

AN EQUILIBRIUM ANALYSIS OF THE SIMULTANEOUS ASCENDING AUCTION

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Abstract

We provide a general Bayes-Nash equilibrium analysis of the simultaneous ascending auction (SAA) when local bidders interested in a single item compete against global bidders with super-additive valuations for combinations of items. We first consider an environment where each local bidder values only a specific item, e.g. when a local bidder wants the license for the region where it has monopoly power. We show that bidders' drop-out levels are constrained efficient in the SAA, but unlike the Vickrey-Clarke-Groves (VCG) mechanism the SAA is not ex post efficient. Furthermore, due to the exposure problem, its revenue may be less than that of the VCG mechanism and may be decreasing in the number of bidders. Our techniques extend to large auctions with many items, and we show that the SAA and VCG mechanisms are efficiency and revenue equivalent in the limit. In contrast, in the sealed-bid second-price auction (SSA), the exposure problem causes inefficiencies even with many items. We also consider an environment where items are substitutes for local bidders so that price arbitrage will occur, as observed in many spectrum auctions. This environment, which combines substitutes and complements, results in an aggravated exposure problem and the SAA performs dramatically worse as a result. Finally, we consider an environment with medium-sized global bidders who demand fewer than all items. We show that global bidders divide the market at low prices when market sharing is feasible while they engage in mutually destructive bidding when there is a fitting problem. In both cases, the SAA results in inefficient outcomes.

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1. Introduction

In recent years, governments around the world have employed auctions to award licenses for the rights to operate in certain markets. The spectrum auctions conducted by the US Federal Communications Commission (FCC) provide a particularly prominent example. In the FCC auctions, telecom firms compete for blocks of frequencies (typically on the order of 10-20MHz) defined over certain geographic areas.¹ The format pioneered by the FCC is the simultaneous ascending auction (SAA), which is a dynamic, multi-round format in which the items are put up for sale simultaneously and the auction closes only when bidding on all items has stopped. The SAA has become the standard to conduct large-scale, large-stakes spectrum auctions and has generated close to \$80 billion for the US Treasury and hundreds of billions worldwide.

An important property of the SAA is that when items are substitutes and bidding is “straightforward,” i.e. in each round of the auction bidders place minimum acceptable bids on those licenses that provide the highest current profits, then prices converge to competitive equilibrium prices and a fully efficient outcome results (Milgrom, 2000; Gul and Stacchetti, 2000). However, in many of the FCC auctions there are synergies between licenses for adjacent geographic regions, and bidders’ values for combinations of licenses exceed the sum of individual license values. For example, the bid regressions reported by Ausubel, Cramton, McAfee, and McMillan (1997) show that the highest losing bid on a license is higher if the bidder who placed the bid has won or eventually wins a license. Bajari and Fox (2009) apply a structural econometrics model to data from FCC auction #5 and find evidence for substantial value complementarities: they estimate that the value of a nationwide package is 69% more than the sum of underlying values.² Value complementarities were considered even more important in the recently conducted FCC auction #73, where potential entrants, e.g. Google, competed against established incumbents such as Verizon and AT&T for highly valuable 700MHz spectrum.³ Most experts believed that an entrant could have a viable business plan only if it would acquire a “national footprint,” i.e. a set of licenses covering the entire United States.

In this paper, we consider an environment where one or more global bidders (entrants) have super-additive values for the licenses, i.e. for global bidders licenses are *complements* rather than substitutes. For this environment an often cited problem of the item-by-item competition that occurs under SAA is that global bidders face an *exposure problem* – when competing

¹The different frequency bands that have been put up for sale in the 73 FCC auctions since 1994 accommodate different usages, including wireless and cellular phone applications, mobile television broadcasting, and air-to-ground communication. See http://wireless.fcc.gov/auctions/default.htm?job=auctions_all.

²FCC auction #5 (also known as the “C-block auction”) was conducted in 1995 and generated over \$10 billion in revenue.

³FCC auction #73 was conducted in 2008 and generated a record \$19 billion in revenue. It was the first combinatorial auction conducted by the FCC, based on hierarchically structured packages (Rothkopf, Pekeč, and Harstad, 1998) and a novel pricing rule (Goeree and Holt, 2010).

aggressively for a package, global bidders may incur a loss when winning only an inferior subset. Foreseeing the possibility of being “exposed,” global bidders may decide to bid cautiously and drop out early, which could adversely affect the auction’s revenue and efficiency.⁴ For example, the counterfactual experiments conducted by Bajari and Fox (2009) demonstrate that in FCC auction #5 only 50% of the total available surplus was captured.⁵ In addition, a substantial body of laboratory evidence documents the negative impact of the exposure problem on the SAA’s performance (see, e.g., Brunner et al., 2010, and references therein).

Despite the potential shortcomings of the item-by-item competition underlying SAA, it has been the preferred choice for most spectrum auctions. Alternatives allowing for package bids were either considered too complex or thought to be prone to “free riding” (Milgrom, 2000).⁶ Furthermore, the familiar Vickrey-Clarke-Groves (VCG) mechanism, which guarantees full efficiency even in the presence of value complementarities, is generally dismissed because of its perverse revenue properties. In particular, the VCG mechanism can lead to non-core outcomes that result in high bidder profits and low seller revenue. Moreover, seller revenue can decrease when more bidders participate. The following three-bidder, two-item example provided by Ausubel and Milgrom (2006) illustrates these shortcomings. Suppose local bidder 1 is interested only in item A , local bidder 2 is interested only in item B , the global bidder 3 is interested only in the package AB , and all bidders’ values (for individual items or the package) are \$1 billion. The VCG mechanism assigns the items efficiently to bidders 1 and 2, but at zero prices!⁷ Besides generating the lowest possible revenue, this outcome is outside the core as the seller and global bidder can form a blocking coalition. Moreover, excluding one of the local bidders, raises the seller’s revenue to \$1 billion. These perverse revenue properties, shown here in a complete-information setting, carry over to the Bayesian framework studied in this paper where bidders’ values are private information (see Example 1 in Section 2.1).

One goal of this paper is to compare SAA and VCG in settings with value complementarities. We first consider an environment where global bidders with super-additive valuations compete with local bidders that each value a different item. For instance, local bidders may be interested only in the license for the region where they have local monopoly power. We characterize

⁴Milgrom (2000) argues that the different per-unit-of-bandwidth prices observed for small and large licenses in the Dutch DCS-1800 auction reflect the exposure problem. A similar observation applies to the recent FCC auction #66, where 12 large (F-band) licenses providing 20MHz of nationwide coverage sold for \$4.2 billion while 734 small (A-band) licenses also providing 20MHz of nationwide coverage went for \$2.3 billion.

⁵Bajari and Fox (2009) show that surplus could have been doubled had the FCC offered large regional licenses or a nationwide package in addition to individual licenses.

⁶In package auctions, small bidders who drop out early (“free ride”) may earn windfall profits when other small bidders remain active and outbid the global bidders. After all, a small bidder’s concern is simply whether *as a group* they meet the threshold set by a global bidder’s package bid. Of course, if all local bidders free ride this threshold may never be met, with adverse effects for the auction’s revenue and efficiency – this is known as the *threshold problem*.

⁷Ausubel and Milgrom (2006) develop this example further in a theorem that shows that bidders’ Vickrey payoffs are the highest payoffs over all points in the core.

equilibrium bidding in the SAA, which allows us to determine the impact of the exposure problem on efficiency and revenue. Our setup is general in that we allow for arbitrary numbers of local and global bidders, arbitrary distributions of local and global bidders' values, and a general convex valuation function to capture global bidders' value complementarities.

We show that individual and socially optimal incentives are aligned in the SAA in that bidders' drop-out levels are *constrained* efficient. In particular, bidders' drop-out levels maximize *expected* welfare, which takes into account information revealed by past drop-out levels as well as prior information about active bidders' values. Due to the residual uncertainty regarding the remaining bidders' values, the SAA is not *ex post* efficient, unlike the VCG mechanism. We show that the efficiency gain of the VCG mechanism does not benefit the seller, however, but merely results in higher profits for the global bidders.

The SAA shares the poor revenue-generating features of the VCG mechanism. Due to the exposure problem the SAA can result in non-core outcomes characterized by low seller revenues – indeed, seller revenue in the SAA may be *less* than in the VCG mechanism. In addition, seller revenue in the SAA can be declining in the number of bidders. Finally, the similarities between the SAA and VCG mechanisms in this environment become even stronger as the number of items grows. The two mechanisms are revenue and efficiency equivalent in the limit. The full efficiency obtained under the SAA is not because the exposure problem is absent in large auctions. We demonstrate that in a sealed-bid version of the SAA, the exposure problem causes global bidders to win fewer items than is socially optimal, even in large auctions.

We also consider the case where local bidders want at most one item and value all items the same, i.e. the items are perfect substitutes for local bidders. For this environment, which combines complements for the global bidders with substitutes for the local bidders, we show that the exposure problem is much worse and the performance of the SAA suffers as a result. In particular, the global bidder's drop-out levels are no longer constrained efficient. Indeed, when the number of items grow large, it may be optimal for the global bidder to drop out immediately with adverse consequences for the efficiency and revenue of the SAA.

Finally, we study a setting with two “medium-sized” global bidders. When the global bidders each need less than half the items, they compete head-to-head only at very low prices after which they compete only with the local bidders. In other words, the global bidders follow a strategy of mutual forbearance and divide the market at low prices. In contrast, when the global bidders each need more than half the items, the equilibrium involves mutually destructive bidding: global bidders may bid above their values and even the winning bidder may incur a loss.⁸ We show that the SAA is generally inefficient whether or not market sharing is feasible. Compared to the VCG mechanism, local bidders are worse off while global bidders benefit.

⁸For a complete-information environment, Bykowsky, Cull, and Ledyard (2000) have argued the possibility of such behavior using the concept of a “local Nash equilibrium.”

Our findings contrast with those for the pure-substitutes environment typically assumed to analyze the SAA (e.g. Milgrom, 2000). Since value complementarities are the rule rather than the exception, our results reinforce the interest of policy makers in further improving the design of auctions to award spectrum licenses, including the possibility of allowing for package bids.

