_{p,q} \in \mathcal{H}$ and $(\mathscr{A}_2)$ is satisfied by taking $\gamma = 36/25$.

4.1 The local variation method

Now, let us describe the local variation method (introduced in [15]) that allows, by using only the values of the function, both to find an approximation of the unique maximizer of a strongly concave real-valued function defined on a finite dimensional space and to obtain an estimation of the distance between the approximation calculated and the (exact) maximizer. Such a method will be employed in our numerical method to approximate the best responses of the players acting in $\Gamma_{p,q}$.

It is worth to highlight that such a method belongs to the class of *coordinate descent methods*, widely used in constrained and unconstrained optimization (see, e.g., [3, Chapter 6], [8, Chapters 1 and 2] or [48, Chapter 3]).

We first illustrate the local variation method to find an approximation of the unique maximizer of a real-valued function f defined on $\mathbb{R}^N$ (following the scheme proposed in [16]), then we generalize such a method to a real-valued function J defined on a general finite dimensional space V_N. So, let us denote by $(\cdot, \cdot)_{\mathbb{R}^N}$ the usual inner product on $\mathbb{R}^N$ and by $\|\cdot\|_{\mathbb{R}^N}$ the Euclidean norm, and consider $f : \mathbb{R}^N \to \mathbb{R}$.

(*Step 0*) Fix an initial point $(z_1^0, z_2^0, \ldots, z_N^0) \in \mathbb{R}^N$ and a range $\epsilon > 0$.

(*Step 1*) Define:

$$\begin{aligned}
\Theta_1 &:= f(z_1^0, z_2^0, z_3^0, \ldots, z_N^0), \\
\Theta_1^+ &:= f(z_1^0 + \epsilon, z_2^0, z_3^0, \ldots, z_N^0), \\
\Theta_1^- &:= f(z_1^0 - \epsilon, z_2^0, z_3^0, \ldots, z_N^0).
\end{aligned}$$

Find the point in the set $\{z_1^0, z_1^0 + \epsilon, z_1^0 - \epsilon\}$ which corresponds to the maximum of the set $\{\Theta_1, \Theta_1^+, \Theta_1^-\}$ and denote it by $z_{1,1}^\epsilon$.

(*Step 2*) Define:

$$\begin{aligned}
\Theta_2 &:= f(z_{1,1}^\epsilon, z_2^0, z_3^0, \ldots, z_N^0), \\
\Theta_2^+ &:= f(z_{1,1}^\epsilon, z_2^0 + \epsilon, z_3^0, \ldots, z_N^0), \\
\Theta_2^- &:= f(z_{1,1}^\epsilon, z_2^0 - \epsilon, z_3^0, \ldots, z_N^0).
\end{aligned}$$

Find the point in the set $\{z_2^0, z_2^0 + \epsilon, z_2^0 - \epsilon\}$ which corresponds to the maximum of the set $\{\Theta_2, \Theta_2^+, \Theta_2^-\}$ and denote it by $z_{1,2}^\epsilon$.
$\vdots$

(*Step i*) Define:

$$\begin{aligned}
\Theta_i &:= f(z_{1,1}^\epsilon, z_{1,2}^\epsilon, \ldots, z_{1,i-1}^\epsilon, z_i^0, z_{i+1}^0, \ldots, z_N^0), \\
\Theta_i^+ &:= f(z_{1,1}^\epsilon, z_{1,2}^\epsilon, \ldots, z_{1,i-1}^\epsilon, z_i^0 + \epsilon, z_{i+1}^0, \ldots, z_N^0), \\
\Theta_i^- &:= f(z_{1,1}^\epsilon, z_{1,2}^\epsilon, \ldots, z_{1,i-1}^\epsilon, z_i^0 - \epsilon, z_{i+1}^0, \ldots, z_N^0).
\end{aligned}$$

Find the point in the set $\{z_i^0, z_i^0 + \epsilon, z_i^0 - \epsilon\}$ which corresponds to the maximum of the set $\{\Theta_i, \Theta_i^+, \Theta_i^-\}$ and denote it by $z_{1,i}^\epsilon$.
$\vdots$

(*Step N*) Define:

$$\begin{aligned}
\Theta_N &:= f(z_{1,1}^\epsilon, z_{1,2}^\epsilon, \ldots, z_{1,N-1}^\epsilon, z_N^0), \\
\Theta_N^+ &:= f(z_{1,1}^\epsilon, z_{1,2}^\epsilon, \ldots, z_{1,N-1}^\epsilon, z_N^0 + \epsilon), \\
\Theta_N^- &:= f(z_{1,1}^\epsilon, z_{1,2}^\epsilon, \ldots, z_{1,N-1}^\epsilon, z_N^0 - \epsilon).
\end{aligned}$$

Find the point in the set $\{z_N^0, z_N^0 + \epsilon, z_N^0 - \epsilon\}$ which corresponds to the maximum of the set $\{\Theta_N, \Theta_N^+, \Theta_N^-\}$ and denote it by $z_{1,N}^\epsilon$. Hence, the vector $z_1^\epsilon := (z_{1,1}^\epsilon, z_{1,2}^\epsilon, \ldots, z_{1,N}^\epsilon)$ is constructed.

(*Step R*) Repeat the steps from 1 to N choosing z_1^ϵ as initial point and the same range ϵ, and get $z_2^\epsilon := (z_{2,1}^\epsilon, z_{2,2}^\epsilon, \ldots, z_{2,N}^\epsilon)$.
$\vdots$

(*Step S*) Continue until obtaining a *stationary* vector $\bar{z}^\epsilon := (\bar{z}_1^\epsilon, \bar{z}_2^\epsilon, \ldots, \bar{z}_N^\epsilon)$, i.e. a vector which satisfies the following inequalities

$$
\begin{cases}
\begin{cases}
\Theta_1^+ \leq \Theta_1 \\
\Theta_1^- \leq \Theta_1 \\
\quad\vdots \\
\Theta_N^+ \leq \Theta_N \\
\Theta_N^- \leq \Theta_N.
\end{cases}
\iff
\begin{cases}
f(\bar{z}_1^\epsilon \pm \epsilon, \bar{z}_2^\epsilon, \ldots, \bar{z}_N^\epsilon) \leq f(\bar{z}_1^\epsilon, \bar{z}_2^\epsilon, \ldots, \bar{z}_N^\epsilon) \\
\qquad\qquad\qquad\vdots \\
f(\bar{z}_1^\epsilon, \bar{z}_2^\epsilon, \ldots, \bar{z}_N^\epsilon \pm \epsilon) \leq f(\bar{z}_1^\epsilon, \bar{z}_2^\epsilon, \ldots, \bar{z}_N^\epsilon).
\end{cases}
\end{cases}
\tag{35}
$$

The existence of a vector verifying (35) is shown in [16]. For the sake of completeness we give below the statement and the proof of such a result.

Lemma 4.1 (Lemme 1.1 in [16]). *Let $f \colon \mathbb{R}^N \to \mathbb{R}$ be a strongly concave function and let $\epsilon > 0$. Then, there exists a vector $\bar{z}^\epsilon$ satisfying* (35), *which is obtained by repeating a finite number of times the steps from* 1 *to* N *of the local variation method.*

Proof. Firstly, it is worth to note that the strong concavity of f implies

$$
\lim_{\|z\|_{\mathbb{R}^N} \to +\infty} f(z) = -\infty.
\tag{36}
$$

Let $(z_k^\epsilon)_k$ be the sequence where $z_k^\epsilon := (z_{k,1}^\epsilon, z_{k,2}^\epsilon, \ldots, z_{k,N}^\epsilon)$ is the vector obtained after repeating k times the steps from 1 to N. Then, the sequence $(z_k^\epsilon)_k$ is necessarily bounded. In fact, by contradiction, if $\lim_{k \to +\infty} \|z_k^\epsilon\|_{\mathbb{R}^N} = +\infty$ then $\lim_{k \to +\infty} f(z_k^\epsilon) = -\infty$, in light of (36). This is not possible since $(f(z_k^\epsilon))_k$ is an increasing sequence, by construction. Therefore $(z_k^\epsilon)_k$ is bounded and, consequently, there exists a constant $C > 0$ such that $|z_{k,i}^\epsilon| \leq C$ for any $i \in \{1, \ldots, N\}$ and $k \in \mathbb{N}$. Given the above and since $z_{k,i}^\epsilon = z_i^0 + m_{k,i}\epsilon$ for some $m_{k,i} \in \mathbb{Z}$, there exists $\bar{k} \in \mathbb{N}$ such that $z_k^\epsilon = z_{\bar{k}}^\epsilon$ for any $k > \bar{k}$ and $z_{\bar{k}}^\epsilon$ necessarily satisfies (35), so the result is proved. $\qquad\square$

