Speculative Bubbles without Stupid Investors*

Milo Bianchi[†] Philippe Jehiel[‡]

May 2, 2007

Abstract

We model speculative behaviors in a market with finite horizon and complete information, where all traders, throughout the game, are aware to be in a bubble. We introduce partially sophisticated investors, who understand only the aggregate selling and buying strategies of other traders along the bubble, without a precise understanding of the timing of these decisions. In the bubble equilibrium, such traders remain invested for too long, since a series of high prices lead them to overestimate the duration of the bubble. Fully rational investors ride the bubble and exit just before the crash. We show that a bubble is more likely to arise and to be longer the greater is the mass of people who can potentially enter the speculative market relatively to those who have already invested. We also show that increasing the share of fully rational investors need not make the bubble disappear. First, a minimal mass of fully rational investors is needed to make other investors aware that the crash is about to occur and it is time to sell. Second, if agents are ambiguity averse, bubbles may last longer when the amount of fully rational traders increases, since rational investors face less ambiguity and so they may be willing to take longer positions in the speculative market.

Keywords: speculative bubbles, bounded rationality.
JEL codes: D84, G12, C72.

*We thank numerous seminar participants and in particular Cedric Argenton, Abhijit Banerjee, Mike Burkart, Gabrielle Demange, Tore Ellingsen, Botond Koszegi, John Moore, Marcus Salomonsson, Johan Walden, Jörgen Weibull, Muhamet Yildiz and Robert Östling for useful discussions. Milo Bianchi gratefully acknowledges financial support from the Knut and Alice Wallenberg Foundation.

[†]Stockholm School of Economics; e-mail: milo.bianchi@hhs.se

[‡]PSE and UCL; e-mail: jehiel@enpc.fr

1 Introduction

A speculative bubble is characterized by an abnormal increase in the price of an asset, followed by a crash. In such situations, the *only* reason behind prices above fundamentals is the expectation that the selling price will be even higher in the near future (Stiglitz, 1990).

These phenomena have been widely documented (see Garber, 1990; Kindleberger, 2005), but their foundation remains largely unclear. Standard economic theory rules out purely speculative behaviors: Rational agents should not invest just before the crash; anticipating this, no one should invest on the day before, and so on... nobody should invest in the first place (Tirole, 1982).

In this paper, we show that speculative bubbles may be explained by considering that some investors have a *partial* understanding of the strategies of other investors, and in particular they fail to understand how these strategies depend on the history of trades. As a result, some investors form erroneous beliefs about the date of the crash, and they are not able to sell just before it. This allows bubbles to arise even in a setting with complete information and in which all investors, throughout the game, are aware that the market is overvalued and bound to crash.

Our approach builds on three related observations. First, private information does not seem a central ingredient in the explanation of bubbles. Second, speculation occurs even if agents are aware to be in a bubble. Third, in such situations, each investor tries to time the market, and this may be hard if the date of the crash does not appear fundamentally different from some other dates. Before previewing our main results, we now consider these arguments more in details.

We are interested in the emergence of bubbles in contexts with complete information both for theoretical and empirical reasons. No-trade theorems tell that private information alone should not lead to trade, as no investor should expect positive gains from trade (Milgrom and Stokey, 1982). Hence, even if agents have private information and hence heterogenous beliefs, say regarding the date of the crash, speculative bubbles should not be sustained. If one adds subjective (and thus erroneous) views about how this information is distributed, then bubbles may occur. However, this line of reasoning raises the issue of where subjective priors come from, and why they survive in equilibrium.[1]

Empirical observations too suggest that asymmetric information may not be the reason for the occurrence of bubbles. First, while it can be argued that in modern times the scope for private information is reduced, speculative bubbles do not seem to have vanished. Many factors, including technology and regulation, now allow virtually all in-

[1] As discussed in Dekel, Fudenberg and Levine (2004), Nash equilibria with subjective priors are not appealing from a learning perspective. If agents can learn the strategies of other agents as a function of their private information, then why these agents cannot learn how private information is distributed? In other words, why don't they learn the strategy of nature?

formation about the economic environment to be *a priori* available.[2] Still, the 1990s have been qualified as "the most tumultuous decade ever" in terms of speculative behaviors (Kindleberger, 2005). Second, bubbles emerge even in experimental asset markets, where the amount of information provided to traders is controlled by design. In a famous study, Smith, Suchanek and Williams (1988) document speculative behaviors in a market with finite horizon, where the structure of the game, and in particular the value of future dividends, is common knowledge. In their interpretation, bubbles arise from strategic uncertainty: agents cannot perfectly assess how other investors act upon the common information, i.e. what are their strategies.[3]

In our model, we follow the insight that some agents may not be able to fully understand their opponents' strategies. Still, we require that they have some understanding of the environment, emphasizing in particular that all investors, throughout the game, understand to be in a bubble. This approach requires that investors are not totally naive, and in particular it rules out explanations based on purely adaptive expectations. If agents were to base their investments simply extrapolating from past trends, then they would expect the price to increase with no bounds, and they would never understand to be in a bubble. We believe that extrapolative traders may exist, but we doubt that they are sufficiently many to produce considerable departures from fundamental values.

This view finds support not only in experimental asset market, but also in the field. Shiller (1989), for example, reports that just before the US stock market crash of October 1987, 84% of institutional investors thought that the market was overpriced; 78% of them thought that this belief was shared by the rest of investors and, still, 93% of them were net buyers.[4] Thus, according to this story, most agents were aware to be in a bubble, and investments were solely driven by speculative motives.[5]

In such situations, the crucial issue is therefore *when* the crash will occur, i.e. the timing of other investors' strategies. This is precisely the dimension over which boundedly rational investors are mistaken: they fail to distinguish different histories of trades when assessing other investors' strategies. Hence, they expect the same "average strategy", in a sense made precise below, to be observed irrespective of previous trades. Their inability to distinguish different histories when assessing future demand and supply should capture the observation that the date of the crash tends to appear very similar to quite a few other days. Investors and analysts typically find it hard to understand in what sense that precise day was so special, and fundamentally different from the day preceding it. Even

[2] Of course, this does not mean that agents process information in an optimal way, as this paper emphasizes.

[3] These conclusions have proven robust to many variations of the original experiment (see the review in Porter and Smith, 2003).

[4] The percentages for individual investors are approximately the same.

[5] Speculative behaviors seem a well documented phenomenon, spanning from the South Sea bubble in 1720 described in Temin and Voth (2004) to the Internet bubble in 2000 considered by Brunnermeier and Nagel (2004).

the systematic analysis by Cutler, Poterba and Summers (1989) concludes that "many of the largest market movements in recent years have occurred on days when there were no major news events".

We formalize these ideas by assuming that partially sophisticated agents understand only the aggregate buy and sell rates along the duration of the speculative market, and they lack a precise perception of the timing of such buy and sell decisions. One reason for this may be that looking at historical data in previous (similar) speculative markets, aggregate information was more accessible or salient (to these agents) than other more detailed statistics about say the daily buy and sell rates.[6]

We further assume that partially sophisticated investors adopt the simplest theory of trade volumes and price dynamics that is compatible with their knowledge.[7] They expect constant buy and sell rates throughout the duration of the speculative market, independently from the history of trades.

In equilibrium, these constant rates match the aggregate intensities averaged over time, as resulting from the actual sell and buy decisions.[8] In this sense, the understanding of partially sophisticated investors is correct. Still, these rates are only a partial representation of their opponents' investment strategies. As a result, these investors may fail to accurately identify the date of the crash, and to sell just before it. This is why the speculative market need not unravel and bubbles may arise.

The bubble equilibrium emerging in our model shares many aspects of a typical speculative phenomenon, as described for example by Kindleberger (2005). The bubble displays first a series of high prices, due to excess demand for speculative stocks. Such high prices are interpreted by boundedly rational investors as a good surprise, from which they infer that the bubble will last further.[9] Thus, they are induced to trade in the speculative market, and to remain invested even longer than they had originally planned. In this way, boundedly rational investors become "euphoric", revise their expectations and overestimate the duration of the bubble. Such expectations can be exploited by fully rational investors, who feed the bubble for a while and exit just before the (endogenous) crash. The massive sale by rational investors reveals that good times are over, boundedly rational investors realize it is time to sell (actually, it may be too late), and this indeed leads to the crash.[10]

[6] See Higgins (1996) for an exposition of the idea of accessibility in psychology, and Kahneman (2003) for an analysis of its implications for decision making.

[7] In line with this view, Wittgenstein (1922) writes: "The process of induction is the process of assuming the simplest law that can be made to harmonize with our experience."

[8] This is a bit reminiscent of the use of linear models in Sargent (1993).

[9] Shiller (2000), documents this strong regularity in investors' attitudes. As the price increases, more people display "bubble expectations", i.e. the belief that, despite the market being overvalued, it will still increase for a while before the crash.

[10] One may now invoke Friedman's critique: as boundedly rational investors on average lose money, in equilibrium they must be wiped away. A number of recent studies show that this argument is not so clear cut (see Blume and Easley, 2006; Kogan, Ross, Wang and Westerfield, 2006). In our model, we can

We show that such a bubble is more likely to exist, and it can last longer, the greater is the mass of people who can potentially enter the speculative market relatively to those already invested in it. This allows a greater excess demand, higher volumes of trade and higher increases in the price. These observations fit well with the empirical literature (e.g. Cochrane, 2002; Kindleberger, 2005), documenting that successful bubbles are characterized by short supply of speculative stocks, initially concentrated in a few hands, and they involve a large number of new, typically inexperienced investors.

The model also provides some novel insight on the relation between bubbles and rationality, showing that in general bubbles need not vanish as one increases the share of rational investors in the market.

As one should expect from the no-trade theorem, bubbles cannot arise if all investors are fully rational. However, we show that some rational investors are required in order to make boundedly rational agents realize that the bubble is bursting earlier than expected. When exiting the market, rational investors create a negative shock which reveals that the crash is about to occur and it is time to sell. In a sense, we stress that rational investors are required to break the bubble at some point, and hence sustain such equilibrium.

Second, we show that, under risk-neutrality, bubbles tend to disappear as the mass of rational investors increases, while this need not be the case if we consider ambiguity averse agents. In the first case, a larger share of rational investors makes the expectations of partially sophisticated traders less optimistic, since in equilibrium they observe more people leaving the market. Moreover, rational investors need to leave earlier when they are many, as they need to find enough investors to buy their stocks. Both effects tend to reduce the maximal duration of the bubble. Under ambiguity aversion, however, these may be counterbalanced by a third one. Rational agents are now more prone to invest than boundedly rational one, since they are less affected by strategic uncertainty. As the mass of rational investor increases, more people are entering the speculative market. Boundedly rational agents may then become more optimistic, hence possibly sustaining longer bubbles.

