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Slightly Altruistic Equilibria in Normal Form Games

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Abstract

We introduce a refinement concept for Nash equilibria (slightly altruistic equilibrium) defined by a limit process and which captures the idea of reciprocal altruism as presented in Binmore (2003). Existence is guaranteed for every finite game and for a large class of games with a continuum of strategies. Results and examples emphasize the (lack of) connections with classical refinement concepts. Finally, it is shown that under a pseudo-monotonicity assumption on a particular operator associated to the game it is possible, by selecting slightly altruistic equilibria, to eliminate those equilibria in which a player can switch to a strategy that is better for the others without leaving the set of equilibria.

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Abstract

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1 Introduction

It is well known that, in case of multiplicity, Nash equilibria may suffer from severe drawbacks. For example, in extensive form games, a Nash equilibrium may describe irrational behavior off the equilibrium path, and, even in normal form games, it may be unstable with respect to perturbations on the strategies or on the payoffs, moreover, it could be possible for coalitions of players to arrange mutually beneficial deviations from a Nash equilibrium (see Chapters 1 and 2 in van Damme (1989) or Chapters 8 and 12 in Fudenberg and Tirole (1991)).

According to this point of view and associated to further (implicit) requirements for self-enforcingness, refinement concepts have been introduced which "rule out" unappealing equilibria of a game (see also van Damme (2002)): the recommendation given by the theory should be stable with respect to deviations caused by some further criterion besides the classical utility maximization. Most of the refinement concepts introduced are based in one way or another on some kind of perturbations or distortions of players' rationality. Following Aumann (1997): *"You must be super-rational in order to deal with my irrationalities. Since this applies to all players, taking account of possible irrationalities leads to a kind of super-rationality for all".* In particular, in normal form games, refinements based on "trembles" (small departures from classical rationality) can be obtained by using (upper semicontinuity-like or lower-semicontinuity-like) stability properties in the definition (see, for example, *perfect equilibria* (Selten (1975), *proper equilibria* (Myerson (1978)), *strictly perfect equilibria* (Okada (1981)), *regular equilibria* (Harsanyi (1973) and Ritzberger (1994)) and *essential equilibria* (Wu and Jiang (1962)).

In this paper we introduce a new refinement concept for normal form games, called *slightly altruistic equilibrium*, which is based on a simple upper semicontinuity-like stability property with respect to a particular class of payoff perturbations (namely based on altruism) and captures, in a static environment, the following idea of reciprocal altruism of D. Hume, as reported in Binmore (2003): *"I learn to do service to another, without bearing him any real kindness, because I foresee, that he will return my service in expectation of another of the same kind, and in order to maintain the same correspondence of good offices with me and others. And accordingly, after I have serv'd him and he is in possession of the advantage arising from my action, he is induced to perform his part,* *as foreseeing the consequence of his refusal."* We use the above idea of reciprocal altruism in an endogenous way in order to define a mechanism that leads to particular Nash equilibria: each agent cares only about himself but his choice corresponds to the limit of choices he would have done in equilibrium if he had slightly cared about the others, provided the others had done the same. In other words, slightly altruistic equilibria are self-enforcing in terms of stability with respect to trembles of players' payoffs towards a kind of altruistic behavior. In fact, in an equilibrium s^* that is not slightly altruistic, there exists at least a player i whose equilibrium strategy is not "rational" whenever he assumes his opponents' behavior might be perturbed in an altruistic way, whatever is the perturbation.

In the definition of slightly altruistic equilibrium, trembles on each player payoff are proportional to the sum of his opponents' payoffs. The idea of modelling altruistic behavior of a player by using the sum (or the weighted sum) of opponents' payoffs has been often considered in game theory and in particular in normal form games. For example, in Fehr and Schmidt (1999) (and in some references therein) it has been shown that this approach is consistent with some experimental evidence whenever altruistic behavior emerges. Moreover, the use of psychological game theory (Geanakoplos, Pearce, and Stacchetti (1989)) allows the characterization of reciprocal altruism in a normal form game by using concepts of equilibrium in particular games derived from the original one (see Rabin (1993) or Falk and Fischbacher (2003) and references therein) where the coefficients of the weighted sum of opponents' payoffs depend explicitly on the beliefs of players about others' intentions. Similar payoff functions have also been used to describe altruistic behavior in dynamic games (for example in Levine (1998), Sethi and Somarathan (2001) or Dufwenberg and Kirchsteiger (2004)). However, we emphasize that all the models quoted above are not suitable for equilibrium selection since neither, in the static models, the prescribed predictions are Nash equilibria of the original game, nor, in the dynamic models, strategies necessarily converge to a Nash equilibrium.

We show that slightly altruistic equilibria exist in all finite games in mixed strategies and in a large class of continuous ones. Moreover, we investigate the connections with other refinement concepts in the context of finite games in mixed strategies. Whenever essential equilibria exist, they are slightly altruistic equilibria; so, perfectness together with *"slightly altruism"* are necessary conditions for essentiality (and hence for regularity) of equilibria. However, counterexamples show that slightly altruistic and perfect equilibria are not related. The same kind of lack of connections is obtained looking at refinement concepts based on properties of robustness with respect to joint mutually beneficial deviations of coalitions of players (for example *strong Nash equilibria* (Aumann (1959)) or *coalition proof equilibria* (Bernheim, Peleg and Whinston (1987)), which, by the way, do not satisfy any general existence theorem even in finite games (see also Ichiishi (1981).

However, we point out that the implicit assumption behind slightly altruistic equilibria (namely the possibility of trembles in the direction of altruism) is a-priori not incompatible with other implicit assumptions associated to the other refinement concepts previously quoted (for example the possibility of other perturbation of rationality or of joint deviations of coalitions). Therefore, it could be possible to restrict the set of outcomes prescribed by a "classical" refinement concept by asking also for "slightly altruism" and this could lead to a sharper selection mechanism, as shown in some examples in Section 3.

Finally, since an essential equilibrium can be Pareto dominated by another Nash equilibrium (even in weakly dominated strategies) as shown in the Example 1.5.2 in van Damme (1989), the degree of "efficiency" embodied in the slightly altruistic equilibrium concept does not necessarily result in collective preferable outcomes, which means that we cannot hope to achieve efficiency in a very strong sense. However, we prove that under suitable assumptions and by selecting slightly altruistic equilibria, it is possible to eliminate equilibria in which a player can switch to a strategy that is better for the others without leaving the set of equilibria. More precisely, we show that the *"pseudo-monotonicity"* of a particular operator associated to the game (the same used in Rosen (1965) to obtain uniqueness of Nash equilibria, under the stronger assumption of *strict monotonicity*) guarantees that the slightly altruistic equilibrium concept satisfies the following property for a refinement concept in normal form games:

Friendliness Property: For every player i and for every selected Nash equilibrium $s = (s_i, s_{-i})$, the strategy s_i maximizes the sum of player i opponents' payoffs on the set of the strategies s'_i of player i such that (s'_i, s_{-i}) is a Nash equilibrium of the game.

This property means that, for every element in the set of solutions, every player has *friendly behavior* as defined in Rusinowska (2002) and therein applied to equilibrium selection in 2-player bargaining models. This definition is based on a lexicographic-like optimality in which the sum of opponents' payoffs gets involved only when a player has reached his maximum level of self-utility and only on the set of Nash equilibria. That is, it is an unconditional kind of altruistic behavior since it does not require any additional beliefs on opponents' behavior other than utility maximization and it is of minimal character.

