Electoral Competition and Incentives to Local Public Good Provision.

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Abstract

Local public good provision from different government levels is subject to many bias coming from the political process; incentives indeed, vary with the size of the beneficiaries' set and costs may affect the results of political competition by reducing total resources available for redistribution. Present work represents a first attempt to look at these issues together; indeed, it considers the situation where politicians have a finite budget to use both for redistributive policies and for the provision of a public good that affects the utility of a fraction of the electorate. In this setting it is not enough that benefits balance costs, in order for the public good to be implemented; the required level of efficiency moreover, is influenced by benefits concentration. If those interested in the public good are less than half of the electorate, concentration increases the efficiency threshold; on the contrary if they amount for more, benefits concentration decreases the required level of efficiency.

Keywords: public goods, redistributive policies, electoral competition, all-pay auctions.

JEL Classification: D72, H41

1 Introduction

The political process deals mainly with problems of resource redistribution; government spending indeed, has grown rapidly in the post war period and has became a decisive element for the welfare of many people. Therefore when resources are limited and the viability of public debt is a concern, funds allocation plays a relevant role in the formation of political consensus.

Redistributive politics in particular, are crucial anytime in a democratic system, candidates compete in an election; in this context indeed, they represent the main tool to gather votes in the electorate and ultimately to pursue politicians' aims (no matter how they are defined).

Two main types of government spending are available to candidates: cash and in kind transfers.

Cash transfers are generally used to redistribute resources across different constituencies and are characterized by a high level of targetability; eligibility criteria indeed, are easily used to identify specific subsets in the electorate.

Public spending in private or public good instead, is usually split among districts and involves redistribution across different geographic areas; resources thus, are not easily targeted to well defined groups of beneficiaries[1].

As a consequence in-kind transfers are less attractive for politicians and must provide benefits exceeding costs in order to be included in candidates' electoral platforms; even if the rate of return is positive though, some investments may be discarded due to strategical reasons. Indeed a trade-off arises between efficiency and targetability of government spending that causes under-provision of local public goods (or of private good provided by the public sector).

Present work focus on one main determinant of this trade-off: the degree of benefits concentration realized by in-kind transfers. This allows to sort among investments that provide the same aggregate benefit but realize different distributions of it, those that mostly suffer of the under-provision problem.

The main scope is answer the following questions:

Which is the minimum rate of return that a public investment must provide in order to be preferred over a cash transfer with the same cost?

How does this rate of returns changes when the number of people that benefits from such government expenditures varies and the degree of benefits concentration is modified?

Current analysis shows that if the beneficiaries form less than half of the electorate, then the efficiency (relative to cash) required of in-kind transfers rises as the beneficiaries' share of electorate falls. If the beneficiaries form more than half of the electorate, then the efficiency required for public investment provision falls as the beneficiaries' share of electorate falls.

A high level of concentration in benefits distribution therefore, is costly (in terms of efficiency) in the first situation and beneficial in the second.

These findings supply a useful insight on why the under-provision problem is particularly severe for some kinds of local public goods.

Consider for instance, the case government spending for education, day-care or health-care in scarcely populated districts.

Investing in marginal areas produces high benefits that accrue to a limited number of voters; electoral competition then, provides incentives to compete in this segment of the electorate using direct cash transfers (or equivalently, lower tax rates) rather than in-kind transfers.

Another example is the provision of non-specific professional training in highly specialized productive areas; such programs produce small benefits for a large number of workers. Benefits concentration is low and the fraction of people that take advantage from public spending usually exceeds half of the electorate.

In this case candidates are more easily induced to include in their platforms training programs that focus on the main economic activity of the area; this

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[1] This classification is introduced in Milesi-Ferretti et al. (2002).

increases benefits concentration and reduces the drawbacks of public spending in terms of targetability.

The first type of investment indeed, requires a higher level of efficiency with respect to the second and thus is more often discarded in favor of cash transfers.

2 Related Literature

Present work can be included in the vast branch of the literature that studies redistribution and special interest politics; an exhaustive overview of this topic is found in Persson and Tabellini (2000).

The main focus of the analysis is on the trade-off between targetability and efficiency that arises when politicians must choose between cash and in-kind transfers and on the problem of public investments under-provision that derives from that.

In particular, the situation where politicians make binding promises during the campaign is considered. The choice on in-kind transfers provision then, is the result of the strategic interactions among competing candidates.

This represent a different point of view with respect to the works of other authors as for instance, Weingast, Shepsle and Johnsen (1981), Baron and Ferejohn (1987 and 1989), and Besley and Coate (2003) that consider politics formation after the elections and study the common pool problem posed by local public good provision.

Current analysis instead, is highly indebted with the seminal contribution of Myerson (1993) that focus on the electoral competition between two parties that must redistribute a given budget among ex-ante identical voters; redistribution in this setting is made using perfectly targetable cash transfers. In the equilibrium of the game candidates choose randomized strategies that identify ex-post advantaged and disadvantaged groups in the electorate.

Recently Lizzeri and Persico (2001) extended Myerson's analysis by considering the case where parties may decide to invest the whole budget in a public good that benefits in the same measure all voters; these authors show that the under-provision problem is less relevant in a majoritarian system rather than in a proportional one.

Present work represents a generalization of previous setting when a proportional system is considered; indeed public good expenses do not absorb the whole budget nor its benefits accrue to all voters in the same measure.

Candidates then, must define also a redistributive program that use perfectly targetable cash transfers; government spending moreover does not involve only pure public goods but is more generally directed to in-kind transfers that benefits a fraction of the electorate.

3 The Model

The political process is described by a sequential game in three stages where two parties compete in a proportional electoral system.

In the first two stages party 1 and party 2 present their platforms to voters; platforms include a decision over the implementation of a local public good, g and a redistribution plan for the public budget.

The local public good is a fixed size one and benefits a fraction of the population that forms a special interest group within the electorate.

In the second stage elections are held.

3.1 Parties

Both party 1 and party 2, try to maximize their vote shares S_i ($i = \{1, 2\}$); these quantities define also the probability that, after the elections, party i platform is implemented. In other words, a probabilistic compromise is the outcome of the political process[2].

Government uses its budget to buy g and to finance the redistribution program; funds allocation is defined by party i platform, with probability S_i.

As in Myerson (1993), the public budget comes from taxation; once in office indeed, the government taxes each voter, n, for 1 dollar.

The electorate is a continuum described by the interval $[0, 1]$; this allows to avoid dealing with the problems posed by a large finite number of voters.

Considering an infinite electorate rises some technical issues about the size of the public budget and the cost and benefits of g. In order to make the analysis simple and get rid of these problems, it is useful to express all the relevant variables in per-capita terms[3].

The taxes levied on each voter form the initial per-capita budget; its size corresponds to the sum paid by a single voter and amounts to 1.

The per-capita share of the benefit generated by the public good is fixed and is equal to U; only a fraction $\lambda \leq 1$ of the electorate though, takes advantage from g. This means that each of those voters receives in facts $\frac{U}{\lambda}$.

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[2] See Sahuguet and Persico (2006) for further details on the probabilistic compromise outcome.

[3] The amount of funds collected by the government is infinite; however as in Myerson (1993) is possible to condisider the infinite electorate as an approximation for a large finite population so that also the public budget is in facts, as a large finite amount of resources with a monetary value proportional to the number of voters.

Consider now the case where the public good is implemented; the budget is reduced by the cost of g. This amount is finite and must be subtracted from an infinite budget so that its impact over it would be null.

The same problem arises further in the definition of the benefits generated by g; since the special interest group is a fraction of an infinite electorate and counts an infinite number of members, the summation of individual benefits gives an infinite quantity.

It is possible though to extend the interpretation adopted for the public budget to both these variables and define total costs and benefits of g as finite quantities; in particular the cost of the public good is proportional to the sum paid by each voter for g. Overall benefits instead are proportional to the size of the lobby and to the utility that a single member of the group gets from the public good.

Total utility received by each beneficiary then, increase as λ decreases; when $\lambda = 1$ the whole electorate is a special interest group and the aggregate benefit U, coincides with the per-capita share accruing to each voter.

The cost of the local public good instead, is always borne by all voters; its per-capita share amounts to $U(1-\xi)$ where $\xi \in (0,1)$ is a parameter that measures the degree of efficiency of the project.

The size of party i redistributable budget is:

$$b_i^r = 1 - g_i$$

where $g_i \in \{0; U(1-\xi)\}$ and is $g_i = U(1-\xi)$ if the local public good is implemented and $g_i = 0$ if is not.

Since the parties cannot present platforms that require public debt to be financed it must be the case that:

$$U(1-\xi) \leq 1$$

Consider now the composition of party i electoral platform; it includes of two elements:

1. Expenses for the local public good, g_i, that amount either to 0 or to $U(1-\xi)$, i.e. $g_i \in \{0; U(1-\xi)\}$.
2. A redistribution plan for b_i^r that specifies for a generic voter, n, the ratio, $x_i^n \in [0,+\infty)$, among the sum that party i platform transfers her and the redistributable budget.

The program must fulfill a balanced budget condition i.e. must hold:

$$\int_0^1 x_i^n \cdot dn = 1$$

with $\qquad i = 1,2 \qquad x_i^n \in [0;+\infty] \qquad$ and $\qquad n \in [0;1]$.

Notice that x_i^n can take any value among the positive real numbers since the budget constraint requires only that party i does not make promises which average more than b_i^r; given that there is a continuum of voters the law of large numbers applies and guarantees that the above condition holds with equality.

Promises included in the electoral platforms are binding; in other words it is assumed that they represent a credible commitment for the parties.

3.2 Voters

The electorate is a continuum described by the interval $[0,1]$.

There are two main types of voters: standard voters and members of the special interest group; the first type is denoted by the letter k and the second one with l. To denote a generic voter, the previous notation, n is used.

3.2.1 Standard voter

A standard voter k derives utility only from money and does not benefit from g; the utility function for the k-th voter, when party i platform is implemented, includes only the cash transfer received through the fiscal system:

$$u^k = x_i^k \cdot b_i^r$$

where $x_i^k \in [0, +\infty)$ is the ratio between the promise that a standard voter receives through the redistribution plan and the per-capita redistributable budget.

Given the assumption of probabilistic compromise, a standard voter gets the following expected utility:

$$E\left[u^k\right] = S_1 \cdot x_1^k \cdot b_1^r + S_2 \cdot x_2^k \cdot b_2^r$$

3.2.2 Special Interest Group

Members of the special interest group amount for a fraction λ of the electorate; the size and location of the group are common knowledge among the parties. Their utility is described by the function:

$$u^l = m + q\left(g_i\right)$$

where m is money, and $q\left(g_i\right)$ is a function defining the benefit derived from the local public good such that $q\left[U\left(1-\xi\right)\right] = \frac{U}{\lambda}$ and $q\left(0\right) = 0$.