1.1. Related Literature

Auctions in which bidders have synergistic values have often been analyzed within a complete-information setting, see, for instance, Szentes and Rosenthal (2003a,b). There are relatively few theoretical papers that apply the standard Bayesian framework of incomplete information. An early exception is Krishna and Rosenthal (1996) who study the simultaneous sealed-bid second-price auction (SSA). Similar to our bidding environment, local bidders in their setup are interested in only one object while global bidders are interested in multiple objects for which they have synergistic values. Krishna and Rosenthal derive an explicit solution for the case of two items and show how it varies with the synergy level. They also discuss the extension to more than two items and provide a numerical comparison of revenue in alternative formats. Other papers that study the SSA include Rosenthal and Wang (1996), who allow for common values and partially overlapping bidder interests, and a more recent paper by Chernomaz and Levin (2008) who use theory and experiments to analyze the SSA and a package bidding variant when local bidders have identical values.

Ascending formats have been analyzed either assuming a clock price that rises in response to excess demand or assuming that bidders can name their own bids (i.e. submit any bid they want). In the latter category, Brusco and Lopomo (2002) demonstrate the possibility of collusive demand-reduction equilibria in the SAA. They find that increasing the number of bidders and objects narrows the scope for collusion. Brusco and Lopomo (2009) analyze the effects of budget constraints. Zheng (2008) shows that jump bidding may serve as a signaling device to alleviate the inefficiencies that result from the exposure problem. Albano, Germano, and Lovo (2006) analyze a (“Japanese style”) clock version of the ascending auction (as we do in this paper) for a setting with only two items. They note the equivalence between the SAA and a “survival auction” and point out that many of the collusive or signaling equilibria that occur when bidders can name their bids do not arise for the clock variant of the SAA.

1.2. Organization

This paper is organized as follows. In Section 2 we provide an equilibrium analysis of the SAA when each local bidder is interested in a different item. We start by considering a single global bidder (Section 2.1) and prove social optimality of the global bidder’s drop-out choices (Section 2.2). In Section 2.3 we extend the result to an arbitrary number of global bidders.

In Section 3 we establish the equivalence of the SAA and VCG mechanisms as the number of items grows large and quantify their advantage over a second-price sealed-bid format (SSA). In Section 4 we demonstrate the poor performance of the SAA in an environment that combines substitutes and complements (Section 4.1) and in an environment with two medium-sized global bidders (Sections 4.2 and 4.3). Section 5 concludes and Appendix A contains all proofs.

2. The Simultaneous Ascending Auction

Consider an environment with $n \geq 1$ local bidders and $K \geq 1$ global bidders who compete for n items labeled $1, \dots, n$. Local bidder i is interested only in acquiring item i , for which she has value v_i . The local bidders' values are identically and independently distributed according to $F(\cdot)$, with support $[0,1]$.⁹ Global bidder j 's value for winning k items is $\alpha(k)V^j$, where the V^j are identically and independently distributed according to $G(\cdot)$ with support $[0, n]$,¹⁰ and $\alpha(k)$ is increasing in k with $\alpha(0) = 0$ and $\alpha(n) = 1$. We define the “marginal values” $V_k^j = (\alpha(n+1-k) - \alpha(n-k))V^j$ for $k = 1, \dots, n$ so that global bidder j 's marginal value of the first item is V_n^j , of the second item is V_{n-1}^j, \dots , and of the n -th item is V_1^j .

Assumption 1 (complementarities). *For global bidder $j = 1, \dots, K$, the marginal values form a non-decreasing sequence $V_n^j \leq V_{n-1}^j \leq \dots \leq V_1^j$. Moreover, we say that*

- (i) *there are “no complementarities” when $V_n^j = \dots = V_1^j = V^j/n$.*
- (ii) *there are “complementarities” when $V_n^j < \dots < V_2^j < V_1^j$.*
- (iii) *there are “extreme complementarities” when $V_n^j = \dots = V_2^j = 0$ and $V_1^j = V^j$.*

The simultaneous ascending auction (SAA) is modeled using n price clocks that tick upward (at equal and constant pace) when two or more bidders accept the current price levels. If at most one bidder accepts the new price then the price clock for that item stops and the bidder who accepted the higher price is allocated the item at the last price accepted by other bidders (ties are resolved randomly). Under this mechanism, local bidders have a dominant strategy to bid up to their values. A global bidder's strategy is complicated by the fact that when competing aggressively for a package, the global bidder may suffer a loss when she is able to win only an inferior subset. Foreseeing the possibility of being “exposed” and incurring a loss, the global bidder may decide to bid cautiously and drop out early, which could adversely affect the auction's revenue and efficiency — this is known as the *exposure problem*.

⁹The assumption of a $[0,1]$ support is without loss of generality since we can rescale local bidders' values.

¹⁰We restrict the support of the global bidders' values to $[0, n]$ because global bidders with values higher than n face no exposure problem and always win all items. The assumption that global bidders' values are identically distributed can be relaxed see Remark 1 below.

2.1. Single Global Bidder

We write $B_k^K(V)$ to denote a global bidder's bidding function when K global bidders and k out of n local bidders are active. A global bidder's optimal drop-out level is then given by $\max\{p, B_k^K(V)\}$, where p denotes the current price level. For ease of notation we simply refer to $B_k^K(V)$ as the optimal drop-out level, where it is implicitly understood that the global bidder drops out immediately when $B_k^K(V) < p$. It will prove useful to introduce the notation $F_k(v_k|p) = 1 - ((1 - F(v_k))/(1 - F(p)))^k$, which is the conditional probability that the minimum of k active local bidders' values is less than v_k given that the minimum is no less than p . In this section we focus on the case of a single global bidder ($K = 1$).

Suppose first that there is only one item for sale ($k = 1$). If the current price level is p and the global bidder chooses a drop-out price level $B_1^1(V)$, her expected profits are

$$\Pi_1^1(V, p) = \int_p^{B_1^1(V)} (V_1 - v_1) dF_1(v_1|p),$$

with v_1 the local bidder's value. The integrand $\pi_1^1(V, v_1) = V_1 - v_1$ is the global bidder's profit if the local bidder drops out at price v_1 . Clearly, the global bidder's expected profit is maximized by choosing the drop-out price $B_1^1(V)$ such that $\pi_1^1(V, B_1^1(V)) = 0$, or $B_1^1(V) = V_1$.

Next, consider the case $k = 2$. Suppose the current price level is p and the global bidder chooses a drop-out price level $B_2^1(V)$ (for both items), her expected profit is non-trivial only when the local bidder with the lower value drops out before $B_2^1(V)$. Once a local bidder drops out, the global bidder faces competition only in a single market and she is willing to bid up to V_1 in this market. The reason is that her profit for the item on which bidding stopped is sunk (i.e. independent of whether or not she wins an additional item). The global bidder's expected profit can be written as

$$\Pi_2^1(V, p) = \int_p^{B_2^1(V)} \left\{ \int_{v_2}^{V_1} (V_1 - v_1) dF_1(v_1|v_2) + (V_2 - v_2) \right\} dF_2(v_2|p),$$

where v_2 (v_1) denotes the lower (higher) of the local bidders' values. The integrand $\pi_2^1(V, v_2) = \int_{v_2}^{V_1} (V_1 - v_1) dF_1(v_1|v_2) + (V_2 - v_2)$ is the global bidder's expected profit conditional on the local bidder with the lower value dropping out at v_2 . The first term arises when the global bidder wins the remaining item, i.e. when $v_1 \geq v_2$ is less than V_1 (since the global bidder bids up to V_1 for the remaining item) in which case the global bidder wins the additional item and pays v_1 for it. The second term indicates that the global bidder profits $V_2 - v_2$ from the item for which bidding stopped first, irrespective of whether she wins the additional item.

Again, the global bidder's optimal drop-out level follows from $\pi_2^1(V, B_2^1(V)) = 0$, which yields $B_2^1(V) = V_2 + \Pi_1^1(V, B_2^1(V))$. Note that the global's profit can be recursively expressed using

the profit for the single local-bidder case: $\Pi_2^1(V, p) = \int_p^{B_2^1(V)} \{\Pi_1^1(V, v_2) + (V_2 - v_2)\} dF_2(v_2|p)$. This recursive relation can be generalized to the case of more than two items.

Proposition 1. *The global bidder's optimal drop-out level solves $B_k^1(V) = V_k + \Pi_{k-1}^1(V, B_k^1(V))$, where the payoffs satisfy the recursive relation*

$$\Pi_k^1(V, p) = \int_p^{B_k^1(V)} \{\Pi_{k-1}^1(V, v_k) + (V_k - v_k)\} dF_k(v_k|p), \quad (2.1)$$

with $\Pi_0^1(V, p) = 0$.

The proposition implies a set of fixed-point equations from which the optimal bids can be solved recursively: $B_1^1(V) = V_1$, $B_2^1(V) = V_2 + \int_{B_2^1(V)}^{B_1^1(V)} (V_1 - v_1) dF_1(v_1|B_2^1(V))$, etc. The intuition behind the bidding functions in Proposition 1 stems from a familiar break-even condition: when a license is marginally won at price $B_n^1(V)$, its value plus the expected payoffs from continuing (knowing all remaining local bidders' values exceed $B_n^1(V)$) must balance this cost.

Example 1. To illustrate, suppose $n = 2$ and the local bidders' values are uniformly distributed. We have $B_1^1(V) = V_1$ and a simple calculation shows that

$$B_2^1(V) = \frac{1}{3} \left(1 + V_1 + V_2 - \sqrt{(2 - V_1 - V_2)^2 - 3(1 - \min(1, V_1))^2} \right) \quad (2.2)$$

For the case of extreme complementarities ($V_2 = 0$, $V_1 = V$), these bidding functions are illustrated by the two left-most lines in the left panel of Figure 1 (the right panel will be discussed in Section 2.3). The left panel also shows the optimal bidding functions $B_n^1(V)$ for higher values of n , which can be used to illustrate some of the perverse revenue properties of the SAA. For instance, revenues of the SAA are (0.33, 0.27, 0.087) with $n = 1, 2, 3$ local bidders respectively. The corresponding revenue numbers for the VCG mechanism are $(\frac{1}{3}, \frac{1}{4}, \frac{1}{10})$. To summarize, the revenue of the SAA can fall as more bidders enter the auction and can be *lower* than that of the VCG mechanism.

