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### *Dynamic Trading and Asset Prices: Keynes vs. Hayek*

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# *Dynamic Trading and Asset Prices: Keynes vs. Hayek*

Giovanni Cespa<sup>♥</sup> and Xavier Vives<sup>♦</sup>

### **Abstract**

We investigate the dynamics of prices, information and expectations in a competitive, noisy, dynamic asset pricing equilibrium model. We look at the bias of prices as estimators of fundamental value in relation to traders' average expectations and note that prices are more (less) biased than average expectations if and only if traders over- (under-) rely on public information with respect to optimal statistical weights. We find that prices are biased in relation to average expectations whenever traders speculate on short-run price movements. In a market with long term traders, over-reliance on public information obtains if noise trade increments are correlated enough and/or there is low enough residual uncertainty in the payoff. This defines a "Keynesian" region; the complementary region is "Hayekian" in that prices are less biased than average expectations in the estimation of fundamental value. The standard case of no residual uncertainty and noise trading following a random walk is on the frontier of the two regions. With short-term traders there typically are two equilibria, with the stable (unstable) one displaying over- (under-) reliance on public information.

**Keywords:** Price bias, long and short-term trading, multiple equilibria, average expectations, higher order beliefs, over-reliance on public information.

**JEL Classification:** G10, G12, G14

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# 1 Introduction

Do investors excessively focus their attention on market aggregate behavior and public information, disregarding their private judgement? Are asset prices aligned with the consensus opinion (average expectations) on the fundamentals in the market? Undeniably, the issues above have generated much debate among economists. In his *General Theory*, Keynes pioneered the vision of stock markets as beauty contests where investors try to guess not the fundamental value of an asset but the average opinion of other investors, and end up chasing the crowd.<sup>1</sup> This view tends to portray a stock market dominated by herding, behavioral biases, fads, booms and crashes (see, for example, Shiller (2000)), and goes against the tradition of considering market prices as aggregators of the dispersed information in the economy advocated by Hayek (1945). According to the latter view prices reflect, perhaps noisily, the collective information that each trader has about the fundamental value of the asset (see, for example, Grossman (1989)), and provide a reliable signal about assets' liquidation values. In this paper, we address the tension between the Keynesian and the Hayekian visions in a dynamic finite horizon market where investors, except for noise traders, have no behavioral bias and hold a common prior on the liquidation value of the risky asset.

In a dynamic market with risk averse short-term traders, differential information, and an independent stock of noisy supply across periods Allen, Morris, and Shin (2006) argue that prices are always farther away from fundamentals than traders' average expectations and display over-reliance on public information. In this case asset prices depend on investors' higher order beliefs (i.e. the average expectation at period  $n$  of the average expectation at  $n + 1$  of ... of the average expectation of the liquidation value in the final period) and systematically depart from fundamentals compared to investors' average expectations. The reason is that in a dynamic market, when traders try to predict their peers' actions, the price heavily weights public information, more than what the optimal statistical weight prescribes to assess the

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<sup>1</sup>Keynes' vision of the stock market as a beauty contest or the situation in which judges are more concerned about the opinion of other judges than of the intrinsic merits of the participants in the contest: "... professional investment may be likened to those newspaper competitions in which the competitors have to pick out the six prettiest faces from a hundred photographs, the prize being awarded to the Competitor whose choice most nearly corresponds to the average preferences of the competitor as a whole; so that each competitor has to pick, not those faces which he himself finds prettiest, but those which he thinks likeliest to catch the fancy of the other competitors, all of whom are looking at the problem from the same point of view." (Keynes, *General Theory*, 1936).

fundamentals, ultimately supporting Keynes' vision of the stock market. The question is thus how general is this finding or whether Keynes always prevail over Hayek. Is it always true that prices are more biased than average expectations in the estimation of fundamentals? Is there always over-reliance on public information?<sup>2</sup>

The central result of this paper is that both the bias of prices in relation to average expectations in the estimation of fundamentals as well as the potential over- or under-reliance on public information are driven by traders' short-term speculative behavior. In a *static* market agents speculate on the difference between the price and the liquidation value, prices are aligned with investors' average expectations about this value, and traders put the optimal statistical weight on public information. In a *dynamic* market, in principle traders with a long or a short horizon *also* speculate on short-run price differences. This may misalign prices and investors' average expectations, potentially leading to over- or under-reliance on public information.

Allowing for a general noise trading process and assuming that residual uncertainty can affect the liquidation value, the following results obtain. When traders have a long horizon short-run speculation is driven by the correlation of noise trade increments and by the potential presence of residual uncertainty in the liquidation value. In the benchmark case with no residual uncertainty affecting the liquidation value, and noise trading following a random walk (i.e. with independent noise trade increments), prices assign the optimal *statistical* weight to public information and equal the average expectations of investors about the fundamental value plus a risk-weighted noise term. In this case traders act as in a static market in every period and the bias of prices and average expectations in the estimation of the fundamental value is the same. When residual pay-off uncertainty and noise trade predictability are added, traders also speculate on short-run price differences, and either over- or under-reliance on public information may occur. Low residual uncertainty and high correlation in noise trading increments tend to generate over-reliance; conversely, high residual uncertainty and low correlation in noise trade increments tend to generate under-reliance on public information. This partitions the parameter space into a Keynesian region, where prices are farther away from fundamentals than average expectations, and a Hayekian region where the opposite occurs. The partition depends on risk tolerance and more risk aversion enlarges the Keynesian region.<sup>3</sup>

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<sup>2</sup>Over-reliance on public information may have deleterious welfare consequences (see, e.g., Vives (1997), Morris and Shin (2002), and Angeletos and Pavan (2007)). In this paper we stay within the bounds of a positive analysis.

<sup>3</sup>It is worth noting that it should be no surprise that in a *noisy* rational expectations equilibrium



With short-term traders there typically are two equilibria ranked by traders' responsiveness to private information. In the equilibrium with low (high) signal responsiveness there is over- (under-) reliance on public information. Again, the first equilibrium may be associated to Keynes and the second one to Hayek. Furthermore, in the former (latter) equilibrium the price adjusts slower (faster) to changes in the fundamentals than the consensus expectations of investors. The "Hayekian" equilibrium is, however, unstable. Finally, if noise trading follows a random walk and there is sufficiently large residual uncertainty in the liquidation value, over-reliance on public information always occurs in equilibrium.

The intuition for our results is as follows. In a dynamic market the relationship between price and fundamentals depends both on the quality of traders' information *and* on their reaction to order imbalances. Suppose a trader observes a positive signal and faces a positive order-flow. Upon the receipt of good news he increases his long position in the asset. On the other hand, his reaction to the order imbalance is either to accommodate it, counting on a future noise trade reversal (and thus acting as a "market-maker"), or to follow the market and further increase his long position anticipating an additional price rise (in this way acting as a "short-term" speculator). The more likely it is that the order imbalance reverts over time due to liquidity traders' behavior, the more actively the trader will want to accommodate it. Conversely, the more likely it is that the imbalance is due to informed speculators' activity, the more the trader will want to follow the market.<sup>4</sup> In the former (latter) case, the trader's speculative position is partially offset (reinforced) by his market making (short-run speculative) position. Thus, the impact of private information on the price is partially sterilized (enhanced) by traders' market-making (short-run speculative) activity. This, in turn, loosens (tightens) the price from (to) the fundamentals in relation to average expectations, yielding over- (under-) reliance on public information.

When long-term traders are in the market, correlation across noise trade increments helps predicting future noise trade shocks, and tilts traders towards accommodating order imbalances. This effect is extreme when the stock of noise traders'

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prices may be less biased than investors' average expectations about the fundamentals. This result depends on the relative weights that in equilibrium traders put on private and public information and, obviously, could not arise in a *fully revealing* equilibrium where such a bias cannot arise.

<sup>4</sup>In this case, indeed, the order imbalance is likely to proxy for upcoming good news that are not yet completely incorporated in the price. There is a vast empirical literature that documents the transient impact of liquidity trades on asset prices as opposed to the permanent effect due to information-driven trades. See e.g. Wang (1994), and Llorente et al. (2002).

demand is independent across periods.<sup>5</sup> The impact of residual uncertainty over the liquidation value, on the other hand, enhances the hedging properties of future positions, boosting traders' signal responsiveness and leading them to speculate more aggressively on short-run price differences. Thus, depending on the intensity of the correlation across noise trade increments, over- or under-reliance on public information occurs. Conversely, absent correlation across noise trade increments and residual uncertainty, traders act as in a static market, and the price assigns the optimal statistical weight to public information. When traders have a short horizon, the lack of future trading opportunities eliminates intertemporal hedging possibilities. This may give rise to self-fulfilling equilibria. In particular, if traders anticipate an order imbalance driven by noise traders (informed traders), they scale down (up) their trading aggressiveness and accommodate it (follow the market), justifying their prediction and enforcing an equilibrium with over- (under-) reliance on public information.

This paper contributes to the recent literature that analyzes the effect of higher order expectations in asset pricing models where traders have differential information, but agree on a common prior over the liquidation value. Bacchetta and van Wincoop (2006) study the role of higher order beliefs in asset prices in an infinite horizon model showing that higher order expectations add an additional term to the traditional asset pricing equation, the higher order “wedge,” which captures the discrepancy between the price of the asset and the average expectations of the fundamentals. According to our results, higher order beliefs do not necessarily enter the pricing equation. In other words, for the higher order wedge to play a role in the asset price we need residual uncertainty to affect the liquidation value or noise trade increments predictability when traders have long horizons; alternatively, traders must display myopic behavior. Nimark (2007), in the context of Singleton (1987)'s model, shows that under some conditions both the variance and the impact that expectations have on the price decrease as the order of expectations increases.

Other authors have analyzed the role of higher order expectations in models where traders hold different initial beliefs about the liquidation value. Biais and Bossaerts (1998) show that departures from the common prior assumption rationalize peculiar trading patterns whereby traders with low private valuations may decide to buy an asset from traders with higher private valuations in the hope to resell it later on

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<sup>5</sup>Indeed, assuming that the stock of noise trade is i.i.d. implies that the gross position noise traders hold in a given period  $n$  completely reverts in period  $n + 1$ . This lowers the risk of accommodating order imbalances in any period, as speculators can always count on the possibility of unwinding their inventory of the risky asset to liquidity traders in the coming round of trade.

during the trading day at an even higher price. Cao and Ou-Yang (2005) study conditions for the existence of bubbles and panics in a model where traders’ opinions about the liquidation value differ.<sup>6</sup> Banerjee et al. (2006) show that in a model with heterogeneous priors differences in higher order beliefs may induce price drift.

Our paper is also related to the literature emphasizing the existence of “limits to arbitrage.” De Long et. al (1990) show how the risk posed by the existence of an unpredictable component in the aggregate demand for an asset can crowd-out rational investors, thereby limiting their arbitrage capabilities.<sup>7</sup> In our setup, it is precisely the risk of facing a reversal in noise traders’ positions that tilts informed traders towards accommodating more eagerly order imbalances. In turn, this effect is responsible for the over-reliance that asset prices place on public information.

Finally, the paper is related to the literature stressing the consequences of traders’ reaction to the aggregate flow of orders. For example, Gennotte and Leland (1990) argue that investors may exacerbate the price impact of trades, yielding potentially destabilizing outcomes, by extracting information from the order flow. In our model, the way in which rational investors interpret aggregate demand realizations is crucial to determine the weight assigned by the price to public information.

The paper is organized as follows: in the next section we present the static benchmark, showing that in this framework the asset price places the optimal statistical weight on public information and is just a noisy version of investors’ average expectations. In section 3 we analyze the dynamic model with long-term traders. In section 4, we then turn our attention to the model with short-term traders. The final section discusses the results and provides concluding remarks.

## 2 The Static Benchmark

Consider a one-period stock market where a single risky asset with liquidation value  $v \sim N(\bar{v}, \tau_v^{-1})$ , and a riskless asset with unitary return are traded by a continuum of risk-averse speculators in the interval  $[0, 1]$  together with noise traders. Speculators have CARA preferences (denote with  $\gamma$  the risk-tolerance coefficient) and maximize

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<sup>6</sup>Kandel and Pearson (1995) provide empirical evidence supporting the non-common prior assumption.

<sup>7</sup>Kondor (2004) shows that limits to arbitrage also occur in a 2-period model where informed traders have market power.

the expected utility of their wealth:  $W_{i1} = (v - p_1)x_{i1}$ .<sup>8</sup> Prior to the opening of the market every informed trader  $i$  receives a signal  $s_{i1} = v + \epsilon_{i1}$ ,  $\epsilon_{i1} \sim N(0, \tau_{\epsilon_1}^{-1})$ , and submits a demand schedule (generalized limit order) to the market  $X_1(s_{i1}, p_1)$  indicating the desired position in the risky asset for each realization of the equilibrium price.<sup>9</sup> Assume that  $v$  and  $\epsilon_{i1}$  are independent for all  $i$ , and that error terms are also independent across traders. Noise traders submit a random demand  $u_1$  (independent of all other random variables in the model), where  $u_1 \sim N(0, \tau_u^{-1})$ . Finally, we make the convention that, given  $v$ , the average signal  $\int_0^1 s_{i1} di$  equals  $v$  almost surely (i.e. errors cancel out in the aggregate:  $\int_0^1 \epsilon_{i1} di = 0$ ).<sup>10</sup>

In this paper we will use the above CARA-normal framework to investigate conditions under which the equilibrium price is farther away from the fundamentals than investors' average expectations. Similarly as in Allen et al. (2006) this occurs whenever

$$|E[p_1 - v|v]| > \left| E \left[ \int_0^1 E[v|s_{i1}, p_1] di - v|v \right] \right|. \quad (1)$$

The above condition holds if the bias of prices in the estimation of  $v$  is larger than the bias of investors' average expectations or, equivalently, if and only if traders assign extra weight to public information in relation to the optimal statistical weight in the estimation of  $v$ . Indeed, at a linear symmetric equilibrium for a given private signal responsiveness  $a_1 > 0$  the equilibrium price can be expressed as

$$p_1 = \alpha_{P1} \left( v + \frac{1}{a_1} u_1 \right) + (1 - \alpha_{P1}) E[v|z_1],$$

where  $\alpha_{P1} = a_1 / (\gamma(\tau_1 + \tau_{\epsilon_1}))$ ,  $\tau_1 \equiv \text{Var}[v|p_1]^{-1} = \tau_v + a_1^2 \tau_u$ , and  $z_1 = a_1 v + u_1$  is informationally equivalent to the price. It thus follows that

$$p_1 - v = (1 - \alpha_{P1})(E[v|z_1] - v) + \alpha_{P1} \frac{1}{a_1} u_1.$$

Owing to normality, on the other hand, one can immediately verify that

$$E[v|s_{i1}, p_1] = \alpha_{E1} s_{i1} + (1 - \alpha_{E1}) E[v|p_1],$$

where  $\alpha_{E1} = \tau_{\epsilon_1} / (\tau_1 + \tau_{\epsilon_1})$  denotes the optimal statistical weight to private information. Noting that  $E[v|p_1] = E[v|z_1]$ , we have

$$\bar{E}_1[v] = \alpha_{E1} v + (1 - \alpha_{E1}) E[v|z_1],$$

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<sup>8</sup>We assume, without loss of generality with CARA preferences, that the non-random endowment of traders is zero.

<sup>9</sup>The unique equilibrium in linear strategies of this model is symmetric.