A member of the group, l, gets the following utility if party i platform is implemented:

$$u_l = x_i^l \cdot b_i^r + q\left(g_i\right)$$

where $x_i^l \in [0, +\infty)$ is the ratio among the sum promised to a lobbyist through the redistribution plan and party i per-capita budget.

Expected utility amounts to:

$$E\left[u_l\right] = S_1 \cdot \left[x_1^l \cdot b_1^r + q\left(g_1\right)\right] + S_2 \cdot \left[x_2^l \cdot b_2^r + q\left(g_2\right)\right]$$

3.3 Timing and Structure of the Game

The game is divided into three stages, played sequentially with the timing described below:

1. Parties announce their decision over the local public good.
2. Parties present the redistribution programs.
3. Elections

The first two stages define the process of platforms presentation.

The choice over local public good provision strategically precedes the definition of the redistribution program; this reflects the different nature of cash and in-kind transfers.

The implementation of a specific public investment indeed, represents a precise commitment for the parties that is hardly changed once is announced.

The definition of a redistribution program through cash transfers instead, is a less binding issue since eligibility criteria are discretional and need not to be precisely stated during the campaign; the program then, can be shaped differently to account for the effects of the decisions over the local public good.

3.3.1 Platforms presentation

Platform presentations includes two stages.

Initially the parties decide simultaneously wether or not to buy g; then both observe the opponent's choice and simultaneously present their redistribution plans for b_1^r and b_2^r.

Parties actions in the first stage, consist only of the choice over the local public good; the strategy space, Σ_i^B, is a function of the cost and the benefits of g:

$$\Sigma_i^B\left(U,\xi,\lambda\right):[0,+\infty)\times[0,1]\times[0,1]\rightarrow\{0;U\left(1-\xi\right)\}$$

Redistribution programs are presented simultaneously after the first step. A strategy for party i ($i=\{1,2\}$), consists in the specification of a function that for each voter n (either lobbyist or not) defines the ratio x_i^n, among the sum that she receives if party i platform is implemented and the available per-capita budget ($b_i^r=1-g_i$).

A strategy for party i, Σ_i^P is a function of its own choice over g and of that of the opponent:

$$\Sigma_i^P(g_i;g_j):\{0,U\left(1-\xi\right)\}\times\{0,U\left(1-\xi\right)\}\rightarrow\{x_i^n\left(n\right)\}$$

where

$$x_i^n\left(n\right):[0,1]\rightarrow[0,+\infty)$$

such that $n=k,l$ and

$$\lambda\int_0^1 x_i^l\cdot dl+(1-\lambda)\int_0^1 x_i^k\cdot dk=1$$

3.3.2 Elections

After platforms presentation, elections are held and votes are cast simultaneously.

A strategy for a standard voter k is the vote itself; in particular it is a function of parties offers and of the size of the redistributable budgets:

$$\Sigma_k \left(x_i^k \cdot b_i^r ; x_j^k \cdot b_j^r\right) = [0,+\infty) \times [0,+\infty) \rightarrow \{1,2\}$$

A strategy for a lobbyist is again the vote; it depends on the definition of parties redistribution plan and on the decisions over g:

$$\Sigma_l \left(g_i ; g_j ; x_i^l \cdot b_i^r ; x_j^l \cdot b_j^r\right) = \{0, U(1-\xi)\} \times \{0, U(1-\xi)\} \times [0,+\infty) \times [0,+\infty) \rightarrow \{1,2\}$$

4 Equilibrium Analysis

The game is sequential and backward induction is used to define the equilibria; the analysis starts then, from the last stage: the elections.

4.1 Elections

When a generic voter n turns to vote, she compares parties platforms to choose which one is better for her.

A standard voter, k, then votes for party i whenever it is the case that $x_i^k b_i^r \geq x_j^k b_j^r$.

A member of the special interest group instead, chooses party i whenever $x_i^l b_i^r + q(g_i) \geq x_j^l b_j^r + q(g_j)$

4.2 Platforms Presentation

Before elections take place and after a decision over the local public good has been taken, both parties present their redistribution programs. In the last subgame then, the allocation of the available budget is defined through the promise of a direct cash transfer to each voter.

An optimal strategy in this context must take into account the outcome of the previous stage of the game; different scenarios emerge depending on the values of g_1 and g_2.

Consider initially, the case where only one party chose g and assume without loss of generality that party 1 is the one to provide it so that $g_1 = U(1-\xi)$.

When this happens no pure strategies equilibria exist for the sub-game; this is a standard result in the literature concerning redistributive politics (see Myerson (1993)) when the whole bunch of resources is freely disposable.

Present setting though, differs from the standard one because some money is bound to a specific location and to a specific distribution; in particular, the benefits deriving from local public good amount in terms of money, to the sum U and accrue to each special interest group member in the fix measure $\frac{U}{\lambda}$.

What turns out to be crucial in this case are only freely disposable resources.

Consider then the size of the budget available to each party for redistribution.

Party 2's resources amount to the whole initial budget and $b_2^r = 1$; party 1 instead, invests in the local public good and its budget is reduced of the cost of g, i.e. is $b_1^r = 1 - U(1-\xi)$. It is the case then that $b_2^r \geq b_1^r$ when $U \geq 0$.

Party 2 then, has always at least as much money as its opponent; as a consequence it can always identify a fraction of the special interest group, ϕ, such that:

$$b_2^r \geq b_1^r + \phi \cdot \lambda \cdot \frac{U}{\lambda}$$

Party 2 plays on this subspace as the favored party in the setting of Sahuguet and Persico (2006) and ignores remaining voters; just as in the case considered by these authors then, only equilibria in mixed strategies survive.

A mixed strategy in this setting, defines the probability attached to every feasible pure strategy, i.e. to each specific redistribution of the public budget over the voters space. Considering a continuum of voters widens a lot the space of pure strategies and makes mixed strategies particularly complicated objects to handle; the same approach used by Myerson (1993) is adopted to deal with this problem.

The analysis focus on a key element of parties strategies: the probability that party i redistribution program awards a generic voter less than the share x of b_i^r; in particular, $F_i^n(x)$ $(n = l, k)$ denotes the cumulate probability function defining the expected fraction of type n voters that receive less than x of b_i^r.

In the definition of the redistribution program, party i chooses two random variables, one for each type of voter; every n type voter receives, then, an offer x that is an independent draw from F_i^n. Present analysis considers only equilibria where the offers to each voter type are realization of the same random variable[4].

Notice now that in this setting party 1 has an advantage over the members of the special interest group; nonetheless party 2 identifies those receiving the benefit $\frac{U}{\lambda}$, given that voters' types are known.

It is possible then, to order the electorate space and split it in two different subspaces: the lobby subspace where special interest group members are placed, and the non-lobby subspace that includes only standard voters. Party 1 counts on an advantage $\frac{U}{\lambda}$ on each individual belonging to the first subspace.

The approach introduced by Sahuguet and Persico (2006) is used to define an equilibrium for the subgame; in particular, the strategic equivalence between the problem where two parties try to maximize their votes by redistributing a given budget in the electorate, and the problem of two players that maximize their expected utility trying to win some objects in an all-pay auction is exploited.

Present framework though, is slightly different from that studied by those authors given that there is an inhomogeneous electorate where one party has a specific advantage on some voters; the following proposition then, is required:

Proposition 1 *Parties problem in the definition of the redistribution plan is strategically equivalent to that of two players competing simultaneously on two*

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[4] The fact that mixed strategies are considered requires to discuss some issues relative to the indipendence assumption over the draws that define parties promises to each voter; given that the electorate is infinite there are also infinite indipendent draws and this poses some technical problems discussed by Myerson (1993) and Alos-Ferrer (2002). Though if we consider the continuum of voters as an approximation for a large finite number, the indipendence assumptions does not cause any problem.

types of independent first-price all-pay auctions where the same object is auctioned and where players' valuations are known.

Proof. *Define* $F_i^l\left(x_i^l\right)$ *and* $F_i^k\left(x_i^k\right)$ *as the cumulate distribution functions for party i promises respectively to a member of the special interest group and to a standard voter; each function specifies the probability that the ratio among the sum promised to a member of the group or to a standard voter, and party i redistributable budget is smaller than* x_i^l *and* x_i^k*.*

The maximization problem of party 1 *is:*

$$\begin{aligned}&Max_{F_1^l,F_1^k}\lambda\int_0^\infty F_2^l\left(x_1^l\cdot\frac{b_1^r}{b_2^r}+\frac{U}{\lambda\cdot b_2^r}\right)dF_1^l\left(x_1^l\right)+\\&+(1-\lambda)\int_0^\infty F_2^k\left(x_1^k\cdot\frac{b_1^r}{b_2^r}\right)dF_1^k\left(x_1^k\right)+\\&+\mu_1\left[1-\lambda\int_0^\infty x_1^l dF_1^l\left(x_1^l\right)-(1-\lambda)\int_0^\infty x_1^k dF_1^k\left(x_1^k\right)\right]\end{aligned}\tag{1}$$

Define now:

$$x_1^l\cdot\frac{b_1^r}{b_2^r}=y_1^l$$

$$x_1^k\cdot\frac{b_1^r}{b_2^r}=y_1^k$$

$$F_1^l\left(y_1^l\cdot\frac{b_2^r}{b_1^r}\right)=\hat{F}_1^l\left(y_1^l\right)$$

$$F_1^k\left(y_1^k\cdot\frac{b_2^r}{b_1^r}\right)=\hat{F}_1^k\left(y_1^k\right)$$

and

$$\frac{b_1^r}{b_2^r}=\rho$$

Party 1 *problem can be rewritten as:*

$$\begin{aligned}&Max_{\hat{F}_1^l,\hat{F}_1^k}\lambda\int_0^\infty F_2^l\left(y_1^l+\frac{U}{\lambda\cdot b_2^r}\right)d\hat{F}_1^l\left(y_1^l\right)+\\&+(1-\lambda)\int_0^\infty F_2^k\left(y_1^k\right)d\hat{F}_1^k\left(y_1^k\right)+\\&+\mu_1\left[\rho-\lambda\int_0^\infty y_1^l d\hat{F}_1^l\left(y_1^l\right)-(1-\lambda)\int_0^\infty y_1^k d\hat{F}_1^k\left(y_1^k\right)\right]\end{aligned}$$

or:

$$\begin{aligned}&(\lambda\cdot\mu_1)\left\{Max_{\hat{F}_1^l}\int_0^\infty\left[\frac{1}{\mu_1}F_2^l\left(y_1^l+\frac{U}{\lambda\cdot b_2^r}\right)-y_1^l\right]\right\}d\hat{F}_1^l\left(y_1^l\right)+\\&+(1-\lambda)\mu_1\left\{Max_{\hat{F}_1^k}\int_0^\infty\left[\frac{1}{\mu_1}F_2^k\left(y_1^k\right)-y_1^k\right]d\hat{F}_1^k\left(y_1^k\right)\right\}+\rho\cdot\mu_1\end{aligned}\tag{2}$$

Up to a linear transformation this setting is equivalent to the problem of a risk-neutral agent that maximizes his expected utility competing on two types of independent all-pay auctions where two identical objects are auctioned.

