A Random Matching Theory*

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ABSTRACT: We develop the theoretical underpinnings of pairwise random matching mechanisms. We formalize the mechanics of matching, and study the links between properties of the different mechanisms and trade frictions. A particular emphasis is placed on providing exact mappings between matching technologies and informational constraints.

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1 Introduction

A large segment of the economic literature is concerned with the study of allocations that arise when markets are not well functioning. A defining characteristic of this literature is its focus on informational and spatial frictions, and the desire to make them explicit by assuming that economic interactions occur in small coalitions. To this end, the literature has traditionally relied on pairwise random matching frameworks. This basic modeling tool has found use in a wide variety of settings, from the study of social norms (as in Ellison [5]), to unemployment (as in Mortensen and Pissarides [11]), to business cycles (as in Diamond and Fudenberg [4]), and to the foundations of monetary theory (as in Kiyotaki and Wright [7]).

A limitation of this literature is the treatment of matching—as a technology—is mostly descriptive and hazily formalized. For example, the mechanics of the economic interactions are generally not made explicit or the map between matching and the frictions assumed to be in place is vague. This tends to prevent a clear understanding of how the matching

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technology impairs market functioning, and consequently the possible allocations. These limitations must be overcome to better formulate models of economies with frictions. An objective economic analysis is thought of as one that focuses on the allocations predicted using a carefully specified physical environment (preferences, technologies, etc.). Thus, a comprehensive theory of exchange cannot be derived by simply assuming that certain economic interactions may or may not take place. Ideally, the theory should clarify how the trading or institutional constraints assumed to be in place originate in the underlying economic environment.

The purpose of this study is to build a solid foundation for random matching models, by means of a fully integrated set-theoretic approach. There are two major contributions. First, the paper provides a rigorous formalization of the *mechanics* of random pairwise matching. To do so, it uses as a starting point the approach to deterministic matching provided by Aliprantis, Camera and Puzzello [1]. By focusing on the *technological* aspects of meeting processes, this study adds to a research discourse focused on advancing our understanding of the inner workings of matching models as done in Ioannides [9] and Gilboa and Matsui [6], for example.

A second contribution of this investigation is it spells out how different matching technologies may facilitate (or obstacle) the exchange of economic resources and information among agents. Particular emphasis is paid to formalizing how the matching technology's properties affect the level of *informational isolation* that exists in economies where agents are randomly paired over time. Research that has taken into consideration these concerns has appeared in the works of Kocherlakota [8], Corbae and Ritter [2] and Corbae, Temzelides, and Wright [3].

The technical procedure that we use to construct any random matching process involves three basic steps. The first step is to specify how to divide the population in each period into spatially separated clusters of agents. To do so, we use partitional correspondences. Then, one must define and calculate all possible ways to form pairs in each cluster. In this case, we resort to using a class of permutation functions—the so-called involutions. Finally, for each period one must specify a probability measure over all possible pairings, for each cluster. This gives us the desired random matching rule for a cluster, and a well-defined random matching process for the entire population in each period. A pairwise random matching framework can then be formalized as a sequence of partitional correspondences, involutions and probability measures. Given these sequences, we can then *explicitly* specify matching histories, and therefore we can rigorously formalize the degree of informational isolation that exists among agents.

The paper is organized as follows. Section 2 introduces the mathematical preliminaries needed for this study. Sections 3 and 4 discuss pairwise random matching in a single period and over time, and characterize matching mechanisms according to the degree of informational isolation they can sustain. In Section 5 we demonstrate how random matching economies can be constructed in which traders are completely anonymous. We then present in Section 6 an application of our theoretical construct to random matching models of money. We offer some final remarks in Section 7.

2 Mathematical Background

This section contains the mathematical notions that will be used extensively in this work. Since our objective is to explain how to match sets of agents, we start by discussing several set-theoretic notions.

If A is an arbitrary set, then $|A|$ denotes its cardinality. As usual, $|A| = \mathcal{N}_0$ means that A is countable and $|A| = \mathfrak{c}$ indicates that the cardinality of A is the continuum. If A is a union of a pairwise disjoint family of sets $\{A_i\}_{i\in I}$, then we denote it by $A = \bigsqcup_{i\in I} A_i$. That is, $A = \bigcup_{i\in I} A_i$ and $A_i \cap A_j = \emptyset$ whenever $i \neq j$. If $A = \bigsqcup_{i\in I} A_i$, then we say that the family $\{A_i\}_{i\in I}$ **partitions** the set A.

Definition 1. *A* ***correspondence*** *ψ from a set X to a set Y assigns to each x in X a subset $\psi(x)$ of Y. We write $\psi\colon X \twoheadrightarrow Y$ to distinguish a correspondence from a function.*

We use the correspondence concept since we intend to divide a population X into separate clusters of agents. To do so, we focus on correspondences with $X = Y$, that is $\psi : X \twoheadrightarrow X$. Furthermore, to formalize the notion of spatial separation of clusters, we consider a special class of correspondences.

Definition 2. *A correspondence $\psi\colon X \twoheadrightarrow X$ is* ***partitional*** *whenever it satisfies the following two properties:*

(a) *$x \in \psi(x)$ for every $x \in X$, and*

(b) *if $y \in \psi(x)$, then $\psi(y) = \psi(x)$.*

*If, in addition, $|\psi(x)| = k$ for all $x \in X$, then we say that ψ is k-**partitional.***

This definition mirrors the one in [12, p. 68]. It states that an agent x always belongs to the cluster $\psi(x)$ and any two clusters either coincide or are disjoint.[1] One can interpret this as meaning that there is spatial separation among clusters. To see this note that, by (b), if some agent y belongs to $\psi(x)$, then x and y must be in the same cluster.

In short, we can use a correspondence ψ to partition the population into subsets of agents called clusters. These clusters can be interpreted as spatially separated groups of agents. Whether and how agents in a cluster can interact with each other, depends on the matching rule in place. To formalize matching rules, we will draw from the mathematical concept of a permutation.

[1] If $z \in \psi(x) \cap \psi(y)$, then $\psi(z) = \psi(x)$ since $z \in \psi(x)$ and $\psi(z) = \psi(y)$ since $z \in \psi(y)$. Thus, $\psi(x) = \psi(y)$.

A **permutation** of a non-empty set X is a one-to-one function ϕ from X onto X. If X is a finite set, say $X = \{x_1, x_2, \ldots, x_k\}$, then a permutation ϕ on X can be represented by a matrix

$$\phi = \begin{pmatrix} x_1 & x_2 & \ldots & x_k \\ y_1 & y_2 & \ldots & y_k \end{pmatrix},$$

where $y_j = \phi(x_j) \in X$ and $y_j \neq y_i$ if $i \neq j$. If ϕ is such that $\phi^2 = \phi \circ \phi = I$ on X, that is, if the function ϕ composed with itself is the identity function, then ϕ is called an **involution**; see [14]. It turns out that involutions formalize the economic concept of bilateral matchings.

3 Random Matching in a Period

In this section, we discuss how to match agents randomly in any representative period. Thus, we omit the time subscript.

We adopt a procedure that involves three separate steps. The first step is to specify how to divide the population into spatially separated clusters of agents. This will require the use of partitional correspondences. Then, we will define and calculate all possible ways to form pairs in each cluster. To do so, we will use involutions. Finally, we will specify a probability measure over all possible pairings for each cluster. This will give us the desired random matching rule for a cluster and a random matching process for the entire population.