We now relate our approach to the existing literature. The basic model is presented in Section 2; the results on the existence and the characterization of bubble equilibria are derived in Section 3. In Section 4, we discuss bubbles and rationality of the market, and in Section 5 we conclude with some policy implications.

1.1 Related Literature

The literature on rational bubbles describes departures from fundamentals in markets with fully rational investors, who have the correct model in mind and make the best use of available information (see Blanchard and Watson, 1982). In such setting, however, the bubble component must be expected to grow over time, hence each agent has to allow that

think of a new population of such investors entering the market in every bubble episode.

with some (small) probability trades will occur at arbitrarily large prices. If one considers an economy with finite wealth (or with wealth that cannot grow as fast as the price of the asset), the maximal bubble has a finite upper bound, and the argument unravels. As shown in Tirole (1982), in a market with finite horizon, purely speculative trade relies on inconsistent plans and as such it is ruled out by rational expectations. The result is fundamental, as it shows that *any* theory of bubbles represents a departure from a fully rational world.

A number of explanation of bubbles are based on investors' private information. Allen, Morris and Postlewaite (1993) show that bubble may arise in a setting with a finite number of trading opportunities and every agent is rational only if one introduces ex-ante inefficiency, private information, short sale constraints and lack of common knowledge of agents' trades. In such setting, bubble may be sustained even if everyone realizes that the stock is overpriced, since other investors' trades cannot be observed and so their beliefs are never common knowledge. A similar point is developed by Abreu and Brunnermeier (2003), where the limit to arbitrage stems from a coordination failure among rational arbitrageurs, who desire to stay invested as long as profitable, while facing different opinions on when the bubble will burst. Here the bubble is fed by some (unmodeled) "behavioral agents" and it can be sustained since again its existence never becomes common knowledge.

In our model too investors are driven by purely speculative motives. However, for the reasons explained in the Introduction, we want to abstract from private information and keep the existence of the bubble and other investors' trades as common knowledge. Moreover, as shown in Tirole (1982), in order to overcome the no-trade theorem one needs to add extra ingredients, like subjective priors or other forms of irrationality. This calls for more structured theories of where these extra ingredients come from (see Bianchi, 2007 for further elaboration on this point.)

Our paper follows the line of incorporating bounded rationality in a setting where no bubble could arise if all investors are fully rational. Our emphasis on cognitive heterogeneity builds on the insights from the wide literature on the limits to information processing[11], which is now being applied also to the study of financial markets (see Hirshleifer, 2001).[12] Along the lines of bounded rationality, De Long, Shleifer, Summers and Waldmann (1990*a*) introduce noise traders, who act like crazy but may make great profits

[11]At least from Herbert Simon (1955) on, these constraints have been widely documented by physiologists (see Rabin, 1998 and Kahneman, 2003). Some of these ideas have recently been explored also in strategic interactions (see Rubinstein, 1998 and the references therein and Jehiel, 1995; Jehiel, 2005; Jehiel and Samet, 2006 for various models of bounded rationality in game theory).

[12]Hirshleifer (2001) states: "Since time and cognitive resources are limited, we cannot analyze the data the environment provides us with optimally. Instead, natural selection has designed minds that implement rules of thumb ('algorithms', 'heuristics', or 'mental modules') selectively to a subset of cues [...]. Such heuristics are effective when applied to appropriate problems. But their inevitable biases can become flagrant when used outside their ideal domain of applicability. [...] People share similar heuristics, those that worked well in our evolutionary past. So on the whole we should be subject to similar biases."

in equilibrium, since rational arbitrageurs are limited by finite horizon and risk aversion. An essential difference with our approach is that expectations and behaviors of noise traders are exogenously given, hence their trading decisions are mechanical. As already emphasized, we depart also from the literature on positive feedback traders (e.g. Cutler, Poterba and Summers, 1990; De Long, Shleifer, Summers and Waldmann, 1990*b*), where some agents form their expectations about future price dynamics only by extrapolating from previous trends. For the reasons exposed in the Introduction, we want to explain bubbles despite all investors understand (and it is actually common knowledge) that a crash will occur.

Lastly, bubbles can be explained if agents have heterogeneous beliefs. In such a word, investors know that there may be others with an higher valuation for the asset, and so they may be willing to speculate, i.e. to pay the asset more than what they would pay if they were forced to hold it forever (Harrison and Kreps, 1978). Scheinkman and Xiong (2003) consider a classical source of multiple priors: overconfidence. Overconfident traders overestimate the precision of their information, i.e. they believe to know "more" than the rest of investors. As such, these agents are willing to weight their private information more than the public information revealed in market prices, and so divergence in opinions about fundamentals may be persistent. This model is silent on how the difference in opinions among traders increases over time, so that one can predict increasingly high prices, and how overconfidence may disappear and so a crash may occur. More generally, we also model speculative trade driven by divergence in expectations, but the difference in opinions stems from the degree of sophistication (how precisely these agents understand others' strategies), rather than subjective priors. In this sense, our work is complementary to these approaches. Moreover, it avoids the issue, already discussed in the Introduction, of modeling equilibria with subjective priors.

2 The model

Our economy is populated by a unitary mass of risk neutral individuals. Initially, a mass K of individuals are endowed with cash, each of them owning $w > 0$, and a mass $(1 - K)$ have stocks, each of them owning one stock. The value of cash is constant over time. Stocks pay no dividend, and their return is given only by changes in the price p_t. That is, we normalize their fundamental value to zero and concentrate on purely speculative assets. For simplicity, we assume that each agent can hold at most one stock and each stock is indivisible.[13] Hence, player i profits are $\pi_i = 0$ if he always holds cash, and $\pi_i = (p_s - p_b)$ if he buys a stock at p_b and sells it at p_s.

At the beginning of each period $t = 1, 2, ...$, individuals can simultaneously submit an order to buy or sell a stock. A price p_t is then announced and orders are cleared.

[13] The substance of our analysis would not change if stocks were perfectly divisible and everyone could spend his entire wealth in stocks.

Borrowing stocks or cash is not allowed, hence the investment option for individual i in each t is simply $\{buy,\ stay\ out\}$ if i holds cash at t, and $\{sell,\ stay\ in\}$ if i holds a stock at t.[14] Let B_t denote the amount of people willing to buy at t, S_t those willing to sell, and $N_t := B_t - S_t$ the net demand. The trading prices at each t is determined according to the rule:

$$p_t = f(N_t, p_{t-1}), \tag{1}$$

where $f : [K-1, K] \times [0, w] \to [0, w]$ is increasing in N_t and $f(0, p_{t-1}) = p_{t-1}$. In case that B_t differs from S_t, buyers/sellers in excess are chosen randomly, so the volume of trade in t is $V_t = \min\{B_t, S_t\}$.

The function $f(\cdot)$ together with the rationing scheme defined above are assumed to be known to all agents, and they can be viewed as a reduced form model of price formation and trade allocation. Our specification assumes that the price in any period t exceeds the one at $t-1$ if and only if there is excess demand at t, where, given borrowing constraints, the trading price can never exceed w. Equating demand and supply in period t is achieved by an anonymous rationing scheme. Whenever trade occurs in this market, we say that there is a bubble.[15]

In each period, people in the speculative market may be hit by a shock and so need to sell immediately. Shocks are i.i.d. among agents; each individual shock occurs with probability $z > 0$, and it is permanent: an agent who exits the market stays out forever.[16] Given that $(1-K)$ is the mass of people in the speculative market,[17] we can define $X := z(1-K)$ as the exogenous exit from speculation in each period, where we assume that $X < K$. We define the amount of exit from the speculative market in period t as E_t, which includes those who sell their stocks and those with cash who decide not to buy at t.[18] Moreover, as it will turn out to be the case in equilibrium, those agents who decide to exit the speculative market, never re-enter. Accordingly, we can define K_t as the mass of

[14] We rule out more sophisticated investment strategies, where for example orders are made conditional on volumes or on other people's orders. Notice however that our analysis does not rely on the exclusion of short selling. As it will become clear, this would tighten the conditions defining our bubble equilibrium, but without changing its structure.

[15] The functioning of our market may remind the one in Kyle (1985), where agents submit a (market) order to buy or sell and the market maker increases the price over time according to excess demand. For simplicity, we consider a market maker who does not take any position, but simply clears the market via rationing. Our analysis would be unaffected if the market maker could take any position. Rationing may capture the common observation that growth stocks are scarce during booms (as documented e.g. in Cochrane, 2002).

[16] The fact that a (small) fraction of investors are hit by a permanent shock avoids that the speculative market increases indefinitely. Such shock can be thought as consequence of unforeseen liquidity needs, and it has not to be confused with noise trade.

[17] This mass never changes over time due to the equalization of demand and supply.

[18] Note that the agents with cash who decide not to buy at t should not be confused with the agents who submit an order to buy but do not get any stock at t due to rationing. Only the former are considered to exit the speculative market.

people in period t who have cash and who have not yet exited the speculative market in period t: these are the only agents who can possibly contribute to the demand in period t.[19] Moreover, we write

$$B_t = \beta_t K_t \text{ and } S_t = \sigma_t(1 - K), \tag{2}$$

where β_t and σ_t are the shares of potential buyers and sellers who want to buy or sell in period t, respectively. Accordingly, we have

$$K_{t+1} = K_t - E_t,$$

where

$$E_t = V_t + (1 - \beta_t)K_t, \tag{3}$$

and thus

$$K_{t+1} = \beta_t K_t - V_t. \tag{4}$$

Observe that if in period t the price drops due to excess supply, then $V_t = \min(B_t, S_t) = B_t$ and equation (4) gives $K_{t+1} = 0$. From then on there is no one left to enter the speculative market, which then closes. In this case, provided that $t > 1$, we say that a crash has occurred.

The amount of available information is the same for everybody. Once realized, trade orders B_t and S_t become common knowledge, hence in particular the mass of cash K_t available in period t is observed by all agents. Moreover, all agents rightly understand that traders who exit the speculative market never re-enter. What varies across individuals is their ability to process information, which determines their expectations about future prices. While some players are fully rational, as in standard economic models, we postulate that some other players have a partial rather than total understanding of the mechanics of these decisions. In particular, they fail to accurately distinguish one history of trades from another when forming their expectation.