The first type of auction is played with probability λ *and the auctioneer increases player* 1*'s bids of the amount* $\frac{U}{\lambda}$*; the second type of auction shows up with complementary probability,* $(1-\lambda)$*, and players are symmetric on it. When player* 1 *submits his bid, he recognizes which type of auction is played.*

Consider now party 2 *maximization problem:*

$$\begin{aligned} & Max_{F_2^l,F_2^k}\lambda\int_0^{\infty}F_1^l\left(x_2^l\cdot\frac{b_2^r}{b_1^r}-\frac{U}{\lambda\cdot b_1^r}\right)dF_2^l\left(x_2^l\right)+ \\ & +(1-\lambda)\int_0^{\infty}F_1^k\left(x_2^k\cdot\frac{b_2^r}{b_1^r}\right)dF_2^k\left(x_2^k\right)+ \\ & +\mu_2\left[1-\lambda\int_0^{\infty}x_2^l dF_2^l\left(x_2^l\right)-(1-\lambda)\int_0^{\infty}x_2^k dF_2^k\left(x_2^k\right)\right] \end{aligned} \tag{3}$$

Since on the lobby subspace, party 1 *has a fixed advantage over each voter, some actions in the set of party* 2 *strategies are strictly dominated; namely every promise made to a member of the group such that* $0<x_2^l\leq\frac{U}{\lambda\cdot b_2^r}$ *is strictly dominated by* $x_2^l=0$*. Any promise smaller than party* 1 *advantage is not useful to obtain votes from the members of the special interest group and results in a mere reduction of party* 2 *budget. It is possible thus, to consider an equivalent setting where these strictly dominated strategies are not considered and rewrite the problem as follows:*

$$\begin{aligned} & Max_{F_2^l,F_2^k}\lambda\int_{\frac{U}{\lambda\cdot b_2^r}}^{\infty}\hat{F}_1^l\left(x_2^l-\frac{U}{\lambda\cdot b_2^r}\right)dF_2^l\left(x_2^l\right)+ \\ & +(1-\lambda)\int_0^{\infty}\hat{F}_1^k\left(x_2^k\right)dF_2^k\left(x_2^k\right)+ \\ & +\mu_2\left[1-\lambda\int_{\frac{U}{\lambda\cdot b_2^r}}^{\infty}x_2^l dF_2^l\left(x_2^l\right)-(1-\lambda)\int_0^{\infty}x_2^k dF_2^k\left(x_2^k\right)\right] \end{aligned}$$

Following the same steps described in the previous case, gives finally:

$$\begin{aligned} & (\lambda\cdot\mu_2)\left\{Max_{F_2^l}\int_{\frac{U}{\lambda\cdot b_2^r}}^{\infty}\left[\frac{1}{\mu_2}\hat{F}_1^l\left(x_2^l-\frac{U}{\lambda\cdot b_2^r}\right)-x_2^l\right]\right\}dF_2^l\left(x_2^l\right)+ \\ & +(1-\lambda)\mu_2\left\{Max_{\hat{F}_2^k}\int_0^{\infty}\left[\frac{1}{\mu_2}\hat{F}_1^k\left(x_2^k\right)-x_2^k\right]dF_2^k\left(x_2^k\right)\right\}+\mu_2 \end{aligned} \tag{4}$$

Up to a linear transformation, this problem is equivalent to that of a risk neutral player that bids on two independent all-pay auctions of the type described above.
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In the alternative setting players don't have a binding budget constraint while parties do.

The inverse of the Lagrangian multipliers attached to the budget constraints in the original framework defines the value of the prize; this implies that players value the object differently when parties have different budgets and face different shadow costs of money. In particular, the player corresponding to the party with the biggest budget, has the highest valuation of the object while that corresponding to the party with less funds available, has the lowest one.

Since the auctions where players bid are independent, each of them can be considered separately; this means further that the distribution of parties promises to standard voters and lobbyists can be considered separately as well.

The auction corresponding to the non-lobby subspace is a standard all-pay one whose equilibrium strategies are known in the literature[5].

The analysis thus focus on the redistribution program on the lobby subspace. The description of the equilibrium for an all-pay auction where one bidder has an advantage over the opponent is skipped; a wider discussion of this argument can be found in the appendix.

Total resources available for redistribution represent the crucial variable that defines how the parties are going to compete in the two subspaces; a different redistributable budget indeed, defines a different shadow cost of money μ_i for each party.

When party 1 includes the public good in its platform, it has also the smallest redistributable budget. In the non-lobby subspace then, party 2 has a lower shadow cost of money and the rate of return that it gets investing in these voters always exceeds that of the opponent i.e. $\frac{1}{\mu_2} \geq \frac{1}{\mu_1}$; this means that it plays as the favored party and gets the biggest fraction of these votes[6].

In the lobby subspace instead, the crucial quantity is $\frac{1}{\mu_2} - \frac{1}{\mu_1} - \frac{U}{\lambda \cdot b_2^r}$; since party 1 has the fixed advantage $\frac{U}{\lambda}$ on every voter, the shadow cost of money for party 2 is increased when it invest in the special interest group subspace.

The gap with the opponent indeed needs to be filled and party 2 must transfer a fraction $\frac{U}{\lambda \cdot b_2^r}$ of its redistributable budget to each individual whose vote it wants to win; the rate of return of investing in the special interest group corresponds to the difference between the inverse of the shadow cost of money and party 1 advantage, i.e. is $\frac{1}{\mu_2} - \frac{U}{\lambda \cdot b_2^r}$. The comparison between $\frac{1}{\mu_1}$ and $\frac{1}{\mu_2} - \frac{U}{\lambda \cdot b_2^r}$ defines the following possible situations:

- $\frac{1}{\mu_2} - \frac{1}{\mu_1} - \frac{U}{\lambda \cdot b_2^r} \geq 0$ party 2 is leader and obtain the main fraction of votes on both subspaces; this happens despite party 1 advantage $\frac{U}{\lambda}$ on every member of the special interest group. The rates of return from the investing in standard voters and lobbyists always exceeds those of the opponent.

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[5] See Baye et al. (1993).

[6] A formal proof for this result can be found in Sahughet and Persico (2006).

- $\frac{1}{\mu_2} - \frac{1}{\mu_1} - \frac{U}{\lambda \cdot b_2^r} < 0$ party 2 is follower on the lobby subspace and leader on the non-lobby one; party 1 rate of return for the investment in the special interest group is higher than that of party 2. Since it is always the case that $\frac{1}{\mu_2} \geq \frac{1}{\mu_1}$ the opposite happen in the standard voters subspace.

The size of party 1 redistributable budget depends on the degree of efficiency of the local public good; this means that the characteristics of g change the equilibrium strategies of the parties.

Current analysis starts from the simplest case of a neutral local public good.

4.2.1 Neutral Public Good

Assume initially that g is neutral and generates benefits equal to its costs so that $\xi = 0$;

Parties' budgets then amount to $b_1^r = 1 - U$ and $b_2^r = 1$.

A first result is stated in the following proposition:

Proposition 2 *Party* 2 *is never leader on the lobby subspace, i.e. it is the case that:*

$$\frac{1}{\mu_2} - \frac{1}{\mu_1} - \frac{U}{\lambda \cdot b_2^r} \leq 0$$

Proof. *Suppose is* $\frac{1}{\mu_2} - \frac{1}{\mu_1} - \frac{U}{\lambda \cdot b_2^r} > 0$ *and consider first the lobby subspace.*

From the analysis of the equivalent all-pay auction setting it is possible to derive the following equilibrium strategies for the parties.

Player 1 *randomizes according to a uniform distribution over the support* $\left[0; \frac{1}{\mu_1}\right]$ *with an atom of probability amounting to* $\left(1 - \frac{\mu_2}{\mu_1}\right)$ *in zero; in particular is:*

$$F_1^l\left(y_1^l\right) = 1 - \frac{\mu_2}{\mu_1} + y_1^l \cdot \mu_2$$

Player 2 *randomizes according to a uniform distribution over the support* $\left[\frac{U}{\lambda \cdot b_2^r}; \frac{1}{\mu_1} + \frac{U}{\lambda \cdot b_2^r}\right]$*; i.e. is:*

$$F_2^l\left(x_2^l\right) = \left(x_2^l - \frac{U}{\lambda \cdot b_2^r}\right) \cdot \mu_1$$

In the non-lobby subspace the equivalent problem boils down to a standard all-pay auction where players have different valuations of the auctioned object. As a consequence, in equilibrium, player 1 *bids zero with probability* $\left(1 - \frac{\mu_2}{\mu_1}\right)$ *and randomizes over the support* $\left[0, \frac{1}{\mu_1}\right]$*, according to a uniform distribution; player* 2 *instead, randomizes according to a uniform distribution on the support* $\left[0, \frac{1}{\mu_1}\right]$[7]*.*

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[7] Further details about this result can be found in Sahuguet and Persico (2006).

Consider now the problem of the parties starting from party 1*; since* $x_1^l = y_1^l \cdot \frac{b_2^r}{b_1^r}$ *it is the case that:*

$$F_1^l\left(x_1^l\right) = 1 - \frac{\mu_2}{\mu_1} + x_1^l \cdot \rho \cdot \mu_2$$

Party 1 *promises zero with probability* $\left(1 - \frac{\mu_2}{\mu_1}\right)$ *and with probability* $\frac{\mu_2}{\mu_1}$ *randomizes according to a uniform distribution over the support* $\left[0; \frac{1}{\mu_1} \cdot \frac{1}{\rho}\right]$*.*

In the non-lobby auction instead is:

$$F_1^k\left(x_1^k\right) = 1 - \frac{\mu_2}{\mu_1} + x_1^k \cdot \rho \cdot \mu_2$$

Party 1 *promises zero with probability* $\left(1 - \frac{\mu_2}{\mu_1}\right)$ *and with complementary probability randomizes according to a uniform distribution over the support* $\left[0; \frac{1}{\mu_1} \cdot \frac{1}{\rho}\right]$*.*

The distribution of party 2 *promises to both types of voters coincide with the distribution of player* 2*'s bids in the all-pay auctions setting.*

Look now at the budget constraints in the original problem; start from party 2*:*

$$1 = \lambda \int_{\frac{U}{\lambda \cdot b_2^r}}^{\frac{1}{\mu_1} + \frac{U}{\lambda \cdot b_2^r}} x_2^l \cdot \mu_1 + (1-\lambda) \int_0^{\frac{1}{\mu_1}} x_2^k \cdot \mu_1 \tag{5}$$

or

$$1 - \frac{U}{b_2^r} = \int_0^{\frac{1}{\mu_1}} x_2^l \cdot \mu_1$$

For party 1 *instead holds:*

$$1 = \lambda \int_0^{\frac{1}{\mu_1} \cdot \frac{1}{\rho}} x_1^l \cdot \mu_2 + (1-\lambda) \int_0^{\frac{1}{\mu_1} \cdot \frac{1}{\rho}} x_1^k \cdot \mu_2 \tag{6}$$

or:

$$\rho = \int_0^{\frac{1}{\mu_1}} x_1^k \cdot \mu_2$$

then it is the case that:

$$\frac{\mu_2}{\mu_1} = \frac{\rho}{1 - \frac{U}{b_2^r}} = \frac{1-U}{1-U} = 1$$

The values for the Lagrangians attached to the parties budget constraints must now be computed.