3.1 Step 1: Spatial Separation Using Clustering Rules

Since we want to deal with matches that are separated in space, we start by taking steps in order to formalize the notion of spatial separation. To this end, we need to introduce the concept of a k-clustering rule. This is a mathematical device that allows us to divide the population X into clusters of k individuals each. Later, we will formalize a notion of spatial separation for these clusters.

Definition 3. *A k-**clustering rule** for a population X is a k-partitional correspondence $\psi\colon X \twoheadrightarrow X$. We call $\psi(x)$ the **cluster** of x.*

Clearly every k-clustering rule ψ induces a partition on the population X by selecting k agents at a time that are placed in separate groups. That is, ψ partitions X into 'slices' or equivalence classes.[2] The family of equivalence classes will be denoted $\{X_s\}_{s\in S}$,

[2]We note that given any partition, there is exactly one equivalence relation on X from which it is derived. An **equivalence relation** on a set X is a binary relation $\sim$ on X satisfying the following three properties: (1) (Reflexivity) $x \sim x$ for every $x \in X$; (2) (Symmetry) If $x \sim y$, then $y \sim x$; and (3) (Transitivity) If $x \sim y$ and $y \sim z$, then $x \sim z$. Given an equivalence relation $\sim$ on a set X and an element $x \in X$, the **equivalence class** of x is the subset of X defined by $[x] = \{y \in X\colon\ y \sim x\}$. Note that $x \in [x]$

where S is the index set of all slices. In other words, for each $s \in S$ there exists some $x \in X$ such that $\psi(x) = X_s$. For instance, if $X = \{a, b, c\}$ and ψ generates the clusters $\psi(a) = \psi(b) = \{a, b\}$ and $\psi(c) = \{c\}$, then $S = \{1, 2\}$ where $X_1 = \{a, b\}$ and $X_2 = \{c\}$.

The natural question at this point is whether we can construct clustering rules on any set. It turns out that not all populations can be partitioned according to a k-clustering rule. The next lemma establishes a basic condition under which this can be done: it requires to partition the population X into k subsets of identical cardinality.

Theorem 4 (Existence of clustering rules). *If $A_1, A_2, \ldots, A_k$ are pairwise disjoint sets having the same cardinality and $X = \bigsqcup_{i=1}^{k} A_i$, then there exists a k-partitional correspondence $\psi\colon X \twoheadrightarrow X$.*

In particular, there exists a k-partitional correspondence $\psi\colon X \twoheadrightarrow X$ such that for each $x \in X$ the set $\psi(x)$ consists of k-elements one from each set A_i.

Proof. Since the sets A_i have the same cardinality, for each $i = 1, \ldots, k-1$, we can find a function $f_i\colon A_i \to A_{i+1}$ which is one-to-one and surjective (onto). We claim the following: If $2 \le j \le k$ and $x \in A_j$, then there exists a unique element $r_x \in A_1$ (called the **root** of x) such that $x = f_{j-1} f_{j-2} \cdots f_1(r_x)$. Indeed, note that the element $r_x = \left(f_1^{-1} \cdots f_{j-1}^{-1}\right)(x)$ satisfies the desired property. That is:

$$\text{If } x = f_{j-1} f_{j-2} \cdots f_1(x_1), \text{ where } x_1 \in A_1 \text{ and } 2 \le j \le k, \text{ then } r_x = x_1\,. \qquad (\star)$$

The uniqueness of r_x should be obvious. If $x \in A_1$, then we let $r_x = x$.

Next, define $\psi\colon X \twoheadrightarrow X$ by $\psi(x) = \{r_x, f_1(r_x), f_2 f_1(r_x), \ldots, f_{k-1} f_{k-2} \cdots f_1(r_x)\}$. It should be clear that $\psi(x)$ contains k elements such that $\psi(x) \cap A_j$ is a singleton for each $j = 1, 2, \ldots, k$. That is, $\psi(x)$ consists of all elements of X that have r_x as their root. (Clearly, $x \in \psi(x)$ and $\psi(x)$ consists exactly of one element from each A_i.) To prove that ψ is a k-partitional correspondence, it remains to be shown that if $y \in \psi(x)$, then $\psi(y) = \psi(x)$. There are two cases to consider:

(1) $y = r_x \in A_1$.

In this case, we have $r_y = r_x$, and so

$$\begin{aligned} \psi(y) &= \{r_y, f_1(r_y), \ldots, f_{k-1} f_{k-2} \cdots f_1(r_y)\} \\ &= \{r_x, f_1(r_x), \ldots, f_{k-1} f_{k-2} \cdots f_1(r_x)\} = \psi(x)\ . \end{aligned}$$

(2) $y \neq r_x$.

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for all $x \in X$, and any two equivalence classes are either disjoint or equal. Given an equivalence relation on X, the collection of all equivalence classes determined by $\sim$ is a partition of X. Thus, studying equivalence relations is equivalent to studying partitions.

Here we have $y = f_{j-1}f_{j-2}\cdots f_1(r_x)$ for some $2 \leq j \leq k$. This, in conjunction with $(\star)$, yields $r_y = r_x$. Thus, as above, $\psi(y) = \psi(x)$, and the proof is finished. ■

An illustration of the clustering rule described in Theorem 4 is shown in Figure 1.

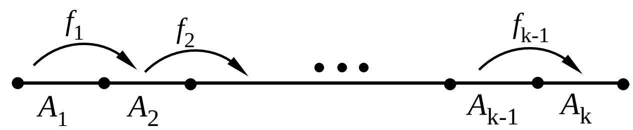

Figure 1: The k-clustering rule of Theorem 4

We note that k-clustering rules, if they exist, are not necessarily unique. This is due to the flexibility in the selection of agents from each of the k sets which define the partition. Clearly, we have many choices over the functions f_i, $1 \leq i \leq k-1$, as long as they are one-to-one and onto. A different k-partitional correspondence is generated by a different choice of any of the f_i.

What if the population X cannot admit k-clustering rules? Then, we can 'normalize' X (as long as it has at least k agents) in such a way that a k-clustering rule can be constructed on a *subset* of X. The remaining agents will be assigned to clusters of one agent each.

Corollary 5. *Let* $X = \left(\bigsqcup_{i=1}^{k} A_i\right) \bigsqcup A_0 = Y \bigsqcup A_0$, *where* $A_0, A_1, \ldots, A_k$ *are nonempty pairwise disjoint sets and* $A_1, \ldots, A_k$ *have the same cardinality. Then we can construct a partitional correspondence* $\psi\colon X \twoheadrightarrow X$ *such that*

(i) ψ *on* Y *is* k*-partitional, and*

(ii) ψ *on* A_0 *is* 1*-partitional.*

Now that we know how to group a population X into clusters of agents, we study how to pair agents in each cluster. In this way, we can also formalize a notion of spatial separation for any economy.

3.2 Step 2: Bilateral Matching Using Involutions

Suppose we have divided the population X according to ψ into the clusters $\{X_s\}_{s\in S}$. We want to pair agents only in each cluster X_s. To do so, we exploit the mathematical concept of a special class of permutations called involutions. These define the bilateral matching rules on any set.

Definition 6. *A* ***bilateral matching rule*** *for a set of agents* Ω *is an involution of* Ω.

Recall that a permutation of Ω is any one-to-one and onto function ϕ on Ω. This means that any permutation can assign an agent to himself. However, such a permutation need not be consistent with the idea of bilateral matching. For example, if $\Omega = \{a, b, c\}$, then a permutation may assign a to b, b to c, and c to a, which clearly is not a matching. Therefore, we need the "involution" restriction: the inverse of the permutation ϕ must coincide with itself or $\phi^2 = I$.