<sup>10</sup>See Section 3.1 in the Technical Appendix of Vives (2007) for a justification of the convention.

where  $\bar{E}_1[v] = \int_0^1 E[v|s_{i1}, p_1] di$ , using our convention  $\int_0^1 s_{i1} di = v$ . Since

$$E[\bar{E}_1[v] - v|v] = (1 - \alpha_{E1})(E[E[v|z_1]|v] - v),$$

and

$$E[p_1 - v|v] = (1 - \alpha_{P1})(E[E[v|z_1]|v] - v),$$

condition (1) holds if and only if the equilibrium price displays over-reliance on public information (in relation to the optimal statistical weight):

$$\alpha_{P1} < \alpha_{E1} \Leftrightarrow a_1 < \gamma\tau_{\epsilon_1}, \quad (2)$$

where the latter equivalence follows immediately from the definitions of  $\alpha_{P1}$  and  $\alpha_{E1}$ .

In the static model it is well known that a unique equilibrium in linear strategies exists in the class of equilibria with price functional of the form  $P_1(v, u_1)$  (see e.g. Admati (1985), Vives (2007)). The equilibrium strategy of a trader  $i$  and the price are given by

$$X_1(s_{i1}, p_1) = \gamma \text{Var}[v|s_{i1}, p_1]^{-1} (E[v|s_{i1}, p_1] - p_1),$$

$$p_1 = \lambda_1 z_1 + (1 - \lambda_1 a_1) \bar{v},$$

with  $z_1 = a_1 v + u_1$ ,  $a_1 = \gamma\tau_{\epsilon_1}$ ,  $\lambda_1 = (\gamma(\tau_1 + \tau_{\epsilon_1}))^{-1}(1 + \gamma\tau_u a_1)$ , and  $\tau_1 \equiv \text{Var}[v|p_1]^{-1} = \tau_v + a_1^2 \tau_u$ . The responsiveness of the strategy of a trader to its signal is given by  $a_1$ , which determines the precision of the price  $\tau_1$  as an estimator of the fundamental value  $v$ . The depth of the market is measured as usual by  $\lambda_1^{-1}$ . Since in equilibrium  $a_1 = \gamma\tau_{\epsilon_1}$ , we can thus conclude that in a static setup, condition (2) can never be satisfied, and the equilibrium price always assigns the optimal statistical weight to public information.<sup>11</sup>

**Remark 1.** There is an alternative, more direct way to verify whether condition (1) is satisfied. Indeed, as traders' aggregate demand is proportional to  $\int_0^1 (E[v|s_{i1}, p_1] - p_1) di$ , imposing market clearing in the above model yields

$$\int_0^1 x_{i1} di + u_1 = \int_0^1 \gamma(\tau_1 + \tau_{\epsilon_1})(E[v|s_{i1}, p_1] - p_1) di + u_1 = 0,$$

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<sup>11</sup>If  $E[u_1]$  is non null, e.g. if  $E[u_1] = \bar{u}_1 > 0$ , we have to replace the price  $p_1$  by the price net of the expected noise component  $\hat{p}_1 = p_1 - \bar{u}_1 \text{Var}[v|s_{i1}, z_1]/\gamma$ . Using this definition it is immediate to verify that also when  $\bar{u}_1 > 0$ , in a static market the equilibrium price assigns the optimal statistical weight to public information.

and solving for the equilibrium price we obtain

$$p_1 = \bar{E}_1[v] + \frac{u_1}{\gamma(\tau_1 + \tau_{\epsilon_1})}. \quad (3)$$

In other words, in equilibrium the price is given by the sum of traders' average expectations and noise (times a constant). As  $u_1$  and  $v$  are by assumption orthogonal, we can therefore conclude that in a static setup the price assigns the optimal statistical weight to public information. To obtain over-reliance on public information, in other words, we need to find conditions under which traders' aggregate demand is no longer proportional to  $\bar{E}_1[v] - p_1$  and this, in a static context with CARA preferences can never happen.

In the following sections we will argue that price over-reliance on public information can be traced to traders' speculative activity on short-run price movements that makes strategies depart from the solution of the static setup.

### 3 A Dynamic Market with Long-Term Traders

Consider now a dynamic extension of the previous model where risk averse traders and noise traders exchange both the risky and the riskless asset during  $N \geq 2$  periods.<sup>12</sup> In period  $N + 1$  the risky asset is liquidated. As before, speculators have CARA preferences and maximize the expected utility of their final wealth:  $W_{iN} = \sum_{n=1}^N \pi_{in} = \sum_{n=1}^{N-1} (p_{n+1} - p_n)x_{in} + (v - p_N)x_{iN}$ . In any period  $n$  an informed trader  $i$  receives a signal  $s_{in} = v + \epsilon_{in}$ , where  $\epsilon_{in} \sim N(0, \tau_{\epsilon_n}^{-1})$ ,  $v$  and  $\epsilon_{in}$  are independent for all  $i, n$  and error terms are also independent both across time periods and traders.

In the general case noise trading follows an AR(1) process  $\theta_n = \beta\theta_{n-1} + u_n$  with  $\beta \in [0, 1]$  and  $\{u_n\}_{n=1}^N$  an i.i.d. normally distributed random process (independent of all other random variables in the model) with  $u_n \sim N(0, \tau_u^{-1})$ . If  $\beta = 1$ ,  $\{\theta_n\}$  follows a random walk and we are in the usual case of independent noise trade increments  $u_n = \theta_n - \theta_{n-1}$  (e.g. Kyle (1985), Vives (1995)). If  $\beta = 0$ , then noise trading is i.i.d. across periods (this is the case considered by Allen et al. (2006)), a plausible assumption only if the time between trading dates is very large. Finally, assume as before that, given  $v$ ,  $\int_0^1 s_{in} di = v$  (a.s.) in any period  $n$ .

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<sup>12</sup>A number of authors have studied competitive, noisy rational expectations equilibria in dynamic markets, see e.g. Brown and Jennings (1989), Grundy and McNichols (1989), He and Wang (1995), Vives (1995), Cespa (2002), and Chapters 8 and 9 in Vives (2007).

### 3.1 A Random Walk Process for Noise Trading

Let us first consider the classical case when noise trade increments  $u_n$  are independent across periods ( $\beta = 1$ ). In any period  $1 < n \leq N$  each informed trader has the vector of private signals  $s_i^n = \{s_{i1}, s_{i2}, \dots, s_{in}\}$  available. It follows from normal theory that the statistic  $\tilde{s}_{in} = (\sum_{t=1}^n \tau_{\epsilon t})^{-1} (\sum_{t=1}^n \tau_{\epsilon t} s_{it})$  is sufficient for the sequence  $s_i^n$  in the estimation of  $v$ . An informed trader  $i$  in period  $n$  submits a limit order  $X_{in}(\tilde{s}_{in}, p^{n-1}, \cdot)$ , indicating the position desired at every price  $p_n$ , contingent on his available information. We will restrict attention to linear equilibria where in every period  $n$  a speculator trades according to  $X_n(\tilde{s}_{in}, p^n) = a_n \tilde{s}_{in} - \varphi_n(p^n)$ , where  $\varphi_n(\cdot)$  is a linear function of the price sequence  $p^n = \{p_1, \dots, p_n\}$ .<sup>13</sup> Let us denote with  $z_n$  the intercept of the  $n$ -th period net aggregate demand  $\int_0^1 \Delta x_{in} di + u_n$ , where  $\Delta x_{in} = x_{in} - x_{in-1}$ . The random variable  $z_n \equiv \Delta a_n v + u_n$  represents the informational addition brought about by the  $n$ -th period trading round, and can thus be interpreted as the informational content of the  $n$ -th period order-flow.

**Lemma 1.** *In any linear equilibrium the sequences  $z^n$  and  $p^n$  are observationally equivalent.*

According to the above lemma in any period  $n$  traders form their estimation of the liquidation value using their private information, summarized by the statistic  $\tilde{s}_{in}$ , and the sequence of public informational additions  $z^n$ . Thus, owing to normality, we have  $E[v|\tilde{s}_{in}, z^n] = (\tau_n + \sum_{t=1}^n \tau_{\epsilon t})^{-1} (\tau_n E[v|z^n] + \sum_{t=1}^n \tau_{\epsilon t} \tilde{s}_{in})$ , where  $E[v|z^n] = \tau_n^{-1} (\tau_v \bar{v} + \tau_u \sum_{t=1}^n \Delta a_t z_t)$ , and  $\tau_n \equiv \text{Var}[v|z^n]^{-1} = \tau_v + \tau_u \sum_{t=1}^n \Delta a_t^2$ .

**Proposition 1.** *In the market with long-term, informed speculators there exists a unique equilibrium in linear strategies. The equilibrium is symmetric. Prices are given by  $p_o = \bar{v}$ ,  $p_{N+1} = v$ , and for  $n = 1, 2, \dots, N$ ,  $p_n = \lambda_n z_n + (1 - \lambda_n \Delta a_n) p_{n-1}$  and strategies are given by:*

$$\begin{aligned} X_n(\tilde{s}_{in}, p^n) &= \gamma \left( \tau_n + \sum_{t=1}^n \tau_{\epsilon t} \right) (E[v|\tilde{s}_{in}, z^n] - p_n) \\ &= a_n (\tilde{s}_{in} - p_n) + \gamma \tau_n (E[v|z^n] - p_n), \end{aligned} \quad (4)$$

where  $a_n = \gamma (\sum_{t=1}^n \tau_{\epsilon t})$ ,  $\Delta a_n \equiv a_n - a_{n-1} = \gamma \tau_{\epsilon n}$ ,  $z_n = \Delta a_n v + u_n$ ,  $z^n = \{z_t\}_{t=1}^n$ ,  $\tau_n = (\text{Var}[v|z^n])^{-1}$ , and  $\lambda_n = (1 + \gamma \tau_u \Delta a_n) / (\gamma (\tau_n + \sum_{t=1}^n \tau_{\epsilon t}))$ .

<sup>13</sup>The unique equilibrium in linear strategies will be shown to be symmetric.

*Proof.* See appendix A.

QED

The equilibrium has a static nature: in every period  $n$  speculators trade “as if” the asset would be liquidated in the following period  $n + 1$ , exploiting all their available information.<sup>14</sup> The intuition is as follows: owing to normality and CARA preferences a trader’s strategy is made of two components:

$$x_{in} = \gamma \frac{E[p_{n+1} - p_n | \tilde{s}_{in}, z^n]}{h_{22}} - \frac{h_{21} E[x_{in+1} | \tilde{s}_{in}, z^n]}{h_{22} \gamma (\tau_{n+1} + \sum_{t=1}^{n+1} \tau_{ct})}, \quad 1 \leq n < N, \quad (5)$$

where  $h_{21} < 0$  and  $h_{22} > 0$  are constants defined in the appendix. The first component in (5) accounts for the forecasted price change, while the second component captures the anticipated future position. Were traders not to expect a change in prices, then their optimal period  $n$  position would be like the one of a static market, and the risk of holding such a position would only be due to the unpredictability of the liquidation value (taking into account risk-aversion):  $(1/\gamma)\text{Var}[v | \tilde{s}_{in}, z^n]$ .<sup>15</sup> If, on the other hand a change in prices is expected, traders optimally exploit short-run price differences. This, in turn, adds two factors to the risk of their period  $n$  position, as traders also suffer from the partial unpredictability of the price change, *and* from the impossibility of determining exactly their future position. However, the opportunity to trade again in the future also grants a hedge against potentially adverse price movements ( $h_{21} < 0$ ). This, in equilibrium, yields a risk-reduction that exactly offsets the risk increase outlined above. As a consequence, even in this case traders’ strategies have a static nature, and  $x_{in} = E[x_{in+1} | \tilde{s}_{in}, z^n]$ .

As one would expect, under the conditions of Proposition 1, it is possible to show that the price assigns the optimal statistical weight to public information *in every* period  $n = 1, 2, \dots, N$ . Indeed, owing to normality, we can express informed traders’ average expectation as a linear combination of the liquidation value  $v$  and the public expectation  $E[v | z^n]$ :

$$\begin{aligned} \bar{E}_n[v] &= \int_0^1 E[v | \tilde{s}_{in}, z^n] di \\ &= \alpha_{En} v + (1 - \alpha_{En}) E[v | z^n], \end{aligned} \quad (6)$$

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<sup>14</sup>Proposition 1 generalizes the equilibrium in Cespa (2002) to the case of  $N \geq 2$  periods.

<sup>15</sup>Intuitively, if given today’s information the asset price is not expected to change, no new private information is expected to arrive to the market and the model collapses to one in which traders hold for two periods the risky asset. Their position, then, naturally coincides with the one they would hold in a static market.



where  $\alpha_{En} \equiv \sum_{t=1}^n \tau_{\epsilon_t} / (\tau_n + \sum_{t=1}^n \tau_{\epsilon_t})$ , for  $n = 1, 2, \dots, N$ . A similar decomposition can also be applied to the equilibrium price for a given responsiveness to private information  $a_n$ :

$$p_n = \alpha_{Pn} \left( v + \frac{1}{a_n} \theta_n \right) + (1 - \alpha_{Pn}) E[v|z^n], \quad (7)$$

where  $\alpha_{Pn} = a_n / (\gamma(\tau_n + \sum_{t=1}^n \tau_{\epsilon_t}))$ . Comparing (6) and (7), as done in Section 2 one can see that in any period  $n$  the equilibrium price displays over-reliance on public information whenever

$$\alpha_{Pn} < \alpha_{En}. \quad (8)$$

However, in equilibrium we have that  $\alpha_{Pn} = \sum_{t=1}^n \tau_{\epsilon_t} / (\tau_n + \sum_{t=1}^n \tau_{\epsilon_t}) = \alpha_{En}$ . Thus, we can conclude:

**Corollary 1.** *In any period  $1 \leq n \leq N$ ,  $E[p_n - v|v] = E[\bar{E}_n[v] - v|v]$ .*

In words: when the market is populated by long-term traders and noise traders' supply shocks follow a random walk, informed traders' demand has a *static* nature, traders assign the optimal statistical weight to public information, and the bias of prices in the estimation of  $v$  is the same as the bias of average expectations. Note that the static nature of demand implies that traders assign the optimal statistical weight to public information in the estimation of  $v$ . This is so since in this case traders are only concerned with the final liquidation value of the asset.

**Remark 2.** As in section 2, a more direct proof of corollary 1 is available. Indeed, market clearing at any period  $n$  is given by

$$\int_0^1 x_{in} di + \theta_n = 0.$$

When  $\beta = 1$  we have that  $\theta_n = \sum_{t=1}^n u_t$  and according to proposition 1, in this case in any period  $n$  long-term traders' aggregate demand is proportional to  $\int_0^1 E[v|\tilde{s}_{in}, z^n] di - p_n$ . Hence, at equilibrium we obtain:

$$\int_0^1 \gamma \left( \tau_n + \sum_{t=1}^n \tau_{\epsilon_t} \right) (E[v|\tilde{s}_{in}, z^n] - p_n) di + \theta_n = 0. \quad (9)$$

Using this condition, we can now express the price as the sum of traders' average expectations and a risk-weighted noise adjustment:

$$p_n = \bar{E}_n[v] + \frac{\theta_n}{\gamma(\tau_n + \sum_{t=1}^n \tau_{\epsilon_t})}. \quad (10)$$

As  $\theta_n$  and  $v$  are by assumption orthogonal, for all  $t = 1, 2, \dots, n$ ,  $E[p_n|v] = E[\bar{E}_n[v]|v]$ , and the result follows.

