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Improving the Quality of the Input in the Term Structure Consistent Models

Javier Giner and Sandra Morini

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DIPARTIMENTO DI SCIENZE ECONOMICHE - UNIVERSITÀ DEGLI STUDI DI SALERNO
Via Ponte Don Melillo - 84084 FISCIANO (SA)
Tel. 089-96 3167/3168 - Fax 089-96 3169 – e-mail: csef@unisa.it

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Abstract

In finance, getting an accurate estimation of the term structure of interest rates is essential because this information is often used as input by other pricing financial models. In this paper, we point out the importance of selecting a suitable estimation of the term structure of interest rates.
To show this fact, we use the Spanish Bond Market to estimate the initial interest rate and forward curves for one day, by using both McCulloch (1975) cubic polynomial splines, and Legendre's polynomials (Morini, 1998). We use these curves as input for pricing pure discount bonds with the Ho and Lee (1986) and Hull and White (1990) models. Then, we find the important result that using an inadequate interest rate curve affects dramatically the behaviour of the dynamic term structure models and, consequently, the estimation of the asset pricing models.

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* University of La Laguna, Tenerife – Canary Islands, Spain

E-mail: jginer@ull.es and smorini@ull.es

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1. Introduction

The estimation of the term structure of interest rates (TSIR) is an important subject in finance. In fact, spot interest rates are the data inputs of many other models, particularly for dynamic interest rates models and asset pricing models. So, any improvement in the term structure estimation models makes better the results obtained by those models. In sum, a good estimation of the term structure of interest rates is the mandatory starting point to analyze adequately dynamic models. Furthermore, it is essential in order to price assets correctly.

McCulloch (1971) was first in applying a direct estimation methodology. It consists in extracting the interest rates from bond prices by using approximation functions. Later on, in 1975, he slightly modified his former technique and he proposes to use cubic polynomial splines instead of quadratic ones. Since then, several authors have proposed similar models, but McCulloch's (1975) model is still one of the most applied. So, in the second section of this paper we introduce the term structure estimation problem and analyze the drawbacks of McCulloch's model. We also propose the use of Legendre's polynomials (Morini, 1998) in the approximation of the discount function.

In the third section, we introduce the Ho and Lee (1986) and Hull and White (1990) dynamic term structure consistent models. These are very important in order to determinate the evolution of the interest rate curve. They can match the initial interest rate curve, using therefore the term structure of interest rates as input.

At last, in the fourth section, we use data for one day of the Spanish Bond Market and bring up the results of using the estimation models introduced at second section into the dynamic models of the third section.

2. Estimating the term structure of interest rates

There are three different ways to characterize the yield curve at initial time *t*. The first representation is by the prices of pure discount bonds $P(t,s)$ where $s \geq t$ is the time to the maturity. Pure discount bonds are bonds that give the holder a single unit cashflow at maturity with no intermediate cashflows. The second is by the spot interest rates $R(t,s)$. It is calculated as a continuous compoundly interest rates to obtain the discount bond prices:

$$P(t,s) = e^{-R(t,s)(s-t)} \tag{1.}$$

We will refer to $R(t,s)$ as the curve of spot interest rates, term structure of interest rates or curve yield. $P(t,s)$ and $R(t,s)$ are the most usual forms to representation of the different interest rates that exist at a moment *t*, typically $t = 0$.

The third formulation is by the forward rate curve. The forward rate $f(t,s)$ at time *t* is the interest rate that we see for an instantaneous period starting at time s and finishing at time $s+h$, with $h > 0$. We can obtain an analytical expression for the forward rates, consistently with the actual spot rates, as a continuous rate of return between two bonds $P(t,s+h)$ and $P(t,s)$ when $h \to 0$:

$$f(t,s) = \frac{1}{P(t,s)} \lim_{h \to 0} \frac{P(t,s) - P(t,s+h)}{h} = \frac{1}{P(t,s)} \frac{-\partial P(t,s)}{\partial s} = \frac{-\partial \ln P(t,s)}{\partial s} \tag{2.}$$

The forward as a function of the spot rates, replacing (1.) in (2.), is:

$$f(t,s) = \frac{\partial [R(t,s)(s-t)]}{\partial s} = R + \frac{\partial R(t,s)}{\partial s}(s-t) \tag{3.}$$

where we can see that forward rates can be calculated as the spot rate plus the slope of the curve yield amplified with the maturity time.