Now that we know what is a bilateral matching rule, we have a very natural way to formalize the notion of spatial separation in the economy.

Definition 7. *A* ***spatially separated economy*** *is a triplet* (X, ψ, ϕ) *such that:*

(a) $\psi \colon X \twoheadrightarrow X$ *is a* k*-clustering rule on* X*, and*

(b) $\phi \colon X \to X$ *is a bilateral matching rule that leaves each cluster* X_s *invariant, that is,* $\phi(X_s) \subseteq X_s$ *for each* $s \in S$*.*

In other words, in our economy clusters of agents are spatially separated if an agent y belonging to a cluster $\psi(x)$ can only meet an agent who also belongs to the cluster $\psi(x)$. We emphasize that a bilateral matching rule necessarily matches every agent to someone else, in his own cluster. Thus, in what follows, we say that a bilateral matching rule on Ω is **exhaustive** if no agent in Ω is unmatched, that is if $\phi(\omega) \neq \omega$ for all $\omega \in \Omega$. Of course, several different bilateral matching rules exist—exhaustive or not. Thus, it is natural to ask how many possible pairings of the k agents in Ω can be accomplished. To answer this question, we need to introduce some notation.

Definition 8. *If* Ω *is a population set, then we shall denote by* $B(\Omega)$ *the collection of all bilateral matching rules on* Ω*.*

For the rest of our discussion in this paper, $\Omega = \{\omega_1, \ldots, \omega_k\}$ will denote a finite set of k agents. Notice that $B(\{\omega_1, \ldots, \omega_k\})$ consists of all possible ways in which the k agents in Ω can be bilaterally arranged—either by pairing them with someone else or with themselves. Since the cardinality of the set of all possible permutations of Ω is $k!$ and $B(\Omega)$ is a subset of the set of all permutations, it follows that $B(\Omega)$ is also a finite set whose cardinality is less than $k!$. The number of possible bilateral matching rules in Ω can be determined recursively as follows.

Lemma 9. *If* $\ell_k = |B(\Omega)|$ *is the number of all possible bilateral matching rules on a set* Ω *with* k *agents, then* $\ell_1 = 1$*,* $\ell_2 = 2$*, and*

$$\ell_{k+1} = \ell_k + k\ell_{k-1} \text{ for } k \geq 2\,.$$

Proof. It is obvious there is only one way to arrange one agent and two ways to arrange two agents. Thus $\ell_1 = 1$ and $\ell_2 = 2$. Now, suppose we have $k+1 \geq 3$ agents in a cluster. There are two possibilities: (1) agent $k+1$ is matched to himself, and (2) agent $k+1$

is matched to someone else. In case (1), according to the definition of possible bilateral matching rules, the remaining k agents can be matched in ℓ_k different ways. In case (2) agent $k+1$ can be matched to any one of the other k agents. The remaining $k-1$ agents can be matched in ℓ_{k-1} different possible ways. Therefore $\ell_{k+1} = \ell_k + k\ell_{k-1}$. ■

We can also calculate the number of possible exhaustive bilateral matching rules for a cluster of k agents.

Lemma 10. *Let n_k and e_k denote the number of possible non-exhaustive and exhaustive bilateral matching rules on a set Ω of $k \geq 2$ agents. Then:*

(a) $n_1 = 1,\ n_2 = 1$ *and* $n_{k+1} = \ell_k + kn_{k-1}$ *for* $k \geq 2$.

(b) $e_1 = 0,\ e_2 = 1$ *and* $e_{k+1} = ke_{k-1}$ *for* $k \geq 2$.

Furthermore, $e_{2k+1} = 0$ *and* $e_{2k} = \frac{(2k)!}{2^k k!}$ *for* $k = 1, 2, 3, \ldots$.

Proof. It is obvious that $n_1 = n_2 = 1$ since in both cases only the identity is non-exhaustive. If $k \geq 2$, then there are two possibilities: agent $k+1$ is matched to himself or to someone else. In the first case the remaining k agents can be paired in ℓ_k different ways. Otherwise, agent $k+1$ can be paired to any of the k agents and the remaining $k-1$ agents can be paired in n_{k-1} different possible non-exhaustive ways. Therefore $n_{k+1} = \ell_k + kn_{k-1}$.

Now notice that the number of possible exhaustive pairings is just the difference between the total number of possible pairings and the number of all possible non-exhaustive pairings. Thus, we have $e_1 = \ell_1 - n_1 = 0$ and $e_2 = \ell_2 - n_2 = 1$. Also,

$$e_{k+1} = \ell_{k+1} - n_{k+1} = k(\ell_{k-1} - n_{k-1}) = ke_{k-1}\,.$$

The latter, in conjunction with $e_1 = 0$, implies that $e_{2k+1} = 0$ for $k = 1, 2, 3, \ldots$. By induction, it is easy to see that $e_{2k} = \frac{(2k)!}{2^k k!}$. If $k = 1$, then clearly $e_2 = \frac{2!}{2\times 1!} = 1$. For the induction step, assume that $e_{2k} = \frac{(2k)!}{2^k k!}$ is true for $k \geq 1$, then we have to show it is true for $k+1$. To see this, note that using the recursive formula for e_{k+1} we have $e_{2(k+1)} = (2k+1)e_{2k} = (2k+1)\frac{(2k)!}{2^k k!} = \frac{(2k+2)!}{2^{k+1}(k+1)!}$. ■

An example may be helpful. Suppose $k = 3$ and $\Omega = \{a, b, c\}$. Then the number of all possible pairings (i.e., bilateral matching rules) is $\ell_3 = n_3 = 4$. The set consisting of all possible bilateral matching rules is $B(\{a, b, c\}) = \{\phi_1, \phi_2, \phi_3, \phi_4\}$, where

$$\phi_1 = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix},\ \phi_2 = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix},\ \phi_3 = \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix},\ \phi_4 = \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}.$$

That is, there are four possible ways to pair the agents a, b and c. We can leave them unmatched, which is the permutation ϕ_4 or we can form pairs leaving one agent unmatched according to ϕ_1, ϕ_2 and ϕ_3. What is interesting is that (as the next table demonstrates) even for relatively small clusters of agents the number of possible pairings is very large.

k	ℓ_k	n_k	e_k
1	1	1	0
2	2	1	1
3	4	4	0
4	10	7	3
5	26	26	0
6	76	61	15
7	232	232	0
8	764	659	105
$\vdots$	$\vdots$	$\vdots$	$\vdots$
20	23,758,664,096	23,103,935,021	654,729,075

An immediate consequence is that we can generate a very large number of possible pairings despite the use of finite clusters of agents. This is very convenient, since it allows us to construct bilateral matches that are random by selecting randomly one out of many possible pairings in each cluster. The mechanics of this are described in the sequel.

3.3 Step 3: Random Pairings Using Probability Measures

We start by formalizing a notion of a random matching.

Definition 11. *A **stochastic bilateral matching rule** on $\Omega = \{\omega_1, \ldots, \omega_k\}$ (or a **stochastic rule**) is simply a probability measure f on $B(\Omega)$.*

Using this formalization we now show how to construct random pairings on Ω in a quite natural way.