For concreteness, we consider a setting with two extreme cognitive types: R and I. R-types are standard rational players, who understand perfectly well the dynamics of trade volumes and the associated prices. I-types instead are less sophisticated: they understand only the average sell and buy rates along the entire duration of the speculative market, as opposed to the exact patterns of these rates. As discussed in the Introduction, a rationale for I-types' limitation may be that the information about aggregate buy and sell rate, derived for example by historical data in similar speculative markets, is more accessible or salient than more detailed statistics, reporting say the daily buy and sell rates.[20]

[19] A crucial ingredient of our analysis is that K_t cannot increase over time. This is due to the fact that the population of potential investors and their capital endowment is fixed and finite. As the price increases, less people can afford the stock, hence K_t decreases. Moreover, in equilibrium, no one is willing to invest when the price decreases. Hence, this structure gives our market a finite horizon.

[20] In this vein, one should view R-types either as having access to better and more intelligible statistics or as being more sophisticated in their information treatment.

We further assume that I-agents adopt the simplest theory of trade volume and price dynamics that is compatible with their knowledge. Specifically, we assume that I-agents expect the same buy and sell rates to occur throughout the duration of the speculative market: Irrespective of the history of trades, they expect a constant fraction of people with cash to buy and a constant fraction of people with stocks to sell.[21] In equilibrium, these rates are required to correctly represent the average buy and sell rates over the entire duration of the speculative market.[22,23]

We denote by r the proportion of R-players, while $(1-r)$ is the proportion of I-players. At the beginning of $t = 1$, cash and stocks are randomly distributed across types. Hence, a mass $r(1-K)$ of R-types is given stocks and a mass rK is given cash. Similarly, I-types with stocks and cash are respectively $(1-r)(1-K)$ and $(1-r)K$.

We can now describe formally an equilibrium in our setup with agents of cognitive types R and I. We start by defining the expectations: for each type $\theta \in \{R, I\}$, we let $B_s^{\theta,t}$ be the expectation of an investor of type θ in period t about the demand in period s. Similarly, $S_s^{\theta,t}$ denotes the same expectation about the supply in period s.

R-agents are fully rational and as such their equilibrium expectations are correct. In particular, if the true buy and sell decisions arising in equilibrium in period s are given by B_s and S_s, R-agents' expectations must satisfy:

$$B_s^{R,t} = B_s \text{ and } S_s^{R,t} = S_s \text{ for every } t \leq s. \tag{5}$$

We turn next to I- agents' expectations. Denoting with $T+1$ the last date in which the speculative market operates (this T will be determined endogenously in equilibrium), we can define the average buy rate $\bar{\beta}$ and the average sell rate $\bar{\sigma}$ for the sequence of buy and sell decisions arising in equilibrium from $t = 1$ to $t = T+1$. That is, average rates are defined by

$$\bar{\beta} = \frac{1}{T+1}\sum_{t=1}^{T+1}\beta_t \text{ and } \bar{\sigma} = \frac{1}{T+1}\sum_{t=1}^{T+1}\sigma_t. \tag{6}$$

Given these average rates $\bar{\beta}$ and $\bar{\sigma}$, and the definition in (2), I- agents' expectations are defined by

$$B_s^{I,t} = \bar{\beta} \cdot K_s^{I,t} \text{ and } S_s^{I,t} = \bar{\sigma} \cdot (1-K) \equiv S^I \text{ for every } t \leq s, \tag{7}$$

[21] Yet, as will become clear, the simplest theory need not correspond to the truth, and accordingly I-agents are rightly viewed as being bounded rational.

[22] The correctness of these statistics is consistent with our interpretation in terms of historical data, which suggests that to the extent that previous markets are similar, they should correspond approximately to the same average patterns.

[23] This idea follows very closely the spirit of Jehiel (2005) who assumes, in the context of extensive form games, that each player i is characterized by a partition of the set of nodes where other players move. This partition is a collection of subsets called analogy classes, and it determines how player i forms expectations about other players' strategies along the game. Player i assesses only the average behavior of his opponents within each analogy class, and expects this same average behavior to be played at each node of the analogy class.

where $K_s^{I,t}$ denotes I-agents' expectations about the amount of cash available in period s, given the cash observed in t. That is, $K_t^{I,t} = K_t$ and, for $s > t$, we have

$$K_s^{I,t} = K_t - \sum_{t'=t}^{t'=s-1} E_{t'}^{I,t}, \tag{8}$$

where, according to the definition (3), we write

$$E_{t'}^{I,t} = \min(B_{t'}^{I,t}, S_{t'}^{I,t}) + (1 - \overline{\beta}) K_{t'}^{I,t}.$$

Given the formulation in equations (5) and (7), agents of all types form expectations about the price dynamics according to the function $f(\cdot)$ defined in equation (1), and the anonymous rationing scheme described above. With these expectations, each agent determines an optimal investment strategy, which specifies for every possible history of B_t and S_t whether this agent should sell his stock, given that he is present on the speculative market, or he should buy a stock, given that he has available cash. An investment strategy profile specifies an investment strategy for every agent in the economy. We can now define an equilibrium in our setting.

Definition 1 ***(Equilibrium)****: An investment strategy profile is an equilibrium if, all along the equilibrium path, each agent's investment strategy is a best response to his expectations, as defined in equations (5) and (7).*

Remarks. Our definition of equilibrium is in the spirit of the rational expectation equilibrium in which, due to the dynamic nature of the interaction, beliefs and investment strategies must be optimally adjusted at every point in time. Yet, the beliefs of I-agents are not necessarily correct, as there is no reason in general that β_t and σ_t be constant over time (in fact, as we shall see, they are not in a bubble equilibrium). Note also that our definition only considers the incentives of agents on the equilibrium path as we do not consider the adjustment of beliefs and strategies after a positive mass of agents have made non-equilibrium decisions.[24]

3 Analysis

We focus on symmetric equilibria in pure strategies, where all investors of a given type and with a given endowment at t follow the same pure strategy at t. Hence, we have

[24] We could easily amend the solution concept to cover off the path optimizations and expectations. Yet, this would make the notation heavier (in particular, the state variable parameterizing the decisions should no longer be the calendar time t but the entire history of buy/sell decisions) without adding much economic insight. Moreover, since each individual agent has a negligible weight (there is a continuum of agents), our notion of equilibrium is in the spirit of the Nash equilibrium where no single agent can on his own move the system away from the equilibrium path.

$B_t \in \{0, rK_t, (1-r)K_t, K_t\}$ and $S_t \in \{X, (1-r)X+r(1-K), rX+(1-r)(1-K), (1-K)\}$. Observe that the existence of an equilibrium is not an issue as there is always the non-bubble equilibrium in which every agent exits the speculative market at the very first period.[25] Our interest lies in showing the possibility of bubble equilibria and characterizing the conditions for such equilibria to exist.

The problem faced by an individual of a given type is the same irrespective of whether he has cash or a stock. That is, player $i \in \theta$ with cash prefers to buy if and only if player $j \in \theta$ with a stock prefers to stay in. Similarly, player $i \in \theta$ with cash prefers to stay out if and only if player $j \in \theta$ with a stock prefers to sell. Hence, the two problems can be treated together. It follows that speculative trade can only occur between players of different types. This is intuitive: trade occurs between people with different needs, described by the liquidity sellers X, or with different expectations, i.e. different types.

If player $i \in \theta$ submits an order to buy at t, it should be that there exists a period $s > t$ such that the expected gain from selling at s is positive. That is, we need

$$q_s^{\theta,t} \cdot p_s^{\theta,t} \geq p_t^{\theta,t}, \tag{9}$$

where $q_s^{\theta,t} = \min\{1; B_s^{\theta,t}/S_s^{\theta,t}\}$ is the expected probability of being able to sell in period s, as perceived in period t, and $p_s^{\theta,t}$ is the corresponding selling price. Otherwise, buying at t would not be profitable.

Our premise in describing the dynamics of our market was that if an agent decides to exit speculation, he never tries to re-enter. The consistency of this premise comes from the fact that we have no increase in the set of potential buyers. Given this, I-players do not re-enter. In fact, their exit at t means that they expect a crash no later than at $t+1$. At $t+1$, even if they do not observe such crash, they expect it no later than at $t+2$, as the amount of available cash has not increased, hence they exit; and so on. R-players too prefer not to re-enter. In fact, if they sell at t, it must be that the price is lower at $t+1$, i.e. $B_{t+1} < S_{t+1}$. Suppose they re-entered at $t+1$, then $B_{t+1} < S_{t+1}$ means that I-investors are selling, but given that they do not re-enter, no one will be willing to buy from $t+2$ on, so it is not optimal to re-enter. The same argument can be replicated in any subsequent period, as long as the market operates. We can express the result more precisely with the following Lemma.

Lemma 1 *Given the expectations and price dynamics considered in Section 2, an agent who sells at t stays out of the speculative market from then on.*

Proof: We prove the result for I-agents first. Suppose $p_{t+1}^{I,t} > p_t^{I,t}$. Then $B_{t+1}^{I,t} > S^I$ and $q_{t+1}^{I,t} = 1$, hence $q_{t+1}^{I,t} \cdot p_{t+1}^{I,t} > p_t^{I,t}$ and i will not sell at t. If $i \in I$ sells at t, then it must be that $p_{t+1}^{I,t} \leq p_t^{I,t}$, i.e. $B_{t+1}^{I,t} = \bar{\beta} K_{t+1}^{I,t} = \bar{\beta}(\bar{\beta} K_t - S^I) < S^I$. Since K_t cannot increase with t, then it must be that $B_s^{I,w} < S^I$ for every $w, s > t$, hence i will never buy.

[25] In this equilibrium, I-agents' expectations are correct, and their decisions to exit immediately is thus rational.