These three different ways to define the term structure are equivalents and can be used alternatively.

The determination of the term structure of interest rates is not an easy task because does not exist pure discount bonds for every time. The majority of government and corporate bonds pay an annual, or semi-annual, coupon. So, the term structure must be estimated from prices of coupon bonds in a way that any maturity time could be calculated. To accomplish this matter, the bond market price P_i is supposed equal to the sum of the current value of all the payments Q_{ij} of the bond:

$$P_i = \sum_{j=1}^{T_i} Q_{ij} D(t_j) \tag{4.}$$

where the continuous time function $D(t_j)$ is the usual notation for the theoretical discount function for pure bonds of maturity t_j, then $D(t_j) = P(0,t_j)$.

If $G(t)$ is a generic approximation function for the discount function, the estimation problem is to solve:

$$P_i = \sum_{j=1}^{T_i} Q_{ij} G(t_j) + \varepsilon_i \tag{5.}$$

where ε_i is a disturbance term which must be added because the price-current value equality is not exact (there are transactions costs, taxes and other factors) and an approximation function is used.

However, not all functions are suitable to approximate the term structure of interest rates. The proposed functions must fulfil at least three requirements related to the properties of the discount function:

$D(0) = 1$. One monetary unit today has one monetary unit value.

$\lim_{t \to \infty} D(t) = 0$. An infinitely distant payment has no current value.

$D(t_1) > D(t_2) > ... > D(t_n)$ for $t_1 < t_2 < ... < t_n$. The more distant the payment is, the less current value it has.

To sum up, the discount function is a non-increasing monotone[1] positive function which maximum value is one. And attending the equations (1.) and (2.), all the interest rates which can be deduced of discount function, spot or forward rates, must be always positives.

McCulloch (1971, 1975) was the first to try to solve the estimation problem (5.) in order to deduce the term structure of interest rates. He proposed to approximate the discount function using a linear combination of m cubic polynomial splines[2]:

$$G(t) = 1 + \sum_{j=1}^{m} \beta_j g_j(t) \tag{6.}$$

The used splines $g_j(t)$ are:

$$g_1(t) = \begin{cases} \dfrac{t^2}{2} - \dfrac{t^3}{6d_2}; & d_1 \le t \le d_2 \\ d_2\left(\dfrac{3t - d_2}{6}\right); & t \ge d_2 \end{cases}$$

$$g_j(t) = \begin{cases} 0; & 0 \le t \le d_{j\text{-}1} \\ \dfrac{(t - d_{j-1})^3}{6h_{j-1}}; & d_{j\text{-}1} \le t \le d_j \\ \dfrac{h_{j-1}^2}{6} + \dfrac{h_{j-1}(t - d_j) + (t - d_j)^2}{2} - \dfrac{(t - d_j)^3}{6h_j}; & d_j \le t \le d_{j+1} \\ (d_{j+1} - d_{j-1})\left(\dfrac{2d_{j+1} - d_j - d_{j-1}}{6} + \dfrac{t - d_{j+1}}{2}\right); & t \ge d_{j+1} \end{cases} \qquad j=2,..., m\text{-}1$$

$$g_m(t) = t \qquad 0 \le t \le T_n$$

where $h_j = d_{j+1} - d_j$ and d_j are the spline nodes.

Indeed observe that for McCulloch's model we have the following properties:

[1] Its first derivative is always negative.