Lemma 12. *Every stochastic rule f on a set $\Omega = \{\omega_1, \ldots, \omega_k\}$ induces a probability measure $F\colon \Omega \times \Omega \to [0, 1]$ via the formula*

$$F(\omega_i, \omega_h) = \sum_{\{\phi \in B(\Omega):\ \omega_i = \phi(\omega_h)\}} f(\phi) = f\left(\{\phi \in B(\Omega):\ \omega_i = \phi(\omega_h)\}\right) .$$

Moreover, the measure F satisfies the following properties:

(i) *For all i and h we have $F(\omega_i, \omega_h) = F(\omega_h, \omega_i)$.*

(ii) *For each fixed $\omega_h \in \Omega$ we have $\sum_{i=1}^{k} F(\omega_i, \omega_h) = 1$.*

(iii) *If k is odd, then $F(\omega_i, \omega_i) > 0$ for some i.*

Moreover, F defines a doubly stochastic matrix[3]

	ω_1	ω_2	$\ldots$	ω_k
ω_1	$F(\omega_1, \omega_1)$	$F(\omega_1, \omega_2)$	$\ldots$	$F(\omega_1, \omega_k)$
ω_2	$F(\omega_2, \omega_1)$	$F(\omega_2, \omega_2)$	$\ldots$	$F(\omega_2, \omega_k)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
ω_k	$F(\omega_k, \omega_1)$	$F(\omega_k, \omega_2)$	$\ldots$	$F(\omega_k, \omega_k)$

Proof. Part (i) follows from the fact that $\omega_i = \phi(\omega_h)$ if and only if $\omega_h = \phi(\omega_i)$. In order to prove part (ii), fix $\omega_h \in \Omega$ and note that $\bigsqcup_{i=1}^{k} \{\phi \in B(\Omega) : \omega_i = \phi(\omega_h)\} = B(\Omega)$. This implies

$$\sum_{i=1}^{k} F(\omega_i, \omega_h) = \sum_{i=1}^{k} f(\{\phi \in B(\Omega) : \omega_i = \phi(\omega_h)\}) = f(B(\Omega)) = 1.$$

To see (iii), observe that if k is odd, then for all $\phi \in B(\Omega)$ there exists some $\omega_i \in \Omega$ such that $\omega_i = \phi(\omega_i)$ and so $B(\Omega) = \bigcup_{i=1}^{k}\{\phi \in B(\Omega)\colon \omega_i = \phi(\omega_i)\}$. This implies that $F(\omega_i, \omega_i) > 0$ holds true for some i. ■

In short, a stochastic rule on Ω selects with probability $f(\phi)$ the pairings specified by the bilateral matching rule $\phi \in B(\Omega)$. Since each ϕ assigns every agent $\omega_i \in \Omega$ to someone in Ω, then we can calculate the probability that ω_i meets ω_h. To do so, we must notice that each ϕ in $B(\Omega)$ can be considered as an independent outcome. Thus, we can define the probability of a match between ω_i and ω_h as $F(\omega_i, \omega_h)$. The latter is computed by adding the probabilities $f(\phi)$ associated to those outcomes in which ω_i meets ω_h. Looking across all possible pairings, this gives rise to the doubly stochastic matrix exhibited in the statement of Lemma 12. Clearly, from such a matrix we can always reconstruct the probability measure f.

Now that we know how to construct random pairings on any finite set of agents Ω, we can formalize a notion of random matching for the entire population.

Definition 13. *A **stochastic bilateral matching process** over a population X relative to a k-clustering rule $\psi\colon X \twoheadrightarrow X$ is a family $\mathcal{F} = \{f_s\}_{s\in S}$ of probability measures, where f_s is a stochastic rule over $B(X_s)$ and $\{X_s\}_{s\in S}$ is the collection of clusters induced by ψ.*

Briefly, here is how we randomly pair agents in our framework. To start with, we use a clustering rule ψ to partition the population X into spatially separated clusters X_s of

[3]Recall that a non-negative real matrix is said to be **doubly stochastic** if each row and each column of the matrix sums up to one; see [10].

k agents each.[4] Once this is done, we must find all possible ways to pair agents within each cluster X_s, which gives rise to the set $B(X_s)$. Given this, we can then specify a probability measure over $B(X_s)$, which is what we call a stochastic rule. The collection of all such rules for $\{B(X_s)\}_{s \in S}$ is called the stochastic bilateral matching process $\mathcal{F}$. Thus, $\mathcal{F}$ induces a family of probability measures $\{F_s\}_{s \in S}$ satisfying the properties in Lemma 12. A single realization of this stochastic process generates a unique match of the population. From our prior discussion, we will say that a stochastic bilateral matching process $\mathcal{F}$ over the population X is exhaustive, if $F_s(\omega, \omega) = 0$ for all agents $\omega \in X_s$ and all $s \in S$. Clearly, this cannot occur if the k-partitional correspondence ψ has k odd; see item (iii) in Lemma 12. We are now ready to discuss how to construct mechanisms that pair agents randomly over time.

4 Random Matching Over Time

Consider discrete time $t = 0, 1, 2, \ldots$. We start the economy by having agents unmatched (period $t = 0$). Then, we say that a sequence $\Psi = \{\psi_t\}_{t=0}^{\infty}$ is a k-**clustering mechanism** if ψ_t is a k-clustering rule for the population X in every period $t \geq 1$. In this way, we can construct random bilateral matches over time by specifying a sequence of bilateral stochastic matching processes.

Definition 14. *A* ***bilateral stochastic matching mechanism*** *(or a* ***stochastic mechanism****) over a population X is a quadruplet $(X, \Psi, \Phi, \mathcal{F})$, where:*

(i) Ψ *is a k-clustering mechanism over X,*

(ii) $\Phi = \{\phi_t\}_{t=0}^{\infty}$ *is a sequence of bilateral matching rules on X such that the triplet (X, ψ_t, ϕ_t) is a spatially separated economy for each t, and*

(iii) $\mathcal{F} = \{\mathcal{F}_t\}_{t=0}^{\infty}$ *is a sequence of stochastic bilateral matching processes such that:*

  (a) $\mathcal{F}_0$ *satisfies $\phi_0(x) = x$ for each $x \in X$, and*

  (b) $\mathcal{F}_t$ *is a stochastic bilateral matching process for ψ_t for each $t \geq 1$.*

Essentially, the stochastic mechanism tells us the probability that in each period $t \neq 0$ an agent x gets matched to someone in his own cluster $\psi_t(x)$. Clearly, every agent meets some other agent at each period $t \neq 0$, i.e., matching is exhaustive, if $\mathcal{F}_t$ is exhaustive at each period $t \neq 0$. We also note that the collection of all deterministic bilateral matching mechanisms is a subset of all stochastic mechanisms, where $\mathcal{F}_t$ induces a family of degenerate probability measures in each period.

[4] The index set S can be countable or uncountable. For example, if $X = [0, 1]$ then there are infinitely many clusters of k agents, and S is uncountable. In fact, since a countable union of countable sets is countable it must be the case that $|S| = \mathfrak{c}$.

It is convenient to call to the agents in $\psi_t(a)$ the **clustermates** of a in period t. Among these agents, there is only one agent $\phi_t(a) \in \psi_t(a)$ who is the **partner** of a in period t. It is useful to introduce the following terminology.