The assumption of pseudo-monotonicity is crucial; in fact, an example in finite games shows that it cannot be dispensed with entirely. Such assumption is widely used in other fields and then considered quite weak, so that our result has a sufficient level of generality even if does not apply to every finite game in mixed strategies.

Summarizing, familiarity with the underlying mathematics of the refinement literatures makes it quite clear that there is the possibility of defining refinements by perturbing agents' preferences in the direction of altruism: this paper explores this idea. So in Section 2, we introduce the concept of slightly altruistic equilibrium and we present an existence result together with some illustrative examples. Section 3 is to establish connections to related refinement concepts, in particular with essential, perfect and strong Nash equilibria; examples are given showing also that the slightly altruistic equilibrium concept may refine the Nash equilibrium concept even when all the other refinement concepts previously quoted do not. Finally, in Section 4, sufficient conditions on the data of the game are given to guarantee that the slightly altruistic equilibrium concept satisfies the Friendliness Property.

2 Definition and existence of slightly altruistic equilibria

In this section we give the definition of slightly altruistic equilibrium and an existence theorem. We also show, with some examples, the interesting level of *effectiveness* of this concept in finite and continuous games. Finally we discuss possible modifications of the concept which seem to be inappropriate, at least in general, because of the following drawbacks: stronger implicit assumptions, lower effectiveness and lack of existence.

2.1 Definition

Let $\Gamma = \{I; S_1, \ldots, S_N; f_1, \ldots, f_N\}$ be a N-player game where $I = \{1, \ldots, N\}$ is the set of players, the strategy set S_i of player i is a subset of $\mathbb{R}^{k(i)}$ and $f_i : S \to \mathbb{R}$ is the payoff function of player i, with $S = \prod_{i \in I} S_i$. Let E be the set of Nash equilibria (see Nash (1950, 1951) of the game Γ; that is a point $s^* \in S$ belongs to E if, for every player i, $f_i(s_i^*, s_{-i}^*) \geq f_i(s_i, s_{-i}^*)$ for all $s_i \in S_i$, where (s_i, s_{-i}^*) denotes the vector $(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_N^*)$.

Definition 2.1: For every $n \in \mathbb{N}$, let ε_n be a positive real number and, for each player i, let $h_{i,n} : S \to \mathbb{R}$ be the function, called ε_n-*altruistic payoff*, defined by:

$$h_{i,n}(s) = f_i(s) + \varepsilon_n \left[\sum_{j \in I \setminus \{i\}} f_j(s) \right] \quad \text{for all } s \in S. \tag{1}$$

For every $n \in \mathbb{N}$, the game

$$\Gamma_n = \{I; S_1, \ldots, S_N; h_{1,n}, \ldots, h_{N,n}\}$$

is called the ε_n-*altruistic game* associated to Γ.

Each $h_{i,n}$ represents the utility function of player i supposed to take into account the sum of the payoffs of the opponents with weight ε_n.
Now we can then introduce the following new concept of refinement:

Definition 2.2: (De Marco and Morgan (2004)). A Nash equilibrium s^* of the game Γ is said to be a *slightly altruistic equilibrium* if there exist a sequence of positive real numbers $(\varepsilon_n)_n$ decreasing to 0 and a sequence of strategy profiles $(s_n)_n \subseteq S$, such that

i) s_n is a Nash equilibrium of Γ_n for every $n \in \mathbb{N}$.

ii) s_n converges to s^* as $n \to \infty$.

We point out that every equilibrium in the ε_n-altruistic games translates the idea of reciprocal altruism reported in the Introduction since in every ε_n-altruistic game each player maximizes his payoffs perturbed by a little amount of the sum of his opponents' payoffs, provided the others do the same.

Before to investigate the properties of the slightly altruistic equilibrium concept, we give two illustrative examples, one for finite games and one for continuous ones.

EXAMPLE 2.3: Consider the following game Γ:

Player 1 \ Player 2	L	R
T	2, 0	1, 0
M	0, 0	1, 3
B	3, 0	0, -1

Γ has three Nash equilibria in pure strategy (T,R), (M,R) and (B,L). Consider the ε_n-altruistic game associated to Γ:

Player 1 \ Player 2	L	R
T	2, $2\varepsilon_n$	1, ε_n
M	0, 0	$1+3\varepsilon_n$, $3+\varepsilon_n$
B	3, $3\varepsilon_n$	$-\varepsilon_n$, -1

For all $n \in \mathbb{N}$, the pure strategy equilibria of the ε_n-altruistic game Γ_n are (B,L) and (M,R). Hence, (B,L) and (M,R) are the two slightly altruistic equilibria in pure strategies for the game Γ.

The previous example shows an easy interpretation of the slightly altruistic equilibrium concept in terms of stability with respect to trembles, but, in this case trembles concern preference relations of the players towards a slightly altruistic behavior rather than strategies. In fact, it turns out that, for an equilibrium that is not slightly altruistic, an equilibrium strategy for a player i might be unstable if player i assumes his opponents' preference relation trembles in an arbitrary way "towards altruism". This is the case for Player 1 in the equilibrium (T,R) in the example above: if Player 1 assumes that Player 2 preference relation may "tremble in the direction of altruism", whatever is the perturbation ε_n of Player 2, he cannot assume that Player 2 will play strategy R as a best reply to his strategy T, since L is the unique best reply to T. Therefore, (T,R) is unstable.

EXAMPLE 2.4: Consider the following game $\Gamma = \{\{1,2\}; [0,1], [0,1]; f_1, f_2\}$, where

$$f_1(s_1, s_2) = -s_1^2 \quad \text{and} \quad f_2(s_1, s_2) = s_1 s_2.$$

The set E of Nash equilibria of Γ is:

$$E = \{0\} \times [0, 1] .$$

Given $\varepsilon_n > 0$, the ε_n-altruistic payoffs are

$$h_{1,n}(s_1, s_2) = -s_1^2 + \varepsilon_n s_1 s_2 \quad \text{and} \quad h_{2,n}(s_1, s_2) = s_1 s_2 - \varepsilon_n s_1^2.$$

For every n, the set of Nash equilibria of the ε_n-altruistic game Γ_n is

$$E_n = \{(0, 0), (\varepsilon_n/2, 1)\}$$

Then, the set E^{SA} of the slightly altruistic equilibria of Γ is given by:

$$E^{SA} = \operatorname{Lim\,sup}_{n\to\infty} E_n = \{(0, 0), (0, 1)\} \subset E$$

where $y \in \operatorname{Lim\,sup}_{n\to\infty} E_n$ if and only if there exists a sequence $(y_k)_k$ converging to y such that $y_k \in E_{n_k}$ for a sequence of integers $(n_k)_k$ and for each $k \in \mathbb{N}$, (see, for example, Aubin and Frankowska (1990)).