2.2. Social Optimality

The optimal drop-out levels of the global bidder shown in the left panel of Figure 1 illustrate the effects of the exposure problem in equilibrium. Consider, for instance, the case of five local bidders and suppose the global bidder is equally strong in expectation, i.e. the global bidder's value for the package is 2.5. When all five local bidders are active, the global bidder drops out when the price for each item is 0.2 (see the lowest line in the left panel of Figure 1), which means that the global bidder drops out at 40% of the package value! This does not necessarily

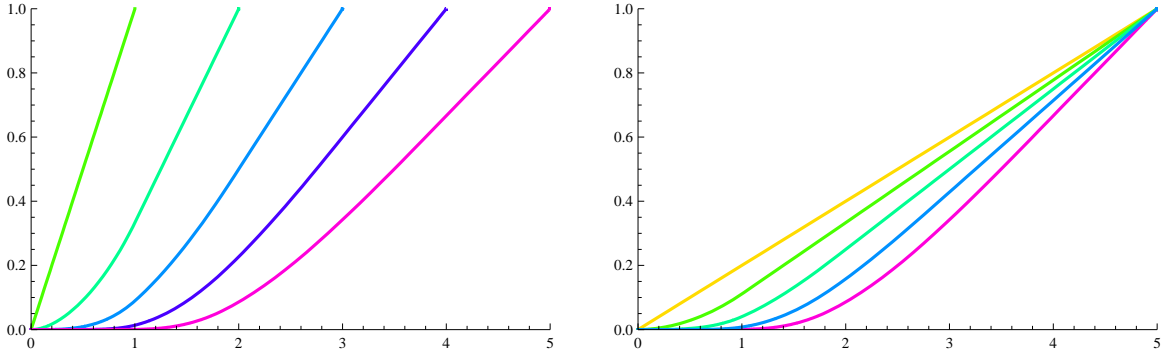


Figure 1. The Global Bidder's Optimal Drop-Out Level When Complementarities are Extreme: $B_n^1(V)$ (left) For $n = 1, \dots, 5$ and $B_n^2(V)$ (right) For $n = 0, \dots, 5$.

mean, however, that efficiency is negatively affected. The lowest line in the left panel of Figure 1 only applies when all five local bidders are active, and if this occurs at an item price of 0.2 then the sum of the local bidders' expected values is 3 (not 2.5). Hence, efficiency may be improved when the global bidder drops out (especially when complementarities are extreme, as in Figure 1, and the global bidder derives no value from winning less than five items).

We next determine what a bidder's drop-out level would be if it were chosen to maximize welfare rather than profit. Since at the the time a bidder drops out the values of other active bidders are not known, we focus on expected welfare where the expectation is taken over active bidders' values. We say that bidders' drop-out levels are *constrained* efficient when they maximize expected welfare, which is based on a mixture of ex post information (values of bidders that have dropped out) and ex ante information (values of active bidders). Due to the residual uncertainty about the values of active bidders, there are necessarily some ex post inefficiencies in the SAA unlike the VCG mechanism where the welfare function is based on only ex post information about bidders' values.

Obviously, for a local bidder the socially optimal drop-out level is equal to her value. To derive the socially optimal drop-out level for the global bidder consider first the case of a single item ($k = 1$). The social planner would choose $B_1^1(V)$ to maximize

$$W_1^1(V, p) = \int_p^{B_1^1(V)} V_1 dF_1(v_1|p) + \int_{B_1^1(V)}^1 v_1 dF_1(v_1|p)$$

where the first (second) term corresponds to the global (local) bidder winning the item. Comparing the expression for welfare to the global's profit $\Pi_1^1(V, p)$ in Section 2.1 shows that $W_1^1(V, p) = \Pi_1^1(V, p) + E(v|v > p)$. In other words, welfare and the global bidder's profit differ only by a constant independent of $B_1^1(V)$. Hence, the drop-out level chosen by a profit-maximizing bidder maximizes welfare. We next generalize this to an arbitrary number of items.

Proposition 2. *Bidders' drop-out levels are constrained efficient.*

As in the benchmark Vickrey-Clarke-Groves (VCG) mechanism, individual and social incentives are aligned in the SAA. The difference is that in the VCG mechanism all values are revealed at once, which allows for a fully efficient outcome. In contrast, the SAA is only constrained efficient because of the residual uncertainty bidders face about the values of other active bidders.

The efficiency gain in the VCG mechanism does not benefit the seller, however, but only the global bidder. Let $W^{SAA}(V)$, $R^{SAA}(V)$, $\Pi^{SAA}(V)$, and $\pi^{SAA}(V)$ denote the expected welfare, expected revenue, expected global bidder's profit, and expected local bidders' total profit under the SAA mechanism, where the (ex ante) expectation is taken over local bidders' values only. Similar definitions apply with respect to the VCG mechanism.

Proposition 3. *The efficiency gain of the VCG mechanism accrues to the global bidder*

$$W^{VCG}(V) - W^{SAA}(V) = \Pi^{VCG}(V) - \Pi^{SAA}(V) \quad (2.3)$$

while the difference in seller's revenue accrues to the local bidders

$$R^{SAA}(V) - R^{VCG}(V) = \pi^{VCG}(V) - \pi^{SAA}(V) \quad (2.4)$$

As we show in Section 3, the differences between the VCG and SAA mechanisms vanish when the number of items grows large. First, we extend the optimality result of Proposition 2 to the case of more than one global bidder.

2.3. Multiple Global Bidders

A global bidder's optimal bidding function when there are multiple global bidders follows from the same 'break even' logic that underlies the result of Proposition 1. First, consider the case of $K = 2$ global bidders and suppose k out the n local bidders are still active: the optimal bid $B_k^2(V)$ is determined by requiring that at this price level the marginal costs and benefits of staying in a little longer (i.e. by bidding as of type $V + \epsilon$) cancel. There are two possible marginal events: one occurs when a local drops out, in which case the other global bidder has a value no less than V . Hence, the continuation profits for a global bidder with value V are zero in this case: $\Pi_{k-1}^2(V, B_k^2(V)) = 0$. Alternatively, the other global bidder drops out, in which case the continuation profits are given by $\Pi_k^1(V, B_k^2(V))$. Furthermore, the global bidder now wins all the $(n - k)$ items for which the local bidders had already dropped out at a price of $B_k^2(V)$ for each item. Finally, when there are $K \geq 3$ global bidders, the only non-vanishing marginal term results from $K - 1$ global bidders dropping out at the same time, which produces the same marginal equation as when $K = 2$.

Proposition 4. *The global bidder's optimal drop-out level satisfies $B_k^K(V) = B_k^2(V)$ for $K \geq 2$ and*

$$\Pi_k^1(V, B_k^2(V)) + \sum_{\ell=k+1}^n (V_\ell - B_k^2(V)) = 0, \quad (2.5)$$

where the $\Pi_k^1(V, p)$ satisfy the recursion relations of Proposition 1.

It is worthwhile pointing out a few cases: $k = 0$ corresponds to the case where all local bidders have dropped out and two (or more) global bidders are active. We then have $B_0^2(V) = V/n$ since at a price of V/n the global bidder is indifferent between winning nothing and winning everything at that price. For $k = n$ we have $\Pi_n^1(V, B_n^2(V)) = 0$ so $B_n^2(V) = B_n^1(V)$. Likewise, for $k = n - 1$ we have $\Pi_{n-1}^1(V, B_{n-1}^2(V)) + V_n = B_{n-1}^2(V)$, which implies $B_{n-1}^2(V) = B_{n-1}^1(V)$ (see Proposition 1). In other words,

$$B_n^2(V) = B_{n-1}^2(V) = B_n^1(V). \quad (2.6)$$

This result may seem surprising given that $B_{n-1}^2(V)$ is determined by the marginal event when the other global bidder drops out, while $B_n^2(V)$ is determined by the marginal event when the other global bidder and a local bidder drop out. Nevertheless, in both scenarios the optimal drop-out level follows from considering the cost and benefit of winning the first item, which yields $B_n^1(V)$ as the optimal drop-out level.

Example 1 (continued). The right panel of Figure 1 shows the global bidder's optimal drop-out levels $B_n^2(V)$ for $0 \leq n \leq 5$ when local bidders' values are uniform and complementarities are extreme. Note that there are five (not 6) lines since $B_5^2(V) = B_4^2(V)$, see (2.6), and that $B_k^2(V) \leq B_{k+1}^1(V) \leq B_k^1(V)$ for all k .

As we show next, the ranking of the global bidders' optimal drop-out levels holds more generally. Competition from other global bidders aggravates the exposure problem and *lowers* a global bidder's optimal drop-out level: $B_k^2(V) \leq B_{k+1}^1(V)$ for all k (see the proof of Proposition 5). The fact that global bidders are more cautious when facing competition from other global bidders does not hurt constrained efficiency. On the contrary, it implies that global bidders who do not have the highest value, and who should therefore not win any items in the optimal allocation, drop out before local bidders that should win items in the optimal allocation do.

Proposition 5. *Bidders' drop-out levels are constrained efficient for any $K \geq 1$.*

3. Large Auctions

In many applications of the SAA the number of items is very large, e.g. in some of the FCC spectrum auctions more than a thousand items are sold. In this section, we show how our

approach extends to large auctions. In particular, we derive the optimal drop-out levels when multiple global bidders are active, $B^K(V)$ for $K \geq 2$, and show that they are less than $B^1(V)$, which applies with only a single global bidder. Since in large auctions there is no residual uncertainty about active local bidders' values, this result implies that the SAA becomes fully efficient in the limit. In addition, we show that in large auctions the SAA generates the same profits for the seller and the bidders as the VCG mechanism.

Let V denote the highest of the global bidders' values and $v_n \leq \dots \leq v_1$ denote the (ordered) local bidders' values. When the highest-value global bidder wins k of n items, welfare is

$$W(k, V) = \sum_{\ell=n-k+1}^n V_\ell + \sum_{\ell=1}^{n-k} v_\ell.$$

In the limit when $n \rightarrow \infty$ the sum of local bidders' values will diverge, and we assume that the highest of the global bidders' values diverges as well, i.e. $V = n\hat{V}$ where \hat{V} is distributed according to $\hat{G}(\hat{V}) = G(V)$ with support $[0,1]$. We can then normalize welfare and profits on a per-item basis. Suppose the highest-value global bidder wins a fraction κ of all items then normalized welfare is $W(\kappa, \hat{V}) = \lim_{n \rightarrow \infty} W(\kappa n, n\hat{V})/n$ and the global bidder's normalized value of winning a fraction κ of all items is $\mathcal{V}(\kappa) = \lim_{n \rightarrow \infty} \alpha(\kappa n)\hat{V}$. Assumption 1 implies that $\alpha(\cdot)$ is convex, and, hence, so is $\mathcal{V}(\cdot)$. To simplify notation, below we write V and $G(\cdot)$ instead of \hat{V} and $\hat{G}(\cdot)$ to indicate the normalized value and its distribution.