Definition 15. *We say that two agents a and b :*

(1) *Share a **direct partner**, if there exist periods $t_1 < t_2 < t_3$ and an agent $c \neq a, b$ such that:*

$$a = \phi_{t_1}(b), \quad b = \phi_{t_2}(c), \quad c = \phi_{t_3}(a) .$$

(2) *Share an **indirect partner**, if there exist periods $t_1 < t_2 < t_3 < \cdots < t_k$ and agents $a_1, a_2, \ldots, a_{k-2}$ different than a and b, where $k \geq 4$, such that:*

$$a = \phi_{t_1}(b), \quad b = \phi_{t_2}(a_1), \quad a_1 = \phi_{t_3}(a_2), \; \ldots \; , a_{k-3} = \phi_{t_{k-1}}(a_{k-2}), \quad a_{k-2} = \phi_{t_k}(a).$$

We have to consider the possibility of having common partners, as this affects the agents' ability to share information with others across periods. For example, in case (1) above, c is a direct partner of a and b. Thus, c can transfer information (or objects) from b to a, after their match is over. In case (2) above, a and b share a succession of indirect partners a_1 through a_{k-2}. Therefore b can provide information to a_1, which in turn can be passed on to agent a down the line in period t_k.

To account for the information that may be available in a match, we need to examine the agents' matching histories. To do so, one must keep track of the clusters to which paired agents belong at each date. We denote by $P_t(a)$ the set of all clustermates of a (including a himself) in periods up to and including t. That is,

$$P_t(a) = \bigcup_{\tau=0}^{t} \psi_\tau(a) .$$

While $P_t(a)$ accounts for all agents that belonged to the same clusters to which a belonged, it excludes agents that were clustermates of a's clustermates and partners, and so on. It turns out there is an easy way to keep track of all these 'indirect' connections among agents by means of a recursive process. Specifically, we denote by $\Pi_t(a)$ the set of a's past and current clusters, the clusters to which a's current clustermates belonged in the past, and so on. In other words, we let

$$\Pi_t(a) = \begin{cases} P_0(a) & \text{for } t = 0 \\ \Pi_{t-1}(a) \bigcup \left[\bigcup_{b \in \psi_t(a)} \Pi_{t-1}(b) \right] & \text{for } t \geq 1. \end{cases}$$

By an inductive argument, we can see that $P_t(a) \subseteq \Pi_t(a)$ and, although $\Pi_t(a)$ is a very large set, it is finite since it is a finite union of finite sets. It is also important to emphasize that $\Pi_t(a)$ does not include agents that a's partners (or clustermates) have been spatially close to after moving away from agent a.

Why do we need all this complex machinery? The reason is now that we know how to match agents over time, we want to be able to discuss how the matching technology in place affects the flow of information in the marketplace. That is, we want to make explicit how different matching mechanisms generate (or remove) obstacles to information flows.

This issue deals with the broadly defined notion of 'anonymity' in trade, which is often seen as a central assumption in several models of matching.[5] The question we need to answer at this point is the following: what does it exactly mean for matched agents to be anonymous? To formalize a notion of anonymity, we need to take two steps. First, we must know how to look into an agent's past. This was already done by introducing the sets $P_t(a)$ and $\Pi_t(a)$, which essentially trace the matching history of each agent in the economy. Second, we need to formalize how these matching histories can be used to define the information that can be available to agent in a match. This will be done next.

Definition 16. *A k-clustering mechanism Ψ on the population X is said to be:*

(1) ***Eventually weakly anonymous****, if for each $a \in X$ there is some $t \geq 1$ such that*

  (i) $\psi_{\tau'}(a) \cap \psi_\tau(a) = \{a\}$ *for all* $\tau', \tau \geq t$, *and*

  (ii) $P_t(a) \bigcap \left[\bigcup_{\tau=t+1}^{\infty} \psi_\tau(a)\right] = \{a\}$.

(2) ***Weakly anonymous****, if for all $a \in X$, all $t \geq 1$ and all $\tau \neq t$ we have*

$$\psi_t(a) \cap \psi_\tau(a) = \{a\} \ .$$

(3) ***Anonymous,*** *if for all $a \in X$, all $t \geq 1$ and all $b \in \psi_{t+1}(a)$ with $b \neq a$ we have*

$$P_t(a) \cap P_t(b) = \emptyset \ .$$

(4) ***Strongly anonymous****, if for all $a \in X$, all $t \geq 1$ and all $b \in \psi_{t+1}(a)$ with $b \neq a$ we have*

$$\Pi_t(a) \cap \Pi_t(b) = \emptyset \ .$$

*We shall say that a stochastic mechanism $(X, \Psi, \Phi, \mathcal{F})$ is **eventually weakly anonymous**, if the k-clustering mechanism Ψ is eventually weakly anonymous. (Analogous properties can be defined for the other notions of anonymity.)*

[5] For instance, anonymity is a prominent feature in the foundations of the money literature. The reason is that the information constraints give value to money since traders cannot base current sales on future repayment; see, e.g., Ostroy [13] and Kocherlakota [8].

This definition allows us to consistently formalize all possible levels of information isolation that can exist in the economy. As a rule, stronger degrees of anonymity provide stricter restrictions on the information flows that can take place among agents; see our companion paper [1] for a more detailed discussion for the case of deterministic matching.

The *eventual weak* anonymity notion captures the idea that the matching mechanisms may allow some agents to repeatedly interact only in the short run. After some period these agents will move out to different clusters. Under *weak* anonymity, instead, clusters cannot be formed with the same agents. It follows that if an agent a is paired to b at some date, then a and b have never met before and will never meet again. However, the possibility exists that b might have met either one of a's past partners or one of a's former clustermates. To remove these possibilities of direct or indirect linkages among agents, we need to add restrictions to the mechanics of matching.

The additional restrictions are progressively formalized in the notions of *anonymous* and *strongly anonymous* matchings. In particular, under strong anonymity we remove all possible direct and indirect links among agents who belong to the same cluster. This reflects a suggestion made by Kocherlakota [8].

What's more, strong anonymity rules out also any *future* direct and indirect links among these agents. This is demonstrated in the following result.

Lemma 17. *Let $(X, \Psi, \Phi, \mathcal{F})$ be a stochastic mechanism. If Ψ is:*

(a) *Anonymous, then no matched agents will share a direct partner over their lifetimes.*

(b) *Strongly anonymous, then no matched agents will share a direct or indirect partner over their lifetimes.*

Proof. (a) Let Ψ be anonymous. Assume by way of contradiction that two agents a and b share a direct partner. Thus, there exist three periods $t_1 < t_2 < t_3$ and an agent c different from a and b such that (i) $a = \phi_{t_1}(b) \in \psi_{t_1}(b)$, (ii) $b = \phi_{t_2}(c) \in \psi_{t_2}(c)$, and (iii) $c = \phi_{t_3}(a) \in \psi_{t_3}(a)$. Clearly, we have

$$t_1 < t_2 \leq t_3 - 1\,. \tag{$\star$}$$

Note that (iii) yields $a = \phi_{t_3}(c) \in \psi_{t_3}(c)$ and so by the anonymity of Ψ, we have

$$P_{t_3-1}(c) \cap P_{t_3-1}(a) = \emptyset\,. \tag{$\star\star$}$$

Using (ii) and ($\star$) we see that $b \in P_{t_3-1}(c)$. Observe that (i) implies $b = \phi_{t_1}(a) \in \psi_{t_1}(a)$, and so from ($\star$) we see that $b \in P_{t_3-1}(a)$. Thus $b \in P_{t_3-1}(c) \cap P_{t_3-1}(a)$ contrary to ($\star\star$). This contradiction establishes the validity of (a).