Now consider R-agents. As above, if $i \in R$ sells at t, then it must be that $p_{t+1} \leq p_t$ (we drop the expectations as these are fully rational), i.e. $B_{t+1} < S_{t+1}$. Hence $K_{t+s} = 0$ and $B_{t+s} = 0$ for every $s > 1$. Since it will be impossible for $i \in R$ to sell after $t+1$, he will not buy. **Q. E. D.**

As we noted earlier, whenever exits from the speculative market are permanent, the price does not recover after having dropped. This implies that condition (9) is also sufficient for an agent i to buy.[26] Moreover, whenever

$$p_{t+1}^{\theta,t} \geq p_t^{\theta,t}, \tag{10}$$

condition (9) *a fortiori* holds. In fact, given the price dynamics in equation (1), $p_{t+1}^{\theta,t} \geq p_t^{\theta,t}$ is equivalent to $q_{t+1}^{\theta,t} = 1$. Hence, condition (9) is satisfied by letting $s = t+1$. Moreover, given Lemma 1, condition (10) is also necessary for i to buy at t. As the price is not expected to recover after having dropped, no investor would buy if condition (10) is violated.

Summing up, we can rewrite condition (10) according to equation (1), and say that a necessary and sufficient condition for $i \in \theta$ to buy/stay in at t is that

$$B_{t+1}^{\theta,t} \geq S_{t+1}^{\theta,t}. \tag{11}$$

Hence, the model endogenously generates short-termism: agents are concerned only with short term speculative gains rather than with fundamentals. Everybody knows that the fundamental value of the asset is zero, any trade is driven by the resale option, and everybody understand that this resale option become worthless as soon as the market displays excess supply.

3.1 The bubble equilibrium

We can now show that, under conditions to be characterized in the next Subsection, there exists an equilibrium of the following form. In each period $t \leq T-1$, everyone tries to enter the speculative market and no one wants no sell; in period T, I-investors buy and R-investors sell; at $T+1$, everyone tries to sell but no one is willing to buy. The crash then occurs and the market closes. We call this the bubble equilibrium.

Along this equilibrium, the buy and sell rates are described as follows:

$$\beta_t = \begin{cases} 1 & \text{for } t \leq T-1, \\ 1-r & \text{for } t = T, \\ 0 & \text{for } t = T+1, \end{cases} \tag{12}$$

[26] Given that each agent can hold at most one stock, he would like to buy when the price is at the minimum and sell when it is at the peak. Hence, in principle, he would wait to buy if he thought the price would drop for a while before recovering. However, this price dynamic is ruled out by the Lemma.

and

$$\sigma_t = \begin{cases} z & \text{for } t \leq T-1, \\ z + r(1-z) & \text{for } t = T, \\ 1 & \text{for } t = T+1. \end{cases} \tag{13}$$

According to equation (6), the expected buy and sell rates induced by this equilibrium for I-agents are given by

$$\bar{\beta} = \frac{T-r}{T+1} \tag{14}$$

and

$$\bar{\sigma} = \frac{Tz + r(1-z) + 1}{T+1}. \tag{15}$$

This allows us to define $B_s^{\theta,t}$ and $S_s^{\theta,t}$ according to equations (5), (6), (7), (8). Besides, given the above specifications, the only variable remaining to endogenize is the duration T of the bubble.

First, as already noticed, Lemma 1 implies that the optimal investment strategy for each player i satisfies a threshold property. If i wants to buy/stay in the market at t, this reveals that he wanted to buy/stay in the market at each $s \leq t$. Similarly, if he wants to sell/stay out at t, then he want to sell/stay out for every $s \geq t$. Hence, the equilibrium T is simply defined by three conditions. First, each player $i \in I$ has to prefer to buy at T. Second, each player $i \in I$ has to prefer to sell at $T+1$. Third, each player $i \in R$ has to prefer to buy at $T-1$.[27] Using equation (11), we can express the three conditions, respectively, in the next Proposition.

Proposition 1 *The above investment strategy profile is an equilibrium if and only if there exists a T is such that*

$$B_{T+1}^{I,T} \geq S_{T+1}^{I,T}, \tag{16}$$

$$B_{T+2}^{I,T+1} \leq S_{T+2}^{I,T+1}, \tag{17}$$

and

$$B_T \geq S_T. \tag{18}$$

In order to express these conditions in terms of our exogenous parameters, notice that, according to equation (7), I-agents invest at t if and only if $\bar{\beta} K_{t+1}^{I,t} \geq \bar{\sigma}(1-K)$, where by (8) we have $K_{t+1}^{I,t} = \bar{\beta} K_t - S^I$. Hence, the optimal investment strategy for I-agents is a function of the available cash in the economy K_t and of their expectations about the future buy and sell rates. Notice that these expectations, as defined in equations (14) and (15), depend on the equilibrium T, and not on the time t.[28] To emphasize this, with some

[27] Notice that the first two conditions imply that each player $i \in R$ prefers to sell at T.

[28] This reasoning should make clear that if they want to buy at T, then they wanted to buy at each $t \leq T$ (recall that K_t does not increase over time).

abuse of notation, we let $\bar{\beta}(T)$ and $\bar{\sigma}(T)$ denote these rates, and we define the function

$$W(T) \equiv \frac{(1-K)\bar{\sigma}(T)(1+\bar{\beta}(T))}{\left(\bar{\beta}(T)\right)^2}. \tag{19}$$

I-investors will then buy at t if and only if

$$K_t \geq W(T).$$

The third condition imposes no excess supply in T, so that p_T exceeds p_{T-1} and hence it is optimal for R-investors to buy at $T-1$. According to equations (12) and (13), $B_T \geq S_T$ writes $(1-r)K_T \geq (z+r(1-z))(1-K)$, where K_T can be computed using the definition in (4). These reasonings lead to the following Lemma.

Lemma 2 *Conditions (16),(17) and (18) are equivalent to*

$$K_T = K - (T-1)X \geq W(T), \tag{20}$$

$$K_{T+1} = (1-r)(K-TX) - r(1-K) \leq W(T), \tag{21}$$

and

$$T \leq \frac{K-r}{X(1-r)}, \tag{22}$$

respectively.

Hence, using Proposition 1 and Lemma 2, we can state the conditions defining the length T of the bubble as follows.

Proposition 2 *Any T satisfying conditions (20), (21) and (22) can be sustained as a bubble equilibrium.*

As we now illustrate, condition (20) and (22) define a maximal T and condition (21) a minimal T which can be sustained in the bubble equilibrium. To show this, we first have to characterize the shape of $W(T)$.[29]

Lemma 3 *$W(T)$ is decreasing and convex in T. Moreover, $W(T) \to 2X$ as $T \to \infty$.*

[29] Doing some algebra, one can see that

$$W(T) = \frac{(T-r)X + (1+r)(1-K)}{(T-r)^2}(2T+1-r).$$

Proof. Simple algebra shows that $W'(T) < 0$ and $W''(T) > 0$ for every T. Moreover, as $T \to \infty$, $\bar{\beta}(T) \to 1$ and $\bar{\sigma}(T) \to z$. Recalling that $X = z(1-K)$, we have $W(T) \to 2X$. **Q. E. D.**

Now, since K_T and K_{T+1} are linearly decreasing in T and they both tend to $-\infty$ as T goes to infinity, each of them can intersect at most twice with $W(T)$. Let T_1 be the largest root solving $K_T = W(T)$ and T_2 be the largest root solving $K_{T+1} = W(T)$.[30] Moreover, define T_3 as

$$T_3 \equiv \frac{K-r}{X(1-r)},$$

and

$$T_{\max} \equiv \min\{T_1, T_3\}.$$

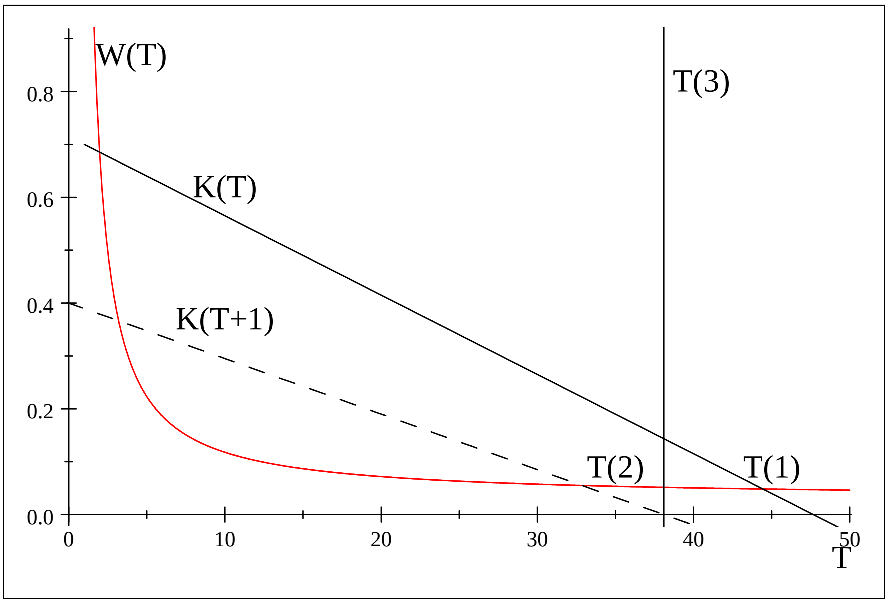
Given the last Lemma, conditions (20) and (22) require that $T \leq T_1$ and $T \leq T_3$ and hence that $T \leq T_{\max}$. This defines the maximum T before which R-players are able to profit from the bubble, selling at its peak: Selling after $T_{\max}$ would imply selling together (or after) I-players, hence during the crash. However, not all $T \leq T_{\max}$ can be sustained as equilibrium. Condition (21) is satisfied for $T \geq T_2$: if R-players sell too early, I-players would not exit and the crash would not occur at $T+1$, hence selling at T would not be optimal.

The structure of this equilibrium seems consistent with a number of empirical observations. First, our market becomes increasingly euphoric (as in Kindleberger, 2005). This is due to the fact that I-investors initially observe much bigger price increases than expected (since more people than expected are willing to invest in the speculative market) and this lead them to revise upwards their expectations about the future demand, and hence the date of the crash, and eventually to remain invested for too long. Second, a number of studies document that major investors do ride bubbles and are often able to earn great profits by exiting at the right time. Moreover, their strategies seem driven by a better understanding of the market rather than to the superior access to information (see e.g. Temin and Voth, 2004 and Brunnermeier and Nagel, 2004).

Before characterizing the conditions for the existence of some T which can be sustained as a bubble equilibrium, it is useful to highlight the structure of conditions (20), (21) and (22) through a numerical example.