The welfare maximizing fraction of items assigned to the highest-value global bidder now follows from $W(V) \equiv \max_{\kappa} W(\kappa, V)$, or, equivalently,

$$W(V) = \max_{0 \leq \kappa \leq 1} \mathcal{V}(\kappa) + \int_{F^{(-1)}(\kappa)}^1 v dF(v), \quad (3.1)$$

where we used that in the limit when n grows large, $v_{(1-\kappa)n}$ is asymptotically normally distributed with mean $F^{(-1)}(\kappa)$ and variance of order $1/n$ (David and Nagajara, 2003). The solution to (3.1) is denoted $\kappa^*(V) = \operatorname{argmax}(\mathcal{V}(\kappa), V)$ so that $W(V) = W(\kappa^*(V), V)$.

In the SAA, local bidders drop out at a known rate, e.g. at price level p a total of $F(p)$ local bidders have dropped out. Suppose there is only one global bidder ($K = 1$). The global bidder's optimal strategy is to bid up to a level $B^1(V)$ that maximizes her per-item profit:

$$\Pi^1(V, p) = \mathcal{V}(F(B^1(V))) - \int_p^{B^1(V)} v dF(v). \quad (3.2)$$

Note that $\Pi^1(V, p) = W(F(B^1(V)), V) - E(v|v > p)$ so the global bidder's optimal drop-out level is simply $B^1(V) = F^{(-1)}(\kappa^*(V))$.

Next consider the case of multiple global bidders. First, let $K = 2$. The optimal drop-out level $B^2(V)$ follows by requiring that the marginal benefits and costs of staying in a little longer (by bidding as of type $V + \epsilon$) cancel. This deviation affects the outcome only when the rival global bidder drops out in between (with probability $\epsilon g(V)$), in which case the net benefit is

$$\mathcal{V}(F(B^2(V))) - B^2(V)F(B^2(V)) + \int_{B^2(V)}^{B^1(V)} (\mathcal{V}'(F(y)) - y)dF(y) = 0.$$

Here the first term reflects the value of the $F(B^2(V))$ items the global bidder wins when her rival drops out, the second term is how much she pays for them, and the third term is her continuation profit when she proceeds to win additional items by bidding up to $B^1(V)$. Integrating this last term and using the definition of $W(V)$ shows that $B^2(V)$ solves

$$B^2(V)F(B^2(V)) + \int_{B^2(V)}^1 ydF(y) - W(V) = 0. \quad (3.3)$$

The left side of (3.3) is strictly increasing in $B^2(V)$ so the solution is unique. We next show that $B^2(V) \leq B^1(V)$. Evaluating the left side of (3.3) at $B^2(V) = B^1(V)$ yields

$$\begin{aligned} B^1(V)F(B^1(V)) - \mathcal{V}(F(B^1(V))) &= \kappa^*(V)F^{(-1)}(\kappa^*(V)) - \mathcal{V}(\kappa^*(V)) \\ &\geq \kappa^*(V)(F^{(-1)}(\kappa^*(V)) - \mathcal{V}'(\kappa^*(V))) \\ &= 0, \end{aligned}$$

where the equality in the first line follows from the definition of $B^1(V)$, the weak inequality in the second line follows from convexity of \mathcal{V} , and the equality in the third line follows from the first-order condition for $\kappa^*(V)$, see (3.1).¹¹ Since the left side of (3.3) is strictly increasing in $B^2(V)$ this implies that $B^2(V) \leq B^1(V)$, with strict inequality when there are complementarities and $\mathcal{V}(\cdot)$ is strictly convex.

Finally, when $K \geq 3$, the marginal equation that determines $B^K(V)$ follows by requiring that the marginal benefits and costs of staying in a little longer (e.g. by bidding as of type $V + \epsilon$) cancel. This deviation affects the outcome only when all rival global bidders drop out in between, and the resulting marginal equation is the same as when $K = 2$.

Example 2. Suppose local bidders' values are uniformly distributed and a global bidder's value of winning a fraction κ of the items is given by $\mathcal{V}(\kappa) = \kappa^\rho V$. When $1 \leq \rho < 2$, the optimal drop-out levels are $B^1(V) = \min(1, (\rho V)^{1/(2-\rho)})$ and $B^K(V) = \min(\sqrt{2V} - 1, \sqrt{2/\rho - 1}(\rho V)^{1/(2-\rho)})$

¹¹To be precise, the equality in the third line holds only for interior solutions. To account for possible boundary solutions, note that for $\kappa^*(V) = 0$ the expression in the second line vanishes. Furthermore, if $\kappa^*(V) = F(B^1(V)) = 1$, then the global bidder's optimal drop-out level is determined solely by the event when other global bidders drop out and $B^1(V) = B^2(V) = \mathcal{V}(1)$.

for $K \geq 2$. When $\rho \geq 2$, $B^1(V) = \mathbf{1}_{V \geq \frac{1}{2}}$ and $B^K(V) = \sqrt{2V-1} \mathbf{1}_{V \geq \frac{1}{2}}$ for $K \geq 2$, where $\mathbf{1}$ denotes an indicator function. With extreme complementarities ($\rho = \infty$), these results are readily extended to general distributions of the local bidders' values: $B^1(V) = \mathbf{1}_{V \geq E(v)}$ and $B^2(V)$ solves $\int_{B^2(V)}^1 F(y)dy = 1 - V$ for $V \geq E(v)$ and is zero otherwise.

Since $B^1(V)$ maximizes welfare the SAA is fully efficient in the limit when the number of items grows large. Surprisingly, the SAA also generates the same revenue as the VCG mechanism in this limit. To glean some insight for this result, note that (3.3) implies

$$B^2(Z)F(B^2(Z)) + \int_{B^2(Z)}^{B^1(V)} ydF(y) = W(Z) - \int_{B^1(V)}^1 ydF(y).$$

On the left side is the SAA payment of the global bidder with the highest value V when the second-highest of the global bidders' values is Z , and on the right side is the corresponding Vickrey payment.

Proposition 6. *With a large number of items, the global bidders' optimal drop-out levels satisfy $B^1(V) = F^{(-1)}(\kappa^*(V))$, where $\kappa^*(V)$ maximizes the welfare in (3.1), and $B^K(V) = B^2(V) \leq B^1(V)$ for $K \geq 2$, where $B^2(V)$ solves (3.3). Moreover:*

$$\begin{aligned} W^{SAA}(V) &= W^{VCG}(V) = W(V) \\ R^{SAA}(V) &= R^{VCG}(V) = \int_0^V W(Z)dG(Z|V)^{K-1} - \int_{B^1(V)}^1 (1 - F(v))dv \\ \Pi^{SAA}(V) &= \Pi^{VCG}(V) = \int_0^V (W(V) - W(Z))dG(Z|V)^{K-1} \\ \pi^{SAA}(V) &= \pi^{VCG}(V) = \int_{B^1(V)}^1 (1 - F(v))dv \end{aligned}$$

where V denotes the highest of the global bidders' values and $G(Z|V) = G(Z)/G(V)$.

Remark 1. Since the global bidders' optimal drop-out levels are independent of the value distribution $G(\cdot)$, Propositions 1 and 4 also apply when there are asymmetries among the global bidders. Furthermore, the equivalence result of Proposition 6 extends to the asymmetric case, with the conditional distribution $G(Z|V)^{K-1}$ replaced by $\prod_{\ell \neq j} G_\ell(Z|V)$ when global bidder j has the highest value V . Note that asymmetries affect the seller's revenue and the winning global bidder's profit but not efficiency or the local bidders' profits.

It is important to point out that the full efficiency result for the SAA is not because the exposure problem is absent in large auctions. Indeed, we next consider a sealed-bid version of the SAA and show that, due to the exposure problem, global bidders generally obtain less than the socially-optimal number of items, even in large auctions.

3.1. Simultaneous Sealed-Bid Auction

Here we follow Krishna and Rosenthal (1996) who consider the sealed-bid, second-price auction (SSA) where local and global bidders submit a single per-unit bid. A local bidder's bid is for a specific item, while a global bidder's bid applies to all items. A local bidder wins her preferred item if her bid exceeds that of the global bidders, in which case she pays the highest of the global bidders' bids. The global bidder with the highest per-unit bid wins all items for which her bid exceeds that of the local bidders (the other global bidders win nothing). The price paid for an item won by the global bidder is the maximum of the other global bidders' per-unit bids and the local bidder's bid for the item. With $K \geq 1$ global bidders, a global bidder's per-item profit when her value is V and she bids b is therefore¹²

$$\Pi(b, V) = \mathcal{V}(F(b))G(B^{(-1)}(b))^{K-1} - \int_0^{B^{(-1)}(b)} \left\{ B(Z)F(B(Z)) + \int_{B(Z)}^b y dF(y) \right\} dG(Z)^{K-1} \quad (3.4)$$

The first term on the right side is the global bidder's value of winning a fraction $F(b)$ of the items times the corresponding win probability. The second term reflects the global bidder's payment: the first term in the integral occurs when the highest of the other global bidders determines the price and the second term occurs when the local bidders determine the price.

From (3.4) it is straightforward to derive the marginal equation that determines a global bidder's optimal bidding function, $B(V)$:

$$(K-1)g(V)\left(\mathcal{V}(F(B(V))) - B(V)F(B(V))\right) + G(V)B'(V)f(B(V))\left(\mathcal{V}'(F(B(V))) - B(V)\right) = 0$$

With only a single global bidder, $K = 1$, this first-order condition is the same as the one that follows by maximizing expected welfare so $B(V) = B^1(V)$. For $K > 1$, however, the highest-value global bidder wins fewer than the socially optimal fraction of items.

Proposition 7. *With a large number of items, the SSA results in an efficient outcome with one global bidder but in inefficient outcomes with multiple global bidders: with complementarities, the global bidder with the highest value wins fewer items than is socially optimal.*

Example 2 (continued). For $\rho = 2$, the global bidders' optimal bids in the SSA are given by

$$B(V) = \left(\frac{2V-1}{V^2}\right)^{\frac{K-1}{2}} \mathbf{1}_{V \geq \frac{1}{2}}$$

which is less than the welfare-maximizing solution $B^1(V) = \mathbf{1}_{V \geq \frac{1}{2}}$ unless $K = 1$.

¹²Here we used that $\sum_{k=1}^n \alpha(k) \binom{n}{k} q^k (1-q)^{n-k}$ limits to $\alpha(nq)$ when n tends to infinity.