We emphasize that the convergence of the local variation method has been shown in [16] for functions defined on a finite dimensional space, whereas, in [42] an error estimation in the case of functions defined on $\mathbb{R}^N$ has been claimed in order to obtain error bounds in zero-sum games. In this paper, having in mind to obtain error estimations for non zero-sum games defined in finite dimensional spaces, we need to prove, preliminarily, error bounds and convergence of the local variation method for functions defined on $\mathbb{R}^N$. So, in the next lemma we address this issue, by exploiting the proof of Théorème 3.1 in [16]. Before proving the result, we recall that, when $f \colon \mathbb{R}^N \to \mathbb{R}$ is a differentiable function and $x \in \mathbb{R}^N$, the Taylor's theorem guarantees

$$
\exists \mathcal{I}_x \subseteq \mathbb{R}^N \text{ s.t.} \quad f(x+h) - f(x) = (\nabla f(x), h)_{\mathbb{R}^N} + r(x,h) \quad \forall h \in \mathcal{I}_x,
\tag{37}
$$

where $\mathcal{I}_x$ is a neighbourhood of 0 depending on x, $\nabla f(x) \in \mathbb{R}^N$ is the gradient of f at x, and the remainder $r(x,h)$ satisfies $\lim_{h \to 0} r(x,h)/\|h\|_{\mathbb{R}^N} = 0$. Moreover, if a differentiable function $f \colon \mathbb{R}^N \to \mathbb{R}$ is strongly concave then there exists $m > 0$ such that

$$
f(x'') - f(x') \leq (\nabla f(x'), x'' - x')_{\mathbb{R}^N} - m\|x'' - x'\|_{\mathbb{R}^N}^2;
\tag{38}
$$

for any $x', x'' \in \mathbb{R}^N$.

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Lemma 4.2. *Let $f\colon \mathbb{R}^N \to \mathbb{R}$ be a differentiable and strongly concave function on $\mathbb{R}^N$. Assume that there exist $C_1 > 0$, $C_0 \geq 0$ and $\tau > 1$ such that*

$$|r(x,h)| \leq C_1 \|h\|_{\mathbb{R}^N}^{\tau} + C_0 \|h\|_{\mathbb{R}^N}^{\tau+1} \quad \text{for any } x \in \mathbb{R}^N \text{ and } h \in \mathcal{I}_x, \tag{39}$$

where r and $\mathcal{I}_x$ are defined in (37).
Let $\epsilon > 0$ and let $z^\epsilon \in \mathbb{R}^N$ be the stationary vector obtained at step S of the local variation method applied to f, i.e. the vector satisfying the inequalities in (35). *Then*

$$\|z^\epsilon - z^{max}\|_{\mathbb{R}^N} \leq \frac{\sqrt{N}(C_1 + \epsilon C_0)}{m} \epsilon^{\tau-1}, \tag{40}$$

where z^{max} is the unique maximizer of f over $\mathbb{R}^N$ and m is the constant related to the strong concavity of f, defined in (38). *Moreover, if $(\epsilon_n)_{n \geq 0} \subseteq\,]0, +\infty[$ is a sequence decreasing to zero, the sequence $(z^{\epsilon_n})_{n \geq 0}$ converges to z^{max}.*

Proof. Let us note that z^ϵ is well-defined in light of Lemma 4.1, and that the last part of the statement follows immediately from (40), as $\lim_{n \to +\infty} \epsilon_n = 0$ and $\tau > 1$. Therefore, we prove only inequality (40). Let $\{e_1, \ldots, e_N\}$ be the standard basis of $\mathbb{R}^N$ and let us fix $i \in \{1, \ldots, N\}$. Since z^ϵ verifies (35), by (37) we have

$$
\begin{aligned}
0 &\geq f(z^\epsilon - \epsilon e_i) - f(z^\epsilon) = -\epsilon(\nabla f(z^\epsilon), e_i)_{\mathbb{R}^N} + r(z^\epsilon, -\epsilon e_i), \\
0 &\geq f(z^\epsilon + \epsilon e_i) - f(z^\epsilon) = \epsilon(\nabla f(z^\epsilon), e_i)_{\mathbb{R}^N} + r(z^\epsilon, \epsilon e_i).
\end{aligned}
\tag{41}
$$

So, in light of (41) and (39)

$$
\begin{aligned}
(\nabla f(z^\epsilon), e_i)_{\mathbb{R}^N} &\geq \frac{r(z^\epsilon, -\epsilon e_i)}{\epsilon} \geq -\frac{C_1\|-\epsilon e_i\|_{\mathbb{R}^N}^{\tau} + C_0\|-\epsilon e_i\|_{\mathbb{R}^N}^{\tau+1}}{\epsilon}, \\
(\nabla f(z^\epsilon), e_i)_{\mathbb{R}^N} &\leq -\frac{r(z^\epsilon, \epsilon e_i)}{\epsilon} \leq \frac{C_1\|\epsilon e_i\|_{\mathbb{R}^N}^{\tau} + C_0\|\epsilon e_i\|_{\mathbb{R}^N}^{\tau+1}}{\epsilon}.
\end{aligned}
\tag{42}
$$

Hence, by (42) and since $\|e_i\|_{\mathbb{R}^N} = 1$ we get

$$|(\nabla f(z^\epsilon), e_i)_{\mathbb{R}^N}| \leq \epsilon^{\tau-1}(C_1 + \epsilon C_0). \tag{43}$$

As z^{max} is the maximizer of f and in light of (38) and (43), then

$$
\begin{aligned}
m\|z^\epsilon - z^{max}\|_{\mathbb{R}^N}^2 &\leq -[f(z^{max}) - f(z^\epsilon)] + (\nabla f(z^\epsilon), z^{max} - z^\epsilon)_{\mathbb{R}^N} \\
&\leq |(\nabla f(z^\epsilon), z^{max} - z^\epsilon)_{\mathbb{R}^N}| \\
&= \left| \left(\nabla f(z^\epsilon), \sum_{i=1}^N (z^{max} - z^\epsilon)_i e_i\right)_{\mathbb{R}^N}\right| \\
&\leq \sum_{i=1}^N |(z^{max} - z^\epsilon)_i| \cdot |(\nabla f(z^\epsilon), e_i)_{\mathbb{R}^N}| \\
&\leq \epsilon^{\tau-1}(C_1 + \epsilon C_0)\|z^{max} - z^\epsilon\|_1 \\
&\leq \sqrt{N}(C_1 + \epsilon C_0)\epsilon^{\tau-1}\|z^{max} - z^\epsilon\|_{\mathbb{R}^N},
\end{aligned}
$$

where $\|\cdot\|_1$ is the 1-norm of $\mathbb{R}^N$ and the last inequality follows from the equivalence of norms in $\mathbb{R}^N$, more precisely from the inequality $\|z\|_p \leq N^{(1/p-1/q)}\|z\|_q$ holding for any $z \in \mathbb{R}^N$ and $p, q \in [1, +\infty[$. Therefore (40) is proved and the proof is complete. $\qquad\square$

Remark 4.3 Let us note that the convergence of the local variation method is guaranteed by assuming only (36), i.e. the coercivity of $-f$, as shown in [16, Théorème 2.1]. In Lemma 4.2 we added further assumptions related to the differentiability of f to obtain also an error estimation result (inequality (40) in the statement of Lemma 4.2).