2.2 Existence

THEOREM 2.5: *Assume that, for every player i, S_i is a compact and convex subset of $\mathbb{R}^{k(i)}$, f_i is continuous on S and concave with respect to every variable s_j on S_j. Then, the game Γ has at least a slightly altruistic equilibrium.*

Proof. For every player i and $m \in \mathbb{N}$, let $h_{i,m}$ be the real valued function defined in S by $h_{i,m}(s) = f_i(s) + \frac{1}{2^m} \sum_{j\in I\setminus\{i\}} f_j(s)$ for all $s \in S$. Since $h_{i,m}$ is continuous on S and concave with respect to the i-th component, the game $\Gamma_m = \{I; S_1, \dots, S_N; h_{1,m}, \dots, h_{N,m}\}$ has at least a Nash equilibrium for every $m \in \mathbb{N}$. Let s_m be a Nash equilibrium of the $\frac{1}{2^m}$-altruistic game Γ_m, S is compact, so there exists a subsequence $(s_{m_n})_n$ of the sequence $(s_m)_m$converging to s^* as $n \to \infty$. For all i and all $s'_i \in S_i$

$$h_{i,m_n}(s_{i,m_n}, s_{-i,m_n}) = f_i(s_{i,m_n}, s_{-i,m_n}) + \frac{1}{2^{m_n}} \sum_{j\in I\setminus\{i\}} f_j(s_{i,m_n}, s_{-i,m_n}) \geq$$

$$f_i(s'_i, s_{-i,m_n}) + \frac{1}{2^{m_n}} \sum_{j\in I\setminus\{i\}} f_j(s'_i, s_{-i,m_n}) = h_{i,m_n}(s'_i, s_{-i,m_n})$$

Then, for all i and all $s'_i \in S_i$, it results that:

$$f_i(s^*_i, s^*_{-i}) = \lim_{n\to\infty} h_{i,m_n}(s_{i,m_n}, s_{-i,m_n}) \geq \lim_{n\to\infty} h_{i,m_n}(s'_i, s_{-i,m_n}) = f_i(s'_i, s^*_{-i})$$

Therefore s^* is a Nash equilibrium of Γ and is a limit of a sequence of Nash equilibria of the ε_n-altruistic games Γ_n, with $\varepsilon_n = 1/2^{m_n}$. □

Obviously, we have:

COROLLARY 2.6: *Every finite game has at least a slightly altruistic equilibrium in mixed strategies.*

2.3 Alternative stability properties based on altruism

In the definition of slightly altruistic equilibria, each player doesn't have a-priori on the relative importance of his opponents, that is, he treats the others fairly in the perturbed games and therefore maximizes the unweighted average of opponents' payoffs. This implicitly means that any two agents, say i_1 and i_2, agree on the relative importance for any two other players, say i_3 and i_4. However, it could happen that players have *exogenous* "a-priori" on others' payoffs. In this case, let $\alpha = (\alpha^i)_{i \in I} \in \mathbb{R}_+^{N(N-1)}$ be the system of a-priori of the game where each $\alpha^i = (\alpha^{i,j})_{j \neq i}$ represents the a-priori of each player i and where $\alpha_{i,j}$ is the relative importance to player i of player j's utility with

$$\sum_{j \neq i} \alpha_{i,j} = 1 \quad \text{and} \quad \alpha_{i,j} \geq 0 \quad \text{for all } j \neq i.$$

Then, the perturbed ε_n-altruistic payoffs would look as follows:

$$\beta_{i,n}^{\alpha} = f_i + \varepsilon_n \left[\sum_{j \in I \setminus \{i\}} \alpha_{i,j} f_j \right] \tag{2}$$

and the corresponding perturbed ε_n-altruistic games as:

$$\overline{\Gamma}_n^{\alpha} = \{I; S_1, \ldots, S_N; \beta_{1,n}^{\alpha}, \ldots, \beta_{N,n}^{\alpha}\} \tag{3}$$

Therefore, it would be possible to select equilibria of the game Γ which are limits of equilibria of games $\overline{\Gamma}_n^{\alpha}$ for a sequence ε_n converging to 0 (*slightly altruistic equilibria with a-priori* α). Of course, this could change predictions (only in games with more than 2 players) but would still guarantee the existence result (simply consider $\beta_{i,n}^{\alpha}$ instead of $h_{i,n}$ in the proof of Theorem 2.5) and the interpretation in terms of stability with respect to trembles. Moreover, in the next sections (Remarks 3.6, 4.4) we show that for this solution concept the same connection with other refinements and a modified version of the Friendliness Property would hold true. However, this solution concept would imply the existence of a system of *a-priori* α which should be exogenous and common knowledge and therefore would require more exogenous information.

More generally, it would be possible to modify the definition and require different stability properties. Let $\varepsilon_n^i = (\varepsilon_n^{i,j})_{j \neq i}$, $\overline{\varepsilon}_n = (\varepsilon_n^i)_{i \in I} \in \mathbb{R}_+^{N(N-1)}$ and consider the game

$$\tilde{\Gamma}_{\overline{\varepsilon}_n} = \left\{ I; S_1, \ldots, S_N; f_1 + \sum_{j \in I \setminus \{1\}} \varepsilon_n^{1,j} f_j, \ldots, f_N + \sum_{j \in I \setminus \{N\}} \varepsilon_n^{N,j} f_j \right\}$$

So:

1) If one would require an equilibrium s^* to be the limit of equilibria of games $\tilde{\Gamma}_{\overline{\varepsilon}_n}$ for at least one sequence $(\overline{\varepsilon}_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+^{N(N-1)}$ converging to 0, then we would obtain a set of equilibria which includes slightly altruistic equilibria. In fact, simply considering, for

every player i, $\varepsilon_n^{i,j} = \varepsilon_n$ for every $j \neq i$ we get slightly altruistic equilibria. Therefore we obtain a selection mechanism having a lower level of effectiveness.

2) If one would require an equilibrium s^* to be the limit of equilibria of games $\tilde{\Gamma}_{\overline{\varepsilon}_n}$ for every sequence $(\overline{\varepsilon}_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+^{N(N-1)}$ converging to 0, then we could not obtain an existence result, as shown in the following example:

Example 2.7: Consider the following three player game.

	L	R
T	1,1,2	0,-1,1
B	1,2,1	0,-1,1

M

	L	R
T	0,0,0	0,-1,0
B	0,0,0	0,-1,0

D

where the first player selects a row, the second a column and the third a matrix. Since L is a dominant strategy for Player 2 and M is a dominant strategy for Player 3 then, denoting with $s_1(T)$ a mixed strategies of Player 1, the set of Nash equilibria of this game is

$$E = \{(s_1(T), L, M) \mid s_1(T) \in [0,1]\}.$$

If one considers a sequence $(\overline{\varepsilon}_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+^6$ converging to 0, where

$$\overline{\varepsilon}_n = (\varepsilon_n^{1,2}, \varepsilon_n^{1,3}, \varepsilon_n^{2,1}, \varepsilon_n^{2,3}, \varepsilon_n^{3,1}, \varepsilon_n^{3,2})$$

and the corresponding game $\tilde{\Gamma}_{\overline{\varepsilon}_n}$, then, for n sufficiently large L is a dominant strategy for Player 2 and M is a dominant strategy for Player 3. So we take into account only the payoff function of Player 1 in the perturbed game $\tilde{\Gamma}_{\overline{\varepsilon}_n}$, given his opponents' strategies L and M:

	L
T	$1 + \varepsilon_n^{1,2} + 2\varepsilon_n^{1,3}$
B	$1 + 2\varepsilon_n^{1,2} + \varepsilon_n^{1,3}$

M

If one considers a sequence $(\overline{\varepsilon}_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+^6$ converging to 0 with $\varepsilon_n^{1,2} < \varepsilon_n^{1,3}$ then the unique equilibrium of $\tilde{\Gamma}_{\overline{\varepsilon}_n}$, for n sufficiently large, is (T, L, M). Otherwise, if one considers a sequence $(\overline{\varepsilon}_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+^6$ converging to 0 with $\varepsilon_n^{1,2} > \varepsilon_n^{1,3}$ then the unique equilibrium of $\tilde{\Gamma}_{\overline{\varepsilon}_n}$, for n sufficiently large, is (B, L, M).

