

# On the reserve price in all-pay auctions with complete information and lobbying games

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## Abstract

In this paper we study the equilibrium of (first-price) all-pay auctions under complete information and a positive reserve price. A seller with some bargaining power should set a positive reserve price (and increases efficiency). This has some implications for the economic theory of lobbying. In particular, once the possibility that the reserve price is optimally set is taken into account, it is dubious if the so-called Exclusion Principle survives (the seller of a political rent might find better to change his reserve price rather than to exclude a lobbyist especially eager to buy). Similarly, the effects of imposing a cap on individual contributions become unclear, since a tightening of the cap should change the optimal reserve price. We also argue that an appealing model should assume that the reserve price is set without knowing about the lobbyist valuations. Future work will have to assess whether the previous results can be confirmed in that setting.

*Keywords:* all-pay auctions, reserve price, economic theory of lobbying.

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## 1. Introduction

Auction models are prototypes of competitive settings, and they are used in several branches of the economic literature. In particular, the so-called (first-price) all-pay auction was used (among others) by Hillman and Riley (1989), Baye et alii (1993) and Che and Gale (1998) to model the lobbying process. This type of auction fits well the lobbying game, since a lobbyist's contribution is not typically returned if her efforts are unsuccessful,<sup>1</sup> and indeed this literature has elaborated a number of interesting results. In particular, Hillman and Riley (1989) prove that, if there is some asymmetry among bidders, the politically contestable rent is not totally dissipated even in the case of a large number of potential contenders. In addition, Baye et alii (1993) show that a politician wishing to maximize political rents may find it in his best interest to exclude certain lobbyists from the "finalists" short list (so-called "Exclusion Principle"), particularly lobbyists valuing most the political prize (in order to rise incentives to spend for the likely losers). Che and Gale (1998) also show that the imposition of an exogenous cap on individual lobbying contributions may have the perverse effect of increasing total expenditure (by increasing competition among lobbyists).

The previous results do contribute to the economic literature on the lobbying process, and in addition also to the pure theory of auction. From the former perspective, it is interesting that some of them (namely the Exclusion Principle and the possibly perverse effect of a cap on expenditure) appear not to hold in the alternative class of models so-called à la Tullock (1980): see Fang (2002). From the latter perspective, it has to be stressed that the quoted literature refers to the case of complete information,<sup>2</sup> which is a somewhat unusual assumption in auction theory. In fact, it is unclear which informational assumptions are made: in particular, the role and the information available to the designer (if any) of the auction is left unexplained.

In this paper we start with placing the (first-price) all-pay auction model with complete information in the context of the auction literature, and derive from this comparison some implications for the economic theory of lobbying (see Boylan, 2000 for a similar approach). In particular, we wish to argue that the previous results are implicitly but sometimes crucially based on the assumption that what would be called the "reserve price" in auction theory is null. In turn, this raises the questions of why the "auctioneer" (say a politician) cannot set a positive reserve price, and of which is the information ex ante available to him. The only consistent explanations seems to be that the politician who receives the lobbies' contributions does not know their preferences, or has very little bargaining power. After characterizing the equilibrium of the all-pay auction with an

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<sup>1</sup> This feature is also shared by other economic and social games, as patent races and sports.

<sup>2</sup> Hillman and Riley (1989: pp. 29-30) also deal with the case of incomplete information among contenders, and Che and Gale (1998: p. 648) claim that their result would hold even under incomplete information if there were asymmetry among bidders.

exogenous (positive) reservation price, we show that in any case the politician would prefer a strictly positive reserve price, which also increases the overall efficiency of the auction outcome (even if it might decrease the efficacy of the lobbying process through higher rent dissipation). This casts some doubts on the “perverse” results concerning the effects of caps on individual spending and lobbyist exclusion, because they might disappear once the possibility that the reserve price is optimally set is taken into account. Finally, we also argue that more robust results (if any) should be based on the explicit assumption that the politician does not know the lobbyists’ valuations. This is left for future work.

The paper is organized as follows: section 2 compares the (first-price) all-pay auction model with complete information with the so-called "standard auction" models. Section 3 characterizes the equilibrium of the former under an exogenously given positive reserve price. Section 4 consider the way an optimal (for the auctioneer) reserve price should be set, and section 5 discusses why this perspective casts some doubts on the Exclusion Principle and the result by Che and Gale (1998) concerning the effect of caps on the lobbies' expenditure. Section 6 briefly concludes.

## 2. Reserve price and auction theory

Consider the following setting:  $n$  (risk-neutral) agents bid for a prize (there is no resale possibility). Bidder  $i$ 's valuation of the prize is  $v_i$  ( $i = 1, n$ ), and we order bidders in such a way that  $v_1 > v_2 > \dots > v_{n-1} > v_n > 0$ .<sup>3</sup> The rules of the auction can include a reserve (minimum) price  $p_r \geq 0$ , i.e., a price below which the prize is not assigned. In an important contribution, Milgrom (1987) puts auction theory in the more general context of bargaining theory, and argues that (winner-pay) standard auctions (namely, the oral ascending or descending-bid and the first and second-price sealed-bid auctions) under complete information (i.e., assuming that every details of the setting is common knowledge to all the participants) lead to efficient outcomes. In particular, Milgrom (1987: Proposition 1, p. 7) argues that, under complete information, the set of perfect<sup>4</sup> equilibrium outcomes of the standard auctions consists of the Core outcomes of the corresponding exchange game (together with the no-trade outcome if  $p_r = v_1$ ), as the minimum price ranges from the "seller"

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<sup>3</sup> The possibility of ties in the valuations is ignored here. This can be justified by assuming that the  $v_i$  are ex-ante continuously independently distributed, and so that case has a priori a zero probability. Moreover, for standard auctions the only relevant implication of ties is that  $v_1 = v_2$  implies full rent dissipation, independently from the level of the reserve price (if not larger than  $v_1$ ) and from the number of contenders (on the contrary, in a all-pay auction ties may imply the existence of multiple Nash equilibria which are not necessarily revenue equivalent: see Baye et alii, 1996 and footnote 13 below).

<sup>4</sup> To avoid technical problems, Milgrom (1987: pp. 5-8) actually works with discrete bid spaces.

evaluation of the good to be sold to the highest "buyer" evaluation.<sup>5</sup> Indeed, while there are many Nash equilibria in those auction games, the only "sensible" (i.e., such that no agent ever uses weakly dominated strategies) seem to be those such that the prize is allocated to agent 1 for her bid of  $r = \text{Max}\{p_r, v_2\}$  (notice that a reserve price  $v_0 < p_r < v_2$ , where  $v_0$  is the seller's valuation of the prize, would have no effect on the auction outcome).