To summarize, the SSA is inefficient even in the limit when the number of items grows large. Global bidders shy away from bidding aggressively because of the exposure problem they face under the sealed-bid mechanism. In contrast, the dynamic SAA allows global bidders to first compete against each other, after which the strongest global bidder competes against the remaining local bidders, resulting in an efficient outcome.

4. Extensions

An important virtue of the SAA is that it facilitates substitution: the items are put up for sale simultaneously and the auction does not close until bidding on all items has stopped. As a result, bidders can switch back and forth between items in response to developing prices without having to worry that certain items are gone because they did not bid on them. This virtue played no role in the analysis above where each local bidder had a specific (regional) interest and after the local bidder dropped out, the market for an item remained inactive regardless of the prices of other items. One extension studied in this section is to assume that items are substitutes for local bidders so that price arbitrage will result in similar prices for the items.

Another extension is to consider market sharing among global bidders. In the analysis above, each global bidder has an interest in all items and the value of each item increases the more items a global bidder wins. As a result, only the global bidder with the highest value obtains any items. In this section, we study the case where each global bidder needs only a subset of the items. We consider the case when market sharing is feasible and the fitting problems that arise when it is not.

As we demonstrate below, these extensions exacerbate the exposure problem. With price arbitrage, the global bidder faces competition from strong local bidders on all items, making it difficult to acquire a large collection of items. When there is a fitting problem, global bidders run the risk of “mutually destructive bidding,” i.e. when global bidders compete fiercely for their desired combination and losers end up paying for inferior subsets. Finally, when market sharing is feasible, global bidders may follow a strategy of mutual forbearance in order to keep prices low, at the expense of efficiency and the local bidders. As we will show, the aggravated exposure problem causes the SAA to perform dramatically worse and, even in large auctions, the SAA and VCG mechanisms are no longer equivalent.

4.1. Substitutes and Complements

In this section we relax the assumption that each local bidder is interested in a different item. Instead we assume local bidders want at most one item and they value all items the same (perfect substitutes). The global bidder has super-additive values as before (complements).

Intuitively, the exposure problem is worse under this setup since local bidders will switch licenses and drive up their prices uniformly until they drop out – the global bidder will thus have to pay the value of the last local bidder that dropped out for all the items she wins.

Proposition 8. *The global bidder's optimal drop-out level $B_k^1(V)$ maximizes $\Pi_k^1(V, p)$, where the payoffs satisfy the recursive relation*

$$\begin{aligned} \Pi_k^1(V, p) &= \int_p^{B_k^1(V)} \{\Pi_{k-1}^1(V, v_k) + (V_k - v_k)\} dF_k(v_k|p) \\ &\quad - (n - k) \int_p^1 \{\min(v_k, B_k^1(V)) - p\} dF_k(v_k|p), \end{aligned} \quad (4.1)$$

with $\Pi_0^1(V, p) = 0$.

Note that if the global bidder is successful in winning an additional item (i.e. when $v_k < B_k^1(V)$) then she gains V_k and pays v_k , and the price increase of the $(n - k)$ items the global bidder would anyhow win is $(v_k - p)$. Otherwise, the price increase for these items is $(B_k^1(V) - p)$.

The profit functions in Proposition 8 are not necessarily concave unlike their counterparts in Proposition 1, which do not have the additional term in the second line of (4.1). As a result, the drop-out levels that maximize profits are not necessarily interior solutions. An exception is when all local bidders are active ($k = n$) since then the second line of (4.1) disappears.

Example 1 continued. Suppose $n = 2$ and local bidders' values are uniformly distributed. For $k = 1$, Proposition 8 implies $\Pi_1^1(V, p) = (B_1^1(V) - p)(V_1 - 1)/(1 - p)$, which is maximized at $B_1^1(V) = \mathbf{1}_{V_1 \geq 1}$ (recall that the bidder drops out at $\max\{p, B_1^1(V)\}$, i.e. the bidder drops out immediately when $B_1^1(V) < p$). Therefore, the continuation profit is $\Pi_1^1(V, p) = (V_1 - 1)\mathbf{1}_{V_1 \geq 1}$ and the first-order condition for $k = 2$

$$B_2^1(V) = V_2 + \Pi_1^1(V, B_2^1(V)),$$

yields $B_2^1(V) = V_2 + (V_1 - 1)\mathbf{1}_{V_1 \geq 1}$. Note that the optimal drop-out levels are lower than those in the original Example 1, which reflects the aggravated exposure problem in this setup. In particular, the global bidder's drop-out levels no longer maximize expected welfare. This example can be extended to more than two local bidders when complementarities are extreme.

Proposition 9. *When $n \geq 2$ local bidders' values are uniformly distributed and complementarities are extreme, the global bidder's optimal drop-out level is given by*

$$B_n^1(V) = \begin{cases} 0 & \text{if } V_1 < n - 1 \\ V - (n - 1) & \text{if } V_1 \geq n - 1 \end{cases}$$

and $B_k^1(V) = 1$ for $1 \leq k < n$ and $V \geq n - 1$.

To summarize, the global bidder drops out of the auction immediately unless her value exceeds $n - 1$.¹³ Clearly, with many items this outcome is very inefficient since the expected value of all the items to the local bidders is only $n/2$. We next generalize this inefficiency result to more general distributions and valuation functions.

Proposition 10. *In large auctions, the global bidder wins fewer items than is socially optimal when there are complementarities.*

Example 2 (continued). Recall that, for $1 \leq \rho < 2$, the socially optimal fraction of items won by the global bidder is $\min(1, (\rho V)^{1/(2-\rho)})$, while in equilibrium, it is only $(\frac{1}{2}\rho V)^{1/(2-\rho)}$.¹⁴ So the global bidder wins fewer items than is socially optimal even with linear valuations (demand reduction). For $\rho \geq 2$, the socially optimal fraction is $\mathbf{1}_{V \geq \frac{1}{2}}$ while in equilibrium the global bidder wins no items *irrespective of her value*.¹⁵ Due to the aggravated exposure problem in this environment with complements and substitutes, the global bidder drops out immediately.

4.2. Fitting Problems

In this section and the next, we assume local bidders' interests are fixed as in Sections 2 and 3 but we relax the assumption that global bidders always value additional items. Instead, we consider large auctions where two global bidders each demand a fraction $\alpha \leq 1$ of the items. Complementarities are extreme, i.e. a global bidder has no value for less than α of the items, nor does the global bidder value additional items beyond an α fraction.

We first study the case $\alpha > 1/2$ so that there is a “fitting problem,” i.e. it is not possible for both global bidders to win their desired fraction of items. As a result, global bidders that drop out have to pay for (worthless) items if fewer than $1 - \alpha$ local bidders are active. This lowers the value of dropping out and may cause global bidders to stay in at prices that exceed their values. Such behavior can be mutually destructive since prices may become so high that even the winning global bidder incurs a loss.

Suppose the global bidder's value is low enough so that, in equilibrium, the global bidder drops out before a fraction α of the local bidders has dropped out. In this case, the optimal drop-out level is akin to that of Example 2 in Section 3. Define $V^*(\alpha) = \alpha E(v | F(v) \leq \alpha)$ then $B^1(V) = F^{(-1)}(\alpha) \mathbf{1}_{V \geq V^*(\alpha)}$. Similarly, from (3.3), $B^2(V)$ is zero when $V < V^*(\alpha)$ and for $V \geq V^*(\alpha)$ it follows from

$$\int_{B^2(V)}^{F^{(-1)}(\alpha)} F(y) dy = \alpha F^{(-1)}(\alpha) - V$$

¹³Even if one of the local bidders drops out at a price of 0, the global bidder still expects to pay n times the expected value of the highest of $n - 1$ local bidders' values. So the expected payment would be $n(n - 1)/n$, which explains why $n - 1$ is the threshold value for the global bidder to start bidding.

¹⁴The global bidder's objective is to maximize $V\kappa^\rho - \kappa^2$, which yields $(\frac{1}{2}\rho V)^{1/(2-\rho)}$ for $1 \leq \rho < 2$.

¹⁵For $\rho \geq 2$, the interior solution is a local minimum, while $\kappa = 0$ yields 0 and $\kappa = 1$ yields $V - 1 \leq 0$.

cf. Example 2. This solution, however, only applies when at least a fraction $1 - \alpha$ of the local bidders are active, which requires $B^2(V) \leq F^{(-1)}(\alpha)$ or $V \leq V^{**}(\alpha) = \alpha F^{(-1)}(\alpha)$.

We next determine $B^2(V)$ for $V \geq V^{**}(\alpha)$. Consider a global bidder with value V who bids as if her value is $V + \epsilon$. The effect of this deviation is twofold. First, with probability $\epsilon g(V)$ the other global bidder drops out in between and the deviation results in a winning payoff of $V - \alpha B^2(V)$ instead of a losing payoff of $-B^2(V)(F(B^2(V)) - \alpha)$. Second, with probability $(1 - G(V))$ the other global bidder has a higher value and the deviation simply raises the costs of the items that are won. Equating the marginal cost and benefit of the deviation yields

$$g(V)\left(V - B^2(V)(2\alpha - F(B^2(V)))\right) = (1 - G(V))\left(B^2(V)(F(B^2(V)) - \alpha)\right)'$$

The right side vanishes at $V = 1$, which implies that $B^2(1) = 1/(2\alpha - 1) > 1$. The intuition is that when the price level exceeds 1, local bidders are no longer active so the global bidder is assigned a fraction $1 - \alpha$ of the items when she drops out or a fraction α of the items when she wins. Hence, a global bidder with $V = 1$ continues to bid as long as the cost of winning an additional $2\alpha - 1$ of the items is less than 1. The global bidders' aggressive bids can cause losses, even for the winning bidder. For example, suppose both global bidders have values close to 1, then the winning bidder's profit is $1 - \alpha/(2\alpha - 1) = -(1 - \alpha)/(2 - \alpha) < 0$, which is also the profit for the losing bidder. The global bidders' aggressive bids do not lower their expected profits though, nor do they necessarily translate into higher seller revenue.

Proposition 11. *Let $1/2 < \alpha < 1$, so that there is a fitting problem. Compared to the VCG mechanism, the SAA is (i) inefficient, (ii) may yield less revenue (and yields less revenue when α is high enough), (iii) hurts the local bidders, and (iv) creates a positive probability of losses for both losing and winning global bidders:*

$$\begin{aligned} W^{VCG} - W^{SAA} &= \int_{V^{**}(\alpha)}^1 \int_{F^{(-1)}(\alpha)}^{B^2(Z)} v dF(v) dG_{\min}(Z) \\ R^{VCG} - R^{SAA} &= \int_{V^{**}(\alpha)}^1 \int_{F^{(-1)}(\alpha)}^{B^2(Z)} \left(v - \frac{1 - F(v)}{f(v)}\right) dF(v) dG_{\min}(Z) \\ \Pi^{VCG} - \Pi^{SAA} &= 0 \\ \pi^{VCG} - \pi^{SAA} &= \int_{V^{**}(\alpha)}^1 \int_{F^{(-1)}(\alpha)}^{B^2(Z)} \frac{1 - F(v)}{f(v)} dF(v) dG_{\min}(Z) \end{aligned}$$

where $G_{\min}(Z) = 1 - (1 - G(Z))^2$.