**Remark 3.** According to (7), and due to traders' risk aversion, in any period the equilibrium price differs from the conditional expectation of the liquidation value given public information:

$$p_n - E[v|z^n] = \frac{\alpha_{Pn}}{a_n} E[\theta_n|z^n].$$

The intuition is as follows: suppose, without loss of generality, that in period  $n$   $E[\theta_n|z^n] > 0$ , then if  $p_n = E[v|z^n]$ , traders would be exposed to the risk of holding an underpriced short position.<sup>16</sup> In order to accept holding such a position, (i.e. to offset such a risk) risk averse traders thus demand a compensation that is proportional to the expected total noisy noise trade in period  $n$ . Such a compensation is larger, the lower is their risk-tolerance and the larger is their uncertainty over the liquidation value:  $\alpha_{Pn}/a_n = 1/(\gamma(\tau_n + \sum_{t=1}^n \tau_{\epsilon_t}))$ .<sup>17</sup>

**Remark 4.** Vives (1995) studies a market with the same informational structure of the one analyzed above with  $\beta = 1$ , but where the asset is priced by a competitive sector of risk-neutral market makers. In this case price setting agents require no premium to hold a risky position (beyond what is due to the severity of the adverse selection problem they face), thus  $\alpha_{Pn} = 0$  and prices are *always* "farther away" from fundamentals than the average expectation among informed traders. The discrepancy between prices and average expectation is larger, the larger is the ratio between private and public precision. This is immediate since in this case the unique equilibrium in linear strategies shares the same informational properties of the one in proposition 1, while the period  $n$  price is given by  $p_n = E[v|z^n]$ . Hence

$$E[p_n - v|v] = \tau_n^{-1} \tau_v (\bar{v} - v) \text{ and } E[\bar{E}_n[v] - v|v] = \frac{\tau_v (\bar{v} - v)}{\tau_n + \sum_{t=1}^n \tau_{\epsilon_t}},$$

and the discrepancy increases in the ratio  $(\tau_n + \sum_{t=1}^n \tau_{\epsilon_t})/\tau_n \equiv \text{Var}[v|\tilde{s}_{in}, z^n]^{-1}/\text{Var}[v|z^n]^{-1}$ . Redefining the benchmark to which prices are compared in terms of the risk-weighted average expectation among *all* the traders in the market, we can show that the equivalence result of corollary 1 is restored. Indeed, suppose that together with the informed traders in the market there also exists a continuum of risk averse (CARA) uninformed traders. In every period  $n$ , the uninformed traders' estimation of the liquidation value

<sup>16</sup>In other words, they would be exposed to the risk of selling the asset for too low a price with respect to its actual liquidation value.

<sup>17</sup>The existence of a premium that affects the equilibrium price in models where risk averse traders price the asset is well-known in the literature (see e.g. Battacharya and Spiegel (1991)).

is given by  $E[v|z^n]$ , and the risk-weighted average expectation among all traders in the market reads as follows:

$$\bar{E}_n[v] = \frac{\int_0^1 \gamma(\tau_n + \sum_{t=1}^n \tau_{\epsilon_t}) E[v|\tilde{s}_{in}, z^n] di + \int_0^1 \gamma^U \tau_n E[v|z^n] dj}{\gamma(\tau_n + \sum_{t=1}^n \tau_{\epsilon_t}) + \gamma^U \tau_n},$$

where  $\gamma^U$  denotes the uninformed traders' risk-tolerance coefficient. Letting  $\gamma^U \rightarrow \infty$  we then obtain that when dealers are risk neutral  $\bar{E}_n[v]$  converges to the equilibrium price  $p_n = E[v|z^n]$ , and in the limit  $E[p_n - v|v] = E[\bar{E}_n[v] - v|v]$ .

### 3.2 Residual Uncertainty and Correlated noise trade Increments: Keynes vs. Hayek

In this section we suppose (i) that the fundamentals are affected by the presence of residual uncertainty, and (ii) that noise trade shocks are serially correlated. Both the presence of residual uncertainty over the liquidation value and noise traders' predictability lead traders to speculate on short-run price movements. This drives a wedge between the equilibrium price and average expectations about the fundamentals which, depending on parameters values, yields either over- or under-reliance on public information.

Assume thus that the liquidation value traders receive at the terminal date is given by  $v + \delta$ , where  $\delta \sim N(0, \tau_\delta^{-1})$  is a noise term independent from all the other random variables in the economy and about which *no* trader possesses private information, and that  $\beta \in [0, 1]$ . The latter assumption implies that noise trade increments are *negatively* correlated, i.e. that a positive (negative) realization of  $\theta_1$  is likely to be followed by a negative (positive) realization of  $\Delta\theta_2$ .<sup>18</sup> Other things equal, this implies that traders find it less risky to accommodate order imbalances in the first period.

Defining the net informational additions  $z_1 = a_1 v + u_1$ , and  $z_2 = \Delta a_2 v + u_2$  ( $\Delta a_2 = a_2 - \beta a_1$ ), in a two-period model we can prove the following result:

**Proposition 2.** *In the 2-period market with long-term, informed speculators, residual uncertainty over the liquidation value, and correlated noise trade increments, equilibria in linear strategies exist. In any linear equilibrium, (1) prices are given by  $p_3 = v$ ,  $p_0 = \bar{v}$ ,  $p_2 = \lambda_2 z_2 + (1 - \lambda_2 \Delta a_2) \hat{p}_1$ , and  $p_1 = \lambda_1 z_1 + (1 - \lambda_1 a_1) \bar{v}$ ; (2) strategies*

<sup>18</sup>To be sure: as  $\theta_2 = u_2 + \beta\theta_1$ ,  $\text{Cov}[\Delta\theta_2, \theta_1] = (\beta - 1)\tau_u^{-1} \leq 0$ , since  $\beta \in [0, 1]$ .

are given by:

$$\begin{aligned} X_2(\tilde{s}_{i2}, p^2) &= a_2(\tilde{s}_{i2} - p_2) + b_2(E[v|z^2] - p_2), \\ X_1(s_{i1}, p_1) &= \frac{a_1(\tau_1 + \tau_{\epsilon_1})}{\tau_{\epsilon_1}}(E[v|s_{i1}, z_1] - p_1) + \frac{(\gamma + h_{21})(\beta\rho - 1)\tau_1}{\gamma(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})}E[\theta_1|z_1], \end{aligned} \quad (11)$$

where  $b_2 = \gamma\tau_2/(1 + \kappa)$ ,  $\lambda_2 = (\gamma(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t}))^{-1}(1 + \gamma\tau_u\Delta a_2 + \kappa)$ ,  $\hat{p}_1 = (\gamma\tau_1 + \beta a_1(1 + \kappa))^{-1}(\gamma\tau_1 E[v|z_1] + \beta(1 + \kappa)z_1)$ ,  $\kappa \equiv (\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})/\tau_\delta$ ,  $\Delta a_2 = a_2 - \beta a_1$ ,  $\rho \equiv a_1(1 + \kappa)/(\gamma\tau_{\epsilon_1})$ ,

$$a_1 = \frac{\gamma\tau_{\epsilon_1}(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})(1 + \kappa)(1 + \gamma\tau_u\Delta a_2)}{(\tau_{\epsilon_2} + (\tau_2 + \tau_{\epsilon_1})(1 + \kappa))(1 + \kappa + \gamma\tau_u\Delta a_2)}, \quad a_2 = \frac{\gamma(\tau_{\epsilon_1} + \tau_{\epsilon_2})}{1 + \kappa}, \quad (12)$$

and

$$\lambda_1 = \alpha_{P1} \frac{1}{a_1} + (1 - \alpha_{P1}) \frac{a_1\tau_u}{\tau_1}, \quad \alpha_{P1} = \frac{\tau_{\epsilon_1}}{\tau_1 + \tau_{\epsilon_1}} + \frac{(\gamma + h_{21})(\beta\rho - 1)\tau_1\tau_{\epsilon_1}}{\gamma(\tau_1 + \tau_{\epsilon_1})(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})},$$

$h_{21}$  (with  $\gamma + h_{21} > 0$ ),  $h_{22} > 0$  are constants defined in the appendix. Furthermore, in equilibrium  $\rho > 1$ .

*Proof.* See appendix A. QED

Due to residual uncertainty, a closed form solution for traders' signal responsiveness is no longer available: the expressions for  $a_1$  and  $a_2$  in (12) constitute a system of non-linear equations whose (potentially multiple) solution(s) must be numerically determined. According to (11) in the second period traders' strategies keep the static property. As a consequence, the price assigns the optimal statistical weight to public information and the bias of the price in the estimation of  $v$  coincides with that of investors' average expectations.<sup>19</sup> However, owing to the residual uncertainty affecting the liquidation value, traders scale down their aggressiveness:  $a_2 < \gamma(\tau_{\epsilon_1} + \tau_{\epsilon_2})$ . In the first period, on the other hand, strategies lose the static property and each investor's position is no longer proportional to  $(E[v|s_{i1}, z_1] - p_1)$ .

Indeed, according to the second of (11), the first period strategy of a trader can be decomposed in two parts. The first part captures the "static" component, while the second part captures a trader's speculative activity on short-run price movements, a term whose sign depends on the magnitude of  $\beta\rho$ . As argued above, for a given expected noise trade shock low values of  $\beta$  encourage informed traders to take the

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<sup>19</sup>Using the expressions provided in the appendix this result is immediate.

other side of the market counting on the noise trade increment future reversion. The parameter  $\rho$  measures the deviation that residual uncertainty induces in traders' first period signal response with respect to the static aggressiveness (note that  $\rho$  depends on  $\beta$ ).<sup>20</sup> In any equilibrium of the market it turns out that  $\rho > 1$ , implying that when residual uncertainty affects the liquidation value, in the first period traders speculate more aggressively on their signal. The intuition is as follows: using the expressions provided in the appendix, we can rewrite a trader's first period signal responsiveness in the following way:

$$a_1 = \frac{\gamma\tau_{\epsilon_1}}{1 + \kappa} \left( \frac{\text{Var}_2[v + \delta] \hat{\lambda}_2}{\text{Var}_2[v]} \frac{\gamma^2(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})}{\lambda_2 (\tau_2 + \tau_{\epsilon_1})(\gamma^2 + \text{Var}_2[v + \delta]\text{Var}_1[x_{i2}](1 - \rho_{1,\{x_{i2}, p_2\}}^2))} \right), \quad (13)$$

where  $\text{Var}_n[Y] = \text{Var}[Y|\tilde{s}_{in}, z^n]$ ,  $\hat{\lambda}_2 = (\gamma(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t}))^{-1}(1 + \gamma\tau_u\Delta a_2)$  denotes the reciprocal of second period market depth in the absence of residual uncertainty, and

$$\rho_{1,\{x_{i2}, p_2\}} = \frac{\text{Cov}[x_{i2}, p_2|s_{i1}, z_1]}{\sqrt{\text{Var}_1[x_{i2}]\text{Var}_1[p_2]}}.$$

The presence of residual uncertainty over the liquidation value produces three effects on a trader's signal responsiveness. First, as  $\text{Var}_2[v]$  increases to  $\text{Var}_2[v + \delta]$ , the first period signal becomes a more valuable source of information, and traders put more weight on it. However, at the same time since the liquidation value is more uncertain, traders in the second period speculate less aggressively, and  $p_2$  becomes more reactive to the upcoming net informational addition  $z_2$  ( $\hat{\lambda}_2$  increases to  $\lambda_2$ ). As traders in the first period also speculate on short-run price differences, this drives down  $a_1$ . Finally, as traders scale down their second period signal aggressiveness, their second period strategy becomes less responsive to private information, and thus more correlated with the second period price.<sup>21</sup> From the point of view of a trader that in the first

<sup>20</sup>In the second period, when short-run price differences cannot be exploited, a trader weights his private information according to  $a_2 = (1 + \kappa)^{-1}\gamma(\sum_{t=1}^2 \tau_{\epsilon_t})$ . In the first period a trader that intends to "buy-and-hold" should trade according to  $(1 + \kappa)^{-1}\gamma\tau_{\epsilon_1}$ . This intuition can be proved formally by using the expressions for a trader's first period strategy provided in the appendix. Indeed, assuming  $\beta = 1$ , plugging  $a_1 = \gamma\tau_{\epsilon_1}/(1 + \kappa)$  in (37) and imposing market clearing, we obtain  $\hat{p}_1 = p_1$ , and  $x_{i1} = E[x_{i2}|s_{i1}, p_1]$  as in the case with no residual uncertainty.

<sup>21</sup>Suppose a trader expects  $p_2 > p_1$ . Then, he increases his first period long position, planning to sell in the second period, and net the short-run profits. However, if  $p_2 < p_1$ , second period sales are suboptimal, and a good hedge calls for holding on to the long position. This, however, is less likely to occur if traders are very reactive to their second period information. In this case, indeed, the second period strategy may unravel the hedge built in the first period, in turn reducing traders' willingness to speculate on their first period information.

period speculates on price differences, this makes  $x_{i2}$  a better hedge against adverse price movements, and has a positive impact on  $a_1$ . The sum of the positive effects more than offset the negative one and  $\rho > 1$ .<sup>22</sup>

As first period signal responsiveness exceeds its static value, accommodating the expected noise trade shock may expose traders to the risk of clearing information-related transactions at an incorrect price. To see this consider the following example. Suppose that conditional on the observation of  $z_1$ , the market expects  $\theta_1 > 0$ , i.e.

$$E[\theta_1|z_1] = \theta_1 + a_1(v - E[v|z_1]) > 0.$$

According to the above expression,  $E[\theta_1|z_1] > 0$  could be due either to a truly positive liquidity shock ( $\theta_1 > 0$ ) or to a liquidation value higher than its market expectation ( $v > E[v|z_1]$ ). In the former case, traders should accommodate the shock, counting on a reversion of the noise trade increment that lowers the risk of holding an unbalanced position in the asset in the first period. However, the higher is  $\beta$ , the lower are the chances that noise trade increments compensate over time and the higher is the possibility that traders sell the asset for too low a price (as  $\rho > 1$ ). Hence, as  $\beta\rho$  increases above 1, traders speculate on a further price increase in the second period, increasing their long position in the asset.<sup>23</sup>

Summarizing, the presence of residual uncertainty and correlated noise trade increments leads traders to speculate on short-run price differences. This, in turn, makes traders' first period strategies depart from the static form of the previous section, implying that the equilibrium price in the first period can no longer be expressed as the sum of traders' average expectations and a risk-weighted noise component. In

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<sup>22</sup>Higher uncertainty over the liquidation value, thus works as a commitment device for a trader's second period strategy. If in the first period the trader could be sure that his second period strategy perfectly hedged adverse price movements, he would more confidently speculate on price differences. However, traders cannot control the information they receive in the second period, and this makes  $x_{i2}$  a less reliable hedge for a short-run, speculative strategy. In this perspective higher uncertainty over the liquidation value ties a trader's hands in the second period: the trader speculates less aggressively on private information and he can also be more confident that  $x_{i2}$  will better serve his first period hedging needs.