Now, let $J\colon V_N \to \mathbb{R}$ where V_N is an N-dimensional real vector space endowed with the inner product $(\cdot,\cdot)_{V_N}$ and related norm $\|\cdot\|_{V_N}$. Moreover, let $\mathcal{B} = \{b_1, b_2, \ldots, b_N\} \subseteq V_N$ be the basis of V_N such that the matrix of $(\cdot,\cdot)_{V_N}$ relative to $\mathcal{B}$ is the identity matrix I_N of size N, i.e. $\mathcal{B}$ is the basis whereby for any $u, v \in V_N$ we have $(u,v)_{V_N} = x^T I_N y$ where $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ and $y = (y_1, \ldots, y_N) \in \mathbb{R}^N$ are the (unique) N-dimensional vectors such that $\sum_{i=1}^{N} x_i b_i = u$ and $\sum_{i=1}^{N} y_i b_i = v$, and where x^T is the transpose of the vector x. The vector $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ such that $u = \sum_{i=1}^{N} x_i b_i \in V_N$ is called *coordinate vector of u relative to $\mathcal{B}$* and we denote by $c_{\mathcal{B}}\colon V_N \to \mathbb{R}^N$ the linear function which associates with each $u \in V_N$ the coordinate vector of u relative to $\mathcal{B}$. By means of $\mathcal{B}$, the inner product $(\cdot,\cdot)_{V_N}$ of V_N can be represented via the usual inner product $(\cdot,\cdot)_{\mathbb{R}^N}$ of $\mathbb{R}^N$. In fact, for any $u, v \in V_N$

$$(u,v)_{V_N} = x^T I_N y = x_1 y_1 + \cdots + x_N y_N = (x,y)_{\mathbb{R}^N}, \tag{44}$$

where $x = (x_1, \ldots, x_N) = c_{\mathcal{B}}(u) \in \mathbb{R}^N$ and $y = (y_1, \ldots, y_N) = c_{\mathcal{B}}(v) \in \mathbb{R}^N$. Consequently, the norm $\|\cdot\|_{V_N}$ of V_N can be represented via the the Euclidean norm $\|\cdot\|_{\mathbb{R}^N}$ of $\mathbb{R}^N$. In fact, in light of (44), for any $u \in V_N$

$$\|u\|_{V_N} = \sqrt{(u,u)_{V_N}} = \sqrt{x_1^2 + \cdots + x_N^2} = \|x\|_{\mathbb{R}^N}, \tag{45}$$

where $x = c_{\mathcal{B}}(u)$.

Let $J\colon V_N \to \mathbb{R}$ be a strongly concave function. Maximizing J over V_N is equivalent to maximize the function $f\colon \mathbb{R}^N \to \mathbb{R}$ defined by

$$f(z_1, \ldots, z_N) := J(z_1 b_1 + \cdots + z_N b_N), \tag{46}$$

in the sense that even f is strongly concave (on $\mathbb{R}^N$) and

$$\{\bar{z}_1 b_1 + \ldots \bar{z}_N b_N\} = \operatorname*{Arg\,max}_{v \in V_N} J(v) \Leftrightarrow \{(\bar{z}_1, \ldots, \bar{z}_N)\} = \operatorname*{Arg\,max}_{(z_1, \ldots, z_N) \in \mathbb{R}^N} f(z_1, \ldots, z_N)$$

$$\left(\text{or, equivalently,} \quad \{w\} = \operatorname*{Arg\,max}_{v \in V_N} J(v) \Leftrightarrow \{c_{\mathcal{B}}(w)\} = \operatorname*{Arg\,max}_{x \in \mathbb{R}^N} f(x) \right),$$

$$\max_{v \in V_N} J(v) = \max_{(z_1, \ldots, z_N) \in \mathbb{R}^N} f(z_1, \ldots, z_N).$$

Before showing the error estimation and the convergence results, it is worth to recall that, when J is differentiable on V_N and given $u \in V_N$, the Taylor's theorem ensures

$$\exists \mathcal{V}_u \subseteq V_N \text{ s.t.} \quad J(u+v) - J(u) = \langle DJ(u), v \rangle_{V_N^* \times V_N} + R(u,v) \quad \forall v \in \mathcal{V}_u, \tag{47}$$

where $\mathcal{V}_u$ is a neighbourhood of 0 depending on u and $R(u,v)$ is the remainder, and furthermore that if a differentiable function $J\colon V_N \to \mathbb{R}$ is strongly concave then there exists $m > 0$ such that

$$J(u'') - J(u') \le \langle DJ(u'), u'' - u' \rangle_{V_N^* \times V_N} - m\|u'' - u'\|_{V_N}^2 \tag{48}$$

for any $u', u'' \in V_N$. Finally, we say that $w^\epsilon \in V_N$ is the point generated by applying the local variation method to J if $c_{\mathcal{B}}(w^\epsilon)$ is the stationary vector obtained at step S of the local variation method applied to the function f defined in (46).

Theorem 4.1. *Let $J\colon V_N \to \mathbb{R}$ be a differentiable and strongly concave function on V_N. Assume that there exist $C_1 > 0$, $C_0 \ge 0$ and $\tau > 1$ such that*

$$|R(u,v)| \le C_1\|v\|_{V_N}^\tau + C_0\|v\|_{V_N}^{\tau+1} \quad \text{for any } u \in V_N \text{ and } v \in \mathcal{V}_u, \tag{49}$$

where R and $\mathcal{V}_u$ are defined in (47).

Let $\epsilon > 0$ and let $w^\epsilon \in V_N$ be the point generated by applying the local variation method to J. Then

$$\|w^\epsilon - w^{max}\|_{V_N} \le \frac{\sqrt{N}(C_1 + \epsilon C_0)}{m} \epsilon^{\tau - 1}, \tag{50}$$

where w^{max} is the unique maximizer of J over V_N, and m is the constant related to the strong concavity of J, defined in (48). Moreover, if $(\epsilon_n)_{n \geq 0} \subseteq]0, +\infty[$ is a sequence decreasing to zero, the sequence $(w^{\epsilon_n})_{n \geq 0}$ converges to w^{max}.

Proof. Let $u, v \in V_N$ be two points verifying (47), whose coordinate vectors relative to $\mathcal{B}$ are, respectively, $x := c_{\mathcal{B}}(u) \in \mathbb{R}^N$ and $h := c_{\mathcal{B}}(v) \in \mathbb{R}^N$, and let $f : \mathbb{R}^N \to \mathbb{R}$ be the function defined in (46). Since J is differentiable and strongly concave over V_N, then f is differentiable and strongly concave over $\mathbb{R}^N$. Moreover, by definition of Gâteaux derivative, the linearity of $c_{\mathcal{B}}$, and (46) we have

$$
\begin{aligned}
\langle DJ(u), v \rangle_{V_N^* \times V_N} &= \langle d_G J(u), v \rangle_{V_N^* \times V_N} \\
&= \lim_{t \to 0} \frac{J(u + tv) - J(u)}{t} \\
&= \lim_{t \to 0} \frac{f(x + th) - f(x)}{t} \\
&= \langle d_G f(x), h \rangle_{\mathbb{R}^{N*} \times \mathbb{R}^N} \\
&= \langle Df(x), h \rangle_{\mathbb{R}^{N*} \times \mathbb{R}^N} = (\nabla f(x), h)_{\mathbb{R}^N}.
\end{aligned}
\tag{51}
$$

Hence, in light of (46), (47), (49) and (45) we get

$$
\begin{aligned}
|f(x + h) - f(x) - (\nabla f(x), h)_{\mathbb{R}^N}| &= |J(u + v) - J(u) - \langle DJ(u), v \rangle_{V_N^* \times V_N}| \\
&= |R(u, v)| \\
&\leq C_1 \|v\|_{V_N}^\tau + C_0 \|v\|_{V_N}^{\tau+1} \\
&= C_1 \|h\|_{\mathbb{R}^N}^\tau + C_0 \|h\|_{\mathbb{R}^N}^{\tau+1}.
\end{aligned}
$$

Furthermore, the constant related to the strong concavity of f is equal to the constant related to strong concavity of J. In fact, for any $x', x'' \in \mathbb{R}^N$, from (46), (48) and (51) it follows

$$
\begin{aligned}
f(x'') - f(x') &= J(u'') - J(u') \\
&\leq \langle DJ(u'), u'' - u' \rangle_{V_N^* \times V_N} - m \|u'' - u'\|_{V_N}^2 \\
&= (\nabla f(x'), x'' - x')_{\mathbb{R}^N} - m \|x'' - x'\|_{\mathbb{R}^N}^2,
\end{aligned}
$$

where $u' \in V_N$ and $u'' \in V_N$ are equal to $c_{\mathcal{B}}^{-1}(x')$ and $c_{\mathcal{B}}^{-1}(x'')$, respectively. Therefore f satisfies the assumptions of Lemma 4.2, and so

$$
\|z^\epsilon - z^{max}\|_{\mathbb{R}^N} \leq \frac{\sqrt{N}(C_1 + \epsilon C_0)}{m} \epsilon^{\tau-1},
\tag{52}
$$

where $z^\epsilon \in \mathbb{R}^N$ is the stationary vector obtained at step S of the local variation method applied to f, i.e. z^ϵ verifies (35), and $z^{max} \in \mathbb{R}^N$ is the unique maximizer of f over $\mathbb{R}^N$. Thus, $z^\epsilon = c_{\mathcal{B}}(w^\epsilon)$ and $z^{max} = c_{\mathcal{B}}(w^{max})$. In light of (45) and (52),

$$
\|w^\epsilon - w^{max}\|_{V_N} = \|z^\epsilon - z^{max}\|_{\mathbb{R}^N} \leq \frac{\sqrt{N}(C_1 + \epsilon C_0)}{m} \epsilon^{\tau-1},
$$

so (50) holds.