3 Connections with other refinement concepts

Aim of this section is to embody the slightly altruistic equilibrium concept in the wide literature of refinements of Nash equilibria. A great part of this literature focuses on refinements based on stability with respect to trembles. Some of these concepts are based on an upper-semicontinuity-like stability property which furnish constructive methods of calculus and the existence (under classical assumptions) of the corresponding solutions (for example *perfect equilibria* and *proper equilibria*). Nevertheless, using a lower semicontinuity-like stability property in the definition, on one hand guarantees a "stronger" stability, on the other hand, may cause a lack of existence of the corresponding solutions (even in mixed extensions of finite games) and gives more difficulties in the computation; this is the case, for example, of *strictly perfect equilibria*, *regular equilibria* and *essential equilibria*. In all the concepts above, trembles are usually intended to represent the possibility of mistakes in the equilibrium play or disturbances in the observation of the payoffs and therefore, stability with respect to such trembles has a specific game theoretic interpretation.

In this section, firstly, we prove that every essential equilibrium is slightly altruistic; this means that perfectness and slightly altruism are necessary conditions for essentiality and regularity of equilibria. However, they are not sufficient conditions because essential equilibria may not exist (see also Example 3.4 below). Moreover, the examples presented below show that slightly altruism is not related to perfectness: neither perfect equilibria are a subset of slightly altruistic equilibria nor slightly altruistic equilibria are a subset of perfect equilibria. The same result is obtained for other refinement concepts based on properties of robustness of the equilibria with respect to joint deviation of coalitions of players. However, by combining ex-post slightly altruism with other refinement concepts, it is possible to obtain a sharper selection procedure which leads, in some examples, to collective preferable outcomes.

Before to investigate the connections between essential equilibria and slightly altruistic equilibria, we recall some definitions.

Let $\Omega = \{I; \Phi_1, \dots, \Phi_N; v_1, \dots, v_N\}$ denote a finite game, where $\Phi_i = \{\varphi_i^1, \dots, \varphi_i^{k(i)}\}$ is the (finite) pure strategy set of player i, $\Phi = \prod_{i \in I} \Phi_i$ and $v_i : \Phi \to \mathbb{R}$ is the payoff function of player i. In this case, we denote with $\Gamma = \{I; S_1, \dots, S_N; f_1, \dots, f_N\}$ its mixed extension where each mixed strategy $s_i \in S_i$ is a vector $s_i = (s_i(\varphi_i))_{\varphi_i \in \Phi_i} \in \mathbb{R}_+^{|\Phi_i|}$ such that $\sum_{\varphi_i \in \Phi_i} s_i(\varphi_i) = 1$ and the expected payoff function $f_i : S \to \mathbb{R}$ is defined by:

$$f_i(s) = \sum_{\varphi \in \Phi} \left[\prod_{i \in I} s_i(\varphi_i) \right] v_i(\varphi) \quad \text{for all } s \in S.$$

Then, let $|\Phi| = K$ denote the cardinality of the set of all pure strategy profiles then every payoff function $v_i : \Phi \to \mathbb{R}$ has finite range, in particular $y_i = (v_i(\varphi))_{\varphi \in \Phi}$ is a K-dimensional vector for every player i. Then it is possible to identify the mixed extension Γ of the game Ω with the point $y = (y_1, \dots, y_n) \in \mathbb{R}^{NK}$. Therefore, denoting with $G(S_1, \dots, S_N)$ the set of N-player finite games with mixed strategy sets $(S_1, \dots, S_N)$, there is a one to one correspondence between $\mathbb{R}^{NK}$ and $G(S_1, \dots, S_N)$. Then, one can define a distance, denoted by $d(\Gamma', \Gamma'')$, between the games Γ' and Γ'' using the classical Euclidean distance between the corresponding vectors in $\mathbb{R}^{NK}$.

Therefore:

Definition 3.1: (Wu and Jiang (1962)). An equilibrium s^* of Γ is said to be *essential* if for every $\eta > 0$ there exists $\delta > 0$ such that for every game Γ' with $d(\Gamma, \Gamma') < \delta$ there exists an equilibrium s' with $d(s^*, s') < \eta$.

Proposition 3.2: *Every essential equilibrium of a finite game in mixed strategies is a slightly altruistic equilibrium.*

Proof. Since s^* is an essential equilibrium for Γ, for every $m \in \mathbb{N}$ there exists $\delta_m > 0$ such that any game Γ' satisfying $d(\Gamma, \Gamma') < \delta_m$ has an equilibrium s' such that $d(s^*, s') < 1/m$. Let ε_m be a positive real number such that the corresponding ε_m-altruistic game Γ_m satisfies $d(\Gamma, \Gamma_m) < \delta_m$. Hence, for every $m \in \mathbb{N}$ there exists an equilibrium s_m of Γ_m, which satisfies $d(s^*, s_m) < 1/m$. Consider a converging subsequence $(s_{m_n})_{n\in\mathbb{N}}$ of the sequence $(s_m)_{m\in\mathbb{N}}$. Then, $\lim_{n\to\infty} s_{m_n} = s^*$ and s^* is a slightly altruistic equilibrium of Γ. $\square$

Therefore, in light of Proposition 2.4.3 in van Damme (1989), every essential equilibrium of a finite game in mixed strategies is perfect and slightly altruistic at the same time. Note also that every regular equilibrium (see Harsanyi (1973), Ritzberger (1994)) is essential.

Before to present some examples showing the (lack of) connections between perfect, weak-Pareto dominant and slightly altruistic equilibria we recall some definitions.

Definition 3.3: A point $y \in Y \subset \mathbb{R}^h$ is a *Pareto point* in Y if it does not exist $z \in Y$ that *weakly Pareto dominates* y, that is if it does not exist $z \in Y$ such that $z - y \in \mathbb{R}^h_+ \setminus \{0\}$. A point $y \in Y \subset \mathbb{R}^h$ is a *weak-Pareto point* if it does not exist $z \in Y$ that *strongly Pareto dominates* y, that is if it does not exist $z \in Y$ such that $z - y \in int\mathbb{R}^h_+$, where $int\mathbb{R}^h_+$ denotes the interior of $\mathbb{R}^h_+$. Finally, $y \in Y \subset \mathbb{R}^h$ is an *ideal point* if $y - z \in \mathbb{R}^h_+$ for all $z \in Y$.
Given a vector-valued function $H : X \to \mathbb{R}^h$, a point $x \in X$ is a *Pareto* (resp.*weak-Pareto* or *ideal*) *solution* for H if $H(x)$ is a Pareto (resp. weak-Pareto or ideal) point in $H(X)$. Then, we say that a strategy profile $s' \in S$ of the game Γ is said to be a *Pareto* (resp. *weak-Pareto* or *ideal*) *profile* in $S' \subseteq S$ if it is a Pareto (resp. weak-Pareto or ideal) solution for the function F defined by $F(s) = (f_1(s), \ldots, f_N(s))$, for all $s \in S$, on the set S'. These definitions will be useful when we will investigate dominance in the set E of Nash equilibria of the game Γ.