In particular, let us indicate with  $b_i$  the bid of agent  $i$ . As it is well known, in the case of a second-price sealed-bid auction  $b_i = v_i$ ,  $i = 1, n$ , does constitute a (weakly) dominant strategy equilibrium in which the prize goes to bidder 1 at a price  $r$  (as soon as  $p_r \leq v_1$ ). In the case of a first-price sealed-bid auction, the typical equilibrium has  $b_1 = r$ , and the strategies of the agents  $j = 3, n$  can be specified arbitrarily, provided that  $b_j \in [p_r, v_j]$  with probability 1 (if  $p_r > v_j$ , agent  $j$  bids arbitrary on  $[0, p_r]$ ). The cumulative distribution function of  $b_2$ ,  $F_2(b_2)$ , is continuous<sup>6</sup> on a support (weakly) included in  $[0, v_2]$  and first-order stochastically dominates  $G(b) = (v_1 - v_2)/(v_1 - b)$ . Notice that in both these auctions (and in the correspondent oral ones)<sup>7</sup> the equilibrium payoffs are  $U_1 = v_1 - r$  and  $U_j = 0$ ,  $j = 2, n$ , while the total payment to the seller is  $r$ . Milgrom (1987: p. 8) then observes that, from the perspective of (cooperative) game theory, the seller's ability to set any particular reserve price and stick to it measures his bargaining power.

What about an (first-price) all-pay auction version of the previous setting? In such a case, bidder  $i$  receives the prize if  $b_i > \text{Max}\{b_{j \neq i}\}$  and in that case her payoff is  $v_i - b_i$ , whereas her payoff is  $-b_i$  if she loses (ties are broken randomly). Assuming  $p_r = 0$ , Hillman and Riley (1989), and Baye et alii (1993) and (1996) show that in the unique Nash equilibrium agent 1 uses the uniform distribution  $F_1(b_1) = b_1/v_2$  on the support  $[0, v_2]$ , while agent 2 uses  $F_2(b_2) = 1 - v_2/v_1 + b_2/v_1$  on the same support (note that this amounts to say that agent 2 randomises between  $b_2 = 0$  and the uniform distribution on  $[0, v_2]$  with probabilities respectively  $1 - v_2/v_1$  and  $v_2/v_1$ ). Agents  $j = 3, n$  bid  $b_j = 0$  with probability 1. The prize is then given to agent 1 with probability  $1 - v_2/(2v_1) > 1/2$  and to agent 2 with probability  $v_2/(2v_1) < 1/2$  (note that in the latter event the result is not ex-post efficient, and thus it would not be stable in the case of a resale opportunity). Agent 1 receives a (expected) payoff of  $E\{U_1|v_1, v_2\} = v_1 - v_2$ , while the (expected) payoffs of the other agents are zero; i.e.,  $E\{U_j|v_1, v_2\} = 0$ ,  $j = 2, n$ . The expected total payment to the seller is  $E\{P|v_1, v_2\} = p(v_1, v_2) = p_1(v_1, v_2) + p_2(v_1, v_2) = v_2/2 + (v_2/v_1)(v_2/2) = v_2(1 + v_2/v_1)/2 < v_2$ , where  $p_i$  is the expected payment of agent  $i = 1, 2$ .

<sup>5</sup> Milgrom (1987: p. 3) also argues that, if the prize can be later resold, the outcome is stable and especially favourable to sellers in a relatively poor bargaining position with respect to the potential buyers.

<sup>6</sup> For  $p_r < v_2$ , agent 2 cannot play  $b_2 \notin [p_r, v_2]$  with a positive probability, since those strategies are (weakly) dominated for her: thus, she should bid between  $p_r$  and  $v_2$  with probability 1. If  $p_r \geq v_2$ , then also  $F_2(\cdot)$  is arbitrary on a support  $\subseteq [0, p_r]$ .

<sup>7</sup> As it is well known, the descending ("Dutch") auction is strategically equivalent to the first-price sealed-bid auction, while in an ascending ("English") auction, similarly to a second-price sealed-bid auction, is clearly a dominant strategy for each bidder  $i$  to stay in the bidding until the price reaches her value  $v_i$  (see e.g. Klemperer, 1999, paragraph 1.3).

The previous results show that the outcome of an (first-price) all-pay auction with a null reserve price does not belong to the Core of the corresponding exchange game (note that the expected social welfare is  $E\{W|v_1, v_2\} = v_1 - v_2 + p(v_1, v_2) < v_1$ ).<sup>8</sup> From the perspective of the economic theory of lobbying, they illustrate the possibility that, even if the number of potential contenders is large, asymmetries among players might imply that the political rent is not fully dissipated (see Hillman and Riley, 1989: pp. 18-19). In addition, note that  $\partial p/\partial v_1 < 0$  and  $\partial p/\partial v_2 > 0$ : indeed, Baye et alii (1993) show that a politician (the seller in the auction) wishing to maximize his political rents should be willing to select the two active lobbyists (the bidders)  $i^*$  and  $i^*+1$  in order to Max  $p(v_i, v_{i+1})$ . This implies that he might find it in his best interest to exclude lobbyists from 1 to  $i^*-1$  from his “finalists short list”, if he is allowed to (there is no point in excluding bidders from  $i^*+2$  to  $n$ ). This can be worthy to him because while the expected payment from any  $i \neq 1$  in the finalists list is necessarily less than the payment expected from 1 in the original auction, the expected payment from  $i+1$  may rise with respect to that of 2 and more than compensate the decrease of the other component of total payment. This is the Exclusion Principle, which is intuitively based on the idea to raise (overall) incentives to spend for the active participants by putting them on more equal foote. More formally, the Exclusion Principle works by raising the equilibrium probability of winning of the less favourite contender (between the two who are active in equilibrium). From the perspective of the economic theory of lobbying, Baye et alii (1993: p. 290) argues that the politician (the seller), under plausible circumstances, has a perverse incentive to preclude lobbyists most valuing the prize from participating in the lobbying game.

The idea of handicapping the favourite is simple, interesting and it has some counterpart both in the auction literature with incomplete information (if agent valuations are not identically distributed: see e.g. Myerson, 1981) and in the sport practice (e.g., in golf competitions).<sup>9</sup> However, note that bidder 1's exclusion decreases (weakly) ex-post efficiency (it gives a positive probability to the allocation of the prize to agent  $j > 2$  and no chance to 1), and tends<sup>10</sup> to decrease the expected social welfare (of course the outcome is not in the Core). Moreover, there might be other, possibly more efficient, ways (for a seller with some power to affect the auction rules) to motivate the less favourite contenders (for example offering, whether possible, multiple (divided) prizes: see Moldovanu and Sela, 2001). One such a way is investigated in Che and Gale (1998). They introduce an (symmetric) exogenous cap  $m$  on bids and show, in a setting with two contenders, that

<sup>8</sup> For the sake of simplicity, in computing social welfare we ignore  $v_0$  and use the gross seller utility given by  $p$  (net utility would be given by  $p - v_0$ ).