The last part of the result follows from (50), since $\lim_{n \to +\infty} \epsilon_n = 0$ and $\tau > 1$. $\qquad \square$

4.2 The Numerical Inverse-Adjusted Best Response Algorithm

We introduce now a numerical method to approximate the unique Nash equilibrium $(\bar{x}_{p,q}, \bar{y}_{p,q})$ of $\Gamma_{p,q}$, referred as *Numerical Inverse-Adjusted Best Response Algorithm via local variation method* and denoted by $(\mathcal{NA})$, which combines the local variation method described in the previous subsection and the algorithm $(\mathcal{A})$ defined in Section 3.

(*Step 0*) Let y_0 be an arbitrary point in Y_q. Having chosen an initial point $\tilde{x}_0 \in X_p$ and a range $\epsilon_0 > 0$, apply the local variation method to the function $F(\cdot, y_0)$ and get the stationary vector $x_0^* \in X_p$.

(*Step 1*) Choosing $\epsilon_1 < \epsilon_0$, apply the local variation method to the function $G(x_0^*, \cdot)$ with initial point y_0 and range ϵ_1, and get the stationary vector $y_1^* \in Y_q$.

Apply the local variation method to the function $F(\cdot, y_1^*)$ with initial point x_0^* and range ϵ_1, and get the stationary vector $\tilde{x}_1^* \in X_p$.

Compute $x_1^* := \nu x_0^* - (\nu - 1)\tilde{x}_1^* \in X_p$, where ν is defined in (33).

(*Step 2*) Choosing $\epsilon_2 < \epsilon_1$, apply the local variation method to the function $G(x_1^*, \cdot)$ with initial point y_1^* and range ϵ_2, and get y_2^*.

Apply the local variation method to the function $F(\cdot, y_2^*)$ with initial point x_1^* and range ϵ_2, and get $\tilde{x}_2^*$.

Compute $x_2^* := \nu x_1^* - (\nu - 1)\tilde{x}_2^*$.
⋮

At the generic step n, with $n > 2$, given $x_{n-1}^* \in X_p$, $y_{n-1}^* \in Y_q$ and ϵ_{n-1}, we come to

(*Step n*) Choosing $\epsilon_n < \epsilon_{n-1}$, apply the local variation method to the function $G(x_{n-1}^*, \cdot)$ with initial point y_{n-1}^* and range ϵ_n, and get $y_n^* \in Y_q$.

Apply the local variation method to the function $F(\cdot, y_n^*)$ with initial point x_{n-1}^* and range ϵ_n, and get $\tilde{x}_n^* \in X_p$.

Compute $x_n^* := \nu x_{n-1}^* - (\nu - 1)\tilde{x}_n^* \in X_p$.
⋮

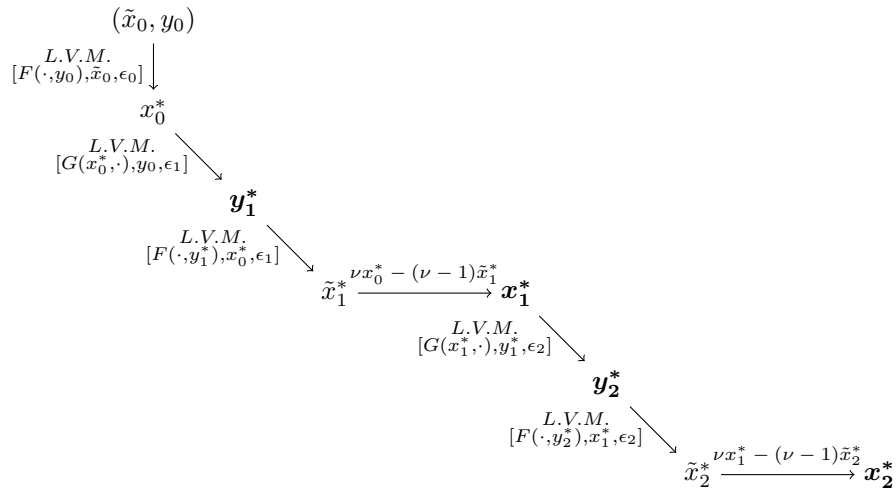Figures 3 and 4 provide a schematization of (*Step 0*)-(*Step 2*) and a schematization of (*Step n*), respectively.

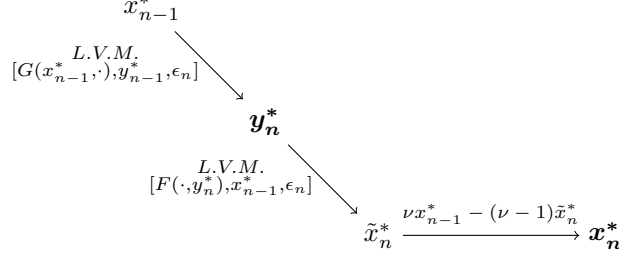Figure 3: (Step 0)-(Step 2) of ($\mathcal{NA}$).

Figure 4: (Step n) of $(\mathcal{NA})$.

The convergence of the numerical algorithm $(\mathcal{NA})$ is shown in the next theorem, whereas the implementation of the algorithm will be the subject of a future research.

Preliminarily, let us remind that the Taylor's theorem applied to $F(\cdot, y)$ at $x \in X_p$ and to $G(x, \cdot)$ at $y \in Y_q$, respectively, guarantees

$$
\begin{cases}
\exists \mathcal{I}_{x,y} \subseteq X_p \text{ neighborhood of 0 depending on } x \text{ and } y \text{ such that} \\
F(x+h, y) - F(x,y) = \langle D_x F(x,y), h \rangle_{X_p^* \times X_p} + R_F(x,h,y) \quad \forall h \in \mathcal{I}_{x,y},
\end{cases} \tag{53a}
$$

$$
\begin{cases}
\exists \mathcal{J}_{y,x} \subseteq Y_q \text{ neighborhood of 0 depending on } y \text{ and } x \text{ such that} \\
G(x, y+k) - G(x,y) = \langle D_y G(x,y), k \rangle_{Y_q^* \times Y_q} + R_G(y,k,x) \quad \forall k \in \mathcal{J}_{y,x},
\end{cases} \tag{53b}
$$

where $R_F(x,h,y)$ and $R_G(y,k,x)$ are the remainders.

Theorem 4.2. *Assume* $\Gamma_{p,q} = \{2, X_p, Y_q, F, G\} \in \boldsymbol{\mathcal{H}}$ *satisfies* $(\mathscr{A}_1)$–$(\mathscr{A}_2)$ *and*

(i) *there exist* $A_1 > 0$, $A_0 \geq 0$ *and* $\alpha > 1$ *such that*

$$
|R_F(x,h,y)| \leq A_1 \|h\|_{X_p}^{\alpha} + A_0 \|h\|_{X_p}^{\alpha+1}, \tag{54}
$$

for any $x \in X_p$, $y \in Y_q$ *and* $h \in \mathcal{I}_{x,y}$, *where* R_F *and* $\mathcal{I}_{x,y}$ *are defined in* (53a);

(ii) *there exist* $B_1 > 0$, $B_0 \geq 0$ *and* $\beta > 1$ *such that*

$$
|R_G(y,k,x)| \leq B_1 \|h\|_{Y_q}^{\beta} + B_0 \|h\|_{Y_q}^{\beta+1}, \tag{55}
$$

for any $y \in Y_q$, $x \in X_p$ *and* $k \in \mathcal{J}_{y,x}$, *where* R_G *and* $\mathcal{J}_{y,x}$ *are defined in* (53b).

Let $\epsilon_0 > 0$ *and* $\epsilon_n = \epsilon_0/2^n$ *for any* $n \in \mathbb{N}$, *and let* $(\tilde{x}_0, y_0) \in X_p \times Y_q$. *Then, the sequence* $(x_n^*, y_n^*)_n \subseteq X_p \times Y_q$ *generated by the numerical algorithm* $(\mathcal{NA})$ *is convergent to the unique Nash equilibrium* $(\bar{x}_{p,q}, \bar{y}_{p,q})$ *of* $\Gamma_{p,q}$.

Proof. The uniqueness of the Nash equilibrium of $\Gamma_{p,q}$ is guaranteed by the assumptions, Theorem 2.1(iii) and Remark 4.2. Moreover, the sequence $(x_n^*, y_n^*)_n \subseteq X_p \times Y_q$ is well-defined.