[2] The number of splines m and the nodes location depend on the sample size. See McCulloch (1971).

$$\lim_{t\to\infty} D(t) = \begin{cases} \infty, \text{if } \beta_m + \sum_{j=1}^{m-1} \beta_j (d_{j+1} - d_{j-1}) > 0 \\ -\infty, \text{if } \beta_m + \sum_{j=1}^{m-1} \beta_j (d_{j+1} - d_{j-1}) < 0 \\ \sum_{j=1}^{m-1} (d_{j+1} - d_{j-1})(-d_{j+1} - d_j - d_{j-1})/6 < 0, \text{ if } \beta_m + \sum_{j=1}^{m-1} \beta_j (d_{j+1} - d_{j-1}) = 0 \end{cases}$$

This means that the model cannot ensure that the discount function decreases monotonically or that it has always a positive value. The consequences of this misspecification on the forward rates are obvious: the forward rates can take negative values.

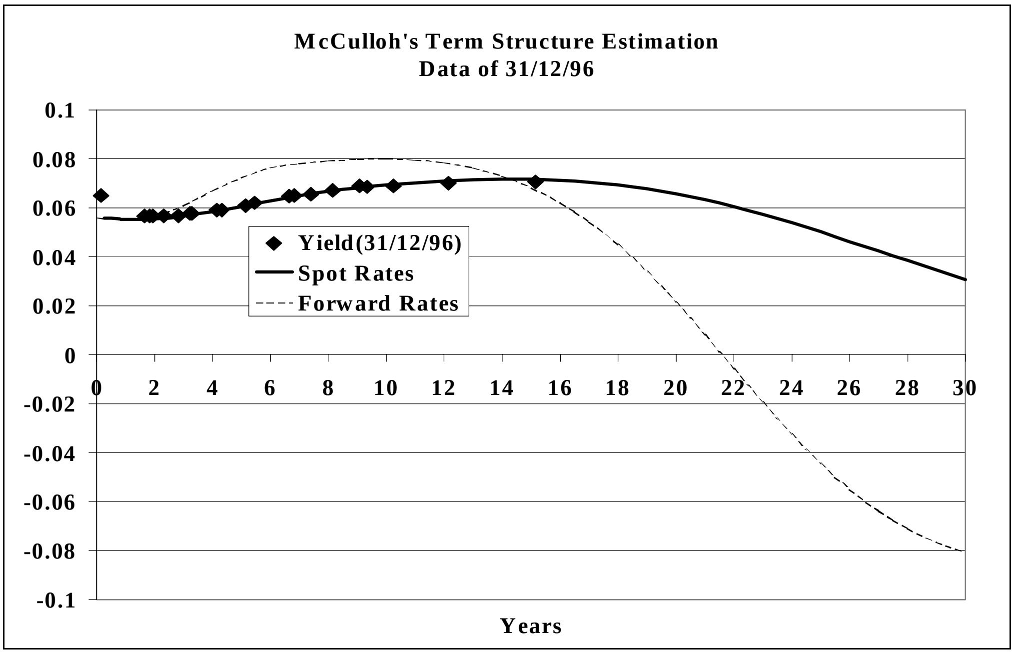

Figure 1 McCulloch's Term Structure Estimation on 1996, December 31th

This problem is shown in figure 1, where we plot the estimated spot curve, the yield to maturity (YTM) and the resulting forward curve for a random selected sample of Spanish Treasury Bonds. The slope of the forward curve for maturities beyond 15 years is very steep, making forward rates at their longest maturities taking negative values, so we must reject this estimation.

Therefore, we propose an alternative way for overall adjustment. We look for a collection of functions that can accomplish the basic properties of the discount function. This family of functions can be the Legendre's polynomials. It has been tested on the Spanish Treasury Bonds Market, during 6 years, on daily samples, with better results than McCulloch's model (Morini, 1998). Then, we can estimate the discount function as a linear combination of Legendre's polynomials, usually, a polynomial of degree 3:

$$G_3(x) = \frac{1}{2}(1-x) + \beta_2 \frac{3}{2}\left[x^2 - 1\right] + \beta_3 \frac{5}{2}\left[x^3 - x\right] \tag{7.}$$

These polynomials are defined for $x \in [-1,1]$, so we must do the change $x = 1 - 2e^{-\alpha t}$ where t is the maturity time, $t \in [0,+\infty]$, and α is the yield of the longest maturity bond.

This functional form is better than the former because $\lim_{t\to\infty} D(t) = 0$, even though it is less flexible than the McCulloch's model, mainly in the short term. We can see this in figure 2.