<sup>9</sup> Sport event organizers are typically interested in some "competitive balance" among players: a famous example comes from the history of the Giro d'Italia ("Tour of Italy"), the Italian most important cycling stage-race. It is reported that at the beginning of the twentieth century cyclist Alfredo Binda was so much stronger than his possible competitors (he had already won the Giro five times) that the organisers paid him not to participate.

<sup>10</sup> A sufficient but not necessary condition is  $v_2 < v_1/2$ .

the Nash equilibrium essentially<sup>11</sup> remains unique. However, while the expected utilities, payments and probability of winning of the agents if  $m \in (v_2/2, v_2)$  are the same than without any cap (the cap has of course no effect if  $m > v_2$ ),<sup>12</sup> for  $m < v_2/2$  the unique equilibrium is  $b_1 = b_2 = m$ , so that the expected total payment ( $2m$ ) increases for  $m \in (v_2(1 + v_2/v_1)/4, v_2/2)$ . The intuitive reason is that if the cap is small enough it reduces the ability of agent 1 to pre-empt his weaker competitor. In this case the prize is allocated randomly either to bidders 1 or 2 with probability 1/2: note that this increases the probability of ex-post inefficiency just because it raises the probability that 2 wins (which motivates her to pay more). Moreover, it always decreases expected welfare with respect to the case of no cap. Che and Gale (1998: p. 648) also show that similar circumstances can arise even if there are more than 2 potential contenders. They argue that limits on individual expenditure (such as the ones imposed by the USA Congress to the lobbies in the case of election campaigns, or due to limited financial endowments) may increase total expenditure and lower social surplus.

To conclude this section, we remember the reader that under incomplete information (i.e., if the valuation of each bidder is private information to herself), the picture of auction theory looks much more complex and the properties of the standard auctions heavily depend on the assumptions made on the informational aspects: see e.g. Klemperer (1999). However, in the benchmark case of (private) valuations independently drawn from a common, strictly increasing and atomless distribution, all (winner-pay) standard auctions (and many non-standard ones as the first-price all-pay auction) yield the same expected revenue to the seller and result in each bidder making the same expected payment as a function of her information. This is the famous Revenue Equivalence Theorem: see e.g. Klemperer (1999: p. 11). A corollary of this result is that, under the same assumptions, all the previous auctions are optimal for the seller if he imposes the optimal reserve price. Under the technical condition that each  $v_i$ ,  $i = 1, n$ , is ex-ante identically and independently distributed according to a cumulative distribution function  $H(v)$  with a monotonic hazard rate (i.e., if  $(1 - H(v))/h(v)$  is a decreasing function of  $v$ ), in a standard auction the optimal reserve price  $\hat{p}_r^*$  for a seller is defined by the condition  $\hat{p}_r^* = v_0 + (1 - H(\hat{p}_r^*))/h(\hat{p}_r^*) > v_0$ , and is then strictly positive: see e.g. Fudenberg and Tirole (1991: paragraph 7.5.1). The optimal reserve price  $\tilde{p}^*(n)$  in an all-pay auction with incomplete information is then given by  $\tilde{p}^*(n) = \hat{p}_r^* H^{n-1}(\hat{p}_r^*) > 0$ : see e.g. Klemperer (1999: p. 45). Note that  $\tilde{p}^*(n)$  is monotonically decreasing with respect to  $n$ , and  $\lim \tilde{p}^*(n) = 0$  for  $n \rightarrow \infty$ .

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<sup>11</sup> There are multiple equilibria for the non-generic cases of  $m = v_2/2$  and  $m = v_1/2$ .

<sup>12</sup> Note that, with a standard auction (and a null reserve price), an increasingly tight cap would decrease monotonically the expected revenue of the seller whenever  $m \leq v_2$  (in the equilibrium the prize is given randomly to some agent  $v_i$ , such that  $v_i > m$ , for a price  $m$ ).

### 3. The (first-price) all-pay auction with a positive reserve price

The results concerning the (first-price) all-pay auction with complete information we quoted in section 2 have been derived assuming a null reserve price. However, together with the Equivalence Revenue Theorem for incomplete information and Milgrom (1987) analysis, at the very least they suggest that it would be interesting to know what happens to the equilibrium outcomes under an exogenously given positive reserve price. This is the goal of this section. In fact, it turns out that the Nash equilibrium is unique. In particular, if  $v_1 \geq p_r \geq v_2$ , very much as in the case of a standard auction, the prize is allocated to agent 1 for here bid of  $p_r$ , while the other agents bid zero with probability 1 (if  $v_1 = p_r$  agent 1 is indifferent to receive the prize and there is another Nash equilibrium in which with probability 1 she bids  $b_1 = 0$  too).

Things are more interesting if  $p_r < v_2$ . Clearly, for each agent  $i$  the set of (weakly) undominated strategies is given by  $\{0 \cup [p_r, v_i]\}$ . Moreover, it can be shown that in equilibrium no bidder plays  $b_i \in (p_r, v_i)$ , and no more than one agent bid  $p_r$ , with a positive probability. This is so because if at least two of them do, both would have an incentive to move the mass slightly higher, so increasing their payoffs (the conditional probability of winning would jump, and so the payoff). If exactly one agent  $i$  has a mass point at some  $b_i \in (p_r, v_i)$ , then no other agent would place density immediately below that bid (it would be better to move that density above the mass point). But then agent  $i$  would do strictly better by moving the mass down (see Che and Gale, 1998: p. 645, Lemma 1, and Hillman and Riley, 1989: pp. 22-23, Proposition 1, for a formal proof). Thus, all equilibrium cumulative distribution function  $F_i(b_i)$  must be continuous on  $(p_r, v_i)$ .

Now note that agent 1 can secure herself a payoff equal to  $v_1 - v_2 > 0$  by bidding  $b_1 = v_2$  with probability 1. It follows that her equilibrium strategy support cannot include  $b_1 \in \{[0, p_r] \cup (v_2, v_1)\}$ . Suppose that there is an agent  $j \neq 1$  who gets in equilibrium a positive expected payoff. Then it must be the case that she bids  $b_j > p_r$  with probability 1 (she cannot neither bid zero nor bid  $p_r$  with positive probability, because otherwise she would get respectively a null and a negative payoff, while she should be indifferent among all bids that belong to the support of her own equilibrium cumulative distribution function  $F_j(b_j)$ ). And it must also be the case that her infimum bid does coincides with the infimum bid of agent 1, say  $\bar{b}$ , because otherwise at least one of them would get a negative payoff by bidding her own infimum bid. In fact, we have found a contradiction, because even by bidding  $\bar{b}$  at least one of them must get a negative payoff. Thus no agent other than 1 can get a positive payoff in the equilibrium, or bid  $p_r$  with a positive probability.