In particular, there can be only a pair of values, μ_1 *and* μ_2*, deriving from the equilibrium strategies for the all-pay auctions that coincides also with the shadow* *cost of money for the parties; this guarantees the uniqueness of the equilibrium in the original setting.*

Solving the integral in party 2 *budget constraint gives:*

$$\frac{1}{\mu_1} = \frac{1}{\mu_2} = 2(1 - U) \tag{7}$$

The initial condition: $\frac{1}{\mu_2} - \frac{1}{\mu_1} - \frac{U}{\lambda \cdot b_2^r} > 0$ *thus, can never be true and must be the case that* $\frac{1}{\mu_1} + \frac{U}{\lambda \cdot b_2^r} - \frac{1}{\mu_2} \geq 0$ ■

Party 1 then, is leader on the lobby subspace; this implies that party 2 promises a positive amount to special interest group members only when is:

$$\frac{1}{\mu_2} \geq \frac{U}{\lambda \cdot b_2^r} \tag{8}$$

The proof descends straightforwardly from the fact that if is $\frac{1}{\mu_2} < \frac{U}{\lambda \cdot b_2^r}$ party 2 rate of return of investing in the lobby subspace is smaller than the fixed cost required to compete on it[8]; when this is the case party 2 offers zero with probability one to the members of the special interest group and competes only for standard voters.

An optimal response for party 1 is to offer zero to the members of the special interest group and concentrate the available budget on the rest of the electorate; the problem then, boils down to the problem studied by Sahuguet and Persico (2006) where two parties with different budgets compete over an homogeneous electorate.

The equilibrium strategies for this case are described below.

Party 1 promises zero with probability $\left(1 - \frac{\mu_2}{\mu_1}\right)$ and with complementary probability randomizes according to a uniform distribution over the support $\left[0; \frac{1}{\mu_1} \cdot \frac{1}{\rho}\right]$; party 2 promises instead, are distributed according to a uniform distribution over the support $\left[0; \frac{1}{\mu_1}\right]$.

The equilibrium values for the Lagrangians are derived from parties budget constraints whose equations are reported below :

$$\rho = (1 - \lambda) \int_0^{\frac{1}{\mu_1}} x_1^k \cdot \mu_2$$

for party 1 and

$$1 = (1 - \lambda) \int_0^{\frac{1}{\mu_1}} x_2^k \cdot \mu_1$$

for party 2.

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[8] In the modified setting when is $\frac{1}{\mu_2} < \frac{U}{\lambda \cdot b_2^r}$ any strictly positive bid smaller than $\frac{U}{\lambda \cdot b_2^r}$ has probability zero to win the object and gives a negative payoff; thus player 2 will either not participate to the auction or bid more than $\frac{U}{\lambda \cdot b_2^r}$.

Solving the integrals gives:

$$\frac{1}{\mu_1} = \frac{2}{(1-\lambda)}$$

and

$$\frac{1}{\mu_2} = \frac{2}{(1-\lambda)} \cdot \frac{1}{\rho}$$

The benefit transferred to a member of the special interest group must be such that:

$$\frac{2}{(1-\lambda)} \cdot \frac{1}{\rho} < \frac{U}{\lambda}$$

Since holds $b_2^r \geq b_1^r$, $(\rho \leq 1)$ parties payoffs are:

$$S_1 = (1-\lambda)\frac{\rho}{2} + \lambda = \underline{S} \tag{9}$$

and

$$S_2 = (1-\lambda)\left(1 - \frac{\rho}{2}\right) = 1 - \underline{S} \tag{10}$$

Proposition 3 *When* $\frac{2}{(1-\lambda)} \cdot \frac{1}{\rho} < \frac{U}{\lambda}$ *party* 1 *payoff is smaller than* $\frac{1}{2}$.

Proof. *In order to have* $S_1 \geq \frac{1}{2}$ *must hold:*

$$\frac{\lambda}{1-\lambda} \geq U$$

The initial condition can be rewritten as:

$$\frac{\lambda}{1-\lambda} < U \cdot \frac{1-U}{2}$$

Notice now that since $0 \leq \frac{1-U}{2} \leq \frac{1}{2}$ *it can never be the case that* $\frac{\lambda}{1-\lambda} \geq U$ *and* $\frac{\lambda}{1-\lambda} < U \cdot \frac{1-U}{2}$ *hold at the same time; therefore if* $\frac{2}{(1-\lambda)} \cdot \frac{1}{\rho} < \frac{U}{\lambda}$ *must be also* $S_1 < \frac{1}{2}$ $\blacksquare$

Consider now the case when is $\frac{1}{\mu_2} \geq \frac{U}{\lambda \cdot b_2^r}$; the following proposition holds.

Proposition 4 *In the unique equilibrium for this sub-game, party* 1 *promises to special interest group members are distributed according to a uniform distribution over the support* $\left[0; \frac{1}{\mu_2} \cdot \frac{1}{\rho} - \frac{U}{\lambda \cdot b_1^r}\right]$*; in the standard voters' subspace moreover, it promises zero with probability* $\left(1 - \frac{\mu_2}{\mu_1}\right)$ *and with complementary probability randomizes according to a uniform distribution over the support* $\left[0; \frac{1}{\mu_1} \cdot \frac{1}{\rho}\right]$.

Party 2 *randomizes according to a uniform distribution over the support* $\left[\frac{U}{\lambda\cdot b_2^r};\frac{1}{\mu_2}\right]$ *in the lobby subspace and uses a uniform distribution on the support* $\left[0,\frac{1}{\mu_1}\right]$ *for the standard voters' subspace.*

Party 1 *payoff,* S_1*, is smaller than* $\frac{1}{2}$*.*

Proof. *When* $\frac{1}{\mu_2}\geq\frac{U}{\lambda\cdot b_2^r}$ *and* $\frac{1}{\mu_2}<\frac{1}{\mu_1}+\frac{U}{\lambda\cdot b_2^r}$*, the analysis of the equivalent all-pay auction setting defines the following equilibrium strategies.*

Player 1 *randomizes according to a uniform distribution over the support* $\left[0;\frac{1}{\mu_2}-\frac{U}{\lambda\cdot b_2^r}\right]$ *with an atom of probability amounting to* $\frac{U}{\lambda\cdot b_2^r}\cdot\mu_2$ *in zero, i.e. is:*

$$\hat{F}_1^l\left(y_1^l\right)=\left(y_1^l+\frac{U}{\lambda\cdot b_2^r}\right)\cdot\mu_2$$

Player 2 *randomizes according to a uniform distribution over the support* $\left[\frac{U}{\lambda\cdot b_2^r};\frac{1}{\mu_2}\right]$*, given that every bid smaller than* $\frac{U}{\lambda\cdot b_2^r}$ *is a strictly dominated action; the equilibrium cumulate distribution function is then:*

$$F_2^l\left(x_2^l\right)=1-\frac{\mu_1}{\mu_2}+x_2^l\cdot\mu_1$$

In the non-lobby subspace the equilibrium strategies do not change: player 1 *bids zero with probability* $\left(1-\frac{\mu_2}{\mu_1}\right)$ *and randomizes over the support* $\left[0,\frac{1}{\mu_1}\right]$*, according to a uniform distribution; player* 2 *instead, randomizes according to a uniform distribution on the support* $\left[0,\frac{1}{\mu_1}\right]$*.*

Consider now the problem of the parties; substituting above for $x_1^l=y_1^l\cdot\frac{b_2^r}{b_1^r}$ *gives:*

$$F_1^l\left(x_1^l\right)=\left(x_1^l+\frac{U}{\lambda\cdot b_1^r}\right)\cdot\rho\cdot\mu_2$$

Party 1 *randomizes according to a uniform distribution over the support* $\left[0;\frac{1}{\mu_2}\cdot\frac{1}{\rho}-\frac{U}{\lambda\cdot b_1^r}\right]$ *with an atom in zero amounting to* $\frac{U}{\lambda\cdot b_2^r}\cdot\mu_2$*. In the non-lobby subspace instead it is the case that:*

$$F_1^k\left(x_1^k\right)=1-\frac{\mu_2}{\mu_1}+x_1^k\cdot\rho\cdot\mu_2$$

Party 1 *promises zero with probability* $\left(1-\frac{\mu_2}{\mu_1}\right)$ *and with complementary probability randomizes according to a uniform distribution over the support* $\left[0;\frac{1}{\mu_1}\cdot\frac{1}{\rho}\right]$.

The distribution of party 2 *promises coincides with the distribution of player* 2*'s bids.*

Consider now the budget constraint in the original problem; look first at party 2*:*

$$1=\lambda\int_{\frac{U}{\lambda\cdot b_2^r}}^{\frac{1}{\mu_2}}x_2^l\cdot\mu_1+(1-\lambda)\int_0^{\frac{1}{\mu_1}}x_2^k\cdot\mu_1 \tag{11}$$

For party 1 *holds:*

$$\rho = \lambda \int_0^{\frac{1}{\mu_2}-\frac{U}{\lambda\cdot b_2^r}} x_1^l \cdot \mu_2 + (1-\lambda) \int_0^{\frac{1}{\mu_1}} x_1^k \cdot \mu_2 \tag{12}$$

then it is the case that:

$$\frac{1}{\mu_1} = \frac{\lambda}{2}\left[\left(\frac{1}{\mu_2}\right)^2 - \left(\frac{U}{\lambda}\right)^2\right] + \frac{1-\lambda}{2}\left(\frac{1}{\mu_1}\right)^2$$

and

$$\frac{1}{\mu_2} = \frac{\lambda}{2}\left[\left(\frac{1}{\mu_2}\right)^2 + \left(\frac{U}{\lambda}\right)^2\right] + \frac{1-\lambda}{2}\left(\frac{1}{\mu_1}\right)^2$$

so that:

$$\frac{1}{\mu_2} - \frac{1}{\mu_1} = \lambda\left(\frac{U}{\lambda}\right)^2$$

The initial condition then, is always verified since is:

$$\frac{1}{\mu_1} + \frac{U}{\lambda\cdot b_2^r} - \frac{1}{\mu_2} = \frac{U}{\lambda}(1-U) \geq 0$$

To prove uniqueness of the equilibrium the values of the Lagrangians must be computed.