The profit-equivalence result for the global bidder stems from the fact that for a global bidder with value $V \geq V^*(\alpha)$, the probability of obtaining α items is $G(V)$ under both mechanisms. The revenue difference is intuitive as well: under SAA, the seller no longer collects the marginal

revenues, $MR(V) \equiv v - (1 - F(v))/f(v)$, for local bidders with values between $F^{(-1)}(\alpha)$ and $B^2(Z)$. These marginal revenues can be negative for low values of v but are positive for high values of v . For example, for uniformly distributed values, the marginal revenues are $MR(v) = 2v - 1 > 0$ for $v > 1/2$. Hence, in the uniform case, the seller's revenue under SAA is always lower than under VCG since $F^{(-1)}(\alpha) = \alpha > 1/2$. More generally, a sufficient (but not necessary) condition for revenues to be higher under VCG is that $\alpha > \max(1/2, F(MR^{(-1)}(0)))$.

4.3. Market Sharing

When $\alpha < 1/2$, market sharing is possible and global bidders have an alternative to dropping out: they can “yield” by not bidding for the same items as the other global bidder and instead compete with the remaining local bidders to obtain a fraction α of the items. Compared to dropping out, yielding results in an extra payoff of

$$V - \int_{B^2(V)}^{F^{(-1)}(2\alpha)} v dF(v) \quad (4.2)$$

i.e. the value of winning the additional items minus their cost. Note that the additional payoff (4.2) is strictly increasing in V . Furthermore, (4.2) is negative when evaluated at $V = V^{**}(\alpha)$ and positive when evaluated at $V = 1$.¹⁶ So there is a unique $V^{**}(\alpha) < V^{***}(\alpha) < 1$ where the global switches from dropping out to yielding.

To determine the first-order condition for $V \geq V^{***}(\alpha)$, consider a global bidder with value V who bids as if her value is $V + \epsilon$. With probability $\epsilon g(V)$ the other global bidder drops out in between and the deviation results in a winning payoff of $V - \alpha B^2(V)$ instead of a yielding payoff of $V - \int_{B^2(V)}^{F^{(-1)}(2\alpha)} v dF(v) - B^2(V)(F(B^2(V)) - \alpha)$. Second, with probability $(1 - G(V))$ the other global bidder has a higher value and the deviation simply raises the costs of the $(F(B^2(V)) - \alpha)$ items that are eventually won. Equating the marginal cost and benefit yields

$$g(V) \left(\int_{B^2(V)}^{F^{(-1)}(2\alpha)} v dF(v) - B^2(V)(2\alpha - F(B^2(V))) \right) = (1 - G(V))(F(B^2(V)) - \alpha)(B^2(V))'$$

The right side vanishes at $V = 1$, which implies that $B^2(1) = F^{(-1)}(2\alpha) < 1$. The intuition is that at a price of $F^{(-1)}(2\alpha)$ a fraction of 2α of the local bidders has dropped out, so global bidders of all types prefer to yield and divide the market rather than to compete further.

To summarize, in equilibrium, the price at which the global bidders divide the market is less than $F^{(-1)}(2\alpha)$. This low price contrasts with the price that would have resulted in a single-price clock auction, for instance, where both global bidders would have to pay $F^{(-1)}(2\alpha)$

¹⁶Recall that $B^2(V^{**}(\alpha)) = F^{(-1)}(\alpha)$ so (4.2) evaluated at $V^{**}(\alpha)$ becomes $\int_{F^{(-1)}(\alpha)}^{F^{(-1)}(2\alpha)} (F(v) - 2\alpha) dv < 0$.

to drive 2α local bidders out of the market. We next demonstrate that the price at which global bidders yield is also too low from a welfare viewpoint. In the VCG mechanism, the global bidder with the lowest value, Z , would obtain a fraction α of the items when

$$Z \geq V^o(\alpha) \equiv \int_{F^{(-1)}(\alpha)}^{F^{(-1)}(2\alpha)} t dF(t)$$

In equilibrium, the lowest-value global bidder obtains a fraction α of the items when $Z \geq V^{***}(\alpha)$, where $V^{***}(\alpha) < V^o(\alpha)$.¹⁷ Hence, there are two value ranges where the lowest-value global bidder obtains too many items. When $V^{**}(\alpha) < Z < V^{***}(\alpha)$ she obtains a strictly positive fraction (although strictly less than α) while it is optimal that she wins nothing. When $V^{***}(\alpha) < Z < V^o(\alpha)$ she obtains an α fraction of the items, while it is optimal that she wins nothing.

Proposition 12. *Let $0 < \alpha \leq 1/2$, so that market sharing is feasible. Compared to the VCG mechanism, the SAA is (i) inefficient, (ii) may yield less revenue, (iii) hurts the local bidders, and (iv) benefits the global bidders who divide the market at low prices:*

$$\begin{aligned} W^{VCG} - W^{SAA} &= \int_{V^{**}(\alpha)}^{V^{***}(\alpha)} \int_{F^{(-1)}(\alpha)}^{B^2(Z)} v dF(v) dG_{\min}(Z) + \int_{V^{***}(\alpha)}^{V^o(\alpha)} (V^o(\alpha) - Z) dG_{\min}(Z) \\ R^{VCG} - R^{SAA} &= \int_{V^{**}(\alpha)}^{V^{***}(\alpha)} \int_{F^{(-1)}(\alpha)}^{B^2(Z)} \left(v - \frac{1 - F(v)}{f(v)}\right) dF(v) dG_{\min}(Z) \\ &\quad + \int_{V^{***}(\alpha)}^{V^o(\alpha)} \int_{F^{(-1)}(\alpha)}^{F^{(-1)}(2\alpha)} \left(v - \frac{1 - F(v)}{f(v)}\right) dF(v) dG_{\min}(Z) \\ &\quad - \int_{V^{***}(\alpha)}^{V^o(\alpha)} \left(Z - \frac{1 - G(Z)}{g(Z)}\right) dG_{\min}(Z) \\ \Pi^{VCG} - \Pi^{SAA} &= - \int_{V^{***}(\alpha)}^{V^o(\alpha)} \frac{1 - G(Z)}{g(Z)} dG_{\min}(Z) \\ \pi^{VCG} - \pi^{SAA} &= \int_{V^{**}(\alpha)}^{V^{***}(\alpha)} \int_{F^{(-1)}(\alpha)}^{B^2(Z)} \frac{1 - F(v)}{f(v)} dF(v) dG_{\min}(Z) \\ &\quad + \int_{V^{***}(\alpha)}^{V^o(\alpha)} \int_{F^{(-1)}(\alpha)}^{F^{(-1)}(2\alpha)} \frac{1 - F(v)}{f(v)} dF(v) dG_{\min}(Z) \end{aligned}$$

where $G_{\min}(Z) = 1 - (1 - G(Z))^2$.

Example 3. Suppose the local and global bidders' values are uniformly distributed. The top-right panel of Figure 2 shows the optimal drop-out level, $B^2(V)$, for $\alpha = 3/4$ and $0 \leq V \leq 1$,

¹⁷Since $V^{***}(\alpha) = \int_{B^2(V^{***}(\alpha))}^{F^{(-1)}(2\alpha)} t dF(t) < \int_{B^2(V^{**}(\alpha))}^{F^{(-1)}(2\alpha)} t dF(t) = V^o(\alpha)$ where the first equality follows from the definition of $V^{***}(\alpha)$, the inequality follows since $V^{***}(\alpha) > V^{**}(\alpha)$, and the second equality follows since $B^2(V^{**}(\alpha)) = F^{(-1)}(\alpha)$.

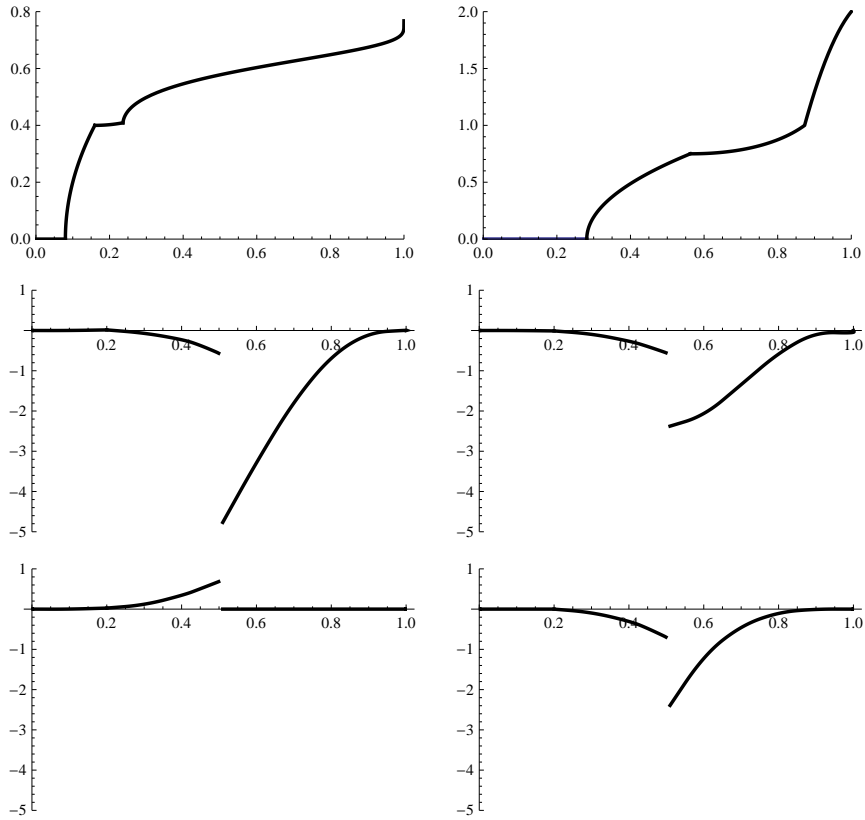


Figure 2. The Optimal Drop-Out Level, $B^2(V)$, for $\alpha = 2/5$ (top left) and $\alpha = 3/4$ (top right). Differences in Welfare (middle left), Revenue (middle right), Global Bidders' Profits (bottom left), and Local Bidders' Profits (bottom right) as Percentages of Maximal Welfare for $0 \leq \alpha \leq 1$.

and illustrates the degree to which mutually destructive bidding can occur. For instance, a bidder with a value of 1 is willing to stay in until the price levels reaches 2, at which point the bidder wins but her profit is $1 - 2(3/4) < 0$. The top-left panel of Figure 2 shows the global bidder's optimal drop-out levels for $\alpha = 2/5$ and $0 \leq V \leq 1$, and illustrates the extent to which global bidders divide the market at low prices. For instance, a global bidder with a value of $V = 1/2$ yields at a per-item price of approximately 0.6, at which point the cost of the $2/5$ fraction of items the global bidder is competing for is roughly half the value.