In addition, any agent different from 1 bidding more than  $p_r$  with a positive probability must have an “infimum” bid ( $\geq p_r$ ) not smaller than  $\bar{b}$  (otherwise she would get less than zero from that bid), and at least one must bid  $\bar{b}$  (otherwise it would pay to someone to move down some density).

Similarly, at least two agents must share the maximum bid, say  $b^+$ , larger than  $p_r$ . Let us now suppose that two agents different from 1, say  $j$  and  $h$ , bid more than  $p_r$  with a positive probability. It must then be the case that:

$$v_j \text{Prob}(j \text{ wins} | b_j = b) - b = v_j \prod_{i \neq j}^n F_i(b) - b = v_h \prod_{i \neq h}^n F_i(b) - b, \quad (1)$$

for any  $b > p_r$  belonging to the support of both  $F_j(\cdot)$  and  $F_h(\cdot)$ . This implies that, for *all* such a  $b$ :

$$\frac{F_h(b)}{F_j(b)} = \frac{v_h}{v_j}, \quad (2)$$

which implies that  $F_h(\cdot)$  *strictly* first-order stochastically dominates  $F_j(\cdot)$  if  $j < h$ . Let  $k$  the largest agent number among those bidding in equilibrium more than  $p_r$  with positive probability. This implies that the maximum bid larger than  $p_r$  belongs to the support of both  $F_1(\cdot)$  and  $F_k(\cdot)$ . In turn, this implies that:

$$v_k \text{Prob}(k \text{ wins} | b_k = b^+) - b^+ = v_k F_1(b^+) - b^+ = v_k - b^+ = 0, \quad (3)$$

but then by bidding  $b^+$  with probability 1 agent 2 would get a positive payoff, unless both  $k = 2$  and  $b^+ = v_2$ .

It follows that in equilibrium only agent 1 and 2 are active, with agent 1 using  $F_1(b_1)$  on a support  $[b^-, v_2]$ , while  $F_2(b_2)$  has possibly support  $\{0 \cup [b^-, v_2]\}$ . Since it must be the case that for any  $b \in [b^-, v_2]$ :

$$v_2 F_1(b) - b = 0, \quad v_1 F_2(b) - b \geq v_1 - v_2, \quad (4)$$

we can conclude that  $b^- = p_r$ , that  $F_2(b_2) = 1 - v_2/v_1 + b_2/v_1$  has in fact the support  $\{0 \cup [p_r, v_2]\}$  (note that this amounts to say that agent 2 randomizes between  $b_2 = 0$  and the uniform distribution on  $[p_r, v_2]$  with probabilities respectively  $1 - (v_2 - p_r)/v_1$  and  $(v_2 - p_r)/v_1$ ), and that agent 1 uses  $F_1(b_1) = b_1/v_2$  on the support  $[p_r, v_2]$  (she randomizes between  $b_2 = p_r$  and the uniform distribution on  $[p_r, v_2]$  with probabilities respectively  $p_r/v_2$  and  $1 - p_r/v_2$ ). The equilibrium cumulative distribution function of agents 1 and 2 are illustrated in Figure 1.

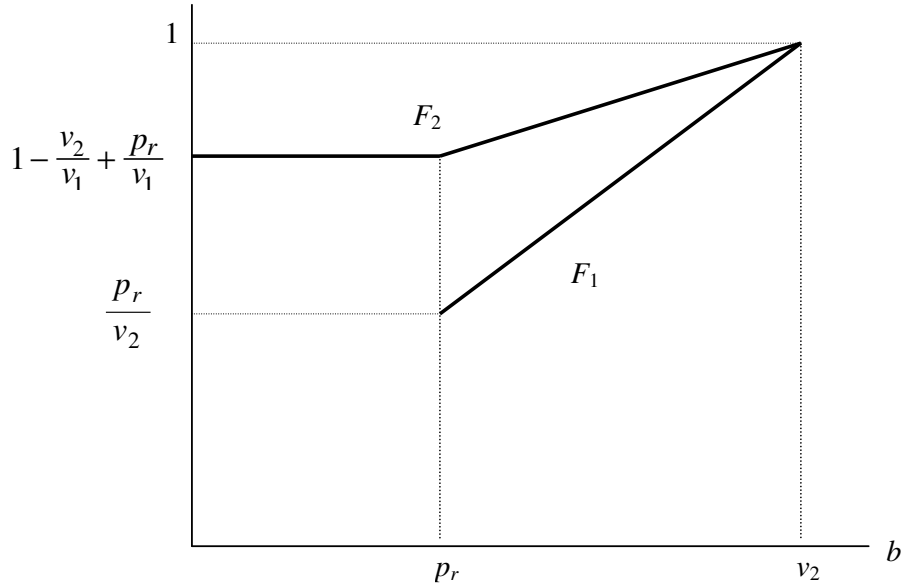


FIGURE 1: The equilibrium distribution functions for  $p_r < v_2$

Note that, again very much as in the case of a standard auction, and exactly as in the case without a positive reserve price, agent 1 receives an (expected) payoff of  $E\{U_1|v_1, v_2, p_r\} = v_1 - v_2$ , while the (expected) payoffs of the other agents are zero. However, the prize is now won by agent 1 with probability  $1 - (v_2^2 - p_r^2)/(2v_1v_2) > 1 - v_2/(2v_1) > 1/2$ ; i.e., the introduction of a positive reserve price raises the probability of an ex-post efficient outcome by raising the probability that the prize is allocated to agent 1. Moreover, the expected total payment to the seller is given by ( $v_2 \geq p_r$ ):

$$E\{P|v_1, v_2, p_r\} = \tilde{p}(v_1, v_2, p_r) = \tilde{p}_1(v_1, v_2, p_r) + \tilde{p}_2(v_1, v_2, p_r) = \frac{v_2^2 + p_r^2}{2v_2} + \frac{v_2^2 - p_r^2}{2v_1} \quad (5)$$

(note that  $\tilde{p}(v_1, v_2, p_r)$  is continuous for any  $v_1 > v_2 \geq p_r \geq 0$ ).<sup>13</sup> Equation (5) shows that, as it should be expected, the payment by agent 1 increases (on expectation), while the payment of agent 2 decreases, with respect to the case of a null reserve price. In particular, the increase is given by  $(p_r^2/2)(v_2^{-1} - v_1^{-1})$ : note that  $\partial \tilde{p} / \partial p_r > 0$  and that  $\tilde{p}(\cdot)$  goes continuously from  $p(\cdot)$  to  $v_2$  as  $p_r$  goes from 0 to  $v_2$ . This is described in Figure 2. Note, finally, that since  $E\{W|v_1, v_2, p_r\} = v_1 - v_2 + \tilde{p}(v_1, v_2, p_r)$ , also the expected social welfare increases with respect to the case of a null reserve price.