Example 2 *Suppose that $z = 0.05$, $r = 0.3$ and $K = 0.7$. The functions $W(T)$ and K_T are plotted below in solid lines, while K_{T+1} is the dashed line. Hence, T_2 is given by the intersection of the dashed line with $W(T)$, while T_1 by the intersection of the solid line with $W(T)$. The vertical line is T_3 (which binds in this example). Substituting our values in equations (20), (21) and (22), we find that (up to integer approximations) $T_1 = 44$, $T_2 = 33$ and $T_3 = 38$.*

[30] In the next Subsection, we give conditions for existence and uniqueness of such T_1 and T_2.



In fact, condition (16) requires $T \in [2, 44]$, condition (17) requires $T \leq 3$ or $T \geq 33$ and condition (18) $T \leq 38$. Hence, in this example, any $T \in [2, 3] \cup [33, 38]$ can be a bubble equilibrium.

3.2 Existence of a bubble equilibrium

In order to investigate the conditions for the existence of some T defining our bubble equilibrium, we now analyze equations (20), (21) and (22) in terms of our exogenous parameters K, z and r. First, since $K_{T+1} < K_T$ for every T, we have that $T_2 < T_1$. Moreover, we can also see that $T_2 < T_3$, since by definition $K_{T_3+1} = 0$ and hence condition (21) holds for sure at $T_3 + 1$. Thus, we can say that

$$T_2 < T_{\max}.$$

This observation, together with Lemma 3, implies that the bubble equilibrium exists if and only if $W(T)$ and K_T intersect at least once, i.e. if there exists a $T_1 \geq 1$ such that $K_{T_1} = W(T_1)$. In fact, when this is the case, $T_{\max}$ can always be sustained as equilibrium. Hence, a *sufficient* condition for the existence of a bubble equilibrium is that $W(T)$ and K_T intersect once, and only once, as it is considered in the following Proposition.

Proposition 3 *If $K \geq W(1)$ then a bubble equilibrium exists. The maximal duration of an equilibrium bubble is given by $T_{\max}$.*

It follows from this Proposition that a bubble equilibrium is more likely to exist when K is large relatively to $W(1)$. Given that $W(1)$ increases in z and r, the condition requires K to be large relatively to z and r. These relations are intuitive: a bubble is more likely to develop when there is a large amount of cash which can potentially be used to fuel it; when a few people are hit by shocks which force them to exit speculation (recall that in equilibrium exits are permanent) and when the number of fully rational investors, who can correctly predict other agents' strategies and hence the exact date of the crash, is not too large.

To see this more precisely, observe that

$$K \geq W(1) \Longleftrightarrow K \geq \frac{((1-r)z(1-K)+(1+r)(1-K))(3-r)}{(1-r)^2}. \tag{23}$$

Rearranging, we can write the previous expression as:

$$K \geq K^*,$$

or equivalently, emphasizing the role of z or r, as

$$z \leq z^* \text{ or } r \leq r^*.$$

Hence, a sufficiently large K is needed to satisfy condition (23), and such condition is more likely to hold for low r and low z.[31] We can sum up with the following Corollary.

Corollary 1 *If $K \geq K^*$, or equivalently $r \leq r^*$ or $z \leq z^*$, then a bubble equilibrium exists.*

The conditions in the last Corollary, as well as the comparative statics emerging in the next Subsection, show that bubble are supported by large K and small z and r. These relations are intuitive and consistent with empirical evidence. A large K is in line with the observation that speculative stocks tend to be initially in short supply, and that bubbles are sustained by the large involvement of new investors. This allows greater excess demand, greater increases in the price and higher volumes of trade (see Cochrane, 2002; Kindleberger, 2005). A small probability of shock z implies that the mass of potential investors decreases slowly, which is consistent with the fact that bubbles tend to display slow booms and sudden crashes (see Veldkamp, 2005). Finally, a small r emphasizes that these episodes tend to attract a large number of inexperienced investors (see Shleifer, 2000, Kindleberger, 2005). However, as we elaborate in the next Section, the relation between bubble and rationality is not so clear cut: an increase in r need not make bubble vanish.

[31] The thresholds K^*, z^* and r^* can be computed with simple algebra from equation (23). One can see that K^* increases in r and z and $K^* \in (3/4, 1)$. Equivalently, one can solve equation (23) for z^* and notice that it is decreasing in r and increasing in K; or solve for r^* and see that it decreases in z and increases in K.

3.3 The maximal equilibrium bubble

As noticed in Proposition 2, there need not be only one T satisfying conditions (20), (21) and (22). In fact, we have seen that, even if $K \geq W(1)$, any T between T_2 and $T_{\max}$ can be sustained as a bubble equilibrium. In general, rational investors need to solve a coordination problem in order to select a T at which they all sell. One natural candidate for such selection is $T_{\max}$, which is the largest T that can be sustained as an equilibrium, i.e. the one maximizing R-players' profits. It is then of interest to characterize such $T_{\max}$, and to investigate how it varies with our exogenous parameters.

First, notice that irrespective of whether or not T_1 exceeds T_3, the comparative statics are clear: both T_1 and T_3 increase in K and decrease in r and z. In fact, $\bar{\sigma}$ increases and $\bar{\beta}$ decreases in z and r, so $W(T)$ decreases in K and increases in z and r. Also, K_T increases in K and decreases in z.[32] Hence, low z and r and high K lead to a large T, which also implies sufficiently large mistakes in players I's expectations and so it allows to sustain a longer equilibrium bubble. We can sum up with the next Proposition.

Proposition 4 *T_{max} increases in K and decreases with z and r.*

In order to further characterize $T_{\max}$, we study under which conditions T_1 or T_3 define the maximal sustainable bubble. In particular, we are interested in the effect of r. Intuitively, one should expect T_1 to be the binding constraint when r is low and T_3 to bind when r is high. When rational investors are many, they need to exit the market at a point in which the cash in the economy is still very high, that is a point in which I-investors would think that the date of the crash is far away. The following Proposition formalizes this intuition.

Proposition 5 *$T_3 < T_1$ if and only if $r > \bar{r}$, where $\bar{r}$ is implicitly defined by*

$$r > \frac{W(T_3) - X}{W(T_3) - X + 1 - K} \equiv G(r). \tag{24}$$

Proof. By definition of T_1, $T_3 < T_1$ if and only if $W(T_3) < K_{T_3}$. By definition of T_3, $K_{T_3}(1-r) = S_T$ and $S_T = X(1-r) + r(1-K)$. Rearranging, we have equation (24). Now notice that $G(r)$ is increasing in $W(T_3)$, and $W(T_3)$ is increasing in r. Moreover, $G(0) > 0$ and $G(1) < 1$. Hence $r > G(r)$ holds for $r > \bar{r}$, where $\bar{r}$ is uniquely defined by $\bar{r} = G(\bar{r})$. **Q. E. D.**

It follows from the last Proposition that if r is smaller than the minimum of $G(r)$, then T_1 binds. This is expressed in the next Corollary.

Corollary 2 *If $r \leq \frac{z}{z+1}$, then $T_{\max} = T_1$.*

[32] These results can be formally obtained by implicitly differentiating the function $W(T) - K_T$.

Proof. As shown in Lemma 3, $W(T) < 2X$ for every T. Since $G(r)$ increases in $W(T)$, we have that $G(r) > \frac{X}{X+(1-K)}$. Recalling the definition $X = z(1-K)$, we write $G(r) > \frac{z}{z+1}$. Hence if $r \leq \frac{z}{z+1}$, then $r < G(r)$, i.e. $T_{\max} = T_1$. **Q. E. D.**

Finally, the Proposition allows to characterize $T_{\max}$ when the probability of liquidity shocks is very small, which is an important special case we will reconsider in the next Section. It turns out that both T_1 and T_3 tend to infinity as z tends to zero, but T_1 exceeds T_3. In fact, recall that by definition T_2 solves $K_{T_2+1} = W(T)$ and T_3 solves $K_{T_3+1} = 0$. Notice that $W(T)$ tends to vanish when T goes to infinity, hence T_3 tends to T_2, which is always smaller than T_1. This argument leads to the next Corollary.

Corollary 3 *If* $z \to 0$, *then* $T_{\max} = T_3$.

Proof. If $z \to 0$, then $X \to 0$, $T_3 \to \infty$ and $W(T_3) \to 0$. Hence, $G(r) \to 0$, and condition (24) always hold. Hence, $T_{\max} = T_3$. **Q. E. D.**

4 Bubbles and rationality

The results above emphasize that bubbles are more likely to be sustained when the mass of rational investors is small. In fact, we have seen that a larger share of rational investors increases $W(T)$, i.e. it makes the expectations of $I-$traders less optimistic, since in equilibrium they observe more people leaving the market. Moreover, it decreases B_T and it increases S_T : rational investors need to leave earlier when they are many, as they need to find enough investors to buy their stocks. Both effects tend to reduce the maximal duration of the bubble.[33]

However, these investors need not always be a stabilizing force in our model. As we now show more in details, a more rational market need not generally display fewer or shorter bubbles.

4.1 Rational investors should not be too many

As in standard models, we cannot have bubbles if all investors are fully rational, i.e. if $r = 1$. In particular, in a bubble equilibrium, r has to be small enough, relative to K and X, in order for all rational players to be able to all sell at T. This requires that R's supply in T does not exceed I's demand, as already expressed in condition (18). The existence of such T requires $T_3 \geq 1$, i.e.

$$r \leq \frac{K-X}{1-X} \equiv r_{\max}.$$

Hence, we can state the following Proposition.

[33] None of these effects are considered in De Long, Shleifer, Summers and Waldmann (1990*b*), where feeback traders have exogenous expectations and no budget constraint. Indeed, they find the opposite result, i.e. more rational investors drive the prices further away from fundamentals.

Proposition 6 *If $r > r_{\max}$ then no bubble equilibrium exists.*

Notice that $r_{\max}$ defines a necessary condition for the existence of a bubble equilibrium. In the previous Section, we characterized a sufficient condition, in terms of the upper bound r^*. As one expects, it can be shown that $r_{\max} \geq r^*$ for each K and z.

4.2 Rational investors should not be too few

As it is clear from the previous analysis, our equilibrium conditions (20), (21), (22) are more likely to be satisfied when the mass of rational investors r is low. Moreover, the longest sustainable bubble $T_{\max}$ decreases with r. We now have a closer look to the extreme case of $r = 0$. First, as implied by Proposition 5, condition (22) never binds. Conditions (20) and (21) write, respectively, as

$$K_T = K - (T - 1)X \geq W(T) \tag{25}$$

and

$$K_{T+1} = (K - TX) \leq W(T), \tag{26}$$

where now $W(T) = (TX+1-K)(2T+1)/T^2$. Indeed, it is easy to see that this induces the largest possible bubble, given K and X. However, this scenario displays a few properties worth discussing further. First, notice that if all agents are of the same type, they share the same expectations and so trade occurs only for exogenous reasons ($V_t = X$ for all $t \leq T$).