(b) Assume Ψ is strongly anonymous and that a and b share an indirect partner. This implies there exist $t_1 < t_2 < t_3 < \cdots < t_k$ and $a_1, a_2, \ldots, a_{k-2}$ different than a and b, where $k \geq 4$, such that:

$$a = \phi_{t_1}(b)\,,\ b = \phi_{t_2}(a_1)\,,\ a_1 = \phi_{t_3}(a_2)\,, \ldots, a_{k-3} = \phi_{t_{k-1}}(a_{k-2})\,,\ a_{k-2} = \phi_{t_k}(a)\,,$$

where we note that $\phi_t(x) \in \psi_t(x)$ for all t and $x \in X$. Clearly,

$$t_1 < t_2 < t_3 < \cdots < t_{k-1} \le t_k - 1. \qquad (\dagger)$$

From $a_{k-2} = \phi_{t_k}(a) \in \psi_{t_k}(a)$ and the strong anonymity of Ψ, we have

$$\Pi_{t_k-1}(a) \cap \Pi_{t_k-1}(a_{k-2}) = \emptyset\ . \qquad (\dagger\dagger)$$

Now note $a_{k-2} \in \Pi_{t_k-1}(a_{k-2})$. It is not difficult to see that $a_{k-2} \in \Pi_{t_k-1}(a)$, since a_{k-2} in t_{k-1} met a_{k-3}, who in t_{k-2} met a_{k-4}, until a_1 in t_2 met b (who in t_1 met a). This implies $a_{k-2} \in \Pi_{t_k-1}(a) \cap \Pi_{t_k-1}(a_{k-2})$, contrary to ($\dagger\dagger$).

Finally, to establish that no matched agents share a direct partner in their lifetimes, use (a) and the fact that strong anonymity implies anonymity. (See also the proof of Lemma 18 below.) ■

As expected, stronger degrees of anonymity imply weaker degrees of anonymity.

Lemma 18. *We have the following implications:*

$$\begin{aligned} \textit{Strong Anonymity} &\implies \textit{Anonymity} \\ &\implies \textit{Weak Anonymity} \\ &\implies \textit{Eventual Weak Anonymity.} \end{aligned}$$

In general, no reverse implication is true.

Proof. First, we show that strong anonymity implies anonymity. Fix some agent $a \in X$. Assume that Ψ is strongly anonymous. For any $b \in \psi_{t+1}(a)$ with $b \neq a$, it follows from

$$P_t(a) \cap P_t(b) \subseteq \Pi_t(a) \cap \Pi_t(b) = \emptyset$$

that $P_t(a) \cap P_t(b) = \emptyset$. This implies that Ψ is anonymous.

Next, we prove that anonymity implies weak anonymity. Suppose Ψ is anonymous but is not weakly anonymous. Then there exist $1 \le t < \tau$ and some agent $a \in X$ such that $\psi_t(a) \cap \psi_\tau(a) \neq \{a\}$. Let $t^* = \tau - 1$, $b \neq a$, and (i) $b \in \psi_t(a)$, (ii) $b \in \psi_{t^*+1}(a) = \psi_\tau(a)$. Clearly $t \le t^*$. The latter in conjunction with (i) implies that $b \in P_{t^*}(a)$. Furthermore, by (ii) and $b \in P_{t^*}(b)$ we have $b \in P_{t^*}(a) \cap P_{t^*}(b)$, which contradicts anonymity for $t = t^*$. Thus, anonymity implies weak anonymity. That weak anonymity implies eventual weak anonymity follows from the definitions. For the fact that no reverse implication holds true, see [1]. ■

The central question at this point is the following: Do random pairings that are strongly anonymous exist? That is to say, is it possible to construct a class of matching mechanisms that can insure total information isolation in every meeting? We answer this challenging question to the positive in the next section.

5 Constructing Anonymous Random Matches

We start by observing that in order to have anonymous stochastic matching mechanisms we need an infinite population X. At date $t = 0$, we partition X in a countable number of sets $A_1, A_2, \ldots$ of identical cardinality.[6] Thus, we have an initial partition of the population $X = \bigsqcup_{n=1}^{\infty} A_n$. Then, in each $t \geq 1$ we divide X into clusters building on this initial partition using k sets at a time.

To do this, we need to describe how to partition the population over time. The construction of these partitions—referred to as a **recursive block-partition**—is described by the recursive method illustrated below. (The brackets below indicate the partition sets.)

Period	Block partition of the population X
0	$X = A_1 \bigsqcup A_2 \bigsqcup A_3 \cdots$
1	$X = \langle A_1 \bigsqcup \cdots \bigsqcup A_k \rangle \bigsqcup \langle A_{k+1} \bigsqcup \cdots \bigsqcup A_{2k} \rangle \bigsqcup \cdots$
2	$X = \langle A_1 \bigsqcup \cdots \bigsqcup A_{k^2} \rangle \bigsqcup \langle A_{k^2+1} \bigsqcup \cdots \bigsqcup A_{2k^2} \rangle \bigsqcup \cdots$
$\vdots$	$\vdots$
t	$X = \bigsqcup_{n=1}^{\infty} \langle A_{(n-1)k^t+1} \bigsqcup A_{(n-1)k^t+2} \bigsqcup \cdots \bigsqcup A_{nk^t} \rangle$
	$= \bigsqcup_{n=1}^{\infty} \bigsqcup_{j=1}^{k^t} A_{(n-1)k^t+j}$
	$= \bigsqcup_{n=1}^{\infty} B_n^t = \langle B_1^t \bigsqcup \cdots \bigsqcup B_k^t \rangle \bigsqcup \langle B_{k+1}^t \bigsqcup \cdots \bigsqcup B_{2k}^t \rangle \bigsqcup \cdots$
	$= \bigsqcup_{n=1}^{\infty} \left\langle B_{kn-(k-1)}^t \bigsqcup \cdots \bigsqcup B_{kn}^t \right\rangle = \bigsqcup_{n=1}^{\infty} B_n^{t+1}$
	$\vdots$

where we have defined $B_n^t = \bigsqcup_{j=1}^{k^t} A_{(n-1)k^t+j}$ for $n = 1, 2, \ldots$ and $t \geq 1$. For example, for $t = n = 1$ then $B_1^1 = \bigsqcup_{j=1}^{k} A_j = \langle A_1 \bigsqcup \ldots \bigsqcup A_k \rangle$.

It should be clear that for each $t \geq 1$ the sets $B_1^t, B_2^t, \ldots$ (called the **blocks** of the population in period t) are pairwise disjoint and have the same cardinality. Moreover, $B_n^{t+1} = \left\langle B_{kn-(k-1)}^t \bigsqcup \cdots \bigsqcup B_{kn}^t \right\rangle$ holds for $n = 1, 2 \ldots$ and $t \geq 1$, so that B_n^{t+1} is a union of k pairwise disjoint sets of identical cardinality. By Theorem 4, we can construct for each n and $t \geq 1$ a k-clustering rule $\psi_{n,t} \colon B_n^{t+1} \twoheadrightarrow B_n^{t+1}$ such that given any $x \in B_n^{t+1}$ the set $\psi_{n,t}(x)$ consists of k agents, one from each of the k blocks $B_{kn-(k-1)}^t, \ldots, B_{kn}^t$. In

[6] This means A_n can be countable or uncountable.

particular, for each t we have a k-clustering rule $\psi_t^* \colon X \twoheadrightarrow X$ defined for each $x \in B_n^{t+1}$ by

$$\psi_t^*(x) = \psi_{n,t}(x)\,.$$

We also let $\psi_0^* = I$, the identity on X.

Definition 19. *Any k-clustering mechanism Ψ^* constructed as above is called **recursive block-invariant**.*

Since spatial separation guarantees that matches at each date occur among agents that belong to the same cluster at that date, we can show that the recursive block-invariant mechanisms insure total informational isolation at each date. Here is the result that formalizes this intuition.