In order to show the result, let us define the following points, associated to $(x_n^*, y_n^*)_n$

$$
z_n := r_2(x_{n-1}^*) \in Y_q \tag{56a}
$$

$$
\tilde{s}_n := r_1(z_n) \in X_p \tag{56b}
$$

$$
s_n := \nu x_{n-1}^* - (\nu - 1)\tilde{s}_n = g^{\nu}(x_{n-1}^*) \in X_p \tag{56c}
$$

$$
\tilde{t}_n := r_1(y_n^*) \in X_p \tag{56d}
$$

$$
t_n := \nu x_{n-1}^* - (\nu - 1)\tilde{t}_n \in X_p, \tag{56e}
$$

25

for any $n \in \mathbb{N}$, where x_0^* is defined in (*Step 0*) of the numerical algorithm $(\mathcal{NA})$. Figures 5 and 6 represent the connections among the sequence $(x_n^*, y_n^*)_n$, the points defined in (56a)–(56e), and the sequence $(x_n, y_n)_n$ generated by the algorithm $(\mathcal{A})$ applied to $\Gamma_{p,q}$. In particular, Figure 5 schematizes such connections for (*Step 0*), (*Step 1*) and (*Step 2*) of the numerical algorithm $(\mathcal{NA})$, whereas, Figure 6 schematizes such connections for (*Step n*), with $n > 2$.
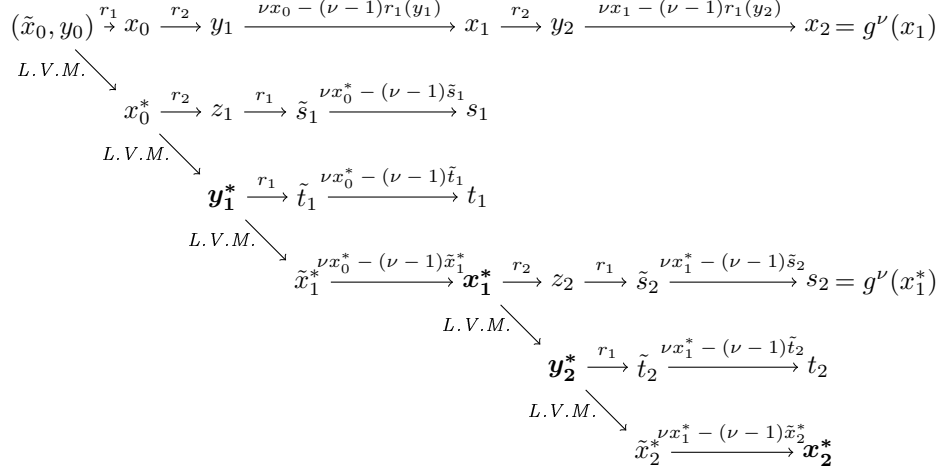
$$(\tilde{x}_0, y_0) \overset{r_1}{\dashrightarrow} x_0 \overset{r_2}{\longrightarrow} y_1 \xrightarrow{\nu x_0 - (\nu - 1)r_1(y_1)} x_1 \overset{r_2}{\longrightarrow} y_2 \xrightarrow{\nu x_1 - (\nu - 1)r_1(y_2)} x_2 = g^\nu(x_1)$$

$$\scriptstyle L.V.M.$$

$$x_0^* \overset{r_2}{\longrightarrow} z_1 \overset{r_1}{\longrightarrow} \tilde{s}_1 \xrightarrow{\nu x_0^* - (\nu - 1)\tilde{s}_1} s_1$$

$$\scriptstyle L.V.M.$$

$$\boldsymbol{y_1^*} \overset{r_1}{\longrightarrow} \tilde{t}_1 \xrightarrow{\nu x_0^* - (\nu - 1)\tilde{t}_1} t_1$$

$$\scriptstyle L.V.M.$$

$$\tilde{x}_1^* \xrightarrow{\nu x_0^* - (\nu - 1)\tilde{x}_1^*} \boldsymbol{x_1^*} \overset{r_2}{\longrightarrow} z_2 \overset{r_1}{\longrightarrow} \tilde{s}_2 \xrightarrow{\nu x_1^* - (\nu - 1)\tilde{s}_2} s_2 = g^\nu(x_1^*)$$

$$\scriptstyle L.V.M.$$

$$\boldsymbol{y_2^*} \overset{r_1}{\longrightarrow} \tilde{t}_2 \xrightarrow{\nu x_1^* - (\nu - 1)\tilde{t}_2} t_2$$

$$\scriptstyle L.V.M.$$

$$\tilde{x}_2^* \xrightarrow{\nu x_1^* - (\nu - 1)\tilde{x}_2^*} \boldsymbol{x_2^*}$$

Figure 5: Representation of $z_n, \tilde{s}_n, s_n, \tilde{t}_n, t_n$, for $n = 1, 2$.

We start by proving that $\lim_{n \to +\infty} \|x_n^* - \bar{x}_{p,q}\|_{X_p} = 0$.
For any $n \in \mathbb{N}$

$$\|x_n^* - \bar{x}_{p,q}\|_{X_p} \le \|x_n^* - x_n\|_{X_p} + \|x_n - \bar{x}_{p,q}\|_{X_p}, \tag{57}$$

where x_n is the first player's strategy generated at (*Step n*) of the algorithm $(\mathcal{A})$ applied to $\Gamma_{p,q}$. Since $\Gamma_{p,q}$ satisfies the assumptions of Theorem 3.1 (in light of Remark 4.2), then

$$\lim_{n \to +\infty} \|x_n - \bar{x}_{p,q}\|_{X_p} = 0. \tag{58}$$

So, focusing only on the first term in the right-hand side of (57), we have

$$\|x_n^* - x_n\|_{X_p} \le \|x_n^* - t_n\|_{X_p} + \|t_n - s_n\|_{X_p} + \|s_n - x_n\|_{X_p}. \tag{59}$$

$$x_{n-1} \overset{r_2}{\longrightarrow} y_n \xrightarrow{\nu x_{n-1} - (\nu - 1)r_1(y_n)} x_n = g^\nu(x_{n-1})$$

$$s_{n-1}$$

$$t_{n-1}$$

$$x_{n-1}^* \overset{r_2}{\longrightarrow} z_n \overset{r_1}{\longrightarrow} \tilde{s}_n \xrightarrow{\nu x_{n-1}^* - (\nu - 1)\tilde{s}_n} s_n = g^\nu(x_{n-1}^*)$$

$$\overset{\scriptstyle L.V.M.}{\scriptstyle [G(x_{n-1}^*, \cdot), y_{n-1}^*, \epsilon_n]}$$

$$\boldsymbol{y_n^*} \overset{r_1}{\longrightarrow} \tilde{t}_n \xrightarrow{\nu x_{n-1}^* - (\nu - 1)\tilde{t}_n} t_n$$

$$\overset{\scriptstyle L.V.M.}{\scriptstyle [F(\cdot, y_n^*), x_{n-1}^*, \epsilon_n]}$$

$$\tilde{x}_n^* \xrightarrow{\nu x_{n-1}^* - (\nu - 1)\tilde{x}_n^*} \boldsymbol{x_n^*}$$
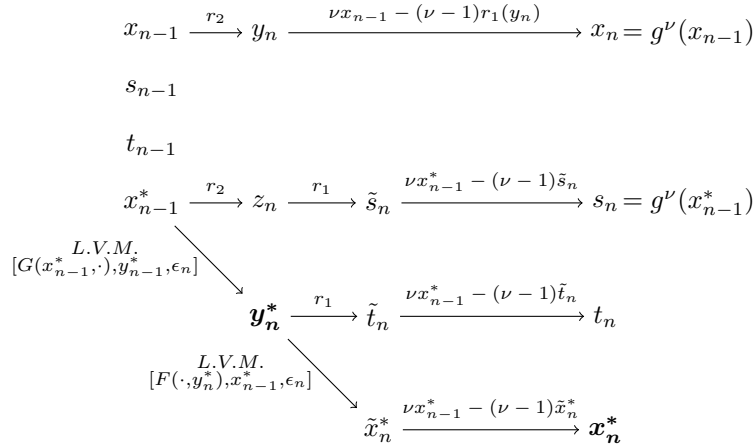
Figure 6: Representation of $z_n, \tilde{s}_n, s_n, \tilde{t}_n, t_n$, for $n > 2$.

Let us analyze the three terms in the right-hand side of (59).