Second, and more interestingly, note that in such scenario I-players never realize that the crash occurs at $T+1$, not even in period $T+1$. By contrast, when there are sufficiently many rational investors, they give a negative shock to the market by exiting at T. At this point, I-investors realize they had overestimated the length of the bubble, i.e. the crash is going to occur earlier than they had predicted at T, when they bought for the last time. That is, after having observed R's exit, I-traders are convinced that a crash is about to occur and they run to sell (actually, it is already too late). This seems a plausible dynamic of a crash (as described for example in Kindleberger, 2005), and, as we now show, this requires that r is not too small. Specifically, consider the following condition:

$$B_{T+1}^{I,T+1} \leq S_{T+1}^{I,T+1}. \tag{27}$$

Condition (27) ensures that, at the beginning of $T + 1$, just before the crash occurs, all investors expect the crash to occur next. An alternative, equivalent, way to interpret condition (27) is that it requires I-investors' expectations about the date of the crash to change between period T and period $T + 1$. In fact, considering condition (27) together with (16), we can see that I-players understand that the plans they made at T are not accurate, and in particular that they have underestimated the amount of exits at T. In fact, they now realize that the crash is going to occur earlier than expected.

Adding (27) to our equilibrium conditions (16), (17) and (18), we can uncover an important role that R-players have in our model. From the previous discussion, it is clear that condition (27) requires some bad shock to occur in period T. Since the only source of such shock is that R-investors decide to exit, we need sufficiently many of them. It follows in particular that if $r = 0$, no equilibrium bubble can satisfy condition (27). I-investors would not receive any bad news along the equilibrium path, hence they would never revise their plans by having to sell earlier than expected.[34] We can state this more precisely with the following Proposition.

Proposition 7 *In a bubble equilibrium where condition (27) holds, we must have* $r > r_{\min}$*, where* $r_{\min}$ *is implicitly defined by*

$$r > \frac{1}{T}.$$

Proof. Condition (27) writes $\bar{\beta}K_{T+1} < S^I$. Together with (20), this require $K_{T+1} = K_T - S_T - rK_T < K_{T+1}^{I,T} = \bar{\beta}K_T - S^I$. That is, $(1-r)K_T + S^I < \bar{\beta}K_T + S_T$.

We now show that $rT > 1$ is equivalent to $(1-r) < \bar{\beta}$ and to $S^I < S_T$. First, by the definition of $\bar{\beta}$, $(1-r) < \bar{\beta}$ writes $(T+1)(1-r) < (T-r)$ which gives $rT > 1$. Second, again by definition, $S^I < S_T$ writes $(T-r)X+(1+r)(1-K) < (T+1)[r(1-K)+X(1-r)]$. Rearranging, this gives $Tr(1-K) - TrX > 1 - K - X$, that is $rT > 1$. Hence requiring $(1-r)K_T + S^I < \bar{\beta}K_T + S_T$ is equivalent to requiring $rT > 1$. Since for $r = 0$, $1/T > 0$, the last expression implicitly defines a unique $r_{\min}$ such that (27) holds only if $r > r_{\min}$. **Q. E. D.**

4.3 Ambiguity Aversion

In our model, uncertainty is purely strategic, i.e. it concerns solely the predictions of what other investors do. Hence, the uncertainty borne by each player depends on his ability to understand his opponents' equilibrium strategies. Since perfectly rational players face no uncertainty, they may be willing to take longer positions than some I-investors. Hence, increasing the share of rational investors need not take the market closer to efficiency. Instead, by increasing the mass of investors in the speculative market, an higher r may make I-investors' expectations more optimistic and hence allow to sustain longer bubbles.[35]

In order to formally explore these intuitions, we enrich our setting by introducing

[34] This result, stressing the need for rational investors to break the bubble at some point, can be viewed as complement to the literature noting that fully rational investors may artificially increase returns and initiate a bubble (De Long et al., 1990*b*).

[35] In particular, in our model, differently from the noise traders literature (De Long et al., 1990*a*), an increase in the risk bearing capacity of the economy drives the prices further away from fundamentals.

ambiguity averse investors.[36] In particular, consider two types of I-investors: L-investors are ambiguity neutral, as modeled in Section 2, and H-investors are ambiguity averse.[37] Specifically, we assume that H-agents know that their model can be wrong, and they consider that predictions of β_t and σ_t can be mistaken by at most ε. That is, they believe that the actual buy rate β_t will be in the interval $[\bar{\beta} - \varepsilon, \bar{\beta} + \varepsilon] \cap [0, 1]$ for every t, and similarly the actual sell rate σ_t in $[\bar{\sigma} - \varepsilon, \bar{\sigma} + \varepsilon] \cap [0, 1]$. Furthermore, these players adopt the most cautious behavior: they assume the worst realizations of β_t and σ_t and choose the optimal strategy given this realization.[38] It follows that, in order to be part of speculation, they require an higher return to be compensated by the increased perceived uncertainty.[39] Investors of type L, who are assumed to be neutral towards ambiguity, can alternatively be thought as being unaware of the fact that they may make mistakes.[40]

We can now replicate our previous analysis, characterizing how in this setting the maximal sustainable bubble varies with the share of rational investors. First, as in equation (20), L-investors buy/stay in at t if and only if

$$K_t \geq \frac{(1-K)\bar{\sigma}(T)(1+\bar{\beta}(T))}{\left(\bar{\beta}(T)\right)^2} \equiv W(T),$$

while H-investors buy/stay in at t if and only if

$$K_t \geq \frac{(1-K)(\bar{\sigma}(T)+\varepsilon)(1+\bar{\beta}(T)-\varepsilon)}{\left(\bar{\beta}(T)-\varepsilon\right)^2} \equiv W(T,\varepsilon).^{41}$$

We can see that $W(T, \varepsilon)$ increases in ε, hence H-investors will always sell before L-investors. We want to define an equilibrium where H-investors sell at some $\tilde{T}$, fully rational investors sell at $T > \tilde{T}$, and L-investors sell at $T + 1$. As above, let r be the mass

[36] Uncertainty (or equivalently ambiguity) describes situations where the perceived likelihood of some event cannot be represented by a probability measure over the possible states of the world. That is, we allow for the possibility that the information perceived by our investors is not accurate enough to provide them with a unique probability measure.

[37] For the reason just mentioned, attitudes toward ambiguity are not relevant for R-investors.

[38] Formally, we are assuming that these investors have a set of probability measures over the possible realizations of β_t and κ_t. Investors compute the minimal expected payoffs conditional on each possible prior, and decide the investment strategy corresponding to the maximum of such payoffs. This idea, which may be thought as an extreme form of uncertainty aversion, was formalized by Gilboa and Schmeidler (1989).

[39] Indeed, many authors have invoked this as a possible resolution to the Equity Premium Puzzle (e.g. Chen and Epstein, 2002; Klibanoff, Marinacci and Mukerji, 2005).

[40] One way to interpret these types is in terms of different degrees of confidence about one own's cognitive ability, with H-investors being cautious and L-investors being (over)confident. Under this perspective, we notice that our previous analysis is implicitly stressing the role of overconfidence, as I-investors do not consider the possibility that their forecast may be inaccurate (see Scheinkman and Xiong, 2003).

[41] This is simply obtained from (20) by replacing $\bar{\beta}(T)$ by $\bar{\beta}(T) - \varepsilon$ and $\overline{\kappa}(T)$ by $\overline{\kappa}(T) + \varepsilon$.

of fully rational investors and $(1-r)$ that of boundedly rational ones. The latter can display either high or low ambiguity aversion, in proportion h and $(1-h)$ respectively. We look for an equilibrium in which

$$\beta_t = \begin{cases} 1 & \text{for } t < \tilde{T} \\ 1-(1-r)h & \text{for } t = \tilde{T} \\ 1 & \text{for } t \in (\tilde{T}, T) \\ 1-\frac{r}{1-h(1-r)} & \text{for } t = T \\ 0 & \text{for } t = T+1, \end{cases} \quad \text{and } \sigma_t = \begin{cases} z & \text{for } t < \tilde{T} \\ z+(1-z)(1-r)h & \text{for } t = \tilde{T} \\ z & \text{for } t \in (\tilde{T}, T) \\ z+(1-z)\frac{r}{1-h(1-r)} & \text{for } t = T \\ 1 & \text{for } t = T+1. \end{cases}$$

That is, for $t < \tilde{T}$ no investor wants to exit, hence the volume of trade is given by liquidity shocks only, occurring with probability z. At $\tilde{T}$, high ambiguity averse agents, with mass $(1-r)h$, leave the speculative market, selling to rational agents R and low ambiguity averse agents L. Since the excess demand is allocated randomly between R and L, and shocks are also random, the proportion of R and L in the speculative market remains constant. That is, for $t \in (\tilde{T}, T)$, the proportion of R-investors in the market is

$$\frac{r}{r+(1-h)(1-r)},$$

and, similarly, the proportion of L-agents is

$$\frac{(1-r)(1-h)}{r+(1-h)(1-r)}.$$

Notice also that for $t \in (\tilde{T}, T)$ everyone in the market is willing to become/remain invested (H-agents have left and they are not considered any more). At T, rational investors exit speculation, selling to L-agents. At $T+1$, L-agents realize that the crash is about to occur and they want to sell, while no one is willing to buy. The crash occurs and the market closes.