Example 3.4: Let us consider the following 2×2-game:

Player 2 Player 1	L	R
T	0 3	0 1
B	3 1	3 1

The expected payoffs are $f_1(s_1, s_2) = 2s_1s_2 + 1$ and $f_2(s_1, s_2) = 3 - 3s_1$, hence, the set of Nash equilibria E is given by:

$$E = ([0, 1] \times \{0\}) \cup (\{1\} \times [0, 1])$$

Note that, for the first player, the strategy $s_1 = 1$ weakly dominates all the others mixed strategies which means that the set of perfect equilibria is $P = \{1\} \times [0, 1]$. However, the unique perfect equilibrium which is a Pareto profile in the set of Nash equilibria is $(1, 1)$, while $(0, 0)$ is a Pareto profile in the set of Nash equilibria which is not a perfect equilibrium. Note also that all Nash equilibria are weak Pareto profiles in the set of all strategy profiles and hence they are all strong Nash equilibria.

Let $h_{1,n}(s_1, s_2) = 2s_1s_2 + 1 + \varepsilon_n(3 - 3s_1)$ and $h_{2,n}(s_1, s_2) = 3 - 3s_1 + \varepsilon_n(2s_1s_2 + 1)$, as defined in (1). If $\varepsilon_n < 2/3$, the set E_n of Nash equilibria of the game Γ_n is given by:

$$E_n = \{(1, 1)\} \cup \left(\{0\} \times \left[0, \frac{3\varepsilon_n}{2}\right]\right).$$

The set E^{SA} of slightly altruistic equilibria is

$$E^{SA} = \text{Lim sup}_{n\to\infty} E_n = \{(1, 1), (0, 0)\}$$

Note that the two slightly altruistic equilibria are the two Pareto profiles in the set of Nash equilibria. Moreover, $(1, 1)$ is the unique equilibrium which is perfect and slightly altruistic. However, it is not essential. In fact, for $\rho > 0$, consider the following perturbed game Γ_ρ:

Player 2 Player 1	L	R
T	0 3	ρ 1
B	3 1	3 1

The set E_ρ of Nash equilibria of Γ_ρ is $E_\rho = [0, 1] \times \{0\}$. Clearly, $d(\Gamma, \Gamma_\rho) \to 0$ as $\rho \to 0$. However, there does not exist a sequence $(s_\rho)_\rho$ of Nash equilibria of the games Γ_ρ converging to $(1, 1)$ as $\rho \to 0$. Hence, $(1, 1)$ is not an essential equilibrium and Γ does not have any essential equilibrium.

Every perfect equilibrium of this game is strictly perfect (Okada (1981)). In fact consider a generic perturbation on the strategy profile set, that is a couple of functions $\eta = (\eta_1, \eta_2)$, where $\eta_1 : \{T, B\} \to]0, 1[$ and $\eta_2 : \{L, R\} \to]0, 1[$ such that:

$$\eta_1(T) + \eta_1(B) < 1 \qquad \eta_2(L) + \eta_2(R) < 1.$$

Let (Γ, η) be the corresponding perturbed game: $(\Gamma, \eta) = \{2; S_1^\eta, S_2^\eta; f_1, f_2\}$ where $S_1^\eta = \{s_1 \in S_1 \mid \eta_1(T) \leq s_1 \leq 1 - \eta_1(T)\}$, $S_2^\eta = \{s_2 \in S_2 \mid \eta_2(L) \leq s_2 \leq 1 - \eta_2(L)\}$ and where the payoff functions $f_1(s_1, s_2) = 2s_1s_2 + 1$ and $f_2(s_1, s_2) = 3 - 3s_1$ are restricted to $S_1^\eta \times S_2^\eta$. Then, the set E_η of Nash equilibria of (Γ, η) is:

$$E_\eta = \{1 - \eta_1(T)\} \times [\eta_2(L), 1 - \eta_2(L)].$$

For every sequence of perturbations $(\eta_n)_n$ converging to zero the corresponding sequence (E_{η_n}) of sets of equilibria of the perturbed games converges in the sense of Painvelé-Kuratowski (see Aubin and Frankowska (1990)) to the set P of the perfect equilibria of the game Γ. Therefore a strictly perfect equilibrium is not always a slightly altruistic equilibrium.

Example 3.5: Consider the following game:

Player 2 Player 1	L	R
T	0 3	0 1
B	2 1	3 1

Let (s_1, s_2) denotes a mixed strategy profile. The expected payoffs are $f_1(s_1, s_2) = 2s_1s_2 + 1$ and $f_2(s_1, s_2) = s_2 [s_1 - 1] - 3s_1 + 3$. The set of Nash equilibria E is given by:

$$E = ([0, 1] \times \{0\}) \cup (\{1\} \times [0, 1])$$

Again, all the equilibria are strong Nash equilibria but the unique perfect equilibrium is $(1, 0)$, which is weakly Pareto dominated by every other equilibria. Observe that $(1, 0)$ is also strictly perfect since for every perturbation on the strategy sets the corresponding perturbed game has a unique Nash equilibrium which gives the maximum probability to the pure strategies T and R.

Let $h_{1,n}(s_1, s_2) = 2s_1s_2 + 1 + \varepsilon_n(s_2 [s_1 - 1] - 3s_1 + 3)$ and $h_{2,n}(s_1, s_2) = s_2 [s_1 - 1] - 3s_1 + 3 + \varepsilon_n(2s_1s_2 + 1)$ be the altruistic payoffs, defined as in (1), with ε_n in $]0, 1[$. The set E_n of Nash equilibria of the game Γ_n is given by:

$$E_n = \left\{(1, 1), \left(\frac{1}{1 + 2\varepsilon_n}, \frac{3\varepsilon_n}{2 + \varepsilon_n}\right), (0, 0)\right\}$$

and the set E^{SA} of slightly altruistic equilibria is:

$$E^{SA} = \text{Lim sup}_{n\to\infty} E_n = \{(1, 1), (0, 0), (1, 0)\}.$$

We observe that in this case the slightly altruistic equilibrium $(1, 0)$ is weakly Pareto dominated.

As already observed, Proposition 3.2 shows that requiring to be an essential equilibrium (lower semicontinuity-like stability) is sufficient to guarantee slightly altruism and perfectness. However, the previous examples show that perfectness and slightly altruism are properties of different nature and can give different results. In fact, in Example 3.4, in the set of all perfect equilibria (which is not countable) there is a unique slightly altruistic equilibrium and the other slightly altruistic equilibrium is in weakly dominated strategies. On the other hand, in Example 3.5, the unique perfect equilibrium is slightly altruistic.

Note also that, even if in finite games in mixed strategies the perfect equilibrium concept satisfies the *Admissibility Property* that is, every equilibrium strategy is weakly undominated (see Chapter 2 in van Damme (1989)), this is not the case for continuous games as shown in Example 2.1 in Simon and Stinchombe (1995). In fact, there exist infinite games such that all Nash equilibria are in weakly dominated strategies so that it is no longer possible to ask for a refinement concept that satisfies both the existence and the Admissibility Property, when not finite games are considered. Moreover, the previous examples show that perfect equilibria are not always Pareto profiles in the set of Nash equilibria while slightly altruistic equilibria may refine the set of strong Nash equilibria, restricting them to those which are also Pareto profiles in the set of Nash equilibria (Example 3.4).