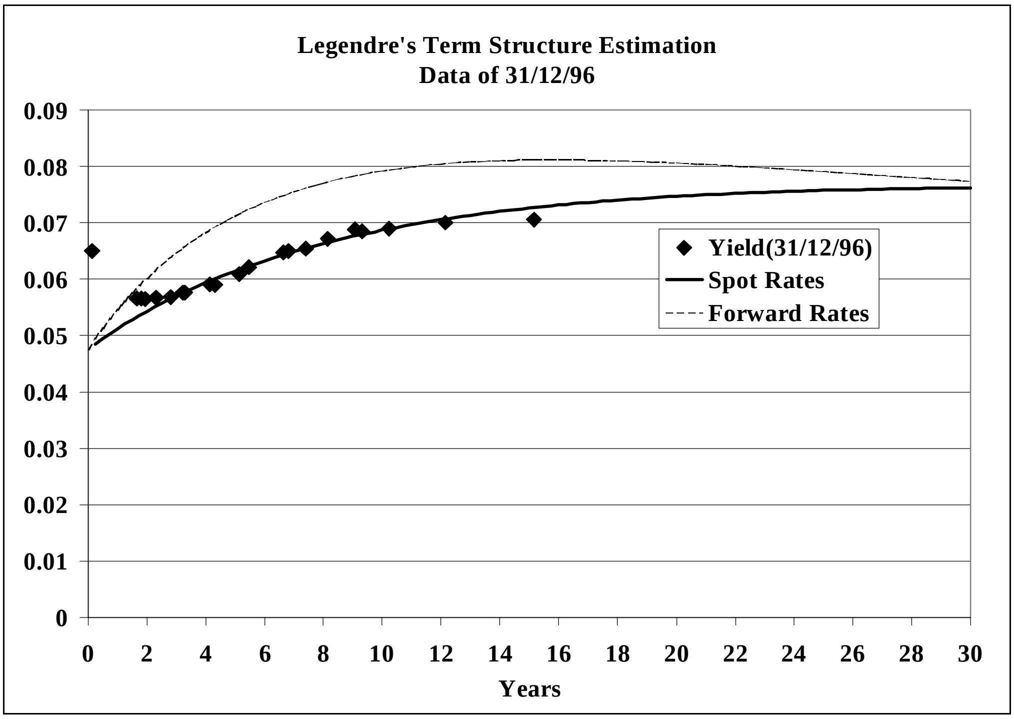

Figure 2 Legendre's Term Structure Estimation on 1996, December 31th

3. Movements of the term structure of interest rates: consistent models

The term structure of interest rates varies each day and these oscillations change the price of financial assets, which also varies continuously. The more affected assets are the derivatives whose pay-off depend on the future value of the interest rates or on the future price of the bonds. So, in order to price accurately these assets, we must model the movements of the term structure of interest rates. The dynamic of the interest rates' term structure can be approached with a different number of continuous models. All of them come from stochastic differential continuous-time equations.

Many authors have proposed different models, which can be classified in many ways. The simplest consist in assembling them as single factor models or as multiple factor models (usually two factors). The features of the models with one factor are derived fron the no - arbitrage assumption of the short term interest rate define completely the yield curve (Strickland, 1996). The short interest rate (typically one-month spot) encloses all the information needed to know the evolution of the interest rate term structure. This is a strong hypothesis and makes that the model does not fit the initial term structure, only the short interest rates. They are named *non-consistent models*.

So, we can distinguish between non-consistent and consistent models. In this paper, we are interested in analyzing *consistent models*. The main characteristic of the term structure consistent models is that they can reproduce, at least, the actual value of the spot interest rates and the initial volatility rates. This sort of models takes explicitly into account the initial term structure interest rates. In this sense the models that we use to estimate the initial term structure are very important.

In order to verify the importance of the initial curve yield estimation, we analyze the behavior of two classical consistent models: Ho and Lee (1986) and Hull and White (1990). Both of them are often used by the practitioners and have close form solution for bond prices[3].