<sup>23</sup>These two types of behavior are reminiscent of the "contrarian" vs. "momentum" type of strategies. Traders' behavior when  $\beta\rho > 1$  is also akin to the "resale option" strategy discussed by Cao and Ou-Yang (2005).

equilibrium we have that

$$\begin{aligned}
p_1 &= \alpha_{P1} \left( v + \frac{\theta_1}{a_1} \right) + (1 - \alpha_{P1}) E[v|z_1] \\
&= \bar{E}_1[v] + \left( \frac{\alpha_{P1} - \alpha_{E1}}{a_1} \right) E[\theta_1|z_1] + \alpha_{E1} \frac{\theta_1}{a_1} \\
&= \bar{E}_1[v] + \frac{(\gamma + h_{21})(\beta\rho - 1)\tau_1\tau_{\epsilon_1}}{a_1\gamma(\tau_1 + \tau_{\epsilon_1})(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})} E[\theta_1|z_1] + \alpha_{E1} \frac{\theta_1}{a_1},
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
\alpha_{P1} &\equiv \frac{\tau_{\epsilon_1}}{\tau_1 + \tau_{\epsilon_1}} + \frac{(\gamma + h_{21})(\beta\rho - 1)\tau_1\tau_{\epsilon_1}}{\gamma(\tau_1 + \tau_{\epsilon_1})(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})} \\
&= \alpha_{E1} + \frac{(\gamma + h_{21})(\beta\rho - 1)\tau_1\tau_{\epsilon_1}}{\gamma(\tau_1 + \tau_{\epsilon_1})(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})}.
\end{aligned} \tag{15}$$

As a consequence the price may display over- or under-reliance on public information. Indeed, according to (15) the weight the price assigns to the fundamentals is the sum of the optimal statistical weight that the average expectation attributes to aggregate private information (i.e.  $\alpha_{E1}$ ), and a term that is proportional to traders' short-run speculative trading intensity. When traders deem their estimation of the noise trade shock to be biased by the presence of information driven trades (i.e. when  $\beta\rho > 1$ ), they “side” with the market (e.g. if  $E[\theta_1|z_1] > 0$ , speculating on a further price increase). This reinforces the optimal statistical weight  $\alpha_{E1}$  with the short-run speculative trading intensity, moving the price *closer* to the fundamentals in relation to average expectations. When, on the other hand, traders count on a reversion of the second period noise trade increment (i.e. when  $\beta\rho < 1$ ), they take the other side of the market (e.g. if  $E[\theta_1|z_1] > 0$ , they absorb the order imbalance), in this way partially *sterilizing* the impact of informed trades on the price. This reduces the optimal statistical weight  $\alpha_{E1}$  and widens the distance between  $p_1$  and  $v$  in relation to average expectations.<sup>24</sup> When  $\beta\rho = 1$  the trading strategy in the first period has a static nature and there is no speculation on short-run price movements.

We thus have :

**Corollary 2.** *In any linear equilibrium of the market with long-term, informed speculators when the liquidation value is affected by residual noise the price displays over-reliance on public information if and only if  $\beta\rho < 1$ .*

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<sup>24</sup>To be sure, when  $\beta\rho < 1$  traders on aggregate hedge their informationally driven *speculative* activity with their *market making* activity. This dampens the effect that the former has on the equilibrium price.

Corollary 2 has a straightforward graphical interpretation. To see this suppose that  $E[E[\theta_1|z_1]|v] > 0$  ( $E[E[\theta_1|z_1]|v] < 0$ ), then if  $\beta\rho < 1$  we have that  $v > E[\bar{E}_1[v]|v] > E[p_1|v]$  ( $v < E[\bar{E}_1[v]|v] < E[p_1|v]$ ). Conversely, if  $\beta\rho > 1$  and  $E[E[\theta_1|z_1]|v] > 0$  ( $E[E[\theta_1|z_1]|v] < 0$ ), we have that  $v > E[p_1|v] > E[\bar{E}_1[v]|v]$  ( $v < E[p_1|v] < E[\bar{E}_1[v]|v]$ ), the price is more firmly tightened to the fundamentals (see figure 1).

[Figure 1 about here.]

As it happens, when  $\beta = 0$  ( $\beta = 1$ ), over- (under-) reliance on public information occurs:

**Corollary 3.** *In any linear equilibrium of the market with long-term, informed speculators when the liquidation value is affected by residual noise, if  $\beta = 0$  ( $\beta = 1$ ) over- (under-) reliance on public information arises.*

*Proof* When  $\beta = 1$ , the condition for under-reliance on public information is that  $\rho > 1$ , which holds always given proposition 2. When, on the other hand  $\beta = 0$ , according to (14) there is always over-reliance on public information. QED

On the other hand if *no* residual uncertainty affects the liquidation value, over-reliance on public information always occurs.

**Corollary 4.** *In any linear equilibrium of the market with long-term, informed speculators when the liquidation value is not affected by residual noise (i.e.  $1/\tau_\delta = 0$ ) there always is over-reliance on public information, for any  $\beta \in [0, 1)$ .*

*Proof* If  $1/\tau_\delta = 0$ ,  $\rho = 1$ , and according to (14) the condition for over-reliance on public information is always satisfied for all  $\beta \in [0, 1)$ . QED

When  $\beta = 0$  due to the noise trade increment mean reverting property, traders in the first period take an important short-run speculative position and absorb the expected liquidity shock at a *discount*. This implies over-reliance on public information. As  $\beta$  approaches 1, traders can decreasingly count on a noise trade increment reversal, and reduce their short-run speculative position. When  $\beta = 1$ , the noise process follows a random walk and the result of corollary 1 is restored.

Numerical simulations show that for any  $1/\tau_\delta > 0$ , there always exists a value of  $\beta$ ,  $\hat{\beta}$  below (above) which only over- (under-) reliance on public information occurs. This allows to define  $\Omega \equiv \{(\beta, 1/\tau_\delta) \in [0, 1] \times \mathfrak{R}_+ | \beta\rho(\beta) = 1\}$  as the set of pairs  $(\beta, 1/\tau_\delta)$



for which the price assigns the optimal statistical weight to public information and first period strategies have a static nature. Values of  $(\beta, 1/\tau_\delta)$  that fall below (above) this locus identify market conditions that yield over- or under-reliance on public information (see figure 2).

[Figure 2 about here.]

The figure divides the parameter space  $(\beta, 1/\tau_\delta)$  into a Keynesian region (below the locus) with over-reliance on public information and a Hayekian region (the rest) where the opposite occurs. Less residual uncertainty and less correlation in noise trading move us towards the Keynesian region. It is interesting to observe also that the Keynesian region gets larger as risk tolerance  $\gamma$  decreases.

**Remark 5.** Given the discussion following proposition 2 we should expect the weight the price assigns to public information and the depth of the first period market to be related. Indeed if  $\tau_\delta$  grows unboundedly it is easy to show that  $\lambda_1|_{\beta=0} < \lambda_1|_{\beta=1}$ . To see this, suppose  $1/\tau_\delta = 0$ , then  $\rho = 1$ , and

$$\alpha_{P1} = \frac{\tau_{\epsilon_1}}{\tau_1 + \tau_{\epsilon_1}} - (1 - \beta) \frac{(\gamma + h_{21})\tau_1\tau_{\epsilon_1}}{\gamma(\tau_1 + \tau_{\epsilon_1})(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})}.$$

Thus,  $\alpha_{P1}(\beta)|_{\beta=0} < \alpha_{P1}(\beta)|_{\beta=1}$ , and since  $\lambda_1 = a_1\tau_u/\tau_1 + \alpha_{P1}\tau_v/(a_1\tau_1)$ , the result follows.

When  $1/\tau_\delta = 0$ , traders' responsiveness to private information does not depend on  $\beta$ . As a consequence the first period depth reaction to a change in  $\beta$  is *completely* determined by the effect that a change in the correlation across noise trade increments has on  $\alpha_{P1}$ . A wider discrepancy between the price and the fundamentals (a smaller  $\alpha_{P1}$ ) proxies for a stronger sterilization of informed trades due to traders' intense market making activity. This dampens the price impact of trades, making the first period market deeper.

When  $\tau_\delta < \infty$  numerical simulations confirm that the first period depth is larger (smaller) when  $\beta = 0$  ( $\beta = 1$ ). Furthermore, as  $\beta$  increases, both  $\alpha_{P1}$  and  $\lambda_1$  grow larger, implying that a *stronger excessive weight on public information occurs in a deeper market*.

**Remark 6.** According to Bacchetta and van Wincoop (2006) in the presence of differential information, higher order expectations add an additional term (the "higher order wedge") to the traditional asset pricing equation. This term emerges as the

difference between the equilibrium price and the asset price that obtains substituting higher order expectations with first order expectations in the pricing equation. We can apply this decomposition to (14) using the fact that  $\bar{E}_1[\bar{E}_2[v]] = \bar{E}_1[v] + ((\tau_1 + \tau_{\epsilon_1})(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t}))^{-1} \tau_{\epsilon_1} \tau_1 (E[v|z_1] - v) = \bar{E}_1[v] - (a_1(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t}))^{-1} \tau_1 \alpha_{E1} (E[\theta_1|z_1] + \theta_1)$ . Denoting with  $p_1^*$  the asset price obtained substituting higher order expectations with first order expectations, the higher order wedge is given by

$$p_1 - p_1^* \equiv \Delta_1 = \bar{E}_1[\bar{E}_2[v] - v] = -\frac{\tau_1 \alpha_{E1}}{a_1(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})} (E[\theta_1|z_1] + \theta_1),$$

and captures the expectational error about the liquidation value due to the presence of noise in the public signal ( $z_1$ ). Using this definition, the equilibrium price can be written as

$$p_1 = \bar{E}_1[v] - \frac{(\gamma + h_{21})(\beta\rho - 1)}{\gamma} \Delta_1 + \left(1 + \frac{(\gamma + h_{21})(\beta\rho - 1)\tau_1}{\gamma(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})}\right) \frac{\alpha_{E1}}{a_1} \theta_1.$$

According to section 3.1 when  $\beta = 1$ , as soon as residual uncertainty vanishes,  $\rho \rightarrow 1$  and the higher order wedge ceases to affect the pricing equation.

## 4 The Market with Short-Term Traders

In the previous section we have argued that a stronger focus on short-run price differences allows over- or under-reliance on public information and relative departures of prices from average expectations of investors to arise in the CARA-normal model with long-term traders. In this section we turn our attention to the analysis of a dynamic model in which traders are *exclusively* concentrated on short-run profit opportunities and  $\beta \in [0, 1]$ . In this case short-run speculation is imposed and we should expect the bias of prices and average expectations not to coincide. In the absence of residual uncertainty over the liquidation value, and as long as  $\beta > 0$ , multiple equilibria arise. In one equilibrium there is over-reliance on public information and in the other one under-reliance occurs. The former equilibrium is the stable one. When  $\beta = 0$  or when  $\beta = 1$  and considerable residual uncertainty affects the liquidation value, over-reliance on public information always arises in the first period price.

Suppose thus that in the dynamic market analyzed in the previous section traders have short-term horizons (i.e. they take a position in period  $n$  and unwind it in period  $n + 1$ ) and that the private information each trader  $i$  receives in every period

$n$  is transmitted to the corresponding trader in period  $n + 1$ .<sup>25</sup> Traders may have a short horizon for incentive reasons, for example. Thus, in every period  $n$  every trader  $i$  maximizes the expected utility of his short-term profits  $\pi_{in} = (p_{n+1} - p_n)x_{in}$ ,  $E[U(\pi_{in})|\tilde{s}_{in}, p^n] = -E[\exp\{-\pi_{in}/\gamma\}|\tilde{s}_{in}, p^n]$ .

## 4.1 A General Pricing Formula

Before turning attention to the linear equilibria of the market, it is worth analyzing the general pricing formula. Assume that  $N = 2$ . Owing to CARA preferences and normality, in the second period a trader's optimal position is given by  $X_2(\tilde{s}_{i2}, p^2) = \gamma(\text{Var}[v|\tilde{s}_{i2}, z^2])^{-1}(E[v|\tilde{s}_{i2}, z^2] - p_2)$ , while the corresponding market clearing equation reads as follows:  $\int_0^1 x_{i2} di + \theta_2 = 0$ . Let  $\text{Var}_n[Y] = \text{Var}[Y|\tilde{s}_{in}, z^n]$ , and  $\bar{E}_n[Y] = \int_0^1 E[Y|\tilde{s}_{in}, z^n] di$ , then the second period equilibrium price is given by

$$p_2 = \bar{E}_2[v] + \frac{\theta_2}{\gamma} \text{Var}_2[v]. \quad (16)$$

Owing to short-term trading horizons, in the first period a trader's optimal position is given by  $X_1(s_{i1}, p_1) = \gamma(\text{Var}[p_2|s_{i1}, p_1])^{-1}(E[p_2|s_{i1}, p_1] - p_1)$ , and the corresponding market clearing equation reads as follows:  $\int_0^1 x_{i1} di + \theta_1 = 0$ . Solving for the first period equilibrium price yields

$$p_1 = \bar{E}_1[p_2] + \frac{\theta_1}{\gamma} \text{Var}_1[p_2].$$

Substituting (16) in the latter equation and using the fact that  $\theta_2 = \beta\theta_1 + u_2$  with  $u_2$  serially uncorrelated, yields

$$p_1 = \bar{E}_1 \left[ \bar{E}_2[v] + \frac{\beta\theta_1}{\gamma} \text{Var}_2[v] \right] + \frac{\theta_1}{\gamma} \text{Var}_1[p_2]. \quad (17)$$

According to (16) and (17), in any period the price of the asset depends on two components: the market average expectation of the market average expected liquidation value and the risk associated with holding a position in the asset (due to the presence of noise traders).<sup>26</sup>

<sup>25</sup>This is without loss of generality. The same qualitative results would arise if traders in period  $n$  did not inherit the information held by their previous period peers.

<sup>26</sup>This is generally true in both periods since in the first period traders anticipate liquidating their position at  $p_2$ , whereas in the second period traders hold the asset until uncertainty is resolved.

Extending this line of reasoning to the  $N$ -period case we obtain

$$\begin{aligned}
p_n = & \bar{E}_n \left[ \bar{E}_{n+1} \left[ \cdots \bar{E}_{N-1} \left[ \bar{E}_N[v] + \frac{\text{Var}_N[v]}{\gamma} \beta^{N-n} \theta_n \right] + \frac{\text{Var}_{N-1}[p_N]}{\gamma} \beta^{N-(n+1)} \theta_n \cdots \right] + \frac{\text{Var}_{n+1}[p_{n+2}]}{\gamma} \beta \theta_n \right] \\
& + \frac{\text{Var}_n[p_{n+1}]}{\gamma} \theta_n, \tag{18}
\end{aligned}$$

for  $1 \leq n \leq N$ .<sup>27</sup> Recall that when  $\beta = 1$  we have that  $\theta_n = \sum_{t=1}^n u_t$ . The above pricing formula coincides with Allen et al. (2006) when  $\beta = 0$ . Indeed, in their framework additional noise shocks do not cumulate over the trading periods. Hence, the price in period  $N$  is simply the average expectation of investors about the fundamental value plus the period  $N$  risk-adjusted noise shock:

$$p_N = \bar{E}_N[v] + \frac{\theta_N}{\gamma} \text{Var}_N[v].$$

In period  $N - 1$  the price is then given by the average expectation of investors about the price in period  $N$  (because investors have a one period horizon) plus the corresponding period, risk-adjusted noise shock:

$$p_{N-1} = \bar{E}_{N-1}[p_N] + \frac{\theta_{N-1}}{\gamma} \text{Var}_{N-1}[p_N].$$

When  $\beta = 0$ ,  $\bar{E}_{N-1}[\theta_N] = 0$ . Hence, we obtain

$$p_{N-1} = \bar{E}_{N-1}[\bar{E}_N[v]] + \frac{\theta_{N-1}}{\gamma} \text{Var}_{N-1}[p_N].$$

This recursive relationship can be iterated backwards to obtain that in any period  $n$

$$p_n = \bar{E}_n[\bar{E}_{n+1}[\cdots \bar{E}_{N-1}[\bar{E}_N[v]] \cdots]] + \frac{\theta_n}{\gamma} \text{Var}_n[p_{n+1}].$$

Thus,  $p_n$  is the average expectation at  $n$  of the average expectation at  $n + 1$  of the average expectation at  $n + 2$  of... the liquidation value in period  $N + 1$ , plus the corresponding period, risk-adjusted noise shock. This is reminiscent of Keynes' vision of the stock market as a beauty contest.