1. By definition of x_n^* in (*Step n*) and by (56e), we get

$$\|x_n^* - t_n\|_{X_p} = (\nu - 1)\|\tilde{x}_n^* - \tilde{t}_n\|_{X_p}. \tag{60}$$

Let us note that $\tilde{x}_n^*$ is the approximation of the maximizer of $F(\cdot, y_n^*)$ over X_p generated by applying the local variation method to $F(\cdot, y_n^*)$ with initial point x_{n-1}^* and range ϵ_n (as represented in Figure 6), whereas $\tilde{t}_n$ is actually such a maximizer, by (56d). In light of assumption (i), from Theorem 4.1 we get

$$\|\tilde{x}_n^* - \tilde{t}_n\|_{X_p} \leq \frac{\sqrt{p}(A_1 + \epsilon_n A_0)}{m_F}\epsilon_n^{\alpha - 1}, \tag{61}$$

where m_F is the constant related to the concavity[3] of F on X_p.

2. In light of (56b)–(56e) and Lemma 2.2(i), we have

$$\begin{aligned}\|t_n - s_n\|_{X_p} &= (\nu - 1)\|\tilde{t}_n - \tilde{s}_n\|_{X_p} \\ &= (\nu - 1)\|r_1(y_n^*) - r_1(z_n)\|_{X_p} \leq \lambda_1(\nu - 1)\|y_n^* - z_n\|_{Y_q}.\end{aligned} \tag{62}$$

Similarly to the previous case, y_n^* is the approximation of the maximizer of $G(x_{n-1}^*, \cdot)$ over Y_q come up by applying the local variation method to $G(x_{n-1}^*, \cdot)$ with initial point y_{n-1}^* and range ϵ_n (as represented in Figure 6), whereas z_n is effectively such a maximizer, by (56a). In light of assumption (ii), from Theorem 4.1 it follows

$$\|y_n^* - z_n\|_{Y_q} \leq \frac{\sqrt{q}(B_1 + \epsilon_n B_0)}{m_G}\epsilon_n^{\beta - 1}, \tag{63}$$

where m_G is the constant related to the concavity[4] of G on Y_q.

3. By definition of x_n, by (56c) and by Theorem 2.1(i) we get

$$\|s_n - x_n\|_{X_p} = \|g^\nu(x_{n-1}^*) - g^\nu(x_{n-1})\|_{X_p} \leq k\|x_{n-1}^* - x_{n-1}\|_{X_p}, \tag{64}$$

where k is defined in (34).

Hence, using (60) to (64) into (59) and since $\epsilon_n \leq \epsilon_0$, we have

$$\|x_n^* - x_n\|_{X_p} \leq k\|x_{n-1}^* - x_{n-1}\|_{X_p} + \lambda_1(\nu - 1)D\epsilon_n^{\beta - 1} + (\nu - 1)C\epsilon_n^{\alpha - 1}, \tag{65}$$

where $D = \sqrt{q}(B_1 + \epsilon_0 B_0)/m_G$ and $C = \sqrt{p}(A_1 + \epsilon_0 A_0)/m_F$. Let $d_n = \|x_n^* - x_n\|_{X_p}$ for any $n \in \mathbb{N}$, and

---

[3]Since $F(\cdot, y)$ is differentiable for any $y \in Y_q$, then F is strongly concave on X_p uniformly on Y_q if and only if there exists a constant $m_F > 0$ such that $F(x'', y) - F(x', y) + m_F\|x'' - x'\|_{X_p}^2 \leq \langle D_x F(x', y), x'' - x'\rangle_{X_p^* \times X_p}$, for any $x', x'' \in X_p$, $y \in Y_q$.

[4]Since $G(x, \cdot)$ is differentiable for any $x \in X_p$, then G is strongly concave on Y_q uniformly on X_p if and only if there exists a constant $m_G > 0$ such that $G(x, y'') - G(x, y') + m_G\|y'' - y'\|_{Y_q}^2 \leq \langle D_y G(x, y'), y'' - y'\rangle_{Y_q^* \times Y_q}$, for any $y', y'' \in Y_q$, $x \in X_p$.

$d_0 = \|x_0^* - x_0\|_{X_p}$. Then by (65) it follows

$$
\begin{aligned}
d_n &\leq k d_{n-1} + \lambda_1 (\nu - 1) D \epsilon_n^{\beta-1} + (\nu - 1) C \epsilon_n^{\alpha-1} \\
&\leq k[k d_{n-2} + \lambda_1 (\nu - 1) D \epsilon_{n-1}^{\beta-1} + (\nu - 1) C \epsilon_{n-1}^{\alpha-1}] \\
&\quad + \lambda_1 (\nu - 1) D \epsilon_n^{\beta-1} + (\nu - 1) C \epsilon_n^{\alpha-1} \\
&\vdots \\
&\leq \lambda_1 (\nu - 1) D [\epsilon_n^{\beta-1} + k \epsilon_{n-1}^{\beta-1} + \cdots + k^{n-1} \epsilon_1^{\beta-1}] \\
&\quad + (\nu - 1) C [\epsilon_n^{\alpha-1} + k \epsilon_{n-1}^{\alpha-1} + \cdots + k^{n-1} \epsilon_1^{\alpha-1}] + d_0 k^n \\
&= \lambda_1 (\nu - 1) D \sum_{m=0}^{n-1} \epsilon_{n-m}^{\beta-1} k^m + (\nu - 1) C \sum_{m=0}^{n-1} \epsilon_{n-m}^{\alpha-1} k^m + d_0 k^n.
\end{aligned}
\tag{66}
$$

The summation $\sum_{m=0}^{n} \epsilon_{n-m}^{\beta-1} k^m$ is the n-th term of the Cauchy product of the two series $\sum_{i=0}^{+\infty} k^i$ and $\sum_{j=0}^{+\infty} \epsilon_j^{\beta-1}$, that is

$$
\left(\sum_{i=0}^{+\infty} k^i \right) \cdot_c \left(\sum_{j=0}^{+\infty} \epsilon_j^{\beta-1} \right) = \sum_{n=0}^{+\infty} \sum_{m=0}^{n} \epsilon_{n-m}^{\beta-1} k^m,
\tag{67}
$$

where $\cdot_c$ denotes the Cauchy product. The two series in the left-hand side of (67) are convergent: in fact, $\sum_{i=0}^{+\infty} k^i < +\infty$ since it is a geometric series with ratio $k < 1$, and $\sum_{j=0}^{+\infty} \epsilon_j^{\beta-1} < +\infty$ as $\epsilon_j = \epsilon_0/2^j$ with $\epsilon_0 > 0$, and $\beta > 1$. Therefore, in light of the Cauchy theorem (see, for example, [30, Theorem 160]), the series in the right-hand side of (67) is convergent, so $\lim_{n \to +\infty} \sum_{m=0}^{n} \epsilon_{n-m}^{\beta-1} k^m = 0$. Analogously, $\lim_{n \to +\infty} \sum_{m=0}^{n} \epsilon_{n-m}^{\alpha-1} k^m = 0$. Given the above and since $\lim_{n \to +\infty} k^n = 0$, by (66) we have

$$
\lim_{n \to +\infty} \|x_n^* - x_n\|_{X_p} = \lim_{n \to +\infty} d_n = 0,
\tag{68}
$$

hence, in light of (57) and (58), the sequence $(x_n^*)_n$ is convergent to $\bar{x}_{p,q}$.

Now, let us prove that $\lim_{n \to +\infty} \|y_n^* - \bar{y}_{p,q}\|_{Y_q} = 0$.
For any $n \in \mathbb{N}$

$$
\|y_n^* - \bar{y}_{p,q}\|_{Y_q} \leq \|y_n^* - z_n\|_{Y_q} + \|z_n - y_n\|_{Y_q} + \|y_n - \bar{y}_{p,q}\|_{Y_q},
\tag{69}
$$

where y_n is the second player's strategy generated at ($Step\ n$) of the algorithm ($\mathcal{A}$) applied to $\Gamma_{p,q}$. Since $\Gamma_{p,q}$ satisfies the assumptions of Theorem 3.1 (in light of Remark 4.2), then

$$
\lim_{n \to +\infty} \|y_n - \bar{y}_{p,q}\|_{Y_q} = 0.
\tag{70}
$$

Hence, let us consider the first and the second term in the right-hand side of (69).