3.1. Ho and Lee (1986)

The Ho and Lee (1986) model sets the continuous time limit of the short rate as:

$$dr = \theta(t)dt + \sigma dz \tag{8.}$$

where dz represents an increment in a Wiener process during a small increment of time dt. $\theta(t)$ is a time-dependent drift reflecting the slope of the initial forward rate curve and σ the volatility parameter of the short-rate process:

$$\theta(t) = \frac{\partial f(t)}{\partial t} + \sigma^2 t \tag{9.}$$

It is this time-dependent function that allows the model to return the observed bond prices, where the partial derivative denotes the slope of the initial forward curve at maturity *t*. The model is single factor and presents a drawback: the interest rates can become negative with a positive probability.

The analytical expression of the bond price at future time $T \geq t$ as a function of the parameters of the process and the short rate at time *T*, $r(T)$, is:

$$P(T,s) = A(T,s)e^{-B(T,s)r(T)} \tag{10.}$$

where $B(T,s) = (s - T)$ and:

[3] A survey about different dynamic models could be found in Vetzal (1994) or Stricklan (1996).

$$\ln A(T,s) = \ln \frac{P(t,s)}{P(t,T)} - B(T,s)\frac{\partial \ln P(t,T)}{\partial T} - \frac{1}{2}\sigma^2 (T-t)B(T,s)^2 \qquad (11.)$$

Using the relations (1.) and (2.) we can rewrite (11.) as:

$$\ln A(T,s) = (T-t)R(t,T) - (s-t)R(t,s) + B(T,s)\, f(t,T) - \frac{1}{2}\sigma^2 (T-t)B(T,s)^2 \qquad (12.)$$

This model fits the dynamics of the whole term structure in a way that is automatically consistent with the initial (observed) term structure of interest rates. Effectively, the shape of the initial curve yield can be obtained choosing $t = T = 0$:

$$\ln A(0,s) = -s\,R(0,s) + s\, f(0,0) = -s\,R(0,s) + s\,r(0) \qquad (13.)$$

where we have used that $f(t,0) = R(t,0) = r(t)$.

Replacing (13.) in (10.), we obtain:

$$P(0,s) = e^{A(0,s)B(0,s)r(0)} = e^{-R(0,s)s+sr(0)}e^{-sr(0)} = e^{-R(0,s)s} \qquad (14.)$$

as expected.

Therefore, this model is consistent with the initial spot rates. And the input data are the spot interest and the forward interest rate functions (equation 12.). So, it is very important to get a fine determination of the initial curve yield.

3.2. Hull and White (1990)

Later in 1990, Hull and White proposed to fit the initial yield curve, a new model that is an extension of the mean-reversion Vasicek (1977) model. The short rate process becomes:

$$dr = [\theta(t) - \alpha r]dt + \sigma dz \qquad (15.)$$

where α is the rate of the mean reversion of the short rate and $\theta(t)$ is a necessary time dependent function to match the initial forward curve:

$$\theta(t) = \frac{\partial f(t)}{\partial t} + \alpha f(t) + \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}) \qquad (16.)$$

In this model, the bond price at future time T is a function of the short rate $r(T)$ and the parameters of the process α and σ:

$$P(T,s) = A(T,s)e^{-B(T,s)r(T)} \qquad (17.)$$

where $B(T,s) = \frac{1}{\alpha}(1 - e^{-\alpha(s-T)})$ and:

$$\ln A(T,s) = \ln\frac{P(t,s)}{P(t,T)} - B(T,s)\frac{\partial \ln P(t,T)}{\partial T} - \frac{1}{4\alpha^3}\sigma^2\left(e^{-\alpha(s-t)} - e^{-\alpha(T-t)}\right)^2\left(e^{2\alpha(T-t)} - 1\right) \tag{18.}$$

or:

$$\begin{aligned}\ln A(T,s) = (T-t)R(t,T) - (s-t)R(t,s) + B(T,s)f(t,T) \\ - \frac{1}{4\alpha^3}\sigma^2\left(e^{-\alpha(s-t)} - e^{-\alpha(T-t)}\right)^2\left(e^{2\alpha(T-t)} - 1\right)\end{aligned} \tag{19.}$$