<sup>13</sup> For  $v_1 = v_2 \geq p_r > 0$  there is more than 1 Nash equilibrium, and equation (5) does not apply to all of them (see footnote 3).

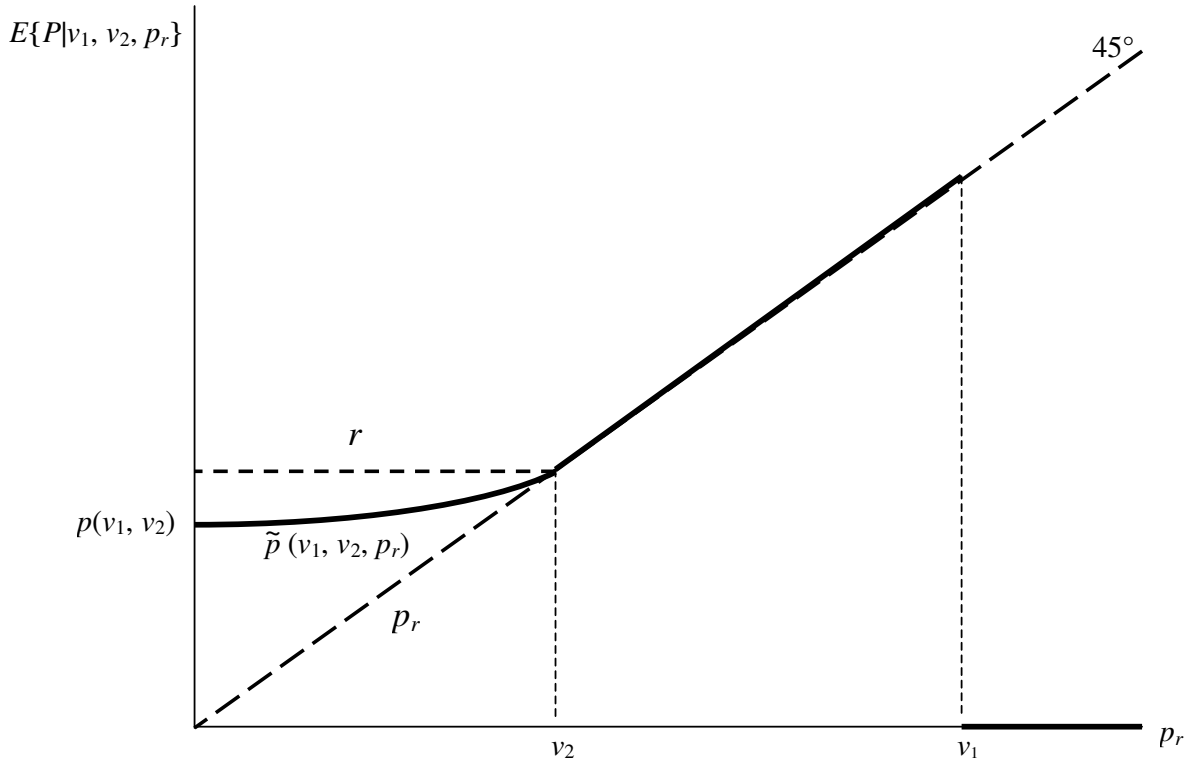


FIGURE 2: The expected revenue as a function of  $p_r$

#### 4. The optimal reserve price

Section 3 shows that the outcomes of the (first-price) all-pay auction with complete information do not coincide with the outcomes of the standard auctions if  $p_r < v_2$ , and are not efficient (they do not belong to the Core). Following Milgrom (1987), we might say that this appears to describe the case of a weak seller in terms of bargaining power (indeed, so weak that he seems ready to accept less than  $v_0$ , unless the latter value is “sufficiently” small; i.e.,  $v_0 < \tilde{p}(v_1, v_2, p_r)$ ). On the contrary, it is clear that a fully-informed strong seller would use  $p_r > v_2$ , and in fact  $p_r = v_1$  (i.e., he would make a take-it-or-leave-it offer to agent 1). Accordingly, we would like to argue that the only assumptions consistent with the use made in the quoted literature of a complete information setting is that the seller, in contrast with the bidders, either is very weak or he does not know their valuations. Let us now consider the latter case, by referring to the Equivalent Revenue Theorem assumptions: namely, we assume that each  $v_i$  is ex-ante identically and independently distributed according to a strictly increasing and atomless distribution function  $H(v)$  with support  $[\underline{v}, \bar{v}]$ , and a monotonic hazard rate.

Since the density function of the joint distribution of the first and second order statistics (see e.g. Mood et alii, 1974: p. 254) is given by:

$$g(v_1, v_2) = (n^2 - n)H^{n-2}(v_2)h(v_1)h(v_2)I_{(v_2, \infty)}(v_1), \quad (6)$$

it follows that the seller should set the reserve price by maximizing:<sup>14</sup>

$$\begin{aligned} P^E(p_r) &= \int_{\underline{v}}^{\bar{v}} \int_{\underline{v}}^{\bar{v}} E\{P|v_1, v_2, p_r\} g(v_1, v_2) dv_1 dv_2 \\ &= (n^2 - n) \int_{\underline{v}}^{\bar{v}} \int_{\underline{v}}^{\bar{v}} E\{P|v_1, v_2, p_r\} H^{n-2}(v_2) h(v_1) h(v_2) dv_1 dv_2 \\ &= (n^2 - n) \left\{ p_r \int_{\underline{v}}^{p_r} \int_{\underline{v}}^{\bar{v}} H^{n-2}(v_2) h(v_1) h(v_2) dv_1 dv_2 + \int_{p_r}^{\bar{v}} \int_{p_r}^{\bar{v}} \tilde{p}(v_1, v_2, p_r) H^{n-2}(v_2) h(v_1) h(v_2) dv_1 dv_2 \right\} \\ &= np_r (1 - H(p_r)) H^{n-1}(p_r) + (n^2 - n) \int_{p_r}^{\bar{v}} \int_{p_r}^{\bar{v}} \tilde{p}(v_1, v_2, p_r) H^{n-2}(v_2) h(v_1) h(v_2) dv_1 dv_2 \end{aligned} \quad (7)$$

with respect to  $p_r$ , where  $P^E(p_r) = E\{E\{P|v_1, v_2, p_r\}\}$  is a continuous and differentiable function.