1. Since $\lim_{n \to +\infty} \epsilon_n = 0$ and $\beta > 1$, then, by (63) we get

$$
\lim_{n \to +\infty} \|y_n^* - z_n\|_{Y_q} \leq \lim_{n \to +\infty} \frac{\sqrt{q}(B_1 + \epsilon_n B_0)}{m_G} \epsilon_n^{\beta-1} = 0.
\tag{71}
$$

2. In light of (56a), the definition of y_n, Lemma 2.2(ii) and (68), we have

$$
\begin{aligned}
\lim_{n \to +\infty} \|z_n - y_n\|_{Y_q} &= \lim_{n \to +\infty} \|r_2(x_{n-1}^*) - r_2(x_{n-1})\|_{Y_q} \\
&\leq \lambda_2 \lim_{n \to +\infty} \|x_{n-1}^* - x_{n-1}\|_{X_p} = 0.
\end{aligned}
\tag{72}
$$

So, $\lim_{n \to +\infty} \|y_n^* - \bar{y}_{p,q}\|_{Y_q} = 0$ by (69) to (72), i.e., the sequence $(y_n^*)_n$ is convergent to $\bar{y}_{p,q}$. Thus, the sequence $(x_n^*, y_n^*)_n$ converges to $(\bar{x}_{p,q}, \bar{y}_{p,q})$ and the proof is complete. $\qquad\square$

Remark 4.4 In the statement of Theorem 4.2, instead of setting $\epsilon_n = \epsilon_0/2^n$ for any $n \in \mathbb{N}$, we could choose any sequence $(\epsilon_n)_n$ such that the series $\sum_{j=0}^{+\infty} \epsilon_j^{\beta-1}$ and $\sum_{j=0}^{+\infty} \epsilon_j^{\alpha-1}$ are convergent.

Remark 4.5 Let us provide a sufficient condition for hypotheses (i) and (ii) in Theorem 4.2. Assumption (i) is satisfied if there exists a constant $A > 0$ such that $\|D_x^2 F(x,y)\|_{\mathcal{L}(X_p, X_p^*)} \leq A$ for any $(x,y) \in X_p \times Y_q$. In fact, since F is twice continuously differentiable, by applying the Taylor's theorem with Lagrange's form of remainder (see, e.g., [4, Formule de Taylor 3.5]), we get

$$|R_F(x,h,y)| = |F(x+h,y) - F(x,y) - \langle D_x F(x,y), h \rangle_{X_p^* \times X_p}| \leq \frac{A}{2}\|h\|_{X_p}^2,$$

for any $x \in X_p$, $y \in Y_q$ and $h \in \mathcal{I}_{x,y}$, where R_F and $\mathcal{I}_{x,y}$ are defined in (53a). Hence, assumption (i) holds setting $A_1 = A/2$, $A_0 = 0$, and $\alpha = 2$. Analogously, assumption (ii) is satisfied if there exists a constant $B > 0$ such that $\|D_y^2 G(x,y)\|_{\mathcal{L}(Y_q, Y_q^*)} \leq B$ for any $(x,y) \in X_p \times Y_q$.

Remark 4.6 In light of Remark 4.5, the games illustrated in Examples 4.1 and 4.2 satisfy the assumptions of Theorem 4.2. Moreover, we emphasize that the unique Nash equilibrium of the weighted potential game in Example 4.2 cannot be approximated through the methods based on the potential function (i.e., the ones exploiting the property that any maximizer of the potential function is a Nash equilibrium of the potential game), since such equilibrium is not a maximizer of the potential function (see [12, Proposition 6]). See, for example, [24, 52] and reference therein for further discussion regarding methods based on the potential function.

4.3 Error bounds and rates of convergence

In this subsection we provide the error estimations for the sequences $(x_n^*)_n$ and $(y_n^*)_n$ generated by the numerical algorithm $(\mathcal{NA})$ introduced in Section 4.2. Let us remind that $(\bar{x}_{p,q}, \bar{y}_{p,q})$ denotes the Nash equilibrium of $\Gamma_{p,q} = \{2, X_p, Y_q, F, G\}$.

Proposition 4.1. *Suppose that the assumptions of Theorem 4.2 hold. Then, there exist $L, M \in \mathbb{R}$ such that*

$$\|x_n^* - \bar{x}_{p,q}\|_{X_p} \leq Lk^n + \frac{M}{(2^{\alpha-1})^n} \quad \text{for any } n \in \mathbb{N}, \tag{73}$$

where k and α are defined, respectively, in (34) and (54).

Proof. Let $n \in \mathbb{N}$, then

$$\|x_n^* - \bar{x}_{p,q}\|_{X_p} \leq \|x_n^* - x_n\|_{X_p} + \|x_n - \bar{x}_{p,q}\|_{X_p}, \tag{74}$$

where x_n is the first player's strategy generated at *(Step n)* of the algorithm $(\mathcal{A})$ applied to $\Gamma_{p,q}$. Let us analyze the two terms in the right-hand side of (74).

1. In light of (66) we know that

$$
\begin{aligned}
\|x_n^* - x_n\|_{X_p} &\leq \lambda_1(\nu-1)D\sum_{m=0}^{n-1} \epsilon_{n-m}^{\beta-1} k^m \\
&\quad + (\nu-1)C\sum_{m=0}^{n-1} \epsilon_{n-m}^{\alpha-1} k^m + k^n\|x_0^* - x_0\|_{X_p},
\end{aligned} \tag{75}
$$

where $D = \sqrt{q}(B_1 + \epsilon_0 B_0)/m_G$ and $C = \sqrt{p}(A_1 + \epsilon_0 A_0)/m_F$. Since x_0^* is the approximation of the maximizer of $F(\cdot, y_0)$ over X_p obtained by applying the local variation method to $F(\cdot, y_0)$ with initial

29

point $\tilde{x}_0$ and range ϵ_0 (as represented in Figure 3), and x_0 is effectively such a maximizer (as defined at (*Step 0*) of the algorithm ($\mathcal{A}$) applied to $\Gamma_{p,q}$), then $\|x_0^* - x_0\|_{X_p} \leq C\epsilon_0^{\alpha-1}$ by Theorem 4.1. Given the above and since $\epsilon_n = \epsilon_0/2^n$, by (75) we have

$$
\begin{aligned}
\|x_n^* - x_n\|_{X_p} &\leq \lambda_1(\nu-1)D\epsilon_0^{\beta-1} \sum_{m=0}^{n-1} \frac{k^m}{(2^{\beta-1})^{n-m}} \\
&\quad + (\nu-1)C\epsilon_0^{\alpha-1} \sum_{m=0}^{n-1} \frac{k^m}{(2^{\alpha-1})^{n-m}} + C\epsilon_0^{\alpha-1}k^n \\
&= \frac{\lambda_1(\nu-1)D\epsilon_0^{\beta-1}}{(2^{\beta-1})^n} \left[\frac{1-(k2^{\beta-1})^n}{1-k2^{\beta-1}}\right] \\
&\quad + \frac{(\nu-1)C\epsilon_0^{\alpha-1}}{(2^{\alpha-1})^n} \left[\frac{1-(k2^{\alpha-1})^n}{1-k2^{\alpha-1}}\right] + C\epsilon_0^{\alpha-1}k^n,
\end{aligned}
\tag{76}
$$

where the equality is obtained by exploiting the sum of the first n terms of geometric series of ratio $k2^{\beta-1}$ and $k2^{\alpha-1}$.

2. In light of Proposition 3.1, recalling that $x_1 = \nu x_0 - (\nu-1)r_1(y_1)$ and $x_0 = r_1(y_0)$, and that r_1 is Lipschitz continuous by Lemma 2.2(i), we get

$$
\begin{aligned}
\|x_n - \bar{x}_{p,q}\|_{X_p} &\leq \frac{k^n}{1-k}\|x_1 - x_0\|_{X_p} \\
&= \frac{k^n}{1-k}\|(\nu-1)(x_0 - r_1(y_1))\|_{X_p} \\
&= \frac{(\nu-1)k^n}{1-k}\|r_1(y_0) - r_1(y_1)\|_{X_p} \\
&\leq \frac{\lambda_1(\nu-1)k^n}{1-k}\|y_1 - y_0\|_{Y_q} \\
&\leq \frac{\lambda_1(\nu-1)k^n}{1-k}[\|y_1 - z_1\|_{Y_q} + \|z_1 - y_1^*\|_{Y_q} + \|y_1^* - y_0\|_{Y_q}].
\end{aligned}
\tag{77}
$$