The middle and bottom panels of Figure 2 illustrate the performance differences of the SAA and VCG mechanisms for $0 \leq \alpha \leq 1$. The middle-left panel shows welfare differences, the middle-right panel shows revenue differences, the lower-left panel shows differences in the global bidders' profits, and the lower-right panel shows differences in local bidders' profits. In each of these four panels, the relevant difference is normalized by the (maximal possible) welfare of the VCG mechanism. Note that both welfare and revenue are lower in the SAA for all values of α , i.e. whether or not market sharing is feasible. Furthermore, the local bidders are always worse off while the global bidders benefit only when market sharing is feasible.

5. Conclusions

We provide a general Bayes-Nash equilibrium analysis of the simultaneous ascending auction (SAA) when global bidders, with super-additive values for combinations of items, compete against smaller bidders interested in a single item. Due to the item-by-item competition in the SAA, global bidders face an *exposure problem* – when competing aggressively for a package, a global bidder may incur a loss when winning an inferior subset. Our equilibrium analysis allows us to quantify the effects of the exposure problem on efficiency and revenue of the SAA, and compare its performance to that of the benchmark Vickrey-Clarke-Groves (VCG) mechanism.

Our setup is very general in that it allows for arbitrary numbers of local and global bidders, arbitrary distributions of local and global bidders’ values, and a general convex valuation function to capture global bidders’ value complementarities. For an environment where each local bidder values a different item, we prove that individual and social optimal incentives coincide in the SAA, as is the case in the VCG mechanism. In particular, bidders’ drop-out levels, which follow from profit-maximizing behavior, are constrained efficient. Unlike VCG, however, the SAA is not ex post efficient because at the time a bidder drops out, she does not know the values of other active bidders.

Importantly, our equilibrium analysis demonstrates that the exposure problem results in perverse revenue properties of the SAA. In particular, the SAA may yield non-core outcomes in which local bidders obtain the items at low prices – prices that may fall short of Vickrey prices. Moreover, the seller’s revenue may decline as more bidders enter the auction. These shortcomings, which are well known for the VCG mechanism (e.g. Ausubel and Milgrom, 2006), were hitherto not known for the SAA simply because a general equilibrium analysis did not exist. The similarity between the SAA and VCG mechanisms becomes even more pronounced in large auctions. When the number of items grow large, as is the case in many FCC spectrum auctions, SAA and VCG yield identical profits for the bidders and the seller.

We also consider an environment where the items are perfect substitutes for local bidders and complements for the global bidder. In this case, local bidders will switch licenses and drive up prices in a uniform manner until they drop out. Such behavior is commonly observed in actual spectrum auctions, e.g. Ausubel and Cramton (2002) note “... in the SAA used for spectrum licenses, there is a strong tendency toward arbitrage of the prices for identical items. Indeed, in the FCC’s Nationwide Narrowband Auction of July 1994, similar licenses were on average priced within 0.3 percent of the mean price for that category of license.” We demonstrate that the exposure problem is much more severe in this case and that the global bidder always wins fewer items than is socially optimal. Indeed, in large auctions it can be optimal for the global bidder to drop out right away, with dramatic effects for the auction’s revenue and efficiency.

Finally, in environments with medium-sized global bidders, the SAA is generally inefficient whether or not market sharing is feasible. When it is, high-value global bidders follow a strategy of mutual forbearance to divide the market at low prices. When it is not, high-value global bidders engage in mutually destructive bidding, which drives out high-value local bidders and may cause even the winning global bidder to incur a loss. In both cases, the SAA yields lower welfare and typically less seller revenue than the VCG mechanism.

To summarize, this paper puts forth several methodological advances to study Bayes-Nash equilibria of the SAA when value complementarities exist. Our techniques apply to a broad array of bidding environments and allow us to quantify the adverse effects of the exposure problem that arises with item-by-item competition. We find that the SAA generally performs much worse than the benchmark VCG mechanism, e.g. when both substitutes and complements play a role, when there are fitting problems, or when global bidders can divide the market. Our findings contrast with the superior performance of the SAA in a substitutes-only environment (e.g. Milgrom, 2000), and reinforce the interest of policy makers in more flexible auction institutions that accommodate bidders' synergistic preferences.

A. Appendix: Proofs

Proof of Proposition 1. Recall that

$$B_k^1(V) = \operatorname{argmax}_b \left\{ \int_p^b (\Pi_{k-1}^1(V, v_k) + (V_k - v_k)) dF_k(v_k|p) \right\} \quad (\text{A.1})$$

The necessary first-order condition is $B_k^1(V) = V_k + \Pi_{k-1}^1(V, B_k^1(V))$, which has a unique solution since the right side is decreasing in $B_k^1(V)$ and the left side is increasing. Moreover, the solution corresponds to a maximum since the second derivative of the objective function in (A.1) evaluated at $b = B_k^1(V)$ is negative. *Q.E.D.*

Proof of Proposition 2. We prove, by induction, that $W_k^1(V, p) = \Pi_k^1(V, p) + kE(v|v > p)$ for all $k \geq 1$. In the main text we have shown it is true for $k = 1$. For $k \geq 2$ we have:

$$\begin{aligned} W_k^1(V, p) &= \int_p^{B_k^1(V)} (W_{k-1}^1(V, v_k) + V_k) dF_k(v_k|p) + \int_{B_k^1(V)}^1 \sum_{i=1}^k v_i dF_k(v_k|p) \\ &= \int_p^{B_k^1(V)} (\Pi_{k-1}^1(V, v_k) + V_k - v_k) dF_k(v_k|p) \\ &\quad + \int_p^{B_k^1(V)} ((k-1)E(v|v > v_k) + v_k) dF_k(v_k|p) + \int_{B_k^1(V)}^1 \sum_{i=1}^k v_i dF_k(v_k|p) \\ &= \Pi_k^1(V, p) + kE(v|v > p) \end{aligned}$$

In the first line, the second term on the right side corresponds to the case where the global bidder drops out before the local bidder with the lowest value among the k active local bidders, in which case all remaining items are awarded to the local bidders. The first term corresponds to the case where the local bidder drops out first (at price level v_k), in which case the social planner optimizes the continuation welfare $W_{k-1}^1(V, v_k)$ with one fewer local bidder. In going from the first to the second line we used the induction hypothesis, and in going from the second to the third line we used the recursive property of the global bidder's profit, see Proposition 1. Since welfare and the global bidder's profit differ only by a constant, $B_k^1(V)$ is chosen in a socially optimal manner. *Q.E.D.*

Proof of Proposition 3. Recall that $W^{SAA}(V) = W_n^1(V, 0)$ and $\Pi^{SAA}(V) = \Pi_n^1(V, 0)$ differ by $nE(v)$, see the proof of Proposition 2. Suppose the VCG mechanism assigns k of the n licenses to the global bidder for which she pays the opportunity cost, which is the sum of the k lowest local bidders' values. Let $\hat{\Pi}^{VCG}$ and \hat{W}^{VCG} denote the global's profit and welfare respectively as a function of the entire profile of bidders' valuations: $\hat{\Pi}^{VCG} = \sum_{\ell=1}^k (V_{n-\ell+1} - v_{n-\ell+1})$ and $\hat{W}^{VCG} = \sum_{\ell=1}^k V_{n-\ell+1} + \sum_{\ell=k+1}^n v_{n-\ell+1}$, so $\hat{W}^{VCG} = \hat{\Pi}^{VCG}(V) + \sum_{k=1}^n v_k$. Taking expectations with respect to local bidders' value shows that the global's expected profit $\Pi^{VCG}(V)$ and expected welfare $W^{VCG}(V)$ differ by $nE(v)$. This establishes (2.3). The equality in (2.4) now follows from the "accounting identity" $R = W - \Pi - \pi$. *Q.E.D.*

Proof of Proposition 4. When $(n - k)$ local bidders have dropped out and k are still active, a global bidder's optimal drop-out level is determined by

$$B_k^2(V) = \operatorname{argmax}_b \left\{ \int_{(B_k^2)^{-1}(p)}^{(B_k^2)^{-1}(b)} \left(\Pi_k^1(V, B_k^2(W)) + \sum_{\ell=k+1}^n (V_\ell - B_k^2(W)) \right) dG(W)^{k-1} \right\} \quad (\text{A.2})$$

The necessary first-order condition yields (2.5), which has a unique solution since the left side is strictly decreasing in $B_k^2(V)$. Moreover, the solution corresponds to a maximum since the second derivative of the objective function in (A.2) evaluated at $b = B_k^2(V)$ is negative. *Q.E.D.*

Proof of Proposition 5. We first prove that competition among global bidders lowers drop-out levels.

Lemma A1. $B_k^2(V) \leq B_{k+1}^1(V)$ for $k = 0, 1, \dots, n-1$.