Using equation (6), we can compute the average buy and sell rates as

$$\bar{\beta} = \frac{1}{T+1}[T-(1-r)h - \frac{r}{1-h(1-r)}] \tag{28}$$

and

$$\bar{\sigma} = \frac{1}{T+1}[1+zT+(1-z)((1-r)h + \frac{r}{1-h(1-r)})]. \tag{29}$$

Similarly to the previous Section, the maximal bubble is defined by two conditions. First, L-players have to buy at T, hence

$$B^{L,T}_{T+1} \geq S^{L,T}_{T+1}, \tag{30}$$

which is a simple variation of (16). Second, R-players have to buy at $T-1$, i.e.

$$B_T \geq S_T,$$

which is condition (18). From the analysis in Section 3, we know that equation (30) can be written as

$$K_T \geq W(T), \tag{31}$$

where now K_T writes:

$$\begin{aligned} K_T &= K - \sum_{t'=1}^{T-1} E_{t'} = K - \sum_{t'=s}^{t'=t} S_{t'} - (1-\beta_{\tilde{T}})K_{\tilde{T}} \\ &= K - (T-1)X - (1-r)h[(1-z)(1-K) + K - (\tilde{T}-1)X]. \end{aligned} \tag{32}$$

In fact, recall that, in equation (3), we defined exit E_t as the mass of agents who submit an order to sell and who do not submit an order to buy in period t. Accordingly, exits from speculation until $T-1$ include liquidity traders $(T-1)X$ and high ambiguity averse traders who have exited at $\tilde{T}$. The latter include those who have decided to sell, who have mass $(1-r)h[(1-z)(1-K)]$, and those who have not bought at $\tilde{T}$, who have mass $(1-r)hK_{\tilde{T}} = (1-r)h[K-(\tilde{T}-1)X]$. Adding these terms, we have the expression in (32).

We are interested in how the longest sustainable bubble $T_{\max}$ varies with r, for given h.[42] In addition to the effects already explored in the previous Section, which lead $T_{\max}$ to unambiguously decrease with r, we now have to consider that the proportion of high ambiguity averse investors H decreases in r. As we now show in more details, this has a series of direct and indirect effects, making it possible that $T_{\max}$ increases in r.

4.3.1 The effects on T_3

Recall that T_3 is the latest period in which all rational investors can exit the market at the peak of the bubble, i.e. the last date with excess demand. That is, T_3 solves $B_T = S_T$. As in the previous Section, an higher r tends to decrease T_3 since it increases the supply S_T and decrease the demand B_T. On the other hand, an higher r also implies that the mass of ambiguity averse people H is lower, and this tends to increase T_3. In fact, a lower H increases K_T, the amount of cash available at T, hence possibly allowing for a greater demand. Moreover, the amount of investors who enter the speculative market at $\tilde{T}$ increases in the amount of investors selling at $\tilde{T}$, i.e. it increases in H. Hence, if H is small, S_T is small as a few rational investors have stocks and B_T is large as many low ambiguity averse still have cash at T. As result, T_3 can be larger.

[42]The effect of h is clear: $T_{\max}$ decreases with h, that is the fraction of people leaving the market early.

To see this more precisely, note that in our equilibrium S_T includes all R-investors with stocks at T and the exogenous sales X, while B_T includes all L-investors with cash at T. That is,

$$S_T = \sigma_t(1-K) = X + (1-z)(1-K)\frac{r}{1-h(1-r)},$$

and

$$B_T = \beta_t K_T = \frac{(1-r)(1-h)}{1-h(1-r)}\{K-(T-1)X-(1-r)h[(1-z)(1-K)+K-(\tilde{T}-1)X]\}.$$

In order to simplify the exposition, from now on we assume that H-investors perceive enough uncertainty to be induced to sell immediately, i.e. that ε is large enough to have $\tilde{T}=1$.[43] Hence $B_T \geq S_T$ defines the following condition:

$$T \leq \frac{1}{X}[K+X-(1-X)(1-r)h-\frac{X-hX(1-r)}{(1-r)(1-h)}-\frac{(1-z)(1-K)r}{(1-r)(1-h)}] \equiv T_3. \quad (33)$$

After simple algebra, we can see that

$$\frac{\partial T_3}{\partial r} = \frac{1}{X}[h(1-X)-\frac{1-K}{(1-r)^2(1-h)}], \quad (34)$$

which is positive when

$$r \leq 1-\sqrt{\frac{1-K}{h(1-h)(1-X)}} \equiv \hat{r}(h). \quad (35)$$

Hence this condition tells that T_3 increases in r when r is sufficiently small. This is due to the fact that, as r increases, $\partial B_T/\partial r$ decreases more than proportionally than $\partial S_T/\partial r$. In fact, the positive effect on K_T, which is weighted by β_t, gets smaller as r increases. We can then state the following Proposition.

Proposition 8 *If $r < \hat{r}(h)$ then $\partial T_3/\partial r > 0$.*

As shown in Section 3.3, T_3 is the constraint defining $T_{\max}$ when z is very small.[44] Hence, the following Corollary follows.

[43] For example H-investors may think that β_t and κ_t are respectively drawn by distributions with mean $\bar{\beta}$ and $\bar{\kappa}$ and full support $[0,1]$. If they are extremely ambiguity adverse, that is the case we are considering, they would assume $\beta_t = 0$ and $\kappa_t = 1$ for all t, hence obviously they will exit as soon as possible.

In other words, given that there is a one-to-one mapping between ε and $\tilde{T}$, we now consider $\tilde{T}$ as an exogenous parameter of the model.

[44] Replicating the analysis of Section 3.3 in this setting gives that $T_3 < T_1$ if and only if r exceeds a threshold, which is implicitly defined by

$$r > \frac{(1-h)(W(T_3)-X)}{(1-h)(W(T_3)-X)+1-K}.$$

Hence, again, if $X \to 0$ the right hand side of the last equation tends to zero, and so $T_3 < T_1$.

Corollary 4 *If $z \to 0$, then $T_{\max}$ increases in r for every $r \leq 1 - \sqrt{\frac{1-K}{h(1-h)}}$.*

4.3.2 The effects on T_1

Recall that T_1 is defined as the latest period in which I-investors believe it is profitable to enter the speculative market. Hence, T_1 is defined by the amount of available cash observed in the economy and by I-investors' expectations about future buy and sell rates. Formally, as seen, T_1 is the largest root solving $W(T) = K_T$. In the previous Section, with risk neutral investors, we had that T_1 unambiguously decreased in r as an higher r made expectations more pessimistic, i.e. it increased $W(T)$. Again, this need not hold now since, by changing r, we also affect the mass H of cautious investors who exit the market immediately.

The effects on T_1 are two. The first, already mentioned, is that the amount of available cash K_T increases in r. Hence, by this effect, an higher r pushes towards a larger T_1. Second, an higher r influences L-players' expectations, as defined in $W(T)$. By decreasing H, it pushes towards more optimistic expectations, i.e. it decreases $W(T)$. In addition, decreasing H has an indirect effect. Given that exits are perceived in relation to the amount of people still in the market, a lower H increases the mass of people in the market at T, hence making the exit of rational players appear smaller. Hence, this may also induce more optimistic expectations and greater T_1.

To identify more precisely these conditions, we recall that $W(T)$ decreases in $\bar{\beta}(T)$ and increases in $\bar{\sigma}(T)$. Moreover, observing equations (28) and (29), we can see that

$$\frac{\partial \bar{\beta}}{\partial r} > 0 \Longleftrightarrow \frac{\partial}{\partial r}[(1-r)h + \frac{r}{1-h(1-r)}] < 0 \Longleftrightarrow \frac{\partial \bar{\sigma}}{\partial r} < 0,$$

and hence

$$\frac{\partial W(T)}{\partial r} < 0 \Longleftrightarrow \frac{\partial}{\partial r}[(1-r)h + \frac{r}{1-h(1-r)}] < 0. \tag{36}$$

Condition (36) requires

$$-h + \frac{1-h}{[1-h(1-r)]^2} < 0,$$

which writes

$$r > \frac{1}{h}(\sqrt{\frac{1-h}{h}} - (1-h)) \equiv \tilde{r}(h). \tag{37}$$

Notice that $\tilde{r}(h)$ decreases with h, where $\tilde{r}(1/2) = 1$ and $\tilde{r}(1) = 0$. The last expression tells that for $W(T)$ decreases in r when h and r are sufficiently large. In fact, as mentioned above, an increase in the mass of rational investors affect expectations in two ways: positively, by decreasing H, and this effect is proportional to h; and negatively, by increasing R, and this effect is proportional to $(1-h)/(1-H)^2$. An high h increases the first effect and decreases the second, hence it makes expectations more optimistic. Moreover, given

h, the negative marginal effect of introducing additional R decreases with r: when r is small the mass of people still in the market at T is small (H is big) and the same amount of exit is perceived as big. Similarly, at $r = 1$ this effect is small. Hence, if $r \geq \tilde{r}(h)$ then an increase in r unambiguously increases T_1 since it increases K_T and decreases $W(T)$.

If $r < \tilde{r}(h)$ instead, the effect is ambiguous, as both K_T and $W(T)$ increase. What matters is then the magnitude of the two effects. The marginal effect on K_T, which tend to increase T_1, is $h[1 - X]$. The marginal effect on $W(T)$ is small when T is large, i.e. when X is small. In fact, as $X \to 0$, $T_1 \to \infty$, $\bar{\beta} \to 1$, $\bar{\sigma} \to 0$ and $\partial W(T)/\partial r \to 0$. Hence the effect on K_T always dominates for sufficiently small values of z and so T_1 increases in r.[45] The next Proposition sums up.

Proposition 9 *If $r \geq \tilde{r}(h)$ or $z \to 0$ then $\partial T_1/\partial r > 0$.*

4.3.3 The general case

Propositions 8 and 9 may not be sufficient for determining how $T_{\max}$ varies with r. In fact, it may be that T_3 is the binding constraint for $r > \tilde{r}(h)$ and T_1 is so for $r < \hat{r}(h)$.[46] One way to show that $T_{\max}$ may increase with r irrespective on whether T_1 or T_3 binds is to choose an r within the interval $(\tilde{r}(h), \hat{r}(h))$, provided this is not empty. Notice that $\hat{r}(h) > \tilde{r}(h)$ can be written as

$$1 - \sqrt{\frac{1-h}{h}} > \sqrt{\frac{h(1-K)}{(1-h)(1-X)}},$$

and this holds for sure as K is sufficiently close to 1.[47] Hence the previous conditions can be jointly satisfied, and so they define a set of sufficient conditions such that the maximal sustainable bubble is locally increasing in r. The next Proposition summarizes this result.

Proposition 10 *If $r \in (\tilde{r}(h), \hat{r}(h))$ then*

$$\frac{\partial T_{\max}}{\partial r} > 0.$$

[45] Formally, one can differentiate $W(T)$ with respect to r and see that

$$\frac{\partial W(T)}{\partial r} \to \frac{2(1-K)}{T+1}[-h + \frac{1-h}{[1-h(1-r)]^2}], \text{ as } T \to \infty.$$

Of course, for this to be relevant we must have $T_{\max} = T_1$, i.e. z small but positive and r very small, as characterized in Section 3.3.