Solving the integral in party 2 *budget constraint gives:*

$$\frac{1}{\mu_1} = 1 - \lambda^2\left(\frac{U}{\lambda}\right)^2 \pm \sqrt[2]{\left[1-\lambda^2\left(\frac{U}{\lambda}\right)^2\right]\left[1-\lambda^2\left(\frac{U}{\lambda}\right)^2 + \lambda\left(\frac{U}{\lambda}\right)^2\right]}$$

Since when $\frac{U}{\lambda} = 0$, *is also* $\frac{1}{\mu_1} = 2$[9], *it must be the case that:*

$$\frac{1}{\mu_1} = 1 - \lambda^2\left(\frac{U}{\lambda}\right)^2 + \sqrt[2]{\left[1-\lambda^2\left(\frac{U}{\lambda}\right)^2\right]\left[1-\lambda^2\left(\frac{U}{\lambda}\right)^2 + \lambda\left(\frac{U}{\lambda}\right)^2\right]}$$

and

$$\frac{1}{\mu_2} = 1 + \sqrt[2]{\left[1-\lambda^2\left(\frac{U}{\lambda}\right)^2\right]\left[1-\lambda^2\left(\frac{U}{\lambda}\right)^2 + \lambda\left(\frac{U}{\lambda}\right)^2\right]}$$

Consider now parties payoffs.

In the standard voters subspace party 1 *gets half of the votes of those who receive a promise* $x_1^k > 0$.*; in the special interest group sub-space instead, party* 1 *gets all the votes of those who receive a zero promise from party* 2 *and half of those who receive a positive promise from both parties.*

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[9] See Sahuguet and Persico (2006) for a formal proof.

Party 1 *then, looses the votes of the member of the special interest group that receive a positive promise from party* 2 *and get at the same time, only the fixed amount* $\frac{U}{\lambda}$ *from its redistribution plan; therefore it must be:*

$$S_1 = (1-\lambda)\cdot\frac{1}{2}\cdot\frac{\mu_2}{\mu_1} + \lambda\cdot\left[1-\frac{\mu_1}{\mu_2}+\frac{U}{\lambda}\cdot\mu_1+\frac{1}{2}\left(\frac{\mu_1}{\mu_2}-\frac{U}{\lambda}\cdot\mu_1\right)\left(1-\frac{U}{\lambda}\cdot\mu_2\right)\right]$$

or

$$S_1 = (1-\lambda)\cdot\frac{1}{2}\cdot\frac{\mu_2}{\mu_1} + \lambda\cdot\left\{1-\frac{1}{2}\left[\frac{\mu_1}{\mu_2}-\left(\frac{U}{\lambda}\right)^2\cdot\mu_2\cdot\mu_1\right]\right\} = \underline{S} \tag{13}$$

and also

$$S_2 = (1-\lambda)\left[1-\frac{1}{2}\cdot\frac{\mu_1}{\mu_2}\right] + \lambda\cdot\frac{1}{2}\left[\frac{\mu_1}{\mu_2}-\left(\frac{U}{\lambda}\right)^2\cdot\mu_2\cdot\mu_1\right] = 1-\underline{S} \tag{14}$$

Notice that $S_1 \leq \frac{1}{2}$ *since substituting for* $\frac{1}{\mu_1}$ *in party* 1 *payoff gives:*

$$\frac{1}{2}\cdot\frac{1-\lambda^2\left(\frac{U}{\lambda}\right)^2}{1-\lambda^2\left(\frac{U}{\lambda}\right)^2+\lambda\left(\frac{U}{\lambda}\right)^2+\sqrt[2]{\left[1-\lambda^2\left(\frac{U}{\lambda}\right)^2\right]\left[1-\lambda^2\left(\frac{U}{\lambda}\right)^2+\lambda\left(\frac{U}{\lambda}\right)^2\right]}} \leq \frac{1}{2}$$

and this is always true for $\lambda \geq 0$ *and* $U \geq 0$. ■

The definition of an equilibrium for the first sub-game requires to consider some other situations and in particular, those where parties choices with respect to g are the same.

Look at party i maximization problem when both platforms include the local public good:

$$\begin{aligned} & Max_{F_i^l,F_i^k}\lambda\int_0^\infty F_j^l\left[x_i^l\cdot\frac{b_i^r}{b_j^r}+\frac{q\left(g_i\right)}{b_2^r}-\frac{q\left(g_j\right)}{b_2^r}\right]dF_i^l\left(x_i^l\right)+ \\ & +(1-\lambda)\int_0^\infty F_j^k\left(x_i^k\cdot\frac{b_i^r}{b_j^r}\right)dF_i^k\left(x_i^k\right)+ \\ & +\mu_i\left[1-\lambda\int_0^\infty x_i^l dF_i^l\left(x_i^l\right)-(1-\lambda)\int_0^\infty x_i^k dF_i^k\left(x_i^k\right)\right]. \end{aligned}$$

Previous expression can be rewritten as:

$$\begin{aligned} & Max_{F_i^l,F_i^k}\lambda\int_0^\infty F_j^l\left(x_i^l\right)dF_i^l\left(x_i^l\right)+ \\ & +(1-\lambda)\int_0^\infty F_j^k\left(x_i^k\right)dF_i^k\left(x_i^k\right)+ \\ & +\mu_i\left[1-\lambda\int_0^\infty x_i^l dF_i^l\left(x_i^l\right)-(1-\lambda)\int_0^\infty x_i^k dF_i^k\left(x_i^k\right)\right]. \end{aligned}$$

and is equivalent to party i maximization problem when both platforms do not include g.

When the choices over the local public good are symmetric, parties equilibrium strategies are the same no matter if g is provided or not.

The maximization problem in the modified setting is:

$$(\lambda \cdot \mu_2)\left\{Max_{F_2^l} \int_0^{\infty} \left[\frac{1}{\mu_2} F_1^l\left(x_2^l\right) - x_2^l\right]\right\} dF_2^l\left(x_2^l\right) + \\ + (1-\lambda)\,\mu_2 \left\{Max_{F_2^k} \int_0^{\infty} \left[\frac{1}{\mu_2} F_1^k\left(x_2^k\right) - x_2^k\right] dF_2^k\left(x_2^k\right)\right\} + \mu_2$$

The players thus, bid on two standard all-pay auctions where the same object is auctioned; in terms of the original problem this means that the parties face an homogeneous electorate and the division in subspaces has no reason to be.

The definition of the equilibrium for the sub-game then, boils down into the problem studied by Sahuguet and Persico (2006); the same unique mixed strategies equilibrium arises defined by the following elements:

- $\frac{1}{\mu_i} = \frac{1}{\mu_j} = 2$;
- Each voter receives from party i ($i = 1, 2$) the promise of a fraction of the unitarian per-capita redistributable budget that depends upon the realization of a random variable distributed as a uniform on the support $\left[0; \frac{1}{\mu_i}\right]$;
- $S_i = S_j = \frac{1}{2}$.

It is possible now to characterize the equilibrium for the first stage of the platform presentation process i.e. the choice over the local public good; this sub-game can be represented in normal form as follows:

$Party 1$			
		0	U
$Party\ 2$	0	$\frac{1}{2}; \frac{1}{2}$	$1 - \underline{\mathrm{S}}; \underline{\mathrm{S}}$
	U	$\underline{\mathrm{S}}; 1 - \underline{\mathrm{S}}$	$\frac{1}{2}; \frac{1}{2}$

Since is $\underline{\mathrm{S}} < \frac{1}{2}$ in all the considered cases, the only Nash equilibrium is the choice not to implement the public good for both parties.

A positive rate of return from g is required in order for it to be included in candidates' platforms.

4.2.2 Efficient Public Good

Consider now the case where the local public good supply benefits exceeding costs and $\xi > 0$. Since the investment in g must give a positive rate of return, it is possible to determine the minimum efficiency threshold that in equilibrium induces the parties to include the local public good in their electoral platforms.

Present section provides a definition of such threshold and considers how it varies when λ changes. This allows to study the effects of benefits concentration on public investments under-provision.

As in the previous case, the analysis of the equilibrium starts from the second stage of the platforms presentation process , i.e. the definition of the redistribution program for the available budget.

A preliminary observation is worthy with respect to this issue. Even if $\xi > 0$, equilibrium strategies and payoffs for the situations where parties' choices are symmetric remain the same; a change arise if only one party includes g in its platform. The analysis of parties equilibrium strategies then is limited to this last case; again it is assumed that $g_1 = U\left(1-\xi\right)$ and $g_2 = 0$.

Since $\xi > 0$, party 1 redistributable budget is increased and amounts now to $\tilde{b}_1^r = 1 - U\left(1-\xi\right)$

This implies further that the shadow cost of money decreases and:

$$\hat{\mu}_1 < \mu_1 \tag{15}$$

where $\hat{\mu}_1$ defines the new value of the Lagrangian attached to party 1 budget constraint.

Moreover since the shadow cost of money is defined with respect to the opponent's budget it is also:

$$\hat{\mu}_2 > \mu_2 \tag{16}$$

where $\hat{\mu}_2$ defines the new value of the Lagrangian attached to party 2 budget constraint.

The definition of an equilibrium for this sub-game requires to consider the possible situations defined by the comparison between $\frac{1}{\hat{\mu}_1}$ and $\frac{1}{\hat{\mu}_2} - \frac{U}{\lambda \cdot b_2^r}$; to this aim notice that also in this case must hold: $\frac{1}{\hat{\mu}_2} \geq \frac{1}{\hat{\mu}_1}$ and $\frac{1}{\hat{\mu}_1} \geq \frac{1}{\hat{\mu}_2} - \frac{U}{\lambda \cdot b_2^r}$.

Party 2 then makes positive promises to special interest group members only if $\frac{1}{\hat{\mu}_2} \geq \frac{U}{\lambda \cdot b_2^r}$.

Notice now that when previous condition holds, the variation in the values of the Lagrangians does not affect the form of players' bids distribution; only the supports change.