An interesting observation by Allen et al. (2006) is that, when averaging over the realizations of noise trading, the price at date  $n$  – the average expectation of the average expectation of the... – will in general not coincide with the period  $n$

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<sup>27</sup>See appendix B.

average expectation of the fundamental value (the price at  $N + 1$ ). In this sense the consensus value of the fundamentals  $\bar{E}_n[v]$  does not coincide with the price  $p_n$ , with the exception of the last period  $n = N$ . The mean price path  $p_n$  gives a higher weight to history – relies more on public information – than the mean consensus path  $\bar{E}_n[v]$ . This is because of the bias towards public information when a Bayesian agent has to forecast the average market opinion knowing that it is formed also on the public information observed by other agents. This also implies that the current price will be always farther away from fundamentals than the average of investors' expectations and that it will be more sluggish to adjust.

However, according to the previous section we know that the weight the price assigns to public information depends on traders' reaction to order imbalances. Indeed, whenever traders deem the order-flow to be mostly liquidity driven, they take the other side of the market, in this way partially sterilizing the effect of informed trades on the price. This reinforces the effect of public information on the price, driving the latter away from the fundamentals compared to the average opinion of traders in the market. If, on the other hand, traders estimate the order-flow to be mostly information driven, they speculate on price momentum reinforcing the impact of informed trades on the price. This weakens the weight on public information, tying the price more firmly to the fundamentals compared to investors' average expectations. Intuitively, the same effects should be at work in a market where traders have short-term horizons. Indeed, in the coming section we formalize this intuition showing that the results in Allen et al. (2006) can be overturned when noise trading is not independent across periods, i.e. examining the case  $\beta \in (0, 1]$  with no residual uncertainty. We also present an example with large residual uncertainty and  $\beta = 1$  where over-reliance on public information always occurs.

## 4.2 Short-Term Trading and the Bias of Prices

Suppose  $N = 2$ , and let  $\beta \in (0, 1]$ , then we can prove the following result:<sup>28</sup>

**Proposition 3.** *In the 2-period market with short-term traders when  $\beta \in (0, 1]$  there exist two symmetric equilibria in linear strategies where: (1) prices are given by  $p_3 = v$ ,  $p_0 = \bar{v}$ ,  $p_2 = \lambda_2 z_2 + (1 - \lambda_2 \Delta a_2) \hat{p}_1$ , and  $p_1 = \lambda_1 z_1 + (1 - \lambda_1 a_1) \bar{v}$ , (2) strategies*

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<sup>28</sup>Proposition 3 extends the multiplicity result in Cespa (2002) to the case of general patterns of noise trading and private information arrival.

are given by:  $X_2(\tilde{s}_{i2}, p^2) = a_2(\tilde{s}_{i2} - p_2) + \gamma\tau_2(E[v|z^2] - p_2)$ ,

$$X_1(s_{i1}, p_1) = \frac{a_1(\tau_1 + \tau_{\epsilon_1})}{\tau_{\epsilon_1}} (E[v|s_{i1}, z_1] - p_1) + \frac{(\beta\rho - 1)\tau_1}{(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})} E[\theta_1|z_1], \quad (19)$$

where  $a_2 = \gamma(\tau_{\epsilon_1} + \tau_{\epsilon_2})$ ,  $a_1$  is given by the (two) real solutions to the quartic equation  $f(a_1) \equiv \lambda_2 a_1(\tau_2 + \tau_{\epsilon_1}) - \gamma\Delta a_2 \tau_u \tau_{\epsilon_1} = 0$ , which satisfy  $0 < a_1^* < \gamma\tau_{\epsilon_1}$ ,  $a_2/\beta < a_1^{**} < \infty$ , and

$$\lambda_1 = \alpha_{P1} \frac{1}{a_1} + (1 - \alpha_{P1}) \frac{a_1 \tau_u}{\tau_1}, \quad \alpha_{P1} = \frac{\tau_{\epsilon_1}}{\tau_1 + \tau_{\epsilon_1}} + \frac{(\beta\rho - 1)\tau_1 \tau_{\epsilon_1}}{(\tau_1 + \tau_{\epsilon_1})(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})}.$$

In any equilibrium  $\hat{p}_1 = (\beta a_1 + \gamma\tau_1)^{-1}(\beta z_1 + \gamma\tau_1 E[v|z_1])$ ,  $\lambda_2 = (\gamma(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t}))^{-1}(1 + \gamma\Delta a_2 \tau_u)$ , and  $\rho = (a_1/\gamma\tau_{\epsilon_1})$ . Therefore,  $a_1^* < a_2/\beta < a_1^{**}$  and  $\lambda_2(a_1^{**}) < 0 < \lambda_2(a_1^*)$ .

*Proof.* See appendix A. QED

Owing to short horizons traders speculate on short-run price movements. However, differently from section 3.2, they cannot count on a second period hedge to protect them from adverse price swings. This induces two (self-fulfilling) equilibria. Along the high trading intensity equilibrium (i.e. when  $a_1 = a_1^{**} > a_2/\beta$ ) agents escalate their response to the private signal and thus anticipate their estimation of the noise trade shock to be biased by the presence of informed trades. As a consequence, they side with the market ( $\beta\rho > 1$ ), and speculate on price momentum. The magnitude of the first period signal aggressiveness ( $a_1^{**} > a_2/\beta$ ), in turn induces a negative second period depth ( $\lambda_2(a_1^{**}) < 0$ ) which limits adverse price movements when traders unwind their position in the second period, justifying their first period speculative behavior. Along the low trading intensity equilibrium (i.e. when  $a_1 = a_1^* < \gamma\tau_{\epsilon_1}$ ) traders scale down their first period response to the signal, and thus anticipate their estimation of the noise trade shock to be due to liquidity trades. As a consequence, they take the other side of the market ( $\beta\rho < 1$ ), and speculate on a noise trade increment reversion. The magnitude of the first period aggressiveness ( $a_1^* < a_2/\beta$ ) induces a positive second period depth ( $\lambda_2(a_1^*) > 0$ ), which again justifies informed traders' first period behavior. Thus, in the absence of a risk neutral market making sector (as in Vives (1995)) short-term trading delivers equilibrium multiplicity.

In Proposition 3 (similarly as in Proposition 2) we denote with

$$\rho \equiv \frac{a_1}{\gamma\tau_{\epsilon_1}},$$

a measure of the deviation that short-term horizons induce in traders' first period signal response with respect to the static response. Depending on the equilibrium that arises  $\beta\rho > (<)1$ , and, as in the previous section it is possible to relate the presence of an over-reliance on public information in the first period to the magnitude of  $\beta\rho$ .<sup>29</sup> Indeed, we have:

$$\begin{aligned}
p_1 &= \alpha_{P1} \left( v + \frac{\theta_1}{a_1} \right) + (1 - \alpha_{P1}) E[v|z_1] \\
&= \bar{E}_1[v] + \left( \frac{\alpha_{P1} - \alpha_{E1}}{a_1} \right) E[\theta_1|z_1] + \alpha_{E1} \frac{\theta_1}{a_1} \\
&= \bar{E}_1[v] + \frac{(\beta\rho - 1)\tau_1\tau_{\epsilon_1}}{a_1(\tau_1 + \tau_{\epsilon_1})(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})} E[\theta_1|z_1] + \alpha_{E1} \frac{\theta_1}{a_1},
\end{aligned} \tag{20}$$

where

$$\alpha_{P1} \equiv \alpha_{E1} + \frac{(\beta\rho - 1)\tau_1\tau_{\epsilon_1}}{(\tau_1 + \tau_{\epsilon_1})(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})}.$$

Using the last equation in (20) we readily realize as before that the existence of a discrepancy between prices and investors' average expectations depends on the sign of  $(\beta\rho - 1)$ . In the low trading intensity equilibrium we have  $a_1^* < \gamma\tau_{\epsilon_1}/\beta \Leftrightarrow \rho < 1/\beta$ , traders thus accommodate the expected noise trade shock and over-reliance on public information occurs. In the high trading intensity equilibrium we have  $a_1^{**} > \gamma\tau_{\epsilon_1}/\beta \Leftrightarrow \rho > 1/\beta$ , traders side with the market and under-reliance on public information arises. Hence, we can conclude that:

**Corollary 5.** *In the market with short-term traders along the low (high) trading intensity equilibrium the first period equilibrium price displays over- (under-) reliance on public information.*

It is again an immediate consequence of (20) that

**Corollary 6.** *In the market with short-term traders when  $\beta = 0$  there exists a unique equilibrium in linear strategies. In this equilibrium over-reliance on public information occurs.*

As  $\beta$  approaches 0, along the high trading intensity equilibrium traders need to speculate increasingly more aggressively to avoid an adverse price swing in the second period, up to the point that if  $\beta = 0$ ,  $a_1^{**}$  should grow unboundedly. However, in this

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<sup>29</sup>In the second period traders act as in a static market and, as argued in the previous section, in this case the price assigns the optimal statistical weight to public information.

case the equilibrium becomes fully revealing,  $p_2 = p_1 = v$ , and traders earn no return from their private information. Hence, they concentrate on the low trading intensity equilibrium. Our model then coincides with Allen et al. (2006) and the first period equilibrium price is always more biased than average expectations in the estimation of the fundamentals (see figure 3).

[Figure 3 about here.]

**Remark 7.** Differently from what happens in the market with long-term traders, when traders have short horizons over- (under-) reliance on public information occurs in a thin (thick) first period market. The intuition is as follows. As argued above over-reliance on public information arises along the low trading intensity equilibrium where, given traders' low aggressiveness, the degree of adverse selection is high. Thus, in this equilibrium the market is thin. Conversely, under-reliance on public information occurs in the high trading intensity equilibrium where traders' speculative aggressiveness yields a faster resolution of the underlying uncertainty, reducing the degree of adverse selection. This, in turn, yields a thick market (see figure 4).

[Figure 4 about here.]

**Remark 8.** In the market with short-term traders it is again possible to apply Bacchetta and Van Wincoop (2006) decomposition to the equilibrium price. As one would expect, along the high (low) trading intensity equilibrium a positive wedge (integrating out the effect of the noise term) ties the price more firmly to the fundamentals in relation to average expectations. Conversely, along the low trading intensity equilibrium the opposite occurs.

**Remark 9.** Summarizing, as soon as additional noise traders' demand shocks become correlated, both over- and under-reliance on public information arises. In a model with  $\beta = 1$ , Cespa (2002) proves that along the high trading intensity equilibrium since  $a_1^{**} > a_2$ , the slope of the aggregate excess demand function is always positive, arguing that as a consequence, the equilibrium in this case is unstable. Using our results we can build a similar argument for a market in which  $\beta \in (0, 1)$ . In particular, given the realization of the first period informational shock  $z_1$ , we can define the second period aggregate excess demand function as follows:

$$XD(p_2) \equiv z_2 + \lambda_2^{-1}(1 - \lambda_2 \Delta a_2) \hat{p}_1 - \lambda_2^{-1} p_2, \quad (21)$$



where  $XD = 0$  when the market is in equilibrium and  $XD \neq 0$  otherwise. Notice that the slope of (21) depends on the sign of  $\lambda_2$ . This can be determined using the equation that defines the first period signal responsiveness:

$$f(a_1) \equiv \lambda_2 a_1 (\tau_2 + \tau_{\epsilon_1}) - \gamma \Delta a_2 \tau_u \tau_{\epsilon_1} = 0.$$

In the high trading intensity equilibrium, we know that  $\Delta a_2 < 0$ , hence for  $f(a_1) = 0$  to be satisfied it must be that  $\lambda_2 < 0$  too. Conversely, along the low trading intensity equilibrium we have  $\Delta a_2 > 0$ , hence for a solution to  $f(a_1) = 0$  to exist,  $\lambda_2$  must also be positive.<sup>30</sup> In the unstable case notice that a price decline (e.g. spurred by a selling pressure) drives the market away from equilibrium. In the low trading intensity equilibrium, the aggregate excess demand function slopes downwards, and the associated equilibrium is stable. Hence, restricting attention to stable equilibria we can conclude that the equilibrium featuring over-reliance on public information in Allen et al. (2006) is robust.

**Remark 10.** A further feature of the equilibrium obtained in proposition 3 is that prices do not necessarily display inertia. In particular, along the high (low) trading intensity equilibrium, the first period price rapidly (slowly) adjusts to the fundamentals. Indeed, as traders “overreact” to their signal, after the first period the price jumps close to  $v$  and then fully adjusts in the following two periods. Thus, differently from Allen et al. (2006) the equilibrium price exhibits inertia if and only if  $\rho < 1$ .

### 4.3 The Effect of Residual Uncertainty

If the liquidation value is affected by residual uncertainty and  $\beta = 1$  we can prove that over-reliance on public information occurs in the first period whenever such uncertainty is sufficiently large. To see this, consider that in the second period the equilibrium strategy of a trader coincides with the one of the model with long-term traders. In the first period

$$X_1(s_{i1}, z_1) = \gamma \frac{E[p_2 | s_{i1}, z_1] - p_1}{\text{Var}[p_2 | s_{i1}, z_1]},$$

where  $E[p_2 | s_{i1}, z_1] = \lambda_2 \Delta a_2 E[v | s_{i1}, z_1] + (1 - \lambda_2 \Delta a_2) \hat{p}_1$ ,  $\text{Var}[p_2 | s_{i1}, z_1] = \lambda_2^2 (\tau_2 + \tau_{\epsilon_1}) / (\tau_u (\tau_1 + \tau_{\epsilon_1}))$ , and  $\lambda_2, \hat{p}_1$  are defined in proposition 2. Thus, identifying the first

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<sup>30</sup>Incidentally, the above equation also shows that  $a_1 > 0$  in equilibrium since if  $a_1 < 0$  both  $\Delta a_2 > 0$  and  $\lambda_2 > 0$ , preventing the existence of a solution to  $f(a_1) = 0$ .

period signal responsiveness:

$$a_1 = \frac{\gamma\tau_{\epsilon_1}\tau_u\Delta a_2}{\lambda_2(\tau_2 + \tau_{\epsilon_1})}.$$

Imposing market clearing and simplifying yields

$$\begin{aligned} p_1 &= \alpha_{P1} \left( v + \frac{\theta_1}{a_1} \right) + (1 - \alpha_{P1}) E[v|z_1] \\ &= \bar{E}_1[v] + \left( \frac{\alpha_{P1} - \alpha_{E1}}{a_1} \right) E[\theta_1|z_1] + \alpha_{E1} \frac{\theta_1}{a_1} \\ &= \bar{E}_1[v] + \frac{(\rho - 1)\tau_1\tau_{\epsilon_1}}{a_1(\tau_1 + \tau_{\epsilon_1})(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})} E[\theta_1|z_1] + \alpha_{E1} \frac{\theta_1}{a_1}, \end{aligned} \quad (22)$$

where

$$\alpha_{P1} \equiv \alpha_{E1} + \frac{(\rho - 1)\tau_1\tau_{\epsilon_1}}{(\tau_1 + \tau_{\epsilon_1})(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})},$$

and  $\rho = a_1(\tau_\delta + \tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})/(\gamma\tau_\delta\tau_{\epsilon_1})$ , denotes a measure of the deviation that the joint effect of residual uncertainty and short-term horizons induce on traders' first period signal aggressiveness with respect to the static aggressiveness. We can thus conclude the following:

**Corollary 7.** *In any linear equilibrium of the market with short-term, informed speculators when the liquidation value is affected by residual noise, over-reliance on public information occurs if and only if  $\rho < 1$ .*

Intuitively, the above condition is more likely to be satisfied whenever the residual uncertainty affecting the liquidation value is sufficiently large. Indeed, as in the market with long-term traders, higher uncertainty over the liquidation value exacerbates the reaction of the second period price to the upcoming net informational addition  $z_2$ . However, owing to short-term horizons, traders cannot count on the improved hedging properties of their second period strategies. Hence, in the face of the increased second period price unpredictability, traders in the first period should scale down their aggressiveness yielding a price which is less anchored to the fundamentals compared to traders' average expectations.<sup>31</sup> This intuition can be formalized by the following

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<sup>31</sup>The increased uncertainty over the second period price in a way "crowds-out" informed traders in the first period. This effect is reminiscent of De Long et al. (1990). Note, however, that in our context all traders – except liquidity traders – are "rational."