Theorem 20 (Existence of strong anonymity). *Every recursive block-invariant mechanism is strongly anonymous.*

Proof. The proof will be based upon the following two properties. For each $n = 1, 2, \ldots$, each $t \geq 0$ and each $0 \leq \tau \leq t$ we have:

(1) $\psi_\tau^*\left(B_n^{t+1}\right) = B_n^{t+1}$, and

(2) $\Pi_\tau(x) \subseteq B_n^{t+1}$ for all $x \in B_n^{t+1}$.

The proof of (1) is by induction on t. For $t = 0$ it is obvious that $\psi_0^*\left(B_n^1\right) = B_n^1$ for all n, since by our definition $\psi_0^*(x) = \{x\}$ for all $x \in X$. Therefore, for the induction step, assume that for some $t \geq 0$ we have $\psi_\tau^*\left(B_n^{t+1}\right) = B_n^{t+1}$ for all n and all $0 \leq \tau \leq t$. We want to prove that for any given n we have $\psi_\tau^*\left(B_n^{t+2}\right) = B_n^{t+2}$ for each $\tau = 0, 1, \ldots, t+1$. Start by observing that by the induction hypothesis $\psi_\tau^*\left(B_n^{t+1}\right) = B_n^{t+1}$ holds true for all $\tau = 0, 1, \ldots, t$. Now note that $B_n^{t+2} = B_{kn-(k-1)}^{t+1} \bigsqcup \cdots \bigsqcup B_{kn}^{t+1}$. But then for each $\tau = 0, 1, \ldots, t$ we have

$$\begin{aligned}
\psi_\tau^*\big(B_n^{t+2}\big) &= \psi_\tau^*\Big(\bigsqcup_{j=kn-(k-1)}^{kn} B_j^{t+1}\Big) = \bigcup_{j=kn-(k-1)}^{kn} \psi_\tau^*\big(B_j^{t+1}\big) \\
&= \bigsqcup_{j=kn-(k-1)}^{kn} B_j^{t+1} = B_n^{t+2}\,.
\end{aligned}$$

Also, by definition $\psi_{t+1}^*\left(B_n^{t+2}\right) = B_n^{t+2}$. Therefore, $\psi_\tau^*\left(B_n^{t+2}\right) = B_n^{t+2}$ holds true for each n and all $\tau = 0, 1, \ldots, t+1$, and the validity of (1) has been established.

The proof of (2) is by induction on τ. For $\tau = 0$ notice that for each $x \in B_n^{t+1}$ we have $\Pi_0(x) = \{x\} \subseteq B_n^{t+1}$. For the inductive step assume that for some $0 \leq \tau < t$ we have $\Pi_\tau(x) \subseteq B_n^{t+1}$ for all $x \in B_n^{t+1}$. We must show that $\Pi_{\tau+1}(x) \subseteq B_n^{t+1}$ for all $x \in B_n^{t+1}$.

Fix $x \in B_n^{t+1}$. From (1) we get $\psi_{\tau+1}^*(B_n^{t+1}) = B_n^{t+1}$, and so $\psi_{t+1}^*(x) \subseteq B_n^{t+1}$. Therefore, each element $y \in \psi_{\tau+1}^*(x)$ belongs to B_n^{t+1}. But then our induction hypothesis yields $\Pi_\tau(y) \subseteq B_n^{t+1}$ for each $y \in \psi_{\tau+1}^*(x)$, and so $\Pi_{\tau+1}(x) = \Pi_\tau(x) \bigcup \big[\bigcup_{y \in \psi_{\tau+1}^*(x)} \Pi_\tau(y)\big] \subseteq B_n^{t+1}$.

We are now ready to show that Ψ^* is strongly anonymous. To this end, assume that $a, b \in X$ satisfy $a \neq b$, and $b \in \psi_{t+1}^*(a)$ with $t \geq 1$. Since $a \in X = \bigsqcup_{n=1}^{\infty} B_n^{t+1}$ there exists a unique natural number n such that $a \in B_n^{t+1}$. Since the correspondence ψ_{t+1}^* restricted to B_n^{t+2} is k-partitional, it follows that there exists some $j \neq n$ such that $b \in B_j^{t+1}$. But then it follows from (2) that $\Pi_t(b) \subseteq B_j^{t+1}$. Using (2) once more we get $\Pi_t(a) \subseteq B_n^{t+1}$. Finally, taking into account that $B_j^{t+1} \cap B_n^{t+1} = \emptyset$ we easily infer that $\Pi_t(a) \cap \Pi_t(b) = \emptyset$, and the proof is finished.[7] ■

This theorem is fundamental, as it demonstrates that (given any infinite population X) a simple matching technique exists that insures complete informational isolation in each match and in each period. The necessary ingredient is an initial partition of the set X composed of countably many pairwise disjoint sets of identical cardinality. For example:

$$\begin{aligned}
X &= (0,1] = \bigsqcup_{n=1}^{\infty} A_n = \bigsqcup_{n=1}^{\infty} \left(\tfrac{1}{n+1}, \tfrac{1}{n}\right] \\
X &= \mathbb{N} = \bigsqcup_{n=1}^{\infty} A_n = \bigsqcup_{n=1}^{\infty} \{n\} \\
X &= \mathbb{N} = \bigsqcup_{n=1}^{\infty} A_n = \bigsqcup_{n=1}^{\infty} \{2n-1, 2n\} \\
X &= (0,\infty] = \bigsqcup_{n=1}^{\infty} A_n = \bigsqcup_{n=1}^{\infty} (n-1, n]\ .
\end{aligned}$$

An example of how to construct a strongly anonymous mechanism can be helpful. Suppose we want to construct clusters of $k = 3$ agents on a population X consisting of the natural numbers. Thus, we can initially partition the population as follows: $X = \bigsqcup_{n=1}^{\infty} \{n\} = \bigsqcup_{n=1}^{\infty} A_n$. That is, each A_n has cardinality one. According to our recursive-block partition, in $t = 0$ we have $B_n^1 = A_n = \{n\}$. For $t = 1$, we have $B_n^2 = B_{3n-2}^1 \sqcup B_{3n-1}^1 \sqcup B_{3n}^1 = \{3n-2, 3n-1, 3n\}$, and so on. An implementation of the recursive-block invariant mechanism Ψ^* is shown in the table below.

[7] Property (1) is related to the notion of invariance with respect to a function. Given a function $f\colon X \to X$, a subset S of X is said to be f-**invariant** if $f(S) \subseteq S$, i.e., $f(x) \in S$ for all $x \in S$. According to this terminology, for each $t \geq 0$, each n and all $\tau = 0, 1, \ldots, t$ the sets B_n^{t+1} are ψ_τ^*-invariant. This implies that one can construct strongly anonymous mechanisms as long as $\psi_\tau^*\left(B_n^{t+1}\right) \subseteq B_n^{t+1}$. The equality $\psi_\tau^*\left(B_n^{t+1}\right) = B_n^{t+1}$ is not necessary for strong anonymity.

t							
0	[1]	[2]	[3]	[4]	[5]	[6]	[7]
1	[**1**,2,3]	[**4**,5,6]	[**7**,8,9]	[**10**,11,12]	[**13**,14,15]	[**16**,17,18]	[**19**,20,21]
2	[**1**,4,7]	[2,5,8]	[3,6,9]	[**10**,13,16]	[11,14,17]	[12,15,18]	[**19**,22,25]
3	[1,10,19]	[2,11,20]	[3,12,21]	[4,13,22]	[5,14,23]	[6,15,24]	[7,16,25]
⋮							

Table 1

That is, in $t = 1$, $\psi_1^*(1) = \psi_1^*(2) = \psi_1^*(3) = \{1, 2, 3\}$. In $t = 2$ we have $\psi_2^*(1) = \psi_2^*(4) = \psi_2^*(7) = \{1, 4, 7\}$, and so on. It is easy to see that agents in any cluster have no direct or indirect links to prior clustermates. That is, this mechanism is strongly anonymous.