Party 1 promises moreover, need to be rescaled for the factor $\tilde{\rho} = \frac{\tilde{b}_1^r}{b_2^r} = 1 - U\left(1-\xi\right)$

Given these observations, the equilibrium strategies for the parties are the following:

- Party 1

  Party 1 randomizes in the standard voters subspace according to the following cumulate probability function:

$$F_1^l\left(x_1^l\right) = \left(x_1^l + \frac{U}{\lambda \cdot \tilde{b}_1^r}\right) \cdot \tilde{\rho} \cdot \tilde{\mu}_2$$

over the support $\left[0; \frac{1}{\tilde{\mu}_2} \cdot \frac{1}{\tilde{\rho}} - \frac{U}{\lambda \cdot \tilde{b}_1^r}\right]$. In the special interest group subspace, instead the cumulate probability function is:

$$F_1^k\left(x_1^k\right) = 1 - \frac{\tilde{\mu}_2}{\tilde{\mu}_1} + x_1^k \cdot \tilde{\rho} \cdot \tilde{\mu}_2$$

Party 1 promises zero with probability $\left(1 - \frac{\tilde{\mu}_2}{\tilde{\mu}_1}\right)$ and with complementary probability randomizes according to a uniform distribution over the support $\left[0; \frac{1}{\tilde{\mu}_1} \cdot \frac{1}{\tilde{\rho}}\right]$.

- Party 2

  Party 2 equilibrium strategy for the lobby subspace is described by:

$$F_2^l\left(x_2^l\right) = 1 - \frac{\tilde{\mu}_1}{\tilde{\mu}_2} + x_2^l \cdot \tilde{\mu}_1$$

  defined over the support $\left[\frac{U}{\lambda \cdot \tilde{b}_2^r}; \frac{1}{\tilde{\mu}_2}\right]$ with an atom in zero amounting to $1 - \frac{\tilde{\mu}_1}{\tilde{\mu}_2} + \frac{U}{\lambda} \cdot \tilde{\mu}_1$. Promises to standard voters finally, are distributed according to the following cumulate probability function:

$$F_2^k\left(x_2^k\right) = x_2^k \cdot \tilde{\mu}_1$$

  over the support $\left[0, \frac{1}{\tilde{\mu}_1}\right]$.

Full characterization of the equilibrium requires to compute the values of $\tilde{\mu}_1$ and $\tilde{\mu}_2$.

Consider the budget constraint in the original problem; look first at party 2:

$$1 = \lambda \int_{\frac{U}{\lambda \cdot \tilde{b}_2^r}}^{\frac{1}{\tilde{\mu}_2}} x_2^l \cdot \tilde{\mu}_1 + (1 - \lambda) \int_0^{\frac{1}{\tilde{\mu}_1}} x_2^k \cdot \tilde{\mu}_1$$

For party 1 instead, holds:

$$\tilde{\rho} = \lambda \int_0^{\frac{1}{\tilde{\mu}_2} - \frac{U}{\lambda \cdot \tilde{b}_2^r}} x_1^l \cdot \tilde{\mu}_2 + (1 - \lambda) \int_0^{\frac{1}{\tilde{\mu}_1}} x_1^k \cdot \tilde{\mu}_2$$

Define now the condition for $g_1 = U(1 - \xi)$ to be an optimal choice for party 1; given that replicating the same choice of the opponent gives a payoff equal to $\frac{1}{2}$, in order for the public good to be chosen must be the case that $S_1 \geq \frac{1}{2}$ i.e.:

$$(1 - \lambda) \cdot \frac{1}{2} \cdot \frac{\mu_2}{\mu_1} + \lambda \cdot \left\{1 - \frac{1}{2}\left[\frac{\mu_1}{\mu_2} - \left(\frac{U}{\lambda}\right)^2 \cdot \mu_2 \cdot \mu_1\right]\right\} \geq \frac{1}{2}$$

An equilibrium where g is included in party 1 platform is characterized then, by a system of three equations in three unknowns ($\frac{1}{\tilde{\mu}_1}$, $\frac{1}{\tilde{\mu}_2}$,ξ) defined as follows:

$$\left\{\begin{array}{c} \frac{1}{\tilde{\mu}_1} = \frac{\lambda}{2}\left[\left(\frac{1}{\tilde{\mu}_2}\right)^2 - \left(\frac{U}{\lambda}\right)^2\right] + \frac{1-\lambda}{2}\left(\frac{1}{\tilde{\mu}_1}\right)^2 \\ \frac{1}{\tilde{\mu}_2}\left(1 + U \cdot \xi\right) = \frac{\lambda}{2}\left[\left(\frac{1}{\tilde{\mu}_2}\right)^2 + \left(\frac{U}{\lambda}\right)^2\right] + \frac{1-\lambda}{2}\left(\frac{1}{\tilde{\mu}_1}\right)^2 \\ (1-\lambda) \cdot \frac{\mu_2}{\mu_1} + \lambda \cdot \left[2 - \frac{\mu_1}{\mu_2} + \left(\frac{U}{\lambda}\right)^2 \cdot \mu_2 \cdot \mu_1\right] = 1 \end{array}\right. \tag{17}$$

In order to find the equilibrium values for these variables express $\frac{1}{\tilde{\mu}_1}$ as a function of $\frac{1}{\tilde{\mu}_2}$ in the first equation to get:

$$\frac{1}{\tilde{\mu}_1} = \frac{1}{1-\lambda} \cdot \left\{1 \pm \sqrt[2]{1 - \lambda\left(1-\lambda\right)\left[\left(\frac{1}{\tilde{\mu}_2}\right)^2 - \left(\frac{U}{\lambda}\right)^2\right]}\right\}$$

Notice that when $\lambda = 0$ present framework reduces to the problem studied in Sahuguet and Persico (2006); in this case is $\frac{1}{\tilde{\mu}_1} = \frac{1}{\tilde{\mu}_2} = 2$ and therefore must be:

$$\frac{1}{\tilde{\mu}_1} = \frac{1}{1-\lambda} \cdot \left\{1 + \sqrt[2]{1 - \lambda\left(1-\lambda\right)\left[\left(\frac{1}{\tilde{\mu}_2}\right)^2 - \left(\frac{U}{\lambda}\right)^2\right]}\right\}$$

Substitute the above value in the third equation and solve for $\frac{1}{\tilde{\mu}_2}$ to get:

$$\frac{1}{\tilde{\mu}_2} = 2\sqrt[2]{1 + \frac{1-\lambda}{\lambda} \cdot U^2}$$

Solve then, for $\frac{1}{\tilde{\mu}_1}$ as a function of λ and U:

$$\frac{1}{\tilde{\mu}_1} = \frac{1}{1-\lambda} \cdot \left\{1 + \sqrt[2]{1 - 4 \cdot \lambda\left(1-\lambda\right)\left[1 + \frac{1-\lambda}{\lambda} \cdot U^2 - \left(\frac{U}{2 \cdot \lambda}\right)^2\right]}\right\}$$

and substitute for $\frac{1}{\tilde{\mu}_2}$ and $\frac{1}{\tilde{\mu}_1}$ in the second equation to obtain the value of ξ:

$$\xi\left(U, \lambda\right) = \frac{1}{2 \cdot U} \cdot \frac{1}{1-\lambda}\left[\sqrt[2]{1 + \frac{1-\lambda}{\lambda} \cdot U^2} \pm \left(1 - 2 \cdot \lambda\right)\right] - \frac{1}{U} \tag{18}$$

Two cases must be distinguished with respect to the above definition of the efficiency parameter:

Case 5 $\lambda \geq \frac{1}{2}$

When this happens and

$$\frac{1}{\tilde{\mu}_2} = 2\sqrt[2]{1 + \frac{1-\lambda}{\lambda} \cdot U^2} \geq \frac{U}{\lambda}$$

or

$$U \leq \frac{2 \cdot \lambda}{2 \cdot \lambda - 1}$$

the minimum level of efficiency required for local public good provision is:

$$\xi\left(U, \lambda\right) = \frac{1}{2 \cdot U} \cdot \frac{1}{1-\lambda}\left[\sqrt[2]{1+\frac{1-\lambda}{\lambda} \cdot U^2} + 4 \cdot \lambda - 3\right]$$

Consider now what happens to the above quantity, if the size of the special interest group increases; this requires to look at $\frac{\delta \cdot \xi(U,\lambda)}{\delta \cdot \lambda}$*, i.e. at the first derivative of the efficiency parameter with respect to* λ*:*

$$\frac{\left(1+\frac{1-\lambda}{\lambda} \cdot U^2\right)^{-\frac{1}{2}}}{2\left(1-\lambda\right)} \cdot \left(\frac{1}{1-\lambda} \cdot \frac{1}{U} + U \cdot \frac{2 \cdot \lambda - 1}{2 \cdot \lambda^2}\right) + \frac{1}{2 \cdot U} \cdot \frac{1}{\left(1-\lambda\right)^2} \geq 0$$

As the number of voters benefitting from the local public good increases, also the degree of efficiency required for the implementation of g *increases; in this case thus, benefits concentration pays.*

Notice finally that in order for g *be implemented it is not enough that the incentive constraint is satisfied, i.e.* $S_1 \geq \frac{1}{2}$ *but it must be the case that also the budget constraint holds so that* $1 - U\left(1-\xi\right) \geq 0$ *and thus:*

$$\xi\left(U\right) \geq \frac{U-1}{U}$$

In other words g *is included in party* 1 *platform if:*

$$\xi \geq \max\left\{\frac{1}{2 \cdot U} \cdot \frac{1}{1-\lambda}\left[\sqrt[2]{1+\frac{1-\lambda}{\lambda} \cdot U^2} + 4 \cdot \lambda - 3\right]; \frac{U-1}{U}\right\} \tag{19}$$

Case 6 $\lambda < \frac{1}{2}$

When this is the case and

$$U \leq \frac{2 \cdot \lambda}{1 - 2 \cdot \lambda}$$

holds also:

$$\xi\left(U, \lambda\right) = \frac{1}{2 \cdot U} \cdot \frac{1}{1-\lambda}\left[\sqrt[2]{1+\frac{1-\lambda}{\lambda} \cdot U^2} - 1\right]$$

Look now at the effects of an increase in the fraction of voters that derive utility from g*, and analyze* $\frac{\delta \cdot \xi(U,\lambda)}{\delta \cdot \lambda}$*:*

$$-\frac{1-\left(1+\frac{1-\lambda}{\lambda} \cdot U^2\right)^{-\frac{1}{2}}}{\left(2 \cdot U\right)\left(1-\lambda\right)} - \frac{\left(1+\frac{1-\lambda}{\lambda} \cdot U^2\right)^{-\frac{1}{2}}}{4 \cdot \lambda^2} \cdot \frac{1-2 \cdot \lambda}{1-\lambda} < 0 \tag{20}$$

As λ *increases the level of efficiency required for the provision of the local public good decreases; benefits concentration in this case does not pay and makes it more difficult for a politician to include* g *in his platform.*

In order for the budget constraint to hold, it must be the case that:

$$\xi \geq \max\left\{\frac{1}{2\cdot U}\cdot\frac{1}{1-\lambda}\left[\sqrt[2]{1+\frac{1-\lambda}{\lambda}\cdot U^2}-1\right];\frac{U-1}{U}\right\} \tag{21}$$

The above statements though, holds when both parties compete over the whole electorate; a different scenario emerges if $\frac{1}{\mu_2} < \frac{U}{\lambda}$.