Proof. For $k = n-1$, this follows from (2.6). To prove the lemma for $k \leq n-2$ note that when $B_k^2(V) = B_{k+1}^1(V)$ the left-side of (2.5) is equal to

$$\Pi_k^1(V, B_{k+1}^1(V)) + \sum_{\ell=k+1}^n (V_\ell - B_{k+1}^1(V)) = \sum_{\ell=k+2}^n (V_\ell - B_{k+1}^1(V)) \leq \sum_{\ell=k+2}^n (V_\ell - V_{k+1}) \leq 0$$

where the first equality follows since $\Pi_k^1(V, B_{k+1}^1(V)) = B_{k+1}^1(V) - V_{k+1}$ (see Proposition 1), the first inequality follows since $B_{k+1}^1(V) \geq V_{k+1}$, and the second inequality follows from Assumption 1. Since the left side of (2.5) is strictly decreasing in $B_k^2(V)$ the above inequality implies that $B_k^2(V) \leq B_{k+1}^1(V)$ for $k = 0, 1, \dots, n-1$. *Q.E.D.*

Next, consider the global bidder with the highest value, V , and suppose in the optimal allocation this global bidder is assigned k^* items. Once other global bidders have dropped out, social optimality follows from Proposition 2, i.e. $B_k^1(V) > v_k$ for $k = n - k^* + 1, \dots, n$ and $B_k^1(V) < v_k$ for $k = 1, \dots, n - k^*$. We need to show that all other global bidders drop out before $B_{n-k^*+1}^1$. This follows since for all $V' < V$ and $k = n - k^* + 1, \dots, n$, we have $B_k^K(V') = B_k^2(V') \leq B_{k+1}^1(V') \leq B_k^1(V') < B_k^1(V) \leq B_{n-k^*+1}^1(V)$. *Q.E.D.*

Proof of Proposition 6. The best global bidder wins $F(B^1(V)) = \kappa^*(V)$ items so welfare is maximized: $W^{SAA}(V) = W(V)$. To determine the best global bidder's profit note that she wins an optimal fraction of items $F(B^1(V))$, which she values at $\mathcal{V}(F(B^1(V)))$, and for which she pays

$$\int_0^V \{B^2(Z)F(B^2(Z)) + \int_{B^2(Z)}^{B^1(V)} y dF(y)\} dG(Z|V)^{K-1}$$

Here the first term in the integral corresponds to the items the best global bidder wins (all at once) when the second-best global bidder drops out at $B^2(Z)$, and the second term corresponds to the items she wins when local bidders subsequently drop out between $B^2(Z)$ and $B^1(V)$. Using (3.1) and (3.3), we can rewrite the global bidder's profit as stated in the proposition. (Note that for $K = 1$, the expression reduces to $W(V) - W(0) = W(V) - E(v)$.) Local bidders with values higher than $B^1(V)$ win an item at a price $B^1(V)$ and the total profits for local bidders as a group therefore are

$$\int_{B^1(V)}^1 (v - B^1(V)) dF(v) = \int_{B^1(V)}^1 (1 - F(v)) dv$$

where we used partial integration. The seller's revenue follows from $R = W - \Pi - \pi$. It is standard to verify the expressions for welfare, revenue, and profits for the VCG mechanism. *Q.E.D.*

Proof of Proposition 7. The expected payoff in (3.4) can be written as

$$\Pi(b, V) = \int_0^{B^{-1}(b)} \{W(V) - B(Z)F(B(Z)) - \int_{B(Z)}^1 y dF(y)\} dG(Z)^{K-1}$$

$$+ G(B^{-1}(b))^{K-1} \int_{B^1(V)}^b (\mathcal{V}'(F(y)) - y) dF(y)$$

The second line is maximized at $b = B^1(V)$ with $B^1(V) = F^{(-1)}(\kappa^*(V))$, as in the single global bidder case. The first line is maximized at $b = B^2(V)$ where $B^2(V)$ follows from (3.3). Furthermore, the derivative of the first line is negative for $b > B^2(V)$ and positive for $b < B^2(V)$. In other words, the first line is quasi-concave in b with a maximum at $b = B^2(V)$. Since $B^2(V) < B^1(V)$, the optimal bid $B(V)$ can never exceed $B^1(V)$ because lowering the bid to $B^1(V)$ would maximize the second line and result in an increase for the first line, which is downward sloping for all $b > B^2(V)$. *Q.E.D.*

Proof of Proposition 8. To establish the recursive relation, note that if at price level p there are k active local bidders then the global bidder will at least win $n - k$ items since $n - k$ local bidders have already dropped out. If the lowest of the active local bidders' values, v_k , is less than $B_k^1(V)$, then the global bidder's gain is the continuation profit $\Pi_{k-1}^1(V, v_k)$ plus $V_k - v_k$ for the additional item she wins, and she has to pay an extra $(n - k)(v_k - p)$ for the items she would anyhow have won. If v_k is greater than $B_k^1(V)$ then the global bidder wins no additional items but the price of the $(n - k)$ items she does win is raised from p to $B_k^1(V)$. *Q.E.D.*

Proof of Proposition 9. Recall that

$$B_k^1(V) = \operatorname{argmax}_b \left\{ \int_p^b (\Pi_{k-1}^1(V, v_k) + (V_k - v_k)) dF_k(v_k|p) - (n - k) \int_p^1 (\min(v_k, b) - p) dF_k(v_k|p) \right\}$$

The derivative of the term between curly brackets, evaluated at $b = B_k^1(V)$, is proportional to

$$\Pi_{k-1}^1(V, B_k^1(V)) + V_k - B_k^1(V) - \frac{n - k}{k} (1 - B_k^1(V)) \quad (\text{A.3})$$

For $k = 1$ we have $V_1 = V$ and $\Pi_0^1 = 0$ so the slope is $V - (n - 1) + (n - 2)B_1^1(V)$, which is positive for $V \geq n - 1$. So profits are increasing in $B_1^1(V)$ and, hence, maximized at $B_1^1(V) = 1$. We next prove, by induction, that for $1 \leq k < n$ and $V \geq n - 1$, the global bidder bids up to 1. Suppose with $k - 1$ active local bidders, the global bidder bids up to 1. To derive $\Pi_{k-1}^1(V, B_k^1(V))$ note that the global bidder pays $B_k^1(V) + (1 - B_k^1(V))(k - 1)/k$ for each of the $k - 1$ additional items she wins plus an additional $(1 - B_k^1(V))(k - 1)/k$ for each of the $n - (k - 1)$ items she is already winning:

$$\Pi_{k-1}^1(V, B_k^1(V)) = V - (n - 1) + (n - k)B_k^1(V) + \frac{n - k}{k} (1 - B_k^1(V))$$

For $k > 1$, (A.3) becomes (recall that complementarities are extreme so $V_{k>1} = 0$)

$$V - (n - 1) + (n - k - 1)B_k^1(V)$$

So the slope is positive and the profit increasing in $B_k^1(V)$ when $V \geq n - 1$ and $k \leq n - 1$. Hence, $B_k^1(V) = 1$ for $1 \leq k < n$ and $V \geq n - 1$. Furthermore, for $k = n$ and $V \geq n - 1$, the optimal bid simply follows from the first-order condition that the slope is zero, i.e. $B_n^1(V) = V - (n - 1)$. *Q.E.D.*

Proof of Proposition 10. When local bidders treat the items as substitutes, the global

bidder's optimal drop-out level satisfies $B(V) = \operatorname{argmax}_b(\mathcal{V}(F(b)) - bF(b))$, compared to $B^1(V) = \operatorname{argmax}_b(\mathcal{V}(F(b)) - \int_0^b v dF(v))$, when each local bidder is interested in a different item. Note that the objective functions in the maximization problems differ by $\int_0^b F(v)dv$ so the solutions differ unless $B(V) = B^1(V) = 0$, which can hold only for low values of V . When $B(V) \neq B^1(V)$, the first optimization problem implies

$$\mathcal{V}(F(B(V))) - B(V)F(B(V)) > \mathcal{V}(F(B^1(V))) - B^1(V)F(B^1(V))$$

and the second optimization problem implies

$$\mathcal{V}(F(B^1(V))) - \int_0^{B^1(V)} v dF(v) > \mathcal{V}(F(B(V))) - \int_0^{B(V)} v dF(v)$$

Hence, we have

$$\begin{aligned} B^1(V)F(B^1(V)) - B(V)F(B(V)) &> \mathcal{V}(F(B^1(V))) - \mathcal{V}(F(B(V))) \\ &> \int_{B(V)}^{B^1(V)} v dF(v) \\ &= B^1(V)F(B^1(V)) - B(V)F(B(V)) - \int_{B(V)}^{B^1(V)} F(v)dv \end{aligned}$$

which implies $B(V) < B^1(V)$.

Q.E.D.

Proof of Proposition 11. The SAA and VCG yield the same allocations, and the same profits, when the lowest of the two global bidders' values, Z , is less than $V^{**}(\alpha)$. When $Z \geq V^{**}(\alpha)$, the lowest-value global bidder bids up to $B^2(Z) > F^{(-1)}(\alpha)$ thereby driving local bidders with values between $F^{(-1)}(\alpha)$ and $B^2(Z)$ out of the market. This explains the welfare difference. Furthermore, if the lowest-value global bidder had dropped out at $F^{(-1)}(\alpha)$ the local bidders' profit would have been $\int_{F^{(-1)}(\alpha)}^1 (y - F^{(-1)}(\alpha)) dF(y) = \int_{F^{(-1)}(\alpha)}^1 (1 - F(y))/f(y) dF(y)$. Instead, the local bidders' profit is $\int_{B^2(Z)}^1 (y - B^2(Z)) dF(y) = \int_{B^2(Z)}^1 (1 - F(y))/f(y) dF(y)$. The equality of the global bidders' profit follows from an envelope argument, since the probability of obtaining α items is the same under the VCG and SAA mechanisms (i.e. the probability that a global bidder with value $V \geq V^*(\alpha)$ obtains α items is $G(V)$). Finally, the expression for the revenue difference follows from the identity: $R = W - \Pi - \pi$.

Q.E.D.

Proof of Proposition 12. The SAA and VCG yield the same allocations, and the same profits, when the lowest of the two global bidders' values, Z , is less than $V^{**}(\alpha)$. The difference in welfare and local bidders' profit when $Z \geq V^{**}(\alpha)$ follows from the discussion in the main text (see also the proof of Proposition 11). To understand the difference in the global bidders' profit, note that the SAA results in a higher probability of obtaining α items than the VCG mechanism. In particular, under the SAA mechanism this probability is $G(V)$ for $V < V^{***}(\alpha)$ and 1 for $V \geq V^{***}(\alpha)$, compared to $G(V)$ for $V < V^o(\alpha)$ and 1 for $V \geq V^o(\alpha)$ under the VCG mechanism. The global bidders' profit difference is therefore

$$\Pi^{SAA} - \Pi^{VCG} = 2 \int_{V^{***}(\alpha)}^1 \int_{V^{***}(\alpha)}^{\min(V, V^o(\alpha))} (1 - G(Z)) dZ dG(V) = \int_{V^{***}(\alpha)}^{V^o(\alpha)} \frac{1 - G(Z)}{g(Z)} dG_{\min}(Z)$$

The expression for the revenue difference follows from the identity: $R = W - \Pi - \pi$.

Q.E.D.

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