Note that  $P^E(\bar{v}) = 0$ , and  $P^E(\underline{v}) > 0$ . A bit of computation shows that:

$$\frac{dP^E(p_r)}{dp_r} = nH^{n-1}(p_r)h(p_r) \left( \frac{1-H(p_r)}{h(p_r)} - p_r \right) + (n^2 - n)p_r \int_{p_r}^{\bar{v}} \int_{p_r}^{\bar{v}} \left( \frac{1}{v_2} - \frac{1}{v_1} \right) H^{n-2}(v_2) h(v_1) h(v_2) dv_1 dv_2, \quad (8)$$

which implies  $dP^E(\bar{v})/dp_r < 0$ , and  $dP^E(\underline{v})/dp_r > 0$  if  $\underline{v} > 0$  (otherwise  $dP^E(\underline{v})/dp_r = 0$ ). Thus the optimal reserve price  $p_r^*(n)$  is larger than  $\underline{v}$ ,<sup>15</sup> smaller than  $\bar{v}$ , and must satisfy  $dP^E(p_r^*)/dp_r = 0$ .

To give an example, suppose that  $H(v)$  is the uniform distribution with support  $[0,1]$ . Then (8) becomes:

$$\begin{aligned} \frac{dP^E(p_r)}{dp_r} &= np_r^{n-1} (1 - 2p_r) + (n^2 - n)p_r \int_{p_r}^1 \int_{p_r}^1 \left( v_2^{n-3} - \frac{v_2^{n-2}}{v_1} \right) dv_1 dv_2 \\ &= np_r^{n-1} (1 - 2p_r) + (n^2 - n)p_r \int_{p_r}^1 \left( v_2^{n-3} - v_2^{n-2} + v_2^{n-2} \ln v_2 \right) dv_2 \\ &= np_r^{n-1} (1 - 2p_r) + np_r [(n-1) \int_{p_r}^1 v_2^{n-3} dv_2 - 2 + p_r^{n-1} + p_r^{n-2} - p_r^{n-1} \ln p_r]. \end{aligned} \quad (9)$$

which is somehow difficult to solve explicitly. However, it is easy to see that  $p_r^*(2)$  must satisfy  $\ln p_r^*(2) = -p_r^*(2)/(1 + p_r^*(2))$ , and thus it is larger than  $E\{v_i\} = 1/2$ . The latter would be the optimal reserve price  $\hat{p}_r^*$  (which does not depend on  $n$ ) in a standard auction under incomplete

<sup>14</sup> For the sake of simplicity, in the following we assume  $v_0 = 0$ .

<sup>15</sup> See Appendix 1 and footnote 17.

information, which in turn is larger than the correspondent optimal reserve price  $\bar{p}^*(2) = 1/4$  of an all-pay auction (see section 2). It can be shown that  $p_r^*(2)$  is on the contrary lower than the price, say  $\bar{p}_r^*(2)$ , a seller in a strong bargaining position might ask (simultaneously) to two agents by a take-it-or-leave-it offer, in the case of uniformly, independently and identically distributed  $v_i$  on  $[0,1]$ . The latter price maximizes  $p(1 - p^2)$  (more generally, the optimal take-it-or-leave-it offer maximizes  $p(1 - H^n(p))$ : see Appendix 1). The obvious reason is that, in contrast with the case of standard auctions, with complete information in an all-pay auction the expected revenue is an increasing (strictly convex) function of  $p_r$  if  $p_r < v_2$ . Similarly, in the case of a take-it-or-leave-it offer, the revenue is exactly  $p_r$  even if  $p_r < v_2$  (see Figure 2). In fact, the previous results hold for any value of  $n \geq 2$  and any cumulative distribution function with a monotonic hazard rate (see Appendix 1). As intuition would suggest, even under complete information the seller strictly gains (in expected terms) by being able to organize an auction, and more by a standard auction than by an all-pay.

## 5. Implications for the economic theory of lobbying

The literature which uses the all-pay auction with complete information to model the lobbying process (Hillman and Riley (1989), Baye et alii (1993) and Che and Gale (1998)) appears to have assumed (somehow implicitly) that the reserve price is null. The previous sections show that this amounts to assume that the seller (the politician) is in a very weak bargaining position, so that the resulting outcomes are not efficient. While this may occasionally be the case, it seems clear that in the general case the politician will be able to set a positive reserve price, since this monotonically increases his expected payoff. Indeed, in the opposite polar case of a politician with a strong bargaining power, we should expect a reserve price at least equal to  $v_2$  (if not  $v_1$ ), and thus an efficient result. In such a case, there will not generally be full rent dissipation (unless in the extreme cases of either  $v_1 = v_2$  or  $p_r = v_1$ ), independently from the number of competitors. But, clearly, the Exclusion Principle does not apply, since it will always be better for the politician to use a reserve price large enough ( $p_r \geq v_2$ ) rather than to exclude from the “finalists short list” some of the lobbyists valuing most the political prize.

However, the situation is more complex if the politician is not strong enough to set a reserve price  $p_r \geq v_2$ . Since  $\partial \bar{p} / \partial v_1 < 0$  and  $\partial \bar{p} / \partial v_2 > 0$ , it is still possible that the exclusion of some agents is in the interest of the politician. In particular, he should choose  $i, j (> i)$  and  $p_r$  in order to maximize  $E\{P|v_i, v_j, p_r\}$  under the “bargaining” constraints he faces. Notice that, if the reserve price that the politician can adopt does not depend on the agents he selects, he will always choose the

largest possible reserve price, say  $p_r^+$ , and also agent  $i+1$  when he chooses agent  $i$  (things are very much as in Baye et alii, 1993). But, in such a case, it cannot be optimal to exclude agent from 1 to  $i-1$  if  $p_r^+ \geq v_{i+1}$  (since  $\tilde{p}(v_i, v_{i+1}, p_r) > p_r$ ), which implies that there will be no exclusion at all if  $p_r^+ \geq v_3$ . Moreover, the assumption that a fully informed politician can credibly exclude some lobbyist from his list while he is unable to ask her a price not larger than her valuation does not seem particularly palatable as a bargaining feature.

Similarly, one might conjecture that a symmetric cap on the individual bids would always be matched by the reserve price set by an informed politician with a bargaining power large enough (i.e.,  $p_r = m$ ). Thus, a cap would always decrease the total spending with respect to the case of no cap, decreasing the efficiency of the auction (but possibly raising overall efficiency if campaign spending is *per se* socially harmful: see Che and Gale, 1998). However, while this is certainly so if  $m \in (v_2, v_1)$ , for smaller (binding) values of  $m$  the effect of a cap on the reserve price optimal for the seller still need to be investigated. In particular, Che and Gale (1998: p. 648) show that, if the reserve price is null, the perverse effect they identified arises when  $v_k/k > m > v_{k+1}/(k+1)$  for some  $k < n$ . In such a case, the revenue expected by the politician is  $km$ , which can be (arbitrary) close to  $v_k$ , and then possibly larger than  $p(v_1, v_2)$ . But, again, a politician with some bargaining power might be able to set  $p_r \geq v_k$ , raising his revenue *at least* (if  $v_2 > p_r$ ) to  $\tilde{p}(v_1, v_2, v_k) > v_k$ . Thus, the imposition of a cap might de facto decrease overall spending, once the optimal reserve price set by the politician is kept into account.