This model, as the previous one, also fits the initial values of the term structure of interest rates. In fact, when setting $t = T = 0$ at (19.), we obtain the expected result:

$$\ln A(0,s) = -s\,R(0,s) + B(0,s)f(0,0) = -s\,R(0,s) + B(0,s)r(0) \tag{20.}$$

Replacing (20.) in (13.):

$$P(0,s) = e^{-R(0,s)s+B(0,s)r(0)}e^{-B(0,s)r(0)} = e^{-R(0,s)s} \tag{21.}$$

as expected.

As before, we see that the input data are also the spot rates function and the forward rates function (equation 19.), so the accuracy of the input function is very important to the quality of the final results.

4. Testing different input estimations into consistent models

It was shown in the previous section that the results of the term structure estimation models are the inputs of consistent dynamic models. In this section we are going to evaluate the importance of selecting a good estimation model in order to get consistent results about interest rates movements and bond pricing. In this sense, Hull (1997) found that an 1% error in the pricing of a bond could generate an option pricing error up to 25%.

To illustrate the problem, we use the Spanish Bond Market Prices for 1996, 31th December and estimate the term structure of interest rates using the McCulloch's model and Legendre's polynomials. We take these estimations as the inputs for Ho and Lee and Hull and White models to calculate the future value (1 year ahead) of different pure discount bonds whose maturity date is within 1, 2, .., 30 years. That is: t=0, T=1 year and s=1, 2, ..., 30 years.

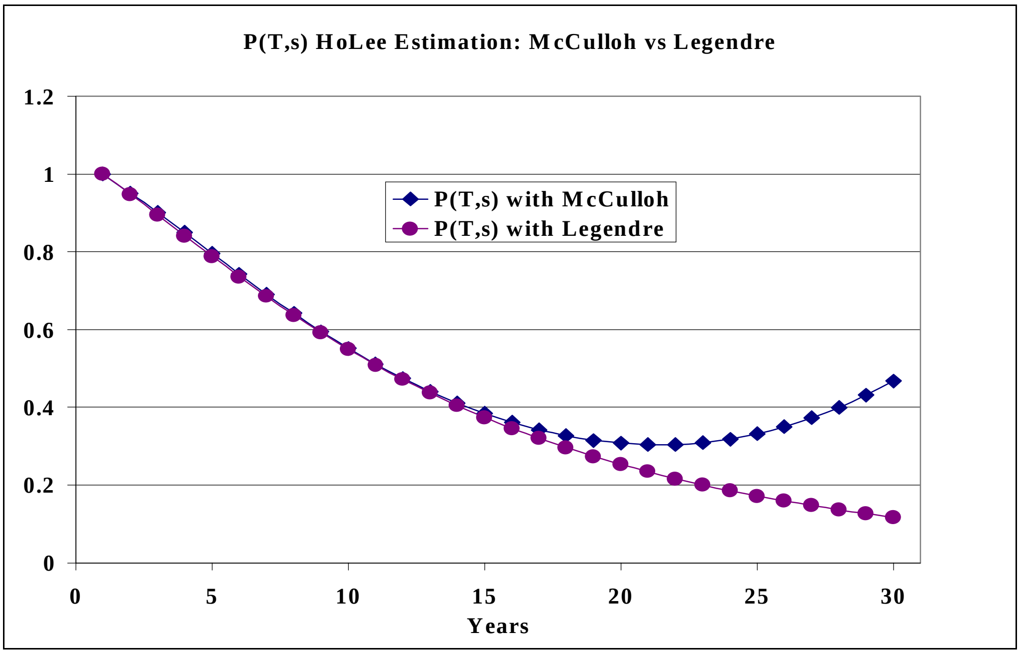

Figure 3. Future pure discount bond prices using Ho and Lee model and different initial term structure estimations.

We assume that the short rate in one year, $r(1)$, is 5 per cent. Also, we need set the parameters of the models. Typical values of the consulted references are short rate volatility $\sigma = 0.01$, and mean reversion parameter $\alpha = 0.10$.