REMARK 3.6: The results previously obtained for slightly altruistic equilibria can be obtained also for slightly altruistic equilibria with a-priori α. Firstly, every essential equilibrium is a slightly altruistic equilibrium with a-priori α; in fact, this result follows by replacing in the proof of Theorem 3.2 the following "given $\delta_m > 0$, let $\varepsilon_m > 0$ be such that the corresponding ε_m-altruistic game Γ_m satisfies $d(\Gamma, \Gamma_m) < \delta_m$" with "given $\delta_m > 0$, let $\varepsilon_m > 0$ be such that the corresponding game $\overline{\Gamma}^\alpha_m$ satisfies $d(\Gamma, \overline{\Gamma}^\alpha_m) < \delta_m$".

Moreover, since in 2-player games the a-priori α do not affect the predictions of the slightly altruistic equilibrium concept, then the same considerations about the lack of connections between slightly altruistic equilibria with a-priori and perfect or strong Nash equilibria hold true.

The next example shows that the slightly altruistic equilibrium concept may refine the Nash equilibrium concept even when all the other refinement concepts previously quoted do not.

EXAMPLE 3.7: Consider the following three player game.

	L	R
T	2,0,-1	2,-1,8
B	0,0,0	2,3,0

M

	L	R
T	2,3,0	3,0,4
B	2,0,0	4,4,-1

D

where the first player selects a row, the second a column and the third a matrix. We restrict our attention to the pure strategies. (B,R,M) and (T,L,D) are the two Nash equilibria in pure strategies. None of them is a strong Nash equilibrium or an essential equilibrium. None of them is in weakly dominated strategies but they both give the same outcome (2,3,0). However, the unique slightly altruistic equilibrium is (T,L,D).

We did not investigate connections with the requirements and the concepts of Strategic Stability (see the seminal paper of Kohlberg and Mertens (1986) or, for example, Hillas et al. (2001) for recent results) in light of two reasons. The first is that the requirements of Strategic Stability imply that the solution concept has to be set-valued while slightly altruistic equilibrium concept deals with classical single-valued solution concepts. The second reason is that the definitions of stable sets also concern some kind of perturbations on the strategy sets. It turns out that stable sets are subsets of the set of trembling hand perfect equilibria, while the set of slightly altruistic equilibria is not of this kind (see the examples above).

4 Friendliness Property

Aim of this section is to investigate sufficient conditions on the data of the game which allow, by playing a slightly altruistic equilibrium, to eliminate equilibria in which a player can switch to a strategy that is better for others without leaving the set of equilibria. That is, we investigate when the slightly altruistic equilibrium concept satisfies the Friendliness Property as defined in the Introduction.

We first illustrate the Friendliness Property by an example:

Example 4.1: Consider the game in Example 3.4. The set of Nash equilibria is

$$E = ([0, 1] \times \{0\}) \cup (\{1\} \times [0, 1])$$

and the slightly altruistic equilibria are $(1, 1), (0, 0)$.
When, for example, Player 1 assumes that Player 2 chooses the strategy $s_2 = 0$, he is indifferent between all the strategies $s_1 \in [0, 1]$, that is Player 1 is indifferent between all the equilibria $[0, 1] \times \{0\}$. However, consider the slightly altruistic equilibrium $(0, 0)$, for $s_1 = 0$ Player 1 maximizes the payoff of Player 2. Analogously, the same situation arises for Player 2 when the equilibrium $(1, 1)$ is played. In other words, in every slightly altruistic equilibrium of this game, given the strategy of his opponent s_{-i}, each player i maximizes the payoff of the opponent over the set of all strategy s_i such that (s_i, s_{-i}) is a Nash equilibrium of the game. This is equivalent to say that in this game the slightly altruistic equilibrium concept satisfies the Friendliness Property.

We have the following:

Theorem 4.2: *Assume that, for every player $i \in I$, S_i is a compact and convex subset of $\mathbb{R}^{k(i)}$, the payoff function f_i is concave with respect to every variable s_j on S_j and continuously differentiable in S. Let $\nabla_{s_i} f_i(s)$ denote the gradient of f_i with respect to s_i in s and assume that the operator $A : S \to \mathcal{R} = \prod_{i\in I} \mathbb{R}^{k(i)}$ defined by*

$$A(s) = (\nabla_{s_1} f_1(s), \nabla_{s_2} f_2(s), \ldots, \nabla_{s_N} f_N(s)) \tag{4}$$

is pseudo-monotone on S, i.e. $\langle A(s), s - t\rangle \geq 0 \Rightarrow \langle A(t), s - t\rangle \geq 0$, $s \in S$, $t \in S$, where

$$\langle A(s), t\rangle = \sum_{i\in I} \langle \nabla_{s_i} f_i(s), t_i\rangle_i \quad \text{for all } s, t \in S \tag{5}$$

and $\langle \cdot, \cdot\rangle$ (resp. $\langle \cdot, \cdot\rangle_i$) denotes the scalar product in $\mathcal{R}$ (resp. in $\mathbb{R}^{k(i)}$).

Then, for every player i and for every slightly altruistic equilibrium $s^* = (s_i^*, s_{-i}^*)$, the strategy s_i^* maximizes $\sum_{j \in I \setminus \{i\}} f_j(\cdot, s_{-i}^*)$ on the set

$$E_i(s_{-i}^*) = \{s_i \in S_i \mid (s_i, s_{-i}^*) \in E\}.$$

That is, the slightly altruistic equilibrium concept satisfies the Friendliness Property.

Proof. Let s^* be a slightly altruistic equilibrium. Then there exist a sequence $(\varepsilon_n)_n \subset \mathbb{R}_+ \setminus \{0\}$ decreasing to 0 and a sequence of Nash equilibria, $(s_n)_n \subseteq S$, of Γ_n such that $s_n \to s^*$ as $n \to \infty$.

Since each s_n is a Nash equilibrium of Γ_n, for every player i, $h_{i,n}(s_{i,n}, s_{-i,n}) \geq h_{i,n}(s_i', s_{-i,n})$ for all $s_i' \in S_i$. S_i is convex then, for every $s_i' \in S_i$, $s_{i,\lambda} = (1-\lambda)s_{i,n} + \lambda s_i' = s_{i,n} + \lambda(s_i' - s_{i,n})$ belongs to S_i and $h_{i,n}(s_{i,n}, s_{-i,n}) \geq h_{i,n}(s_{i,\lambda}, s_{-i,n})$ for all $\lambda \in]0,1[$. Therefore:

$$\lim_{\lambda \to 0^+} \frac{h_{i,n}(s_{i,\lambda}, s_{-i,n}) - h_{i,n}(s_{i,n}, s_{-i,n})}{\lambda} = \langle \nabla_{s_i} h_{i,n}(s_n), s_i' - s_{i,n} \rangle_i \leq 0$$

Hence $\langle \nabla_{s_i} h_{i,n}(s_n), s_{i,n} - s_i' \rangle_i \geq 0$ which means that

$$\left\langle \nabla_{s_i} f_i(s_n) + \varepsilon_n \left[\sum_{j \in I \setminus \{i\}} \nabla_{s_i} f_j(s_n) \right], s_{i,n} - s_i' \right\rangle_i \geq 0 \quad \text{for all } s_i' \in S_i.$$

Since this inequality holds for every player i

$$\sum_{i \in I} \left\langle \nabla_{s_i} f_i(s_n) + \varepsilon_n \left[\sum_{j \in I \setminus \{i\}} \nabla_{s_i} f_j(s_n) \right], s_{i,n} - s_i' \right\rangle_i \geq 0 \quad \text{for all } s' \in S. \tag{6}$$