In fact, we have already argued that perhaps a more consistent assumption is that the seller (the politician) does not know the contenders valuations when he sets the reserve price. In such a setting, both the effect of excluding a potential participant and the effect of imposing a cap on the individual bids need to be re-evaluated. In particular, results from the case of the standard auctions under incomplete information appear to suggest that the seller should not find useful to exclude symmetric participants. In particular, Bulow and Klemperer (1996) proved that an English auction with  $n+1$  bidders and *no* reserve price is always more profitable than *any* negotiation with just  $n$  participants. To grasp the idea, consider the case of a seller that can either make a take-it-or-leave-it offer to a buyer whose valuation for the prize is uniformly distributed on  $[0,1]$ , or to run a all-pay auction with complete information for two participants whose valuations are identically and independently uniformly distributed on the same support  $[0,1]$ . The first opportunity yields him an expected payoff of 0,25 (as we know, he should ask a “reserve price” equal to  $1/2$ ). In the second case he should expect the larger payoff of:

$$P^E(0) = 2 \int_0^1 \int_{v_2}^1 \tilde{p}(v_1, v_2, 0) dv_1 dv_2 = \frac{5}{18} \quad (10)$$

(the seller would get  $\hat{P}^E = 1/3$  if he could use a standard auction), and even more by using a positive reserve price. Can this result be generalized? In the previous case of uniformly distributed valuations it can be show that  $\partial P^E(p_r^*(n))/\partial n > 0$  for  $n \in [2,3]$  (see Appendix 2). Accordingly, in such a case, it is never profitable to exclude one over three potential (symmetric) lobbyists.

While we leave this issue to future work, we have to mention that an assessment of the Exclusion Principle should rather refer to the case of asymmetric distributions of the valuations. However, even in those cases, we conjecture that it might well be that the seller will generally not find profitable to exclude the contender “more eager” to buy, once he can set optimally the reserve price. Here, we content ourselves with the following examples. Suppose that there are three possible participants, and everybody knows that valuations are  $v_2 = 40$ ,  $v_3 = 38$ ,<sup>16</sup> and that  $v_1$  is distributed on  $[47, \bar{v}]$  according to a cumulative distribution function  $H_1(v_1)$ . It is also common knowledge that the agents know the realized value of  $v_1$ . If  $p_r = 0$ , the seller should expect an equilibrium revenue of less than  $p(47, 40) \approx 37,02$  when all participate. If agent 1 is excluded, this gives the seller an expected payoff of 37,05 but, clearly, he can do better by excluding none, setting  $p_r = 47$  and receiving that amount for sure. In fact, if  $1/h_1(47) > 47$ , he should set the higher reserve price such that  $p_r = (1 - H_1(p_r))/h_1(p_r)$ , and expects an even larger revenue equal to  $(1 - H_1(p_r))^2/h_1(p_r)$ . Similarly, suppose that, while it is common knowledge that  $v_3 = 38$  and that  $v_1$  and  $v_2$  are known to the agents, the seller just knows that  $H_1(v_1)$  and  $H_2(v_2)$  have the same support  $[40, \bar{v}]$ , with  $H_1(v) \leq H_2(v)$ . Again, by excluding agent 1 but setting no reserve price the seller cannot achieve more than 37,05. However, he should expect no less than 40 by optimally setting his reserve price  $p_r \geq 40$  and excluding no agent. Moreover, whatever reserve price larger than 40 he might set, he would get a larger expected payoff by excluding agent 2 rather than agent 1.

Finally, let us go back to the case in which both  $v_2$  and  $v_3$  are independently, identically and uniformly distributed on  $[0,1]$ , while  $v_1$  is distributed on the same support with  $H_1(v) < v$  for  $0 < v < 1$ . By excluding agent 1 and optimally setting his reserve price for the remaining two agents the seller cannot achieve more than:

$$\hat{P}^E\left(\frac{1}{2}\right) = \frac{1}{4} + 2 \int_{1/2}^1 (v_2 - v_2^2) dv_2 = \frac{5}{12}, \quad (11)$$

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<sup>16</sup> These numbers are taken by Baye et alii (1993: p. 293).

where  $\hat{P}^E(1/2)$  is the payoff he could get by running a standard auction. However, by excluding no agent and optimally setting his reserve price he can get more than:

$$\bar{P}^E\left(\frac{1}{2}\right) = \frac{1}{2}\left(1 - \frac{1}{4}H_1\left(\frac{1}{2}\right)\right) > \frac{7}{16} > \frac{5}{12}, \quad (12)$$

where  $\bar{P}^E(1/2)$  is the payoff he could get by a take-it-or-leave-it offer.

## 6. Conclusions.

In this paper we have characterized the equilibrium of (first-price) all-pay auctions under complete information and a positive reserve price. Even in these auctions, a seller with some bargaining power should be happy to set a positive reserve price, since this is profitable for him (and increases overall efficiency). Moreover, once the possibility that the reserve price is optimally set is taken into account, it is unclear if some interesting recent findings in the economic theory of lobbying still apply. In particular, in order to increase his total revenue, the seller (of a political rent) might find better to adapt his reserve price rather than to exclude a lobbyist especially eager to buy. Similarly, the effects of imposing a cap on the individual campaign contributions become dubious, since a tightening of the cap, that might increase the overall spending, should also change the optimal reserve price. Finally, we have also argued that, in partial contrast to the literature, an appealing model should assume that the reserve price (if any) is set without knowing about the lobbyist valuations. Future work will have to assess if the previous results can be confirmed in that setting.