We estimate two models, Ho and Lee and Hull and White, in order of generalize the conclusions of this work. The results are presented in figures 3 and 4. In figure 3 we calculate the future value (1 year ahead) of pure discount bonds $P(T,s)$ with Ho and Lee model. We can clearly see that, when McCulloch's estimation is used, the price of pure discount bonds is not decreasing in time, so is erroneous. On the other hand, when the term structure is estimated by Legendre's polynomials, it presents the right behaviour. Indeed, we can measure a relative difference of 2.8% between future pure discount bonds prices at s=15 years, depending on the initial curve yield estimations, McCulloh or Legendre's ones.

The same result has been found for the Hull and White future values of pure discount bonds, presented on figure 4. When McCulloch's estimation is used, the price of pure discount bonds is erroneous at longest maturities, but not Legendre's estimation. And, it exists a relative difference of 2.4% between future pure discount bonds prices at s=15 years, depending on the initial curve yield estimations, that it's an important difference.

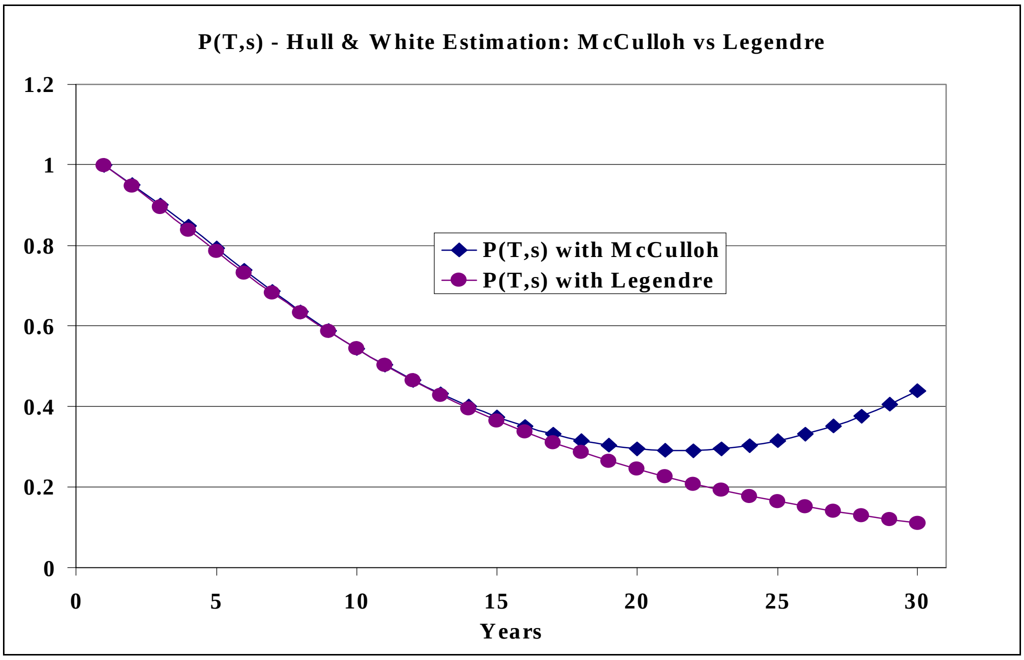

Figure 4. Future pure discount bond prices using Hull and White model and different initial term structure estimations.

5. Conclusions

In this paper we compared McCulloch's estimation model with a linear combination of Legendre's polynomials in the framework of the consistent dynamic models. The results show that the researchers must pay special attention to the quality of the term structure estimation models because their results are used as input in the consistent dynamic models.

An accuracy determination of initial curve yield is very important. A little deviation or bad specification in spot rates can produce an enormous error in the valuation of bonds, derivatives or whatever financial instrument linked with interest rates. Moreover, the forward rates are necessary in most of the term structure consistent models, and these values are very sensible to little differences between the spot rates.

This study does not invalidate the McCulloch's model but it just reveals the problems of applying this model in the long term where it yiels negative forward rates. In other hand, Legendre's polynomials are not the final solution because they could have estimation problems in the short term due to their low flexibility.

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