6 An Application: Matching Models of Money

Here we demonstrate how the theoretical construction we have developed can be used to provide an explicit and rigorous foundation to the existing matching literature. To do so, we focus on the monetary literature, where the desire to make trading frictions explicit is prominent. This desire explains why a large segment of modern monetary theory is now based on the so-called “search-theoretic” models of money. These are models in which infinitely-lived agents are assumed to meet randomly and pairwise over time. In such economies it is assumed that agents cannot be paired more than once and cannot observe the trading histories of others. These, as well as additional conditions on preferences and technologies, make trading frictions explicit and provide a definite medium-of-exchange role to fiat money. We now look at two representative pairwise-matching models of this literature.

6.1 A Prototypical Random Matching Model of Money

The seminal paper in this literature is Kiyotaki and Wright [7]. This paper describes a discrete-time monetary economy with a continuum of infinitely lived agents. The population is constant and is assumed to have mass one. Agents can be one of three types, in equal proportions. It is assumed that these agents are bilaterally matched. While the matching technology is not formalized, the paper contains the following description of the *outcome* of the matching process:

> “..., each period, agents are matched randomly in pairs and must decide whether or not to trade bilaterally, without the benefit of an auctioneer or

some other outside authority to impose any arrangement. Trade always entails a one-for-one swap of inventories, given the physical environment, and occurs if and only if mutually agreeable (there is no credit since a given pair will meet again with probability 0)."

We now show how to formalize a matching process that satisfies such a description and explain how to insure that matching is done in such a manner that credit trades cannot take place at all. That is, not only—as described in the original paper—every pair meets again with probability zero (which we called weak anonymity), but *also* we show how to insure that every pair does not share past partners, etc. In brief, we construct matches in which agents are completely isolated from an informational standpoint.

Here are the steps one needs to take in order to construct a matching à la Kiyotaki–Wright. The first step is to select a population with infinitely many agents. To do so, let for instance $X = \mathbb{N} = \{1, 2, 3, \ldots\}$. The second step, is to divide the population in three types of agents in equal "proportions." Therefore, we let agents $\{1, 4, 7, \ldots\}$ be of type I, agents $\{2, 5, 8, \ldots\}$ be of type II, and agents $\{3, 6, 9, \ldots\}$ be of type III.

The third step, is to insure that each agent has probability $\frac{1}{3}$ to meet an agent of any type in each period. To do so, we restrict attention to k-clustering rules which include multiples of three. In this way we can have an equal number of agents of each type in each cluster. Then we choose a probability measure over the set of ℓ_k matching rules, such that each agent has probability $\frac{1}{3}$ to be matched to any type.

An example may be helpful. Suppose $k = 3$ and we have formed the cluster $\Omega = \{1, 2, 3\}$. Then the number of all possible pairings (i.e., bilateral matching rules) is $\ell_3 = 4$. The set which lists all possible bilateral matching rules is $B(\{1, 2, 3\}) = \{\phi_1, \phi_2, \phi_3, \phi_4\}$, where

$$\phi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \ \phi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \ \phi_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \ \phi_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

If we consider the probability measure f on $B(\Omega)$ defined by $f(\phi_1) = f(\phi_2) = f(\phi_3) = \frac{1}{3}$ and $f(\phi_4) = 0$, then each type has equal probability of being matched to any of the three types.

We emphasize, as we did earlier, that the number of bilateral matching rules ℓ_k grows large very quickly. In short, the probability of any given matching rule being chosen drops rapidly to zero as k grows. That is, the chance of meeting any one agent drops to zero rapidly as the size of the cluster grows.

To insure matches are anonymous, we use our recursive block-invariant clustering mechanism. To give an illustration of such a clustering mechanism, consider agent 1 and the clusters to which he belongs over time as shown in Table 1. In period $t = 0$ agent 1 is by himself. In period $t = 1$ he is in a cluster with agents 2 and 3. In period $t = 2$

agent 1 is in a cluster with agents 4 and 7, and so on. In general, in date $t > 0$ agent 1 is in a cluster with agents $3^{t-1} + 1$ and $2 \times 3^{t-1} + 1$. Therefore, agents in this economy are matched randomly and once matched they will never meet again.

6.2 A Directed-search Matching Model of Money

Our formulation of stochastic matching mechanisms is quite general. For example, it can be employed to formalize environments where agents are periodically matched to a specific type of agent, as in Corbae et al [3]. There, if $X = \{a, b, c\}$, parameters exist such that the following deterministic matching mechanism can endogenously emerge in equilibrium:

$t = 1$	$\phi(a) = b$ and $\phi(c) = c$
$t = 2$	$\phi(b) = c$ and $\phi(a) = a$
$t = 3$	$\phi(a) = c$ and $\phi(b) = b$
$t = 4$	$\phi(a) = b$ and $\phi(c) = c$
$\vdots$	$\vdots$

In short, the authors define in [3] a three-period matching cycle, where every agent stays unmatched for one period after two consecutive matches. Using our machinery, we can formalize such a matching mechanism as follows. First, at each date we partition the population into a single all-encompassing cluster, that is, $\psi_t(a) = \psi_t(b) = \psi_t(c) = X$.

Second, we specify a degenerate probability distribution over the set of all possible matching rules $B(\{a, b, c\})$, in each period. The possible matching rules are:

$$\phi_1 = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}, \ \phi_2 = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}, \ \phi_3 = \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}, \ \phi_4 = \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}.$$

Specifically, in every period t we have $f(\phi_4) = 0$, while in $t = 1, 4, 7, ...$ we have $f(\phi_1) = 1$, in $t = 2, 5, 8, ...$ we have $f(\phi_2) = 1$, and in $t = 3, 6, 9, ...$ we have $f(\phi_3) = 1$.

Finally, to achieve the desired cyclicality in matching, we specify a time-dependent stochastic matrix that is defined by

$M_1(F) =$

	a	b	c
a	0	1	0
b	1	0	0
c	0	0	1

, $M_2(F) =$

	a	b	c
a	1	0	0
b	0	0	1
c	0	1	0

, $M_3(F) =$

	a	b	c
a	0	0	1
b	0	1	0
c	1	0	0

, ...

Consequently, agents are bilaterally matched in a deterministic way at each point in time having the same information sets.

7 Final Remarks

In this paper we have presented a rigorous theoretical formalization of a general class of technologies that support random pairwise interactions in any population. Especially, the central focus of the analysis has been to demonstrate how different properties of the matching technology give rise to different degrees of informational frictions.

By developing a comprehensive theoretical approach to random matching, we contribute to building more solid foundations for a research discourse centered around the study of allocations in decentralized trading environments. Identifying an exact map between matching technologies and the frictions impinging on the trading process can improve the formulation of economic models whose main trait is markets that are not well functioning. In particular, once preferences, production technologies, and a suitable equilibrium concept are defined, this study is a useful starting point for a deeper understanding of how informational and spatial constraints might affect the possible allocations.

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