When this is the case party 2 gives up competing for the votes of special interest group members; the voting game then, boils down to a standard electoral competition of the type described in Sahuguet and Persico (2006), where the electorate size is $(1-\lambda)$. Parties problem is equivalent to that of two players bidding on a single all-pay auction where no advantages are awarded to any player.

In equilibrium party 1 promises zero with probability $\left(1-\frac{\tilde{\mu}_2}{\tilde{\mu}_1}\right)$ and with complementary probability randomizes according to a uniform distribution over the support $\left[0;\frac{1}{\tilde{\mu}_1}\cdot\frac{1}{\tilde{\rho}}\right]$; party 2 promises instead, are distributed according to a uniform distribution over the support $\left[0;\frac{1}{\tilde{\mu}_2}\right]$.

Parties payoffs are:

$$S_1 = \frac{1-\lambda}{2}\cdot\frac{\tilde{\mu}_2}{\tilde{\mu}_1}+\lambda = \bar{S}$$

and

$$S_2 = (1-\lambda)\left(1-\frac{1}{2}\cdot\frac{\tilde{\mu}_2}{\tilde{\mu}_1}\right) = 1-\bar{S}$$

The system of equations that define an equilibrium where the public good is included in parties platforms is the following:

$$\left\{\begin{array}{c} 1 = (1-\lambda)\int_0^{\frac{1}{\tilde{\mu}_1}} x_2^k\cdot\tilde{\mu}_1 \\ \tilde{\rho} = (1-\lambda)\int_0^{\frac{1}{\tilde{\mu}_1}} x_1^k\cdot\tilde{\mu}_2 \\ \frac{1-\lambda}{2}\cdot\frac{\tilde{\mu}_2}{\tilde{\mu}_1}+\lambda \geq \frac{1}{2} \end{array}\right. \tag{22}$$

Solve first for $\frac{1}{\tilde{\mu}_1}$ in the first equation to get:

$$\frac{1}{\tilde{\mu}_1} = \frac{2}{1-\lambda}$$

Substitute now this result in the third equation and solve for $\frac{1}{\tilde{\mu}_2}$:

$$\frac{1}{\tilde{\mu}_2} = \frac{2}{1-\lambda}\cdot\frac{1}{1-U\,(1-\xi)}$$

Consider the last equation when the above values for $\frac{1}{\tilde{\mu}_1}$ and $\frac{1}{\tilde{\mu}_2}$ are used; solving for ξ gives:

$$\xi\,(U,\lambda) \geq 1-\frac{1}{U}\cdot\frac{\lambda}{1-\lambda} \tag{23}$$

Solve finally for $\frac{1}{\tilde{\mu}_2}$ to have:

$$\frac{1}{\tilde{\mu}_2} = \frac{2}{1 - 2 \cdot \lambda}$$

and notice that the present case is possible only if $\lambda < \frac{1}{2}$.

The initial condition $\frac{1}{\tilde{\mu}_2} < \frac{U}{\lambda}$, can be rewritten now as:

$$\frac{2 \cdot \lambda}{1 - 2 \cdot \lambda} < U$$

Look at the first derivative of $\xi(U, \lambda)$ with respect to λ to see what happens as the size of the special interest group with respect to the whole electorate increases:

$$\frac{\delta \cdot \xi(U, \lambda)}{\delta \cdot \lambda} = -\frac{1}{U} \cdot \frac{1}{1 - \lambda} - \frac{\lambda}{(1 - \lambda)^2} < 0 \tag{24}$$

Also in this different setting, if $\lambda < \frac{1}{2}$, benefits concentration does not pay.

In order for the inclusion of g in party 1 platform to be feasible it is required that:

$$\xi \geq \max \left\{ 1 - \frac{1}{U} \cdot \frac{\lambda}{1 - \lambda}; \frac{U - 1}{U} \right\} \tag{25}$$

The function $\xi(U, \lambda)$, then, has a point of minimum in $\lambda = \frac{1}{2}$; indeed for a given U, the level of efficiency required for the implementation of g decreases as the beneficiaries form half of the electorate and then starts to increase.

Given previous observations, the game in normal form can be described as follows:

$Party 1$			
		0	$U(1-\xi)$
$Party$ 2	0	$\frac{1}{2}; \frac{1}{2}$	$1 - \bar{S}; \bar{S}$
	$U(1-\xi)$	$\bar{S}; 1 - \bar{S}$	$\frac{1}{2}; \frac{1}{2}$

where $\bar{S} \geq \frac{1}{2}$.

As a consequence buying the local public good is a weakly dominant strategy when both the budget and the incentive constraint hold.

5 Final Remarks

Government spending that provides benefits accruing to a fraction of the electorate is subject to some constraints if available resources are limited and redistribution includes also direct cash transfers; the efficiency of public projects in particular, is a major issue and it is not enough that benefits balance costs.

The fact that some resources are bounded to a specific location represents an important drawback in the electoral competition; in order to be included in candidates' electoral platforms, an investment in in-kind transfers then, must give a positive rate of return.

Present analysis defines a minimum level of efficiency characterizing public projects that are chosen by parties competing in a proportional election; this threshold varies with the degree of benefits concentration.

If a party binds some of its redistributable resources to a specific segment of the electorate through the provision of in-kind transfers, the opponent finds these votes more costly to win. Public investments work as a deterrent and reduce competition; the higher is the per-capita benefit distributed the stronger is this effect.

The opponent in particular, gives up contending some of the beneficiaries' votes and shifts resources to the rest of the electorate. The vote share of the other party over this segment then increases.

On the other hand benefits distributed to voters that are ignored by the opponent become redundant; these resources moreover cannot be allocated to other segments of the electorate and are wasted.

Benefits concentration thus, produces two opposite effects on vote shares; a positive one that derives from a lower degree of competition in the beneficiaries' subset and a negative one that depends on resource misallocation.

If the number of people that takes advantage from the local public good increases also the incentives of the opponent to compete on it increase; indeed the fraction of votes of the beneficiaries on the total gets larger.

As a consequence when the special interest group form less than half of the electorate, the degree of competition over the special interest group subspace is low; the positive effect on vote shares is weak.

The misallocation problem, instead, is particularly relevant since per-capita benefits are high. Benefits concentration exacerbate the negative effect and increases the rate of return required for the implementation of the local public good.

When the special interest group forms more than half of the electorate, the misallocation problem is less relevant because benefits are more widely distributed. On the other hand opponent's incentives to compete for the votes of the special interest group are increased and it takes a higher per-capita benefit to obtain a significant reduction in competition. As a consequence benefits concentration proves to be beneficial in this case.

From previous observations descends further, that there is an optimal size for the set of beneficiaries of a local public good, and it coincide with half of the electorate; in that point indeed, the function that defines the efficiency threshold reaches its minimum for each amount of benefits generated.

6 References

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7 Appendix

7.1 Two Players All-pay Auctions where a Bidder has an Advantage over the Opponent

Consider the case where two risk-neutral players, 1 and 2, compete in a first-price all-pay auction with complete information and one player has an advantage over its opponent; without loss of generality, suppose that every bid of player 1 is increased of a fixed quantity $\alpha_1 > 0$. This additional amount is paid by the auctioneer.

Players do not have budget constraints and can bid any amount $x_i \geq 0$.

Player i's valuation of the auctioned item is:

$V_i > 0$ with $i = 1, 2$.

Different equilibria emerge depending on players' valuations of the auctioned item and on the size of player 1's advantage; the crucial element is the comparison between $V_1 + \alpha_1$ and V_2.

Proposition 7 *When is:*

$V_2 < \alpha_1$

player 1 *and player* 2 *bid zero with probability one. Player* 1*'s payoff amounts to* V_1 *while that of player* 2 *is zero.*

Proof. *The proof descends straightforwardly from the fact that any strictly positive bid smaller than* α_1 *has zero probability to win and gives player* 2 *a strictly negative payoff; any bid* $x_2 \geq \alpha_1$ *on the other hand, requires player* 2 *to pay a sum that exceeds his valuation of the auctioned item and again gives a negative expected payoff.*

A zero bid is a dominant strategy for player 2 *and gives a null payoff.*

Since player 2 *does not submit positive bids, player* 1 *can always get the object by bidding zero with probability one. Any offer strictly bigger than zero cannot increase the probability to win and decreases player* 1*'s payoff.*

A zero bid is a dominant strategy also for player 1 *and gives a payoff equal to the valuation of the object* V_1*.* ■

If is $V_2 \geq \alpha_1$, two main situations are possible:

- $V_2 > V_1 + \alpha_1$
- $V_2 \leq V_1 + \alpha_1$

Consider the first case; the following proposition holds:

Proposition 8 *If is:*

$V_2 > V_1 + \alpha_1$

then player 1*'s equilibrium distribution has an atom in zero amounting to* $\left(1 - \frac{V_1}{V_2}\right)$ *and is a uniform distribution over the support* $[0, V_1]$ *with probability* $\frac{V_1}{V_2}$*; expected payoff is zero.*

Player 2 *instead, randomizes over the support* $[\alpha_1, V_1 + \alpha_1]$ *according to a uniform distribution; his equilibrium payoff is:*

$V_2 - V_1 - \alpha_1 > 0$

Proof. *Successive rounds of elimination of strictly dominated strategies allow to restrict the intervals where players randomize with positive probability to* $[0, V_1]$ *for player* 1 *and to* $[\alpha_1, V_1 + \alpha_1]$ *for player* 2.

Player 1 *indeed, never bids more than his own valuation of the object since this would give a negative payoff; bidding zero then, strictly dominates* $x_1 > V_1$.

Consider now player 2.

Every strictly positive bid smaller than α_1 *is strictly dominated by bidding zero; any of such offer in facts, has zero probability to win and gives a negative payoff.*

Since player 1 *never bids more than* V_1, *player* 2 *can get the auctioned item with probability one by bidding* $x_2 \geq V_1 + \alpha_1$; *every bid strictly greater than* $V_1 + \alpha_1$ *though, is strictly dominated by* $x_2 = V_1 + \alpha_1$ [10]. *In both cases indeed, player* 2 *wins with probability one but choosing* $x_2 = V_1 + \alpha_1$ *involves a smaller payment to the auctioneer and results in a higher payoff.*

Player 2 *then, can always secure himself a payoff of* $V_2 - V_1 - \alpha_1 > 0$.

The argument developed by Hillman and Riley (1987)[11] *allows to exclude that in equilibrium, player* 1 *submits a strictly positive offer,* κ, *in the set* $[0, V_1]$, *with some strictly positive probability; when this happen indeed, player* 2*'s probability to win the auction rises discontinuously at* $x_2 = \kappa$; *then there is some* $\varepsilon > 0$ *such that player* 2 *bids on the interval* $[\kappa - \varepsilon, \kappa]$ *with zero probability. This means that player* 1 *is better off by moving the mass of probability down from* κ *to* $\kappa - \varepsilon$.