Let $B : S \to \mathcal{R}$ the operator defined by:

$$B(s) = \left(\sum_{j \in I \setminus \{1\}} \nabla_{s_1} f_j(s), \sum_{j \in I \setminus \{2\}} \nabla_{s_2} f_j(s), \ldots, \sum_{j \in I \setminus \{N\}} \nabla_{s_N} f_j(s) \right) \tag{7}$$

then

$$\langle B(s), t \rangle = \sum_{i \in I} \left\langle \sum_{j \in I \setminus \{i\}} \nabla_{s_i} f_j(s), t_i \right\rangle_i$$

and therefore equation (6) can be written as

$$\langle (A + \varepsilon_n B)(s_n), s_n - s' \rangle \geq 0 \quad \text{for all } s' \in S. \tag{8}$$

Let s'' be a Nash equilibrium of Γ then, as before, it follows that

$$\langle \nabla_{s_i} f_i(s''), s_i'' - s_{i,n} \rangle_i \geq 0 \quad \text{for every player } i$$

therefore $\langle A(s''), s'' - s_n \rangle \geq 0$. In light of the pseudo-monotonicity of the operator A,

$$\langle A(s_n), s'' - s_n \rangle \geq 0$$

then, from (8),

$$\varepsilon_n \langle B(s_n), s_n - s'' \rangle \geq \langle A(s_n), s'' - s_n \rangle \geq 0 \quad \text{for all } s'' \in E \tag{9}$$

where E is the set of Nash equilibria of Γ. Hence (9) implies that

$$\langle B(s_n), s_n - s'' \rangle \geq 0 \quad \text{for all } s'' \in E.$$

Passing to the limit for $n \to \infty$ we have

$$\langle B(s^*), s^* - s'' \rangle \geq 0 \quad \text{for all } s'' \in E. \tag{10}$$

For every player i, let $g_i : S \to \mathbb{R}$ be the function defined by:

$$g_i(s) = \sum_{j \in I \setminus \{i\}} f_j(s) \quad \text{for all } s \in S.$$

From the assumptions, g_i is concave with respect to every variable so, for every s_i' in S_i,

$$g_i(s_i', s_{-i}^*) - g_i(s_i^*, s_{-i}^*) \leq \langle \nabla_{s_i} g_i(s^*), s_i' - s_i^* \rangle_i \,.$$

For every $s_i'' \in E_i(s_{-i}^*) = \{s_i \in S_i \mid (s_i, s_{-i}^*) \in E\}$ and in light of (10),

$$\langle B(s^*), s^* - s'' \rangle = \sum_{j \in I} \left\langle \nabla_{s_j} g_j(s^*), s_j^* - s_j'' \right\rangle_j = \langle \nabla_{s_i} g_i(s^*), s_i^* - s_i'' \rangle_i \geq 0$$

therefore $g_i(s_i'', s_{-i}^*) - g_i(s_i^*, s_{-i}^*) \leq 0$.

Since this inequality holds for every slightly altruistic equilibrium then the slightly altruistic equilibrium concept satisfies the Friendliness Property. □

Example 4.3: Consider the game in Example 3.4. The payoff functions are $f_1(s_1, s_2) = 2s_1s_2 + 1$ and $f_2(s_1, s_2) = 3 - 3s_1$.
Since

$$\frac{\partial f_1}{\partial s_1}(s) = 2s_2 \quad \text{and} \quad \frac{\partial f_2}{\partial s_2}(s) = 0$$

we have that:

$$\langle A(\sigma), \sigma - \tau \rangle = \frac{\partial f_1}{\partial s_1}(\sigma)\,[\sigma_1 - \tau_1] = 2\sigma_2(\sigma_1 - \tau_1) \geq 0 \;\Rightarrow \sigma_1 - \tau_1 \geq 0 \;\Rightarrow$$

$$\Rightarrow \langle A(\tau), \sigma - \tau \rangle = \frac{\partial f_1}{\partial s_1}(\tau)\,[\sigma_1 - \tau_1] = 2\tau_2(\sigma_1 - \tau_1) \geq 0$$

Then the operator A is pseudo-monotone and the assumptions of Theorem 4.2 are satisfied. Note that, in this example, the operator A is not monotone, where A is said to be monotone if $\langle A(\sigma) - A(\tau), \sigma - \tau \rangle \leq 0$ for all σ, τ in S, in fact:

$$\langle A(\sigma) - A(\tau), \sigma - \tau \rangle = \frac{\partial f_1}{\partial s_1}(\sigma)\,[\sigma_1 - \tau_1] - \frac{\partial f_1}{\partial s_1}(\tau)\,[\sigma_1 - \tau_1] = 2\,[\sigma_2 - \tau_2]\,[\sigma_1 - \tau_1]$$

Given $(\sigma_1, \sigma_2) = (1, 1)$ and $(\tau_1, \tau_2) = (0, 0)$, we have:

$$\langle A(\sigma) - A(\tau), \sigma - \tau \rangle = 2.$$

The assumption of pseudo-monotonicity cannot be drooped in the Theorem 4.2 as shown below. Consider the game in Example 3.5, where the operator A is not pseudo-monotone. In fact, recalling that the payoff functions are $f_1(s_1, s_2) = 2s_1 s_2 + 1$ and $f_2(s_1, s_2) = s_2\,[s_1 - 1] - 3s_1 + 3$, we have

$$\frac{\partial f_1}{\partial s_1}(s) = 2s_2 \quad \text{and} \quad \frac{\partial f_2}{\partial s_2}(s) = s_1 - 1.$$

Given $\sigma = (0, 0)$ and $\tau = (1, 1)$,

$$\langle A(\sigma), \sigma - \tau \rangle = \frac{\partial f_1}{\partial s_1}(\sigma)\,[\sigma_1 - \tau_1] + \frac{\partial f_2}{\partial s_2}(\sigma)\,[\sigma_2 - \tau_2] =$$

$$= 2\sigma_2(\sigma_1 - \tau_1) + (\sigma_1 - 1)(\sigma_2 - \tau_2) = 1,$$

while

$$\langle A(\tau), \sigma - \tau \rangle = \frac{\partial f_1}{\partial s_1}(\tau)\,[\sigma_1 - \tau_1] + \frac{\partial f_2}{\partial s_2}(\tau)\,[\sigma_2 - \tau_2] =$$

$$= 2\tau_2(\sigma_1 - \tau_1) + (\tau_1 - 1)(\sigma_2 - \tau_2) = -2$$

Therefore, Theorem 4.2 does not hold and we can see that in the slightly altruistic equilibrium $(1, 0)$, the equilibrium strategy of Player 1, that is $s_1 = 1$, does not maximize the function $f_2(\cdot, 0)$ in $[0, 1]$. Hence, in this game the slightly altruistic equilibrium concept does not satisfy the Friendliness Property.

Observe that, in order to obtain uniqueness of Nash equilibria, the operator A defined in (4) has been used in Rosen (1965) under the stronger condition (of strict monotonicity):

$$\langle A(\sigma) - A(\tau), \sigma - \tau \rangle < 0 \quad \text{for all } \sigma, \tau \in S,\ \sigma \neq \tau$$

In Rosen (1965), the functions f_i satisfying the assumptions of strictly monotonicity of the operator A are said to be *diagonally strictly concave.* Similarly, when A is pseudo-monotone, the functions f_i will be said to be *diagonally pseudo-concave.* Pseudo-monotone operators have been introduced in Brezis, Nirenmberg and Stampacchia (1972) and then applied in many fields (see also Lignola and Morgan (1999)). Finally, recall that in 1-player games the pseudo-monotonicity assumption on the operator A coincides with the *pseudo-concavity* of the payoff function (Mangasarian (1965), see also, for example, Avriel et al. (1988)).