## Appendix 1

In the case of a standard auction the seller would set the reserve price by maximizing:

$$\begin{aligned} \hat{P}^E(p_r) &= np_r(1 - H(p_r))H^{n-1}(p_r) + (n^2 - n) \int_{p_r}^{\bar{v}} \int_{p_r}^{\bar{v}} v_2 H^{n-2}(v_2) h(v_1) h(v_2) dv_1 dv_2 \\ &= np_r(1 - H(p_r))H^{n-1}(p_r) + (n^2 - n) \int_{p_r}^{\bar{v}} v_2 H^{n-2}(v_2) [1 - H(v_2)] h(v_2) dv_2. \end{aligned} \quad (A.1)$$

Note that  $\hat{P}^E(p_r) > P^E(p_r)$  for any  $p_r < \bar{v}$  (as it is clear from Figure 2). Again,  $\hat{P}^E(\bar{v}) = 0$ , and  $\hat{P}^E(\underline{v}) > 0$ . Computation shows that:

$$\frac{d\hat{P}^E(p_r)}{dp_r} = nH^{n-1}(p_r)h(p_r)\left(\frac{1 - H(p_r)}{h(p_r)} - p_r\right), \quad (A.2)$$

which implies  $d\hat{P}^E(\bar{v})/dp_r < 0$  and  $d\hat{P}^E(\underline{v})/dp_r = 0$ . Thus, also in the case of standard auctions, the optimal reserve price  $\hat{p}_r^*$  is generally<sup>17</sup> in the interior of the support  $[\underline{v}, \bar{v}]$ . Note that  $\hat{p}_r^*$  does not depend on the number of agents and, under the assumption of a monotonic hazard rate, it is uniquely identified by the condition  $\hat{p}_r^* = (1 - H(\hat{p}_r^*)) / h(\hat{p}_r^*)$ . This is, of course, just a consequence the Equivalence Revenue Theorem (clearly, the revenue in A.1 does coincide with that expected by a seller using a (winner-pay) second-price sealed-bid auction under incomplete information). Under the assumption of uniformly distributed valuations on  $[0,1]$ , (A.2) then implies  $\hat{p}_r^* = 1/2$ . More generally, note that the right-hand side in (A.2) does coincide with the first term in the right-hand side in (8), while the second term in the latter expression is always positive for  $p_r \in (\underline{v}, \bar{v})$ . It follows that necessarily  $p_r^*(n) > \hat{p}_r^*$  for all  $n \geq 2$  if the hazard rate is monotonic.

Similarly, by using a take-it-or-leave-it offer the seller would maximize:

$$\begin{aligned} \bar{P}^E(p_r) &= np_r(1-H(p_r))H^{n-1}(p_r) + (n^2 - n)p_r \int_{p_r}^{\bar{v}} \int_{p_r}^{\bar{v}} H^{n-2}(v_2)h(v_1)h(v_2)dv_1dv_2 \\ &= p_r(1-H^n(p_r)) \end{aligned} \quad (\text{A.3})$$

with respect to  $p_r$ . Note that  $\bar{P}^E(p_r) < P^E(p_r)$  for any  $p_r < \bar{v}$ ,  $\bar{P}^E(\bar{v}) = 0$  and  $\bar{P}^E(\underline{v}) = \underline{v}$  (also notice that  $\partial \bar{P}^E(p_r) / \partial n > 0$  for all  $p_r \in (\underline{v}, \bar{v})$ ). Computation shows that:

$$\frac{d\bar{P}^E(p_r)}{dp_r} = nH^{n-1}(p_r)h(p_r)\left(\frac{1-H(p_r)}{h(p_r)} - p_r\right) + (n^2 - n) \int_{p_r}^{\bar{v}} \int_{p_r}^{\bar{v}} H^{n-2}(v_2)h(v_1)h(v_2)dv_1dv_2, \quad (\text{A.4})$$

which implies  $d\bar{P}^E(\bar{v})/dp_r < 0$ , and  $d\bar{P}^E(\underline{v})/dp_r > 0$ . Thus, once again the optimal “reserve” price  $\bar{p}_r^*$  is in the interior of the support  $[\underline{v}, \bar{v}]$ . Note that the first term in the right-hand side in (A.4) does coincide with the first term in the right-hand side in (8), while the second term in the latter expression is always smaller than the correspondent term in (A.4) for  $p_r \in (\underline{v}, \bar{v})$ . It follows that necessarily  $\bar{p}_r^*(n) > p_r^*(n)$  for all  $n \geq 2$  if the hazard rate is monotonic. Obviously,  $\bar{P}^E(\bar{p}_r^*(n)) < P^E(p_r^*(n)) < \hat{P}^E(\hat{p}_r^*)$ .

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<sup>17</sup> The optimal reserve price  $\bar{p}_r^*$  is equal to  $\underline{v}$  only if  $1/h(\underline{v}) \leq \underline{v}$ .

## Appendix 2

By taking the derivative of  $P^E$  with respect to  $n$  and using  $E\{P|v_1, v_2, p_r\} \geq p_r$  one gets, for any  $p_r$ :

$$\frac{\partial P^E(p_r)}{\partial n} = \int_{\underline{v}}^{\bar{v}} \int_{\underline{v}_2}^{\bar{v}_2} [(2n-1) + (n^2-n) \ln H(v_2)] E\{P|v_1, v_2, p_r\} H^{n-2}(v_2) h(v_1) h(v_2) dv_1 dv_2. \quad (\text{A.5})$$

Since the identity:

$$(n^2-n) \int_{\underline{v}}^{\bar{v}} \int_{\underline{v}_2}^{\bar{v}_2} H^{n-2}(v_2) h(v_1) h(v_2) dv_1 dv_2 \equiv 1 \quad (\text{A.6})$$

implies:

$$(2n-1) \int_{\underline{v}}^{\bar{v}} \int_{\underline{v}_2}^{\bar{v}_2} H^{n-2}(v_2) h(v_1) h(v_2) dv_1 dv_2 = -(n^2-n) \int_{\underline{v}}^{\bar{v}} \int_{\underline{v}_2}^{\bar{v}_2} H^{n-2}(v_2) \ln H(v_2) h(v_1) h(v_2) dv_1 dv_2, \quad (\text{A.7})$$

it follows from (A.5) that:

$$\frac{\partial P^E(p_r)}{\partial n} = \int_{p_r, v_2}^{\bar{v}} \int_{\underline{v}}^{\bar{v}} [(2n-1) + (n^2-n) \ln H(v_2)] \tilde{p}(v_1, v_2, p_r) H^{n-2}(v_2) h(v_1) h(v_2) dv_1 dv_2. \quad (\text{A.8})$$

Assuming that  $H(\cdot)$  is the uniform distribution on the support  $[0,1]$ , (A.8) becomes

$$\frac{\partial P^E(p_r)}{\partial n} = \int_{p_r, v_2}^1 \int_{\underline{v}}^1 [(2n-1) + (n^2-n) \ln v_2] \tilde{p}(v_1, v_2, p_r) v_2^{n-2} dv_1 dv_2, \quad (\text{A.9})$$

which is always positive when evaluated at  $p_r \geq \frac{1}{2}$  and  $3 \geq n \geq 2$ . Since we know that  $p_r^*(n) \geq \frac{1}{2}$  in the cases of identically, independently and uniformly distributed valuations, the Envelope Theorem implies that  $P^E(p_r^*(3)) > P^E(p_r^*(2))$ .

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