Discontinuities in player 1*'s mixed strategies at other points than zero are ruled out; an analogous argument excludes also discontinuities in player* 2*'s mixed strategies at other points than* α_1.

Notice that it cannot be the case that player 1 *places an atom of probability in zero and that, at the same time, player* 2*'s distribution has an atom of probability in* α_1; *if this happens indeed, each player can increase his probability to get the object by a finite amount, by bidding slightly more than* 0 *for player* 1 *and than* α_1 *for player* 2. *There is then,a profitable deviation for both players and this cannot be in equilibrium.*

Given that player 1 *and player* 2 *never place at the same time an atom of probability respectively in zero and* α_1, *and given that players randomize continuously on the sets of pure undominated strategies, the probability of ties is null.*

Since only equilibria in mixed strategies survive, it is useful then, to focus on the function denoted by $F_i(x_i)$ $(i = 1, 2$ *) that defines the probability that player* i *bids less than* x_i.*; in equilibrium both players must obtain from each pure strategy over which they randomize with positive probability, the same expected payoff.*

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[10] Notice that when player 2 bids $\frac{1}{V_1} + \alpha_1$, player 1 is indifferent between submitting an identical bid or zero with probability one; I assume that the second choice is always preferred.

[11] See also Baye, Kovenock and de Vries (1996), Bertoletti (2005), Che and Gale (1998) and Ellingsen (1991).

Consider then, players' expected payoffs starting from player 1*:*

$$E\left[U_1\left(x_1,x_2\right)\right]=\left(V_1-x_1\right)F_2\left(x_1+\alpha_1\right)-x_1\left[1-F_2\left(x_1+\alpha_1\right)\right] \tag{A1}$$

player 2*'s payoff instead, is:*

$$E\left[U_2\left(x_2,x_1\right)\right]=\left(V_2-x_2\right)F_1\left(x_2-\alpha_1\right)-x_2^l\left[1-F_1\left(x_2-\alpha_1\right)\right] \tag{A2}$$

Since player 2 *can always secure himself a positive payoff, must be* $E\left[U_2\left(x_2,x_1\right)\right]>0$*; setting* $x_2=\alpha_1$ *in* $A2$*, requires that* $F_1\left(0\right)>0$*.*

Player 1 *instead can get at most zero; if indeed were* $E\left[U_1\left(x_1,x_2\right)\right]>0$*, setting* $x_1=0$ *in* $A1$*, would require* $F_2\left(\alpha_1\right)>0$*.*

In equilibrium though, it cannot be the case that $F_1\left(0\right)>0$ *and* $F_2\left(\alpha_1\right)>0$ *at the same time; thus it must be* $F_2\left(\alpha_1\right)=0$ *and* $E\left[U_1\left(x_1,x_2\right)\right]=0$*.*

Since his equilibrium payoff is zero, player 1 *never spends more than* V_1*; this requires* $F_1\left(V_1\right)=1$*. Setting* $x_2=V_1+\alpha_1$ *in* $A2$ *gives:*

$$\left(V_2-V_1-\alpha_1\right)F_1\left(V_1\right)-\left(V_1+\alpha_1\right)\left[1-F_1\left(V_1\right)\right]=V_2-V_1-\alpha_1$$

This leads to the following equilibrium condition:

Player 1:

$$\left(V_1-x_1\right)F_2\left(x_1+\alpha_1\right)-x_1\left[1-F_2\left(x_1+\alpha_1\right)\right]=0 \tag{A3}$$

or

$$F_2\left(x_1+\alpha_1\right)=\frac{x_1+\alpha_1}{V_1}-\frac{\alpha_1}{V_1}$$

Since every bid x_1 *is equivalent to a bid of player* 2 *amounting to* $x_1+\alpha_1$*, is also:*

$$F_2\left(x_2\right)=\frac{x_2-\alpha_1}{V_1} \tag{A4}$$

Player 2 *then, randomizes in equilibrium according to a uniform distribution over the interval* $\left[\alpha_1;V_1+\alpha_1\right]$*.*

Player 2

$$\left(V_2-x_2\right)F_1\left(x_2-\alpha_1\right)-x_2\left[1-F_1\left(x_2-\alpha_1\right)\right]=V_2-V_1-\alpha_1 \tag{A5}$$

or

$$F_1\left(x_2-\alpha_1\right)=1-\frac{V_1}{V_2}+\frac{x_2-\alpha_1}{V_2}$$

Since every bid x_2 *is equivalent to a bid of player* 1 *amounting to* $x_2-\alpha_1$*, must be:*

$$F_1\left(x_1\right)=1-\frac{V_1}{V_2}+\frac{x_1}{V_2} \tag{A6}$$

Player 1*'s equilibrium distribution has an atom of probability amounting to* $\left(1-\frac{V_1}{V_2}\right)$ *that is placed in zero.*

Player 1 *thus, randomizes according to a uniform over the interval* $[0, V_1]$ *with probability* $\frac{V_1}{V_2}$ *and bids zero with probability* $\left(1-\frac{V_1}{V_2}\right)$*.* ■

Consider now the case where:

$V_2 \leq V_1 + \alpha_1$.

Proposition 9 *If is:*

$V_2 \geq \alpha_1$

and

$V_2 \leq V_1 + \alpha_1$

player 1 *randomizes in equilibrium, over the support* $[0, V_2-\alpha_1]$ *according to the following cumulate distribution function:*

$$F_1(x_1) \left\{ \begin{array}{ll} \frac{\alpha_1}{V_2} & \text{if } x_1 = 0 \\ \frac{x_1+\alpha_1}{V_2} & \text{if } x_1 > 0 \end{array} \right\}$$

His equilibrium payoff amounts to:

$V_1 + \alpha_1 - V_2 \geq 0$.

Player 2 *instead, randomizes over the support* $[\alpha_1, V_2] \cup \{0\}$ *according to the cumulate distribution function defined below:*

$$F_2(x_2) \left\{ \begin{array}{ll} 1-\frac{V_2}{V_1}+\frac{\alpha_1}{V_1} & \text{if } x_2 = 0 \\ 1-\frac{V_2}{V_1}+\frac{x_2}{V_1} & \text{if } x_1 \geq \alpha_1 \end{array} \right\}$$

He gets in equilibrium, a null payoff.

Proof. *Successive rounds of elimination of strictly dominated strategies, restrict the intervals where players randomizes with positive probability to* $[0, V_2-\alpha_1]$ *for player* 1 *and to* $[\alpha_1, V_2] \cup \{0\}$ *for player* 2*.*

Indeed player 2 *never submits strictly positive bids smaller than* α_1*, because this would give him a negative expected payoff; the same happens for bids exceeding* V_2*. These actions are strictly dominated by bidding zero.*

Player 1 *instead, bids* $x_1 > V_2 - \alpha_1$ *with probability zero since player* 2*'s maximum bid is* V_2*; any of such actions is strictly dominated by* $x_1 = V_2 - \alpha_1$*. In both cases indeed, player* 1 *wins the item with probability one but bidding* $x_2 = V_1 + \alpha_1$ *involves a smaller payment to the auctioneer and provides a higher payoff.*

Player 1 *then, can always secure himself a payoff of* $V_1 + \alpha_1 - V_2 > 0$*.*

The same arguments used in the previous case rules out discontinuities, at other points than zero for player 1 *and at other points than zero or* α_1 *for player* 2*.*

Consider now players' expected payoffs; player 1 *always gets* $E[U_1(x_1, x_2)] > 0$*; this requires setting* $x_1 = 0$ *in (2) that* $F_2(\alpha_1) > 0$*.*

Player 2 *cannot get in equilibrium, more than zero; indeed setting* $E[U_2(x_2, x_1)] > 0$ *requires* $F_1(-\alpha_1) > 0$ *when* $x_2 = 0$*, but this cannot be since no negative bids are allowed.*

Setting instead $x_2 = \alpha_1$ *in (2) requires* $F_1(0) > 0$*; this implies that is* $F_1(0) > 0$ *and* $F_2(\alpha_1) > 0$ *at the same time. Player* 1 *and player* 2 *then,* *could increase of a finite amount their probability of winning by bidding respectively slightly more than* 0 *and slightly more than* α_1*; this though, is not possible in equilibrium. As a consequence must be* $E\left[U_2\left(x_2, x_1\right)\right] = 0$*.*

Given that his equilibrium payoff is zero, player 2 *never bids more than* V_2 *so that* $F_2\left(V_2\right) = 1$*; setting now* $x_1 = V_2 - \alpha_1$ *in* $A1$ *gives:*

$$\left(V_1 - V_2 + \alpha_1\right) F_2\left(V_2\right) - \left(V_2 - \alpha_1\right)\left[1 - F_2\left(V_2\right)\right] = V_1 - V_2 + \alpha_1$$

The equilibrium conditions in this setting change as follows:

Player 2:

$$\left(V_2 - x_2\right) F_1\left(x_2 - \alpha_1\right) - x_2^l\left[1 - F_1\left(x_2 - \alpha_1\right)\right] = 0 \tag{A7}$$

or

$$F_1\left(x_2 - \alpha_1\right) = \frac{x_2 - \alpha_1}{V_2} + \frac{\alpha_1}{V_2}$$

since every bid x_2 *is equivalent to a bid of player* 1 *amounting to* $x_2 - \alpha_1$ *is also:*

$$F_1\left(x_1\right) = \frac{x_1 + \alpha_1}{V_2} \tag{A8}$$

No negative bids are allowed; therefore player 1*'s equilibrium distribution has an atom in zero amounting to* $\frac{\alpha_1}{V_2}$*; for* $x_1 \in \left(0, V_2 - \alpha_1\right]$ *instead, the cumulative distribution function is* $F_1\left(x_1\right) = \frac{x_1 + \alpha_1}{V_2}$ *.*

Player 1:

$$\left(V_1 - x_1\right) F_2\left(x_1 + \alpha_1\right) - x_1\left[1 - F_2\left(x_1 + \alpha_1\right)\right] = V_1 + \alpha_1 - V_2 \tag{A9}$$

or

$$F_2\left(x_1 + \alpha_1\right) = 1 - \frac{V_2}{V_1} + \frac{x_1 + \alpha_1}{V_1}$$

Since every bid x_1 *is equivalent to a bid of player* 2 *amounting to* $x_1 + \alpha_1$*, holds finally:*

$$F_2\left(x_2\right) = 1 - \frac{V_2}{V_1} + \frac{x_2}{V_1} \tag{A10}$$

Player 2*'s equilibrium distribution has an atom of probability in zero amounting to* $1 - \frac{V_2}{V_1} + \frac{\alpha_1}{V_1} \geq 0$*; Player* 2 *moreover, randomizes according to the cumulative distribution function* $\frac{x_2}{V_1}$ *over the support* $\left[\alpha_1, V_2\right]$*.* ■