**Proposition 4.** *In any linear equilibrium of the market with short-term traders, when the liquidation value is affected by the presence of residual uncertainty, for sufficiently large residual uncertainty over-reliance on public information always arises in the first period.*

*Proof.* See appendix A.

QED

Numerical simulations show that for low values of  $\tau_\delta$  the equilibrium where the price displays over-reliance on public information is unique.<sup>32</sup>

## 5 Discussion and Concluding Remarks

In this paper we have studied under what conditions asset prices display over- or under-reliance on public information and are more or less biased than traders' average expectations in the estimation of fundamentals in a variety of market contexts. We argued that for over- or under-reliance on public information to occur traders need to speculate on short-run price differences. If, instead, agents only care about the liquidation value of the asset, their strategies have a static nature. Hence, traders assign the optimal statistical weight to public information and prices equal the average expectations of investors plus a risk-weighted noise component. This is the case when long-term traders populate the market, there is no residual uncertainty in the liquidation value, and noise trading follows a random walk. When traders speculate on short run price differences, because of noise trade increments persistence and/or uncertainty in the liquidation value of the asset, then they may over- or under-weight public information and prices may be more or less biased than average expectations. According to the degree of correlation of noise trading and the relevance of residual uncertainty we find in fact a Keynesian region where the former occurs and a Hayekian region where the latter occurs. When traders have a short horizon multiple equilibria arise, with one equilibrium displaying over-reliance and the other one under-reliance on public information. In this case the equilibrium displaying over-reliance is the stable one.

Our paper thus shows that in contrast to the results put forth by Allen et al. (2006), there are situations where equilibrium prices may have less bias in estimating fundamentals than the consensus opinion of traders. Indeed, the flavor of Keynes' beauty contest allegory obtains when traders focus their short-term activities on the

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<sup>32</sup>Conversely, as  $\tau_\delta$  increases, the multiplicity result of the previous section is restored.

exploitation of noise trade predictability. This happens in the stable equilibrium of the market with short-term traders and, with long-term traders, when noise trade increments are very correlated and residual uncertainty on the liquidation value is low. Otherwise, an alternative result, more in line with Hayek (1945)'s view of the market arises when informed trades *drive* the order-flow and speculators short-term trading activity reinforces the weight the price assigns to fundamental information. This may happen with short-term traders in an unstable equilibrium, and with long-term traders when noise trade increments are not very correlated and the residual uncertainty affecting the liquidation value is high.

## References

- [1] Admati, A. R. (1985). A noisy rational expectations equilibrium for multi-asset securities markets. *Econometrica* 53(3), 629–657.
- [2] Allen, F., S. Morris, and H. S. Shin (2006). Beauty contests and iterated expectations in asset markets. *Review of Financial Studies* 19(3), 719–752.
- [3] Angeletos, G. M. and A. Pavan (2007). Efficiency and welfare in economies with incomplete information. *Econometrica* 75(4), 1103–1142.
- [4] Bacchetta, P. and E. van Wincoop (2006). Higher order expectations in asset pricing. *Working Paper*.
- [5] Banerjee, S., Kaniel, R. and I. Kremer (2007). Price Drift as an Outcome of Differences in Higher Order Beliefs. *MIMEO, Stanford University and Duke University*.
- [6] Bhattacharya, U. and M. Spiegel (1991). Insiders, outsiders, and market breakdowns. *Review of Financial Studies* 4(2), 255–282.
- [7] Biais, B. and P. Bossaerts (1998, April). Asset prices and trading volume in a beauty contest. *Review of Economic Studies* 65(2), 307–340.
- [8] Brown, D. P. and R. H. Jennings (1989). On technical analysis. *Review of Financial Studies* 2(4), 527–551.
- [9] Cao, H. H. and H. Ou-Yang (2005, October). Bubbles and panics in a frictionless market with heterogeneous expectations. *Working Paper*.
- [10] Cespa, G. (2002). Short-term investment and equilibrium multiplicity. *European Economic Review* 46(9), 1645–1670.
- [11] De Long, B., Shleifer, A., Summers, L. H. and R. J. Waldmann (1990). Noise trader risk in financial markets. *Journal of Political Economy* 98(4), 703–738.
- [12] Gennotte, G. and H. Leland (1990). Market liquidity, hedging, and crashes. *American Economic Review* 80(5), 999–1021.
- [13] Grossman, S. J. (1989). *The Informational Role of Prices*. Cambridge: MIT Press.
- [14] Grundy, B. D. and M. McNichols (1989). Trade and the revelation of information through prices and direct disclosure. *Review of Financial Studies* 2(4), 495–526.

- [15] Hayek, F. A. (1945). The use of knowledge in society. *American Economic Review* 35(4), 519–530.
- [16] He, H. and J. Wang (1995). Differential information and dynamic behavior of stock trading volume. *Review of Financial Studies* 8(4), 919–972.
- [17] Hellwig, M. F. (1980). On the aggregation of information in competitive markets. *Journal of Economic Theory* 22(3), 477–498.
- [18] Kandel, E. and N. D. Pearson (1995). Differential interpretation of public signals and trade in speculative markets. *The Journal of Political Economy* 103(4), 831–872.
- [19] Kondor, P. (2004). Rational trader risk. *Working Paper LSE*.
- [20] Kyle, A. (1985). Continuous auctions and insider trading. *Econometrica* 53(6), 1315–1336.
- [21] Llorente, G. Michaely R., Saar, G. and J. Wang (2002). Dynamic volume-return relation of individual stocks. *Review of Financial Studies* 15(4), 1005–1047.
- [22] Morris, S. and H.S. Shin (2002). The social value of public information. *American Economic Review* 92, 1521–1534.
- [23] Nimark, K. P. (2007). Dynamic higher order expectations. *Working paper*.
- [24] Singleton, K. J. (1987). Asset prices in a time-series model with disparately informed, competitive traders in *New approaches to monetary economics*, W. A. Burnett and K. J. Singleton, Eds. Cambridge University Press.
- [25] Shiller, R. J. (2005). *Irrational Exuberance*. Princeton, New Jersey: Princeton University Press.
- [26] Vives, X. (1995). Short-term investment and the informational efficiency of the market. *Review of Financial Studies* 8(1), 125–160.
- [27] Vives, X. (1997). Learning from others: a welfare analysis. *Games and Economic Behavior* 20(2), 177–200.
- [28] Vives, X. (2007). *Information and Learning in Markets*. Forthcoming, Princeton University Press.
- [29] Wang, J. (1994). A model of competitive stock trading volume. *Journal of Political Economy* 102(1), 127–168.

# Appendix A

## *Proof of lemma 1*

Consider the first period. In any linear equilibrium market clearing yields  $\int_0^1 a_1 s_{i1} - \varphi_1(p_1) di + \theta_1 = a_1 v - \varphi_1(p_1) + \theta_1 = 0$  or, denoting with  $z_1 = a_1 v + \theta_1$  the informational content of the first period order-flow,  $z_1 = \varphi_1(p_1)$ , where  $\varphi_1(\cdot)$  is a linear function. Hence,  $z_1$  and  $p_1$  are observationally equivalent. Suppose now that  $z^{n-1} = \{z_1, z_2, \dots, z_{n-1}\}$  and  $p^{n-1} = \{p_1, p_2, \dots, p_{n-1}\}$  are observationally equivalent and consider the  $n$ -th period market clearing condition:  $\int_0^1 X_n(\tilde{s}_{in}, p^{n-1}, p_n) di + \theta_n = 0$ . Adding and subtracting  $\sum_{t=1}^{n-1} a_t v$ , the latter condition can be rewritten as follows:

$$\sum_{t=1}^n z_t - \varphi_n(p^n) = 0,$$

where  $\varphi_n(\cdot)$  is a linear function, and  $z_t = \Delta a_t v + u_t$  denotes the informational content of the  $t$ -th period order-flow. As by assumption  $p^{n-1}$  and  $z^{n-1}$  are observationally equivalent, it follows that observing  $p_n$  is equivalent to observing  $z_n$ . QED

## *Proof of proposition 1*

Given the past trading history, in the last trading period ( $N$ ) each informed trader maximizes the expected utility of his last period profit  $\pi_{iN} = (v - p_N)x_{iN}$ . Owing to CARA and normality of the random variables his optimal strategy is:

$$X_N(\tilde{s}_{iN}, p^N) = a_N(\tilde{s}_{iN} - p_N) + \gamma\tau_N(E[v|z^N] - p_N), \quad (23)$$

where  $a_N = \gamma(\sum_{t=1}^N \tau_{\epsilon_t})$ . Imposing market clearing yields

$$\int_0^1 a_N(\tilde{s}_{iN} - p_N) + \gamma\tau_N(E[v|z^N] - p_N) di + \theta_N = 0.$$

Adding and subtracting  $\sum_{n=1}^{N-1} a_n v$  to the above, and rearranging:

$$(1 + \gamma\tau_u \Delta a_N) z_N + \sum_{n=1}^{N-1} (1 + \gamma\tau_u \Delta a_n) z_n + \gamma\tau_v \bar{v} = (a_N + \gamma\tau_N) p_N.$$

Denoting with  $\lambda_N = (1 + \gamma\tau_u \Delta a_N)/(a_N + \gamma\tau_N)$  the  $N$ -th period reciprocal of market depth we can solve for  $p_N$  obtaining

$$p_N = \lambda_N z_N + (1 - \lambda_N \Delta a_N) \hat{p}_{N-1}, \quad (24)$$

where  $\hat{p}_{N-k} = \lambda_{N-kl}z_{N-k} + (1 - \lambda_{N-kl}\Delta a_{N-1})\hat{p}_{N-(k+1)l}$ , and  $\hat{\lambda}_{N-k} = (1 + \gamma\tau_u\Delta a_{N-k}) / (a_{N-k} + \gamma\tau_{N-k})$ , for  $k = 1, \dots, N-1, \dots$ . Substituting (23) into the trader's objective function yields:

$$E[U(\pi_{iN}) | \tilde{s}_{iN}, z^N] = -\exp\left\{-\frac{x_{iN}^2}{2\gamma^2(\tau_N + \sum_{t=1}^N \tau_{\epsilon_t})}\right\}.$$

In period  $N-1$  the trader then maximizes

$$E[U(\pi_{iN-1} + \pi_{iN}) | \tilde{s}_{iN-1}, z^{N-1}] = \tag{25}$$

$$-E\left[\exp\left\{-\frac{1}{\gamma}\left((p_N - p_{N-1})x_{iN-1} + \frac{x_{iN}^2}{2\gamma(\tau_N + \sum_{t=1}^N \tau_{\epsilon_N})}\right)\right\} | \tilde{s}_{iN-1}, z^{N-1}\right].$$

Denote with  $\phi_{iN-1}$  the term in parenthesis in the above equation:  $\phi_{iN-1} = (p_N - p_{N-1})x_{iN-1} + x_{iN}^2 / (2\gamma(\tau_N + \sum_{t=1}^N \tau_{\epsilon_N}))$ . As one can easily check  $\phi_{iN-1}$  is a quadratic form of the random vector  $Z = (x_{iN} - \mu_1, p_N - \mu_2)'$ , which is normally distributed (conditionally on  $\tilde{s}_{iN-1}, z^{N-1}$ ) with zero mean and variance covariance matrix

$$\Sigma = \frac{1}{(\tau_{N-1} + \sum_{t=1}^{N-1} \tau_{\epsilon_t})\tau_u} \times$$

$$\begin{pmatrix} (\Delta a_N - \gamma\tau_{\epsilon_N})^2\tau_u + (\tau_{N-1} + \sum_{t=1}^{N-1} \tau_{\epsilon_t})(1 + \gamma^2\tau_u\tau_{\epsilon_N}) & \cdot \\ \lambda_N((\gamma\tau_{\epsilon_N} - \Delta a_N)\Delta a_N\tau_u - (\tau_{N-1} + \sum_{t=1}^{N-1} \tau_{\epsilon_N})) & \lambda_N^2(\tau_N + \sum_{t=1}^{N-1} \tau_{\epsilon_t}) \end{pmatrix},$$

(i.e.  $\mu_1 = E[x_{iN} | \tilde{s}_{iN-1}, z^{N-1}] = \gamma(\tau_N + \sum_{t=1}^N \tau_{\epsilon_N})(1 - \lambda_N\Delta a_N)(E[v | \tilde{s}_{iN-1}, z^{N-1}] - \hat{p}_{N-1})$  and  $\mu_2 = E[p_N | \tilde{s}_{iN-1}, z^{N-1}] = \lambda_N\Delta a_N E[v | \tilde{s}_{iN-1}, z^{N-1}] + (1 - \lambda_N\Delta a_N)\hat{p}_{N-1}$ ):

$$\phi_{iN-1} = c + b'Z + Z'AZ,$$

where  $c = (\mu_2 - p_{N-1})x_{iN-1} + \mu_1^2 / (2\gamma(\tau_N + \sum_{t=1}^N \tau_{\epsilon_N}))$ ,  $b = (\mu_1 / (\gamma(\tau_N + \sum_{t=1}^N \tau_{\epsilon_N})), x_{iN-1})'$ , and  $A$  is a  $2 \times 2$  matrix with  $a_{11} = 1 / (2\gamma(\tau_N + \sum_{t=1}^N \tau_{\epsilon_N}))$  and the rest zeroes. Owing to a well-known property of multivariate normal random variables the objective function (25) can then be rewritten as

$$E[U(\pi_{iN-1} + \pi_{iN}) | \tilde{s}_{iN-1}, z^{N-1}] = \tag{26}$$

$$-|\Sigma|^{-1/2} |\Sigma^{-1} + 2/\gamma A|^{-1/2} \times \exp\left\{-1/\gamma\left(c - \frac{1}{2\gamma}b'(\Sigma^{-1} + (2/\gamma)A)^{-1}b\right)\right\}.$$