Since y_1^* is the approximation of the maximizer of $G(x_0^*, \cdot)$ over Y_q generated by applying the local variation method to $G(x_0^*, \cdot)$ with initial point y_0 and range ϵ_1 (as represented in Figure 3), and z_1 is actually such a maximizer (in light of (56a)), then $\|z_1 - y_1^*\|_{Y_q} \leq D\epsilon_1^{\beta-1}$ by Theorem 4.1. Given the above, by definition of y_1 in (*Step 1*) of the algorithm ($\mathcal{A}$) applied to $\Gamma_{p,q}$ and the definition of z_1 in (56a), it follows

$$
\begin{aligned}
\|y_1 - z_1\|_{Y_q} + \|z_1 - y_1^*\|_{Y_q} &\leq \|r_2(x_0) - r_2(x_0^*)\|_{Y_q} + D\epsilon_1^{\beta-1} \\
&\leq \lambda_2 C\epsilon_0^{\alpha-1} + D\frac{\epsilon_0^{\beta-1}}{2^{\beta-1}},
\end{aligned}
\tag{78}
$$

where the last inequality holds in light of Lemma 2.2(ii), inequality $\|x_0^* - x_0\|_{X_p} \leq C\epsilon_0^{\alpha-1}$ proved in the previous point, and the definition of ϵ_1. Hence, (77)–(78) imply

$$
\|x_n - \bar{x}_{p,q}\|_{X_p} \leq \frac{\lambda_1(\nu-1)k^n}{1-k}\left[\lambda_2 C\epsilon_0^{\alpha-1} + D\frac{\epsilon_0^{\beta-1}}{2^{\beta-1}} + \|y_1^* - y_0\|_{Y_q}\right].
\tag{79}
$$

Finally, by using (76) and (79), from (74) we get

$$
\begin{aligned}
\|x_n^* - \bar{x}_{p,q}\|_{X_p} &\leq \frac{(\nu-1)C\epsilon_0^{\alpha-1}}{(2^{\alpha-1})^n}\left[\frac{1-(k2^{\alpha-1})^n}{1-k2^{\alpha-1}}\right] + C\epsilon_0^{\alpha-1}k^n \\
&\quad + \frac{\lambda_1(\nu-1)k^n}{1-k}\left[\lambda_2 C\epsilon_0^{\alpha-1} + D\frac{\epsilon_0^{\beta-1}}{2^{\beta-1}} + \|y_1^* - y_0\|_{Y_q}\right] \\
&= Lk^n + \frac{M}{(2^{\alpha-1})^n},
\end{aligned}
$$

where

$$L = C\epsilon_0^{\alpha-1}\left[\frac{2-\nu-k2^{\alpha-1}}{1-k2^{\alpha-1}}\right] + \frac{\lambda_1(\nu-1)}{1-k}\left[\lambda_2 C\epsilon_0^{\alpha-1} + D\frac{\epsilon_0^{\beta-1}}{2^{\beta-1}} + \|y_1^* - y_0\|_{Y_q}\right],$$
$$M = \frac{(\nu-1)C\epsilon_0^{\alpha-1}}{1-k2^{\alpha-1}},$$
$$(80)$$

therefore the result is proved. $\qquad\square$

Proposition 4.2. *Suppose that the assumptions of Theorem 4.2 hold. Then, there exist $L', M' \in \mathbb{R}$ and $W > 0$ such that*

$$\|y_n^* - \bar{y}_{p,q}\|_{Y_q} \le L'k^{n-1} + \frac{M'}{(2^{\alpha-1})^{n-1}} + \frac{W}{(2^{\beta-1})^n} \quad \text{for any } n \in \mathbb{N}, \tag{81}$$

where k, α and β are defined, respectively, in (34), (54) and (55).

Proof. Let $n \in \mathbb{N}$, then

$$\|y_n^* - \bar{y}_{p,q}\|_{Y_q} \le \|y_n^* - z_n\|_{Y_q} + \|z_n - \bar{y}_{p,q}\|_{Y_q}, \tag{82}$$

where z_n is defined in (56a).
Recall that $\bar{y}_{p,q} = r_2(\bar{x}_{p,q})$, by definition of Nash equilibrium. Then, in light of (63), (56a) and Lemma 2.2(ii) we have

$$\|y_n^* - z_n\|_{Y_q} + \|z_n - \bar{y}_{p,q}\|_{Y_q} \le D\epsilon_n^{\beta-1} + \|r_2(x_{n-1}^*) - r_2(\bar{x}_{p,q})\|_{Y_q}$$
$$\le D\epsilon_n^{\beta-1} + \lambda_2\|x_{n-1}^* - \bar{x}_{p,q}\|_{X_p}, \tag{83}$$

where $D = \sqrt{q}(B_1 + \epsilon_0 B_0)/m_G$.
Since $\epsilon_n = \epsilon_0/2^n \le \epsilon_0$ and by applying Proposition 4.1, from (82)–(83) we get

$$\|y_n^* - \bar{y}_{p,q}\|_{Y_q} \le D\frac{\epsilon_0^{\beta-1}}{(2^{\beta-1})^n} + \lambda_2\left[Lk^{n-1} + \frac{M}{(2^{\alpha-1})^{n-1}}\right]$$
$$= L'k^{n-1} + \frac{M'}{(2^{\alpha-1})^{n-1}} + \frac{W}{(2^{\beta-1})^n},$$

where $L' = \lambda_2 L$, $M' = \lambda_2 M$ with L and M explicitly stated in (80), and $W = D\epsilon_0^{\beta-1} > 0$. $\qquad\square$

Error estimations proved in Propositions 4.1 and 4.2 allow also to derive the rate and the order of convergence of the sequences $(x_n^*)_n$ and $(y_n^*)_n$. Before stating the results, we remind that a sequence $(z_n)_n$ converges *R-linearly* to $\bar{z}$ in a finite dimensional space Z (see, e.g., [48, pp. 28–30]) if the sequence $(\|z_n - \bar{z}\|_Z)_n$ is dominated by a sequence converging linearly to 0, that is if there exists a sequence of nonnegative real numbers $(\zeta_n)_n$ converging to 0 and a constant $t \in\,]0, 1[$ such that

$$\|z_n - \bar{z}\|_Z \le \zeta_n \quad \text{and} \quad \zeta_{n+1} \le t \cdot \zeta_n, \quad \text{for any } n \text{ sufficiently large.}$$

Proposition 4.3. *Suppose that the assumptions of Theorem 4.2 hold and let $T = \min\{k^{-1}, 2^{\alpha-1}\}$ and $Q = \min\{k^{-1}, 2^{\alpha-1}, 2^{\beta-1}\}$. Then*

(i) *the sequence $(x_n^*)_n$ exhibits $O(T^{-n})$-rate of convergence;*
(ii) *the sequence $(y_n^*)_n$ exhibits $O(Q^{-n})$-rate of convergence;*
(iii) *the sequence $(x_n^*)_n$ converges R-linearly to $\bar{x}_{p,q}$;*
(iv) *the sequence $(y_n^*)_n$ converges R-linearly to $\bar{y}_{p,q}$.*

Proof. First we note that $T \ge Q > 1$ since $k \in\,]0, 1[$, $\alpha > 1$ and $\beta > 1$.
From (73) it follows that $\|x_n^* - \bar{x}_{p,q}\|_{X_p} \le \xi_n$ for any $n \in \mathbb{N}$, where $\xi_n := (|L| + |M|)T^{-n}$. Hence, (i) holds. Moreover, $(\xi_n)_n$ converges to 0 and

$$\lim_{n\to\infty} \frac{\xi_{n+1}}{\xi_n} = \frac{1}{T} \in\,]0, 1[,$$

therefore (iii) is proved.

Analogously, from (81) it follows that $\|y_n^* - \bar{y}_{p,q}\|_{y_p} \le \chi_n$ for any $n \in \mathbb{N}$, where $\chi_n := (|L'|k^{-1} + |M'|2^{\alpha-1} + W)Q^{-n}$. Hence, (ii) holds. Moreover, $(\chi_n)_n$ converges to 0 and

$$\lim_{n \to \infty} \frac{\chi_{n+1}}{\chi_n} = \frac{1}{Q} \in]0,1[,$$

therefore (iv) is proved. $\qquad\square$

Remark 4.7 The same arguments used in Proposition 4.3 ensure that the sequence of strategy profiles $(x_n^*, y_n^*)_n$ exhibits $O(Q^{-n})$-rate of convergence and it converges R-linearly to $(\bar{x}_{p,q}, \bar{y}_{p,q})$ in $X_p \times Y_q$.

To conclude, we highlight that the error bounds proved in Propositions 4.1 and 4.2 and the rates of convergence derived in Proposition 4.3 crucially depend on the fact that, in order to make the discussion easier to follow, we chose in the statement of Theorem 4.2 the sequence of ranges $(\epsilon_n)_n$ with $\epsilon_n = \epsilon_0/2^n$. However, improvements in the error estimations and in the rates of convergence could be achieved by choosing other suitable sequences of ranges satisfying the requirements in Remark 4.4.

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