[46] Indeed, in Section 3.3, we showed that T_3 is binding when r is large.

[47] Recall that we are considering $h > 1/2$, which is needed to have $r \geq \tilde{r}(h)$. More precisely, we require $K \geq 1 - (1-X)(1-y)^2y^2$, where $y \in (0,1)$ is defined as $y = \sqrt{(1-h)/h}$.

To sum up, the possibility that $T_{\max}$ increases in r is in general assured when r is not "too big" nor "too small". For small r, the constraint defining $T_{\max}$ is T_1. However, T_1 may not increase in r if r is too small. In this case, the amount of people in the market would be small, exit would be perceived as big and hence expectations would be more pessimistic. On the other hand, for large r, the constraint defining $T_{\max}$ is T_3. However, T_3 does not increase in r if r is too large. In fact, in this case, the amount of people buying at T would be small, hence the positive effect of having more cash available at T would be small. Hence, r has to take some intermediate values, within the interval defined in the last Proposition.

5 Conclusion

In this paper, we have modeled speculative behaviors in a finite game where it is common knowledge that the stock market is overvalued and that a crash is bound to occur. In our setting, traders face the same common information, but they differ in the ability to process it, and hence in the accuracy of their forecasts. We have shown that the existence of bubbles in such market does not require investors to be irrational, and we have provided a simple, possibly plausible, structure to "partially sophisticated" investors, emphasizing failures in understanding the timing of other investors' strategies. The resulting bubble equilibrium displays a number of features which have been documented in the empirical literature: high short-term returns attracting many new, typically unexperienced, investors, forming euphoric expectations about the length of the bubble. These investors provide the capital necessary to fuel the bubble, hence allowing high volumes of trade and further price increase. At some point, fully rational investors exit speculation and the crash occurs. Moreover, the analysis has emphasized in several ways that the relation between bubbles and rationality of the market is multifaceted, thereby qualifying the view that deviations from fundamentals tend to vanish if one increases the amount of rational traders in the market.

Our model is deliberately very simple, and surely open to many extensions and modifications. However, with due *caveats*, some policy implications can be drawn. First, the general idea behind our approach is that information availability *per se* need not be the solution for market efficiency. Instead, we emphasize information *accessibility*, which focuses on whether information is presented in a way to ease its interpretation. In our model, information is complete, but bubbles arise as some people face limitations, due to cognitive ability, lack of experience or alike, in processing all the relevant aspects of such information. In this sense, a regulator concerned with market stability, or with people losing too much money, may consider issues of simplicity and interpretability of information, or even of information overload, rather than just increasing the amount of potentially available information.

Another way in which information is not necessarily promoting efficiency in our setting is that "news" may have a destabilizing effect. In fact, as seen, partially sophisticated investors get excited by observing unexpected increases in the price, which lead them to overestimate the duration of the bubble and stay invested for too long. According to our approach, media can play an important role in stimulating or undermining a speculative phenomenon.[48] If media make information more accessible, in the sense just described, they have a stabilizing effect as they allow inexperienced investors to be sufficiently informed to avoid exploitation. If instead media are just reporting short-term high returns, they can distort perceptions, and hence asset prices, by making people euphoric (as it happens in our model). Under this light, we can see that I-investors would benefit from ignoring "news" and committing to an investment strategy in $t = 1$. In fact, having observed apparently good realizations in the early phase of the bubble, these investors revise their expectations in the wrong way. If such commitment were possible, the bubble would not arise (or it would be much shorter).

Third, we have seen that having a larger fraction of rational traders need not lead closer to market efficiency. This is due to their willingness to ride the bubble and, in the case of ambiguity aversion, to ride it longer than some not fully rational, but cautious, investors. In our model, increasing the risk-bearing capacity in the economy drives the prices further away from fundamentals, as people who perceive less uncertainty are willing to speculate more. Efficiency would instead be achieved by making investors aware that their predictions can be imprecise, and that incongruous observations may not be the result of chance, but rather of a wrong model. Hence, in this sense, the really stabilizing force in our setting does not come from arbitrageurs, but from people considering that they can possibly make mistakes.

References

Abreu, D. and Brunnermeier, M. K. (2003), 'Bubbles and crashes', *Econometrica* **71**(1), 173–204.

Allen, F., Morris, S. and Postlewaite, A. (1993), 'Finite bubbles with short sale constraints and asymmetric information', *Journal of Economic Theory* **61**(2), 206–229.

Bianchi, M. (2007), 'Speculative bubbles and rationality - a review of the theory', Mimeo: Stockholm School of Economics.

Blanchard, O. J. and Watson, M. W. (1982), Bubbles, rational expectations and financial markets, NBER Working Papers 0945, National Bureau of Economic Research, Inc.

[48] The strong relation between media coverage and "abnormal" returns has been recently documented e.g. by Dyck and Zingales (2003) and Veldkamp (2006).

Blume, L. and Easley, D. (2006), 'If you're so smart, why aren't you rich? belief selection in complete and incomplete markets', *Econometrica* **74**(4), 929–966.

Brunnermeier, M. K. and Nagel, S. (2004), 'Hedge funds and the technology bubble', *Journal of Finance* **59**(5), 2013–2040.

Chen, Z. and Epstein, L. (2002), 'Ambiguity, risk, and asset returns in continuous time', *Econometrica* **70**(2), 1403–1443.

Cochrane, J. H. (2002), 'Stocks as money: Convenience yield and the tech-stock bubble', NBER Working Papers No. 8987.

Cutler, D. M., Poterba, J. M. and Summers, L. H. (1990), 'Speculative dynamics and the role of feedback traders', *American Economic Review* **80**(2), 63–68.

Cutler, D., Poterba, J. and Summers, L. (1989), 'What moves stock prices?', *Journal of Portfolio Management* **15**(3), 4–12.

De Long, J. B., Shleifer, A., Summers, L. H. and Waldmann, R. J. (1990*a*), 'Noise trader risk in financial markets', *Journal of Political Economy* **98**(4), 703–38.

De Long, J. B., Shleifer, A., Summers, L. H. and Waldmann, R. J. (1990*b*), 'Positive feedback investment strategies and destabilizing rational speculation', *Journal of Finance* **45**(2), 379–395.

Dekel, E., Fudenberg, D. and Levine, D. K. (2004), 'Learning to play bayesian games', *Games and Economic Behavior* **46**(2), 282–303.

Dyck, A. and Zingales, L. (2003), 'The media and asset prices', Mimeo: Harvard Business School and University of Chicago.

Garber, P. M. (1990), 'Famous first bubbles', *Journal of Economic Perspectives* **4**(2), 35–54.

Gilboa, I. and Schmeidler, D. (1989), 'Maxmin expected utility with nonunique prior', *Journal of Mathematical Economics* **18**, 141–153.

Harrison, J. M. and Kreps, D. M. (1978), 'Speculative investor behavior in a stock market with heterogeneous expectations', *Quarterly Journal of Economics* **92**(2), 323–36.

Higgins, E. T. (1996), Knowledge activation: Accessibility, applicability and salience., *in* E. T. Higgins and A. E. Kruglanski, eds, 'Social psychology: Handbook of basic principles', New York: Guilford., pp. 133–168.

Hirshleifer, D. (2001), 'Investor psychology and asset pricing', *Journal of Finance* **56**(4), 1533–1597.

Jehiel, P. (1995), 'Limited horizon forecast in repeated alternate games', *Journal of Economic Theory* **67**, 497–519.

Jehiel, P. (2005), 'Analogy-based expectation equilibrium', *Journal of Economic Theory* **123**(2), 81–104.

Jehiel, P. and Samet, D. (2006), 'Valuation equilibria', Mimeo: UCL and PSE.

Kahneman, D. (2003), 'Maps of bounded rationality: Psychology for behavioral economics', *American Economic Review* **93**(5), 1449–1475.

Kindleberger, C. (2005), *Manias, panics, and crashes: a history of financial crises*, Wiley and Sons.

Klibanoff, P., Marinacci, M. and Mukerji, S. (2005), 'A smooth model of decision making under ambiguity', *Econometrica* **73**(6), 1849–1892.

Kogan, L., Ross, S., Wang, J. and Westerfield, M. (2006), 'The price impact and survival of irrational traders', *The Journal of Finance* **61**(1), 195–229.

Kyle, A. S. (1985), 'Continuous auctions and insider trading', *Econometrica* **53**(6), 1315–35.

Milgrom, P. and Stokey, N. (1982), 'Information, trade and common knowledge', *Journal of Economic Theory* **26**(1), 17–27.

Porter, D. P. and Smith, V. L. (2003), 'Stock market bubbles in the laboratory', *Journal of Behavioral Finance* **4**(1), 7–20.

Rabin, M. (1998), 'Psychology and economics', *Journal of Economic Literature* **36**(1), 11–46.

Rubinstein, A. (1998), *Modeling Bounded Rationality*, The MIT Press.

Sargent, T. (1993), *Bounded Rationality in Macroeconomics*, Oxford University Press.

Scheinkman, J. A. and Xiong, W. (2003), 'Overconfidence and speculative bubbles', *Journal of Political Economy* **111**(6), 1183–1219.

Shiller, R. (2000), 'Measuring bubble expectations and investor confidence', *Journal of Psychology and Financial Markets* **1**(1), 49–60.

Shiller, R. J. (1989), *Market Volatility*, MIT Press, Cambridge MA.

Shleifer, A. (2000), *Inefficient Markets*, Oxford University Press.

Simon, H. A. (1955), 'A behavioral model of rational choice', *Quarterly Journal of Economics* **69**(1), 99–118.

Smith, V. L., Suchanek, G. L. and Williams, A. W. (1988), 'Bubbles, crashes, and endogenous expectations in experimental spot asset markets', *Econometrica* **56**(5), 1119–51.

Stiglitz, J. E. (1990), 'Symposium on bubbles', *Journal of Economic Perspectives* **4**(2), 13–18.

Temin, P. and Voth, H.-J. (2004), 'Riding the south sea bubble', *American Economic Review* **94**(5), 1654–1668.

Tirole, J. (1982), 'On the possibility of speculation under rational expectations', *Econometrica* **50**(5), 1163–1182.

Veldkamp, L. (2005), 'Slow boom, sudden crash', *Journal of Economic Theory* **124**(2), 230–257.

Veldkamp, L. (2006), 'Media frenzies in markets for financial information', *American Economic Review* **96**(3), 577–601.