Maximizing the above function with respect to  $x_{iN-1}$  yields

$$x_{iN-1} = \gamma \frac{\mu_2 - p_{N-1}}{h_{22}} - \frac{h_{21}\mu_1}{h_{22}\gamma(\tau_N + \sum_{t=1}^N \tau_{\epsilon_N})}, \tag{27}$$



where  $h_{ij}$  denotes the  $ij$ -th term of the symmetric matrix  $H = (\Sigma^{-1} + (2/\gamma)A)^{-1}$ :

$$\begin{aligned} h_{11} &= \frac{\gamma^2 \text{Var}_{N-1}[x_{iN}]}{D}, \\ h_{12} &= \frac{\gamma^2 \text{Cov}_{N-1}[x_{iN}, p_N]}{D}, \\ h_{22} &= \frac{\gamma^2 \text{Var}_{N-1}[p_N] + \text{Var}_N[v](\text{Var}_{N-1}[x_{iN}]\text{Var}_{N-1}[p_N] - \text{Cov}_{N-1}[x_{iN}, p_N]^2)}{D}, \end{aligned}$$

and  $\text{Var}_n[Y] = \text{Var}[Y|\tilde{s}_{in}, z^n]$ ,  $\text{Cov}_{N-1}[X, Y] = \text{Cov}[X, Y|\tilde{s}_{N-1}, z^{N-1}]$ ,  $D = \gamma^2 + \text{Var}_{N-1}[x_{iN}]\text{Var}_N[v]$ . Identifying the  $N - 1$ -th period trading aggressiveness:

$$\begin{aligned} a_{N-1} &= \frac{\sum_{t=1}^{N-1} \tau_{\epsilon_t}}{h_{22}(\tau_{N-1} + \sum_{t=1}^{N-1} \tau_{\epsilon_t})} (\gamma \lambda_N \Delta a_N - h_{21}(1 - \lambda_N \Delta a_N)) \\ &= \gamma \left( \sum_{t=1}^{N-1} \tau_{\epsilon_t} \right), \end{aligned}$$

since  $(\gamma \lambda_N \Delta a_N - h_{21}(1 - \lambda_N \Delta a_N))/h_{22} = \gamma(\tau_{N-1} + \sum_{t=1}^{N-1} \tau_{\epsilon_t})$ . Substituting  $a_{N-1}$  in (27) and rearranging

$$\begin{aligned} &X_{N-1}(\tilde{s}_{iN-1}, z^{N-1}) \\ &= \left( \frac{\gamma \lambda_N \Delta a_N - h_{21}(1 - \lambda_N \Delta a_N)}{h_{22}} \right) (E[v|\tilde{s}_{iN-1}, z^{N-1}] - \hat{p}_{N-1}) - \frac{\gamma}{h_{22}}(p_{N-1} - \hat{p}_{N-1}) \\ &= \gamma \left( \tau_{N-1} + \sum_{t=1}^{N-1} \tau_{\epsilon_t} \right) (E[v|\tilde{s}_{iN-1}, z^{N-1}] - \hat{p}_{N-1}) - \frac{\gamma}{h_{22}}(p_{N-1} - \hat{p}_{N-1}). \end{aligned} \quad (28)$$

Interpreting  $\hat{p}_{N-1}$  as the “static” period  $N - 1$  equilibrium price (i.e. the price that would arise if the asset was liquidated in period  $N$ ), a trader  $i$ 's strategy is thus the sum of his static position and the adjustment he makes to exploit short-run price movements. For example, suppose that  $E[v|\tilde{s}_{iN-1}, z^{N-1}] - \hat{p}_{N-1} > 0$ , but that  $p_{N-1} > \hat{p}_{N-1}$ , then the trader scales down his period  $N - 1$  static position to avoid buying the asset for too high a price. Imposing market clearing yields:

$$\int_0^1 \gamma \text{Var}[v|\tilde{s}_{iN-1}, z^{N-1}]^{-1} (E[v|\tilde{s}_{iN-1}, z^{N-1}] - \hat{p}_{N-1}) di - \frac{\gamma}{h_{22}}(p_{N-1} - \hat{p}_{N-1}) + \theta_{N-1} = 0. \quad (29)$$

Notice that

$$\begin{aligned}
& \int_0^1 \gamma \text{Var} [v | \tilde{s}_{iN-1}, z^{N-1}]^{-1} (E [v | \tilde{s}_{iN-1}, z^{N-1}] - \hat{p}_{N-1}) di = \\
& = \int_0^1 \gamma \left( \tau_v \bar{v} + \tau_u \sum_{n=1}^{N-1} \Delta a_n z_n + \sum_{n=1}^{N-1} \tau_{\epsilon_n} s_{in} \right) - \left( \gamma \tau_v \bar{v} + \sum_{n=1}^{N-1} (1 + \gamma \tau_u \Delta a_n) z_n \right) di \\
& = a_{N-1} v - \sum_{n=1}^{N-1} z_n \\
& = -\theta_{N-1}.
\end{aligned}$$

Hence, since  $\gamma/h_{22} \neq 0$ , (29) is satisfied if and only if  $\hat{p}_{N-1} = \hat{\lambda}_{N-1} z_{N-1} + (1 - \hat{\lambda}_{N-1} \Delta a_{N-1}) \hat{p}_{N-2} = p_{N-1}$ : traders' aggregate static position is just enough to hold the total noisy supply that has accumulated up to period  $N-1$ . Therefore,  $\hat{\lambda}_{N-1} = \lambda_{N-1}$ , and

$$\begin{aligned}
X_{N-1}(\tilde{s}_{iN-1}, z^{N-1}) & = \gamma \left( \tau_{N-1} + \sum_{t=1}^{N-1} \tau_{\epsilon_t} \right) (E[v | \tilde{s}_{iN-1}, z^{N-1}] - p_{N-1}) \\
& = a_{N-1} (\tilde{s}_{iN-1} - p_{N-1}) + \gamma \tau_{N-1} (E[v | z^{N-1}] - p_{N-1}). \quad (30)
\end{aligned}$$

Plugging (30) into the term in parenthesis in the exponential of (26) we can now evaluate the  $N-1$ -th period objective function of a trader  $i$  at the optimum:

$$c - \frac{1}{2\gamma} b' (\Sigma^{-1} + 2/\gamma A)^{-1} b = \frac{x_{iN-1}^2}{2\gamma(\tau_{N-1} + \sum_{t=1}^{N-1} \tau_{\epsilon_N})},$$

given that, as one can check, in equilibrium

$$\mu_2 - p_{N-1} = \frac{\lambda_N \Delta a_N}{\gamma(\tau_{N-1} + \sum_{t=1}^{N-1} \tau_{\epsilon_N})} x_{iN-1},$$

and  $\mu_1^2 = x_{iN-1}^2$ . The  $N-2$ -th objective function of a trader is then given by

$$\begin{aligned}
& E [U(\pi_{iN-2} + \pi_{iN-1}) | \tilde{s}_{iN-2}, z^{N-2}] = \quad (31) \\
& -E \left[ \exp \left\{ -\frac{1}{\gamma} \left( (p_{N-1} - p_{N-2}) x_{iN-2} + \frac{x_{iN-1}^2}{2\gamma(\tau_{N-1} + \sum_{t=1}^{N-1} \tau_{\epsilon_N})} \right) \right\} | \tilde{s}_{iN-2}, z^{N-2} \right].
\end{aligned}$$

Comparing (31) with (25) the form of the objective function in the recursion between the  $N-1$ -th and the  $N-2$ -th period looks exactly as the one in the recursion between the  $N$ -th and the  $N-1$ -th period. Thus, the trader's optimal strategy in

period  $N-2$  is given by  $x_{iN-2} = a_{N-2}(\tilde{s}_{iN-2} - p_{N-2}) + \gamma\tau_{N-2}(E[v|\tilde{s}_{iN-2}, z^{N-2}] - p_{N-2})$ , where  $a_{N-2} = \gamma(\sum_{t=1}^{N-2} \tau_{\epsilon t})$ , and the corresponding equilibrium price is given by:

$$p_{N-2} = \lambda_{N-2}z_{N-2} + (1 - \lambda_{N-2}\Delta a_{N-2})\hat{p}_{N-3},$$

where  $\lambda_{N-2} = (1 + \gamma\tau_u\Delta a_{N-2})/(a_{N-2} + \gamma\tau_{N-2})$ . Iterating this procedure until the first period, we obtain that  $x_{in} = a_n(\tilde{s}_{in} - p_n) + \gamma\tau_n(E[v|z^n] - p_n)$ ,  $a_n = \gamma\sum_{t=1}^n \tau_{\epsilon t}$ , and  $\hat{p}_n = p_n$  for all  $1 \leq n \leq N$ .

Note that since at equilibrium  $\lambda_n = (a_n + \gamma\tau_n)^{-1}(1 + \gamma\tau_u\Delta a_n)$ , and  $\hat{p}_n = p_n$ , evaluating  $\mu_1$  at equilibrium we obtain

$$\begin{aligned} \mu_1 &= E[x_{iN}|\tilde{s}_{iN-1}, z^{N-1}] \\ &= \gamma \left( \tau_N + \sum_{t=1}^N \tau_{\epsilon t} \right) (1 - \lambda_N\Delta a_N)(E[v|\tilde{s}_{iN-1}, z^{N-1}] - p_{N-1}) \\ &= \gamma \left( \tau_{N-1} + \sum_{t=1}^{N-1} \tau_{\epsilon t} \right) (E[v|\tilde{s}_{iN-1}, z^{N-1}] - p_{N-1}) \\ &= x_{iN-1}. \end{aligned}$$

Hence,  $x_{iN-1} = E[x_{iN}|s_{iN-1}, z^{N-1}]$ . Now, define  $\mu'_1 = E[x_{iN-1}|\tilde{s}_{iN-2}, z^{N-2}] = \gamma(\tau_{N-1} + \sum_{t=1}^{N-1} \tau_{\epsilon t})(1 - \lambda_{N-1}\Delta a_{N-1})(E[v|\tilde{s}_{iN-2}, z^{N-2}] - p_{N-2})$ , and since  $\lambda_{N-1} = (a_{N-1} + \gamma\tau_{N-1})^{-1}(1 + \gamma\tau_u\Delta a_{N-1})$ , evaluating  $\mu'_1$  at equilibrium yields:

$$\begin{aligned} \mu'_1 &= \gamma \left( \tau_{N-2} + \sum_{t=1}^{N-2} \tau_{\epsilon t} \right) (E[v|\tilde{s}_{iN-2}, z^{N-2}] - p_{N-2}) \\ &= x_{iN-2}. \end{aligned}$$

Thus  $x_{iN-2} = E[x_{iN-1}|\tilde{s}_{iN-2}, z^{N-2}]$ . Iterating this procedure it follows that  $x_{in} = E[x_{in+1}|\tilde{s}_{in}, z^n]$ .

Finally, in any period  $n$  the response to private information is equal across traders. Hence, the equilibrium is symmetric. QED

### *Proof of proposition 2*

To find the equilibria of the model we proceed as illustrated in the proof of proposition 1. In the second period, a trader behaves like in a static model and trades according to

$$\begin{aligned} X_2(\tilde{s}_{i2}, z^2) &= \gamma \frac{E[v|\tilde{s}_{i2}, z^2] - p_2}{\text{Var}[v + \delta|\tilde{s}_{i2}, z^2]} \\ &= a_2(\tilde{s}_{i2} - p_2) + b_2(E[v|z^2] - p_2), \end{aligned} \tag{32}$$

where

$$a_2 = \frac{\gamma(\tau_{\epsilon_1} + \tau_{\epsilon_2})}{1 + \kappa}, \quad (33)$$

$b_2 = \gamma\tau_2/(1 + \kappa)$ , and  $\kappa = (\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})/\tau_\delta$ . Imposing market clearing, we find that the second period equilibrium price is given by

$$p_2 = \lambda_2 z_2 + (1 - \lambda_2 \Delta a_2) \hat{p}_1,$$

where  $\lambda_2 = (\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})^{-1}(1 + \gamma\tau_u \Delta a_2 + \kappa)$ ,  $\Delta a_2 = a_2 - \beta a_1$ , and

$$\hat{p}_1 = \frac{\gamma\tau_1 E[v|z_1] + \beta(1 + \kappa)z_1}{\gamma\tau_1 + \beta a_1(1 + \kappa)}.$$

Notice that  $\alpha_{P2} = (\gamma\tau_\delta(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t}))^{-1}a_2(1 + \kappa) = \alpha_{E2}$ . Hence, in the second period the price assigns the optimal statistical weight to public information. Substituting (32) in the second period objective function, a trader in the first period maximizes

$$E[U(\pi_{i1} + \pi_{i2})|s_{i1}, z_1] = -E \left[ \exp \left\{ -\frac{1}{\gamma} \left( (p_2 - p_1)x_{i1} + \frac{x_{i2}^2 \text{Var}[v + \delta|\tilde{s}_{i2}, z^2]}{2\gamma} \right) \right\} |s_{i1}, z_1 \right]. \quad (34)$$

Let  $\phi_{i1} = (p_2 - p_1)x_{i1} + x_{i2}^2 \text{Var}[v + \delta|\tilde{s}_{i2}, z^2]/(2\gamma)$ . The term  $\phi_{i1}$  is a quadratic form of the random vector  $Z = (x_{i2} - \mu_1, p_2 - \mu_2)'$ , which is normally distributed (conditionally on  $s_{i1}, p_1$ ) with zero mean and variance covariance matrix

$$\Sigma = \begin{pmatrix} \text{Var}[x_{i2}|s_{i1}, z_1] & \text{Cov}[x_{i2}, p_2|s_{i1}, z_1] \\ \text{Cov}[x_{i2}, p_2|s_{i1}, z_1] & \text{Var}[p_2|s_{i1}, z_1] \end{pmatrix}, \quad (35)$$

where

$$\begin{aligned} \text{Var}[x_{i2}|s_{i1}, z_1] &= \frac{(\Delta a_2(1 + \kappa) - \gamma\tau_{\epsilon_2})^2 \tau_u + (\tau_1 + \tau_{\epsilon_1})((1 + \kappa)^2 + \gamma^2 \tau_u \tau_{\epsilon_2})}{(\tau_1 + \tau_{\epsilon_1})\tau_u(1 + \kappa)^2}, \\ \text{Cov}[x_{i2}, p_2|s_{i1}, z_1] &= \lambda_2 \left( \frac{(\gamma\tau_{\epsilon_2} - (1 + \kappa)\Delta a_2)\Delta a_2 \tau_u - (1 + \kappa)(\tau_1 + \tau_{\epsilon_1})}{(\tau_1 + \tau_{\epsilon_1})\tau_u(1 + \kappa)} \right), \\ \text{Var}[p_2|s_{i1}, z_1] &= \lambda_2^2 \left( \frac{\tau_2 + \tau_{\epsilon_1}}{(\tau_1 + \tau_{\epsilon_1})\tau_u} \right), \end{aligned}$$