Finally, note that the Friendliness Property has been extensively investigated as a selection device in the general framework of N-player normal form games also by the authors in De Marco and Morgan (2007).

REMARK 4.4: Theorem 4.2 could be modified for slightly altruistic equilibria with a-priori α in a natural way. In fact, under the same assumptions of Theorem 4.2 on the payoffs and the strategy sets, in every slightly altruistic equilibrium with a-priori α s^*, the strategy s_i^* maximizes the weighted sum of opponents' payoffs $g_i^\alpha = \sum_{j \in I \setminus \{i\}} \alpha_{i,j} f_j$ on the set $E_i(s_{-i}^*)$. The proof comes directly from the proof of Theorem 4.2 by replacing (in the proof) the functions $h_{i,n}$ with the functions $\beta_{i,n}^\alpha$ as defined in (2), the operator B as defined in (7) with the operator $B^\alpha : S \to \mathcal{R}$ defined by:

$$B^\alpha(s) = \left(\sum_{j \in I \setminus \{1\}} \alpha_{1,j} \nabla_{s_1} f_j(s), \sum_{j \in I \setminus \{2\}} \alpha_{2,j} \nabla_{s_2} f_j(s), \dots, \sum_{j \in I \setminus \{N\}} \alpha_{N,j} \nabla_{s_N} f_j(s) \right) \qquad (11)$$

and the functions g_i with the functions g_i^α.

5 Conclusions

In this paper we define a refinement concept for situations in which (a little amount of) altruism may emerge. The evidence coming out from the examples shows that this concept leads to different predictions with respect to other classical refinement concepts which are based on different assumptions (trembles in the choice of the equilibrium strategies or not binding pre-play communication between players) and that, if players are supposed to be altruist (even in a faintly way), such predictions seem to be reasonable (also in light of the Friendliness Property). However, there are no theoretical or psychological reasons which state that an altruism assumption is not compatible with one of the assumptions associated to other refinement concepts and, therefore, they could be taken simultaneously into account.

References

[1] AUBIN J.P. AND H. FRANKOWSKA (1990): Set Valued Analysis. *Boston: Birkhauser.*

[2] AUMANN R. (1959): Acceptable Points in General Cooperative n-person games, *in: Contributions in the Theory of Games Princeton: Princeton University Press.*

[3] AUMANN R. (1997): Rationality and Bounded Rationality, *Games and Economic Behavior, 21, 2- 14.*

[4] AVRIEL M., DIEWERT W.E., SCHAIBLE S. AND I. ZANG (1988): Generalized Concavity. *New York: Plenum Publishing Corporation.*

[5] Bernheim B.D., B. Peleg and M.D. Whinston (1987): Coalition-Proof Nash Equilibria I. Concepts, *Journal of Economic Theory, 42, 1-12.*

[6] Binmore K. (2003): Plenary Lecture: "Natural Justice (Game Theory and Morality)", *in "XV Italian Meeting on Game Theory and Applications Conference Book.*

[7] Brézis, H., Nirenberg, L. and G. Stampacchia (1972): A remark on Ky Fan's minimax principle, *Bollettino Unione Matematica Italiana, 6, 293-300.*

[8] Dufwenberg M. and G. Kirchsteiger (2004): A Theory of Sequential Reciprocity, *Games and Economic Behavior, 47, 268 - 298.*

[9] De Marco G. and J. Morgan (2004): A Selection Mechanism for Nash Equilibria, *Preprint 62/2004 Dipartimento di Matematica e Statistica, Università di Napoli Federico II*

[10] De Marco G. and J. Morgan (2007): Friendliness and Reciprocity in Equilibrium Selection, *International Game Theory Review, 9(4), to appear.*

[11] Falk A. and U. Fischbacher (2005): A Theory of Reciprocity, *Games and Economic Behavior, article in press.*

[12] Fehr E. and K. Schmidt (1999): A Theory of Fairness, Competition, and Cooperation, *Quaterly Journal of Economics, 114, 817 - 868.*

[13] Fudenberg D. and J. Tirole (1991) Game Theory, *Cambridge, Mass.: MIT Press.*

[14] Geanakoplos J., D. Pearce, and E. Stacchetti (1989): Psychological Games and Sequential Rationality, *Games and Economic Behavior, 1, 60-79.*

[15] Harsanyi J.C. (1973): Oddness of the Number of Equilibrium Points: a New Proof, *International Journal of Game Theory, 2, 235-250.*

[16] Hillas J., M. Jansen, J. Potters and D. Vermeulen (2001), On the Relation Among Some Definitions of the Strategic Stability, *Mathematics of Operations Research, 26, 611-635.*

[17] Ichiishi T. (1981): A Social Coalitional Equilibrium Existence Theorem, *Econometrica, 49, 369-377.*

[18] Kohlberg E. and J.F.Mertens (1986), On Strategic Stability of Equilibria, *Econometrica, 54, 1003-1039.*

[19] Levine D.(1998): Modeling Altruism and Spitefulness in Experiments, *Review of Economic Dynamics, 1, 593-622.*

[20] Lignola B and J. Morgan (1999): Generalized variational inequalities with pseudomonotone operators under perturbations, *Journal of Optimization Theory and Applications, 101 , 213-220.*

[21] Mangasarian, O. L. (1965), Pseudo-convex functions, *Journal of the Society for Industrial and Applied Mathematics, Series A: Control, 3, 281-290.*

[22] Myerson R.B. (1978): Refinements of the Nash Equilibrium Concept, *International Journal of Game Theory, 7, 73-80.*

[23] Nash J. (1950): Equilibrium Points in N-Person Game, *Proceedings of the National Academy of Sciences of the U.S.A., 36, 48-49.*

[24] Nash J. (1951): Non-Cooperative Games, *Annals of Mathematics, 54, 286-295.*

[25] Okada A. (1981): On Stability of Perfect Equilibrium Points, *International Journal of Game Theory, 10, 67-73.*

[26] Ritzberger K. (1994): The Theory of Normal Form Games from the Differentiable Viewpoint, *International Journal of Game Theory, 23, 207-236.*

[27] Rosen J.B. (1965): Existence and Uniqueness of Equilibrium Points for Concave N-Person Games, *Econometrica, 3, 520–534.*

[28] Rusinowska A. (2002): Refinements of Nash Equilibria in view of Jealous and Friendly Behavior of Players, *International Game Theory Review, 4, 281-299.*

[29] Selten R. (1975): Reexamination of the Perfectness Concept for Equilibrium Points in Estensive Games, *International Journal of Game Theory, 4, 25-55.*

[30] Sethi R. and E. Saman (2001): Preference Evolution and Reciprocity, *Journal of Economic Theory, 97, 273-297.*

[31] Simon L.K. and M.B. Stinchombe (1995): Equilibrium Refinements for Infinite Normal-Form Games, *Econometrica, 63, 1421-1443.*

[32] van Damme E. (1989): Stability and Perfection of Nash Equilibria, *Berlin: Springer-Verlag.*

[33] van Damme E. (2002): Strategic Equilibrium. *Chapter 41 in Handbook of Game Theory with Economic Applications, Volume 3, Elsevier Science Publishers (North-Holland), Amsterdam.*

[34] Wu W.T. and J.H. Jiang (1962): Essential Equilibrium Points of n-person Cooperative Games, *Science Sinica, 11, 1307-1322.*