(i.e.  $\mu_1 = E[x_{i2}|s_{i1}, z_1] = (a_2 + b_2)(1 - \lambda_2 \Delta a_2)(E[v|s_{i1}, z_1] - \hat{p}_1)$  and  $\mu_2 = E[p_2|s_{i1}, z_1] = \lambda_2 \Delta a_2 E[v|s_{i1}, z_1] + (1 - \lambda_2 \Delta a_2) \hat{p}_1$ ):

$$\phi_{i1} = c + b'Z + Z'AZ,$$

where  $c = (\mu_2 - p_1)x_{i1} + \mu_1^2 \text{Var}[v + \delta|\tilde{s}_{i2}, z^2]/(2\gamma)$ ,  $b = (\mu_1 \text{Var}[v + \delta|\tilde{s}_{i2}, z^2]/\gamma, X_{i1})'$ , and  $A$  is a  $2 \times 2$  matrix with  $a_{11} = \text{Var}[v + \delta|\tilde{s}_{i2}, z^2]/(2\gamma)$  and the rest zeroes. We can now rewrite the objective function (34) as

$$E[U(\pi_{i1} + \pi_{i2}) | s_{i1}, z_1] = \tag{36}$$

$$- |\Sigma|^{-1/2} |\Sigma^{-1} + 2/\gamma A|^{-1/2} \times \exp \left\{ -1/\gamma \left( c - \frac{1}{2\gamma} b' (\Sigma^{-1} + 2/\gamma A)^{-1} b \right) \right\}.$$

Maximizing the above function with respect to  $x_{i1}$  yields

$$x_{i1} = \gamma \frac{\mu_2 - p_{N-1}}{h_{22}} - \frac{h_{21} \mu_1 \text{Var}[v + \delta|\tilde{s}_{i2}, z^2]}{\gamma h_{22}}, \tag{37}$$

where  $h_{ij}$  denotes the  $ij$ -th term of the symmetric matrix  $H = (\Sigma^{-1} + 2/\gamma A)^{-1}$ :

$$\begin{aligned} h_{11} &= \frac{\gamma^2 \text{Var}_1[x_{i2}]}{D}, \\ h_{12} &= \frac{\gamma^2 \text{Cov}_1[x_{i2}, p_2]}{D}, \\ h_{22} &= \frac{\text{Var}_1[p_2](\gamma^2 + \text{Var}_2[v + \delta] \text{Var}_1[x_{i2}](1 - \rho_{1,\{x_{i2}, p_2\}}^2))}{D}, \end{aligned}$$

and  $\text{Var}_n[Y] = \text{Var}[Y|\tilde{s}_{in}, z^n]$ ,  $\text{Cov}_1[X, Y] = \text{Cov}[X, Y|s_{i1}, z_1]$ ,

$$\rho_{1,\{x_{i2}, p_2\}}^2 = \frac{\text{Cov}_1[x_{i2}, p_2]^2}{\text{Var}_1[p_2] \text{Var}_1[x_{i2}]},$$

$D = \gamma^2 + \text{Var}_1[x_{i2}] \text{Var}_2[v + \delta]$ . Substituting in (37) the expressions for  $\mu_1$  and  $\mu_2$  and rearranging we obtain

$$\begin{aligned} X_1(s_{i1}, p_1) &= \frac{a_1(\tau_1 + \tau_{\epsilon_1})}{\tau_{\epsilon_1}} (E[v|s_{i1}, z_1] - \hat{p}_1) - \frac{\gamma}{h_{22}} (p_1 - \hat{p}_1) \\ &= \frac{a_1(\tau_1 + \tau_{\epsilon_1})}{\tau_{\epsilon_1}} (E[v|s_{i1}, z_1] - p_1) + \frac{(\gamma + h_{21})(1 - \lambda_2 \Delta a_2)}{h_{22}} (\hat{p}_1 - p_1), \end{aligned} \tag{38}$$

where  $a_1$  denotes the 1st period trading aggressiveness:

$$\begin{aligned} a_1 &= \frac{\tau_{\epsilon_1} (\gamma \lambda_2 \Delta a_2 - h_{21} (1 - \lambda_2 \Delta a_2))}{h_{22} (\tau_1 + \tau_{\epsilon_1})} \\ &= \frac{\gamma \tau_{\epsilon_1} (\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t}) (1 + \kappa) (1 + \gamma \tau_u \Delta a_2)}{(1 + \kappa + \gamma \tau_u \Delta a_2) (\tau_{\epsilon_2} + (\tau_2 + \tau_{\epsilon_1}) (1 + \kappa))}. \end{aligned} \tag{39}$$

Both (33) and (39) are implicit solutions for  $a_1$  and  $a_2$ . Thus, equilibria must be determined via numerical methods.<sup>33</sup> Rearranging the first period equilibrium strategy

<sup>33</sup>If  $\tau_\delta \rightarrow \infty$ , residual noise vanishes,  $\kappa \rightarrow 0$  and  $a_n \rightarrow \gamma(\sum_{t=1}^n \tau_{\epsilon_t})$  as in proposition 1.

yields the second of (11). Using this expression and imposing market clearing yields  $p_1 = \lambda_1 z_1 + (1 - \lambda_1 a_1) \bar{v}$ , where

$$\lambda_1 = \frac{\alpha_{P1}}{a_1} + \frac{(1 - \alpha_{P1})a_1 \tau_u}{\tau_1},$$

and the expression for  $\alpha_{P1}$  is given by (15). In order to obtain the expression for the first period equilibrium price given in (14), we impose market clearing on (38) obtaining  $\tau_{\epsilon_1}^{-1} a_1 (\tau_1 + \tau_{\epsilon_1}) \int_0^1 (E_1[v] - \hat{p}_1) di - (\gamma/h_{22})(p_1 - \hat{p}_1) + \theta_1 = 0$ . Simplifying the latter condition yields

$$\frac{\gamma \tau_1 (\beta \rho - 1)}{(\gamma \tau_1 + \beta a_1 (1 + \kappa))} E[\theta_1 | z_1] - \frac{\gamma}{h_{22}} (p_1 - \hat{p}_1) = 0, \quad (40)$$

which can be easily rearranged to obtain (14). Notice also that according to the above market clearing equation the first period equilibrium price will differ from the static solution as long as  $\rho \neq 1/\beta$ .

To show existence note that (12) defines a system of non-linear equations. Let us denote with  $f(a_1, a_2) = 0$  the equation defining  $a_2$ , and with  $g(a_1, a_2) = 0$  the equation defining  $a_1$ . Both  $f(\cdot)$  and  $g(\cdot)$  are continuous. In particular, it is easy to check that  $f(a_1, a_2) = (\tau_\delta + \sum_{t=1}^2 \tau_{\epsilon_t})(a_2^3 \tau_u - 2a_2^2 a_1 \tau_u + a_2(\tau_1 + a_1^2 \tau_u)) - \gamma \tau_\delta (\sum_{t=1}^2 \tau_{\epsilon_t}) = 0$  is a nondegenerate cubic in  $a_2$ , given that  $(\tau_\delta + \sum_{t=1}^2 \tau_{\epsilon_t}) \tau_u > 0$ , and always admits a real solution for any  $a_1$ . Furthermore, since  $\partial f / \partial a_2 = (\tau_\delta + \sum_{t=1}^2 \tau_{\epsilon_t})(3a_2^2 \tau_u - 4a_2 a_1 \tau_u + \tau_1 + a_1^2 \tau_u)$  and the discriminant associated to this quadratic equation in  $a_1$  can be shown to be negative, we have that  $\partial f / \partial a_2 \neq 0$  and the solutions to the cubic equation are continuous in  $a_1$ .<sup>34</sup> Hence, denoting by  $a_2(a_1)$  a (real) solution to the cubic we have that

$$\lim_{a_1 \rightarrow 0} a_2(a_1) = \bar{a}_2 > 0, \quad \lim_{a_1 \rightarrow \infty} a_2(a_1) = 0.$$

We can now verify that a real solution always exists to the equation  $g(a_1, a_2(a_1)) = 0$ . Indeed,

$$\begin{aligned} \lim_{a_1 \rightarrow 0} g(a_1, a_2(a_1)) &= \gamma \tau_\delta \tau_{\epsilon_1} (1 + \bar{a}_2 \gamma \tau_u) \left( \tau_v + \bar{a}_2^2 \tau_u + \sum_{t=1}^2 \tau_{\epsilon_t} + \tau_\delta \right) > 0, \\ \lim_{a_1 \rightarrow \infty} g(a_1, a_2(a_1)) &= -\infty, \end{aligned}$$

and the result follows.

<sup>34</sup>Indeed, as one can check  $\Delta \equiv 16a_1^2 \tau_u^2 - 12\tau_u(\tau_1 + a_1^2 \tau_u) = -(8a_1^2 \tau_u + 12\tau_v) \tau_u < 0$ .

We are now left with the task of proving that in any linear equilibrium  $\rho > 1$ . Notice that in any equilibrium  $a_1 > 0$ , hence if  $1 + \gamma\tau_u\Delta a_2 > (<)0$ , then also  $1 + \kappa + \gamma\tau_u\Delta a_2 > (<)0$ .<sup>35</sup> Notice also that if  $\Delta a_2 < 0$  then  $1 + \kappa + \gamma\tau_u\Delta a_2 < 0$ . To see this last point, compute  $\Delta a_2$  using (33) and (39):

$$\Delta a_2 = \frac{\gamma}{D} \left( \tau_{\epsilon_2}(1 + \kappa + \gamma\tau_u\Delta a_2)(\tau_{\epsilon_2} + \tau_{\epsilon_1} + \tau_2(1 + \kappa)) - \tau_{\epsilon_1}\kappa(1 + \kappa)\gamma\tau_u\Delta a_2 \left( \tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t} \right) \right), \quad (41)$$

where  $D = (1 + \kappa + \gamma\tau_u\Delta a_2)(\tau_{\epsilon_2} + (\tau_2 + \tau_{\epsilon_1})(1 + \kappa))(1 + \kappa)$ . Suppose that  $\Delta a_2 < 0$  but that  $(1 + \kappa + \gamma\tau_u\Delta a_2) > 0$ , then given (41) this is impossible.

To prove our claim start by assuming that  $\Delta a_2 > 0$ , then using (39) we can directly check whether  $\rho < 1$  since as one can see

$$a_1 < \frac{\gamma\tau_{\epsilon_1}}{1 + \kappa} \Leftrightarrow \tau_{\epsilon_2}(1 + \gamma\tau_u\Delta a_2 + \kappa) + (1 + \kappa) \left( \tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t} \right) \gamma\tau_u\Delta a_2 < 0,$$

which is clearly impossible. If, on the other hand  $\Delta a_2 < 0$ , given what we have said above for  $\rho < 1$  we need

$$\tau_{\epsilon_2}(1 + \gamma\tau_u\Delta a_2 + \kappa) + (1 + \kappa) \left( \tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t} \right) \gamma\tau_u\Delta a_2 > 0,$$

which is again impossible.<sup>36</sup> Therefore, in any linear equilibrium  $\rho > 1$ .

QED

### *Proof of proposition 3*

In the second period a trader speculates according to  $X_2(\tilde{s}_{i2}, p^2) = (\gamma/\text{Var}[v|\tilde{s}_{i2}, z^2]) \times (E[v|\tilde{s}_{i2}, z^2] - p_2)$ . Imposing market clearing yields

$$\begin{aligned} \int_0^1 x_{i2} di + \theta_2 &= \\ &= a_2(v - p_2) + \gamma\tau_2(E[v|z^2] - p_2) + \theta_2 \\ &= (a_2 - \beta a_1)v + \beta a_1 v + u_2 + \beta\theta_1 - (a_2 + \gamma\tau_2)p_2 + \gamma\tau_2 E[v|z^2] = 0, \end{aligned} \quad (42)$$

<sup>35</sup>For suppose  $a_1 < 0$ , then  $\Delta a_2 > 0$  and both  $1 + \gamma\tau_u\Delta a_2 > 0$  and  $1 + \gamma\tau_u\Delta a_2 + \kappa > 0$ , implying  $a_1 > 0$ , a contradiction.

<sup>36</sup>Suppose  $\tau_{\epsilon_2} = 0$ , then  $a_1$  must satisfy  $(a_1(1 + \kappa) - \gamma\tau_{\epsilon_1}) + \gamma\tau_u\Delta a_2(a_1 - \gamma\tau_{\epsilon_1}) = 0$ . It is easy to see that a solution to this equation is  $a_1 = a_2 = \gamma\tau_{\epsilon_1}/(1 + \kappa)$ . Then,  $\alpha_{P1} = \alpha_{E1}$  and the first period price assigns the optimal statistical weight to public information. The intuition is straightforward: if  $\tau_{\epsilon_2} = 0$  the second period price is just a noisy version of  $p_1$ , traders do not expect any price variation that justifies a rebalancing of their speculative position in the second period, and the absence of over-reliance on public information when  $n = 2$  also extends to the first period.

where  $a_2 = \gamma(\tau_{\epsilon_1} + \tau_{\epsilon_2})$ , and  $\theta_2 = u_2 + \beta\theta_1$ . Let  $\Delta a_2 = a_2 - \beta a_1$ , then the above market clearing condition can be rewritten as

$$z_2 + \beta z_1 + \gamma\tau_2 E[v|z^2] = (a_2 + \gamma\tau_2)p_2,$$

where  $z_2 = \Delta a_2 v + u_2$ . Rearranging it yields

$$p_2 = \lambda_2 z_2 + (1 - \lambda_2 \Delta a_2) \hat{p}_1,$$

where  $\lambda_2 = (a_2 + \gamma\tau_2)^{-1}(1 + \gamma\tau_2 \Delta a_2)$ , and

$$\hat{p}_1 = \frac{\gamma\tau_v \bar{v} + (\beta + \gamma\tau_u a_1) z_1}{\beta a_1 + \gamma\tau_1}.$$

In the first period owing to short-term horizons  $X_1(s_{i1}, p_1) = (\gamma/\text{Var}[p_2|s_{i1}, z_1])(E[p_2|s_{i1}, z_1] - p_1)$ , where  $E[p_2|s_{i1}, z_1] = \lambda_2 \Delta a_2 E[v|s_{i1}, z_1] + (1 - \lambda_2 \Delta a_2) \hat{p}_1$ , and  $\text{Var}[p_2|s_{i1}, z_1] = \lambda_2^2 (\tau_2 + \tau_{\epsilon_1}) / ((\tau_1 + \tau_{\epsilon_1}) \tau_u)$ . Identifying the first period signal responsiveness yields

$$a_1 = \gamma \frac{\Delta a_2 \tau_u \tau_{\epsilon_1}}{\lambda_2 (\tau_2 + \tau_{\epsilon_1})}. \quad (43)$$

Let  $f(a_1) \equiv a_1 \lambda_2 (\tau_2 + \tau_{\epsilon_1}) - \gamma \Delta a_2 \tau_u \tau_{\epsilon_1}$ . The solutions to the quartic  $f(a_1) = 0$  clearly identify the equilibria of the model. Notice that  $f(0) = -a_2 \gamma \tau_{\epsilon_1} \tau_u < 0$ , while

$$f(\gamma\tau_{\epsilon_1}) = \frac{\tau_{\epsilon_1} ((2(1-\beta) + \beta^2) \gamma^2 \tau_{\epsilon_1}^2 \tau_u + \tau_{\epsilon_1} (1 + (1-\beta) \gamma^2 \tau_{\epsilon_2} \tau_u) + \tau_v)}{\tau_{\epsilon_2} + (2(1-\beta) + \beta^2) \gamma^2 \tau_{\epsilon_1}^2 \tau_u + \gamma^2 \tau_{\epsilon_2}^2 \tau_u + \tau_{\epsilon_1} (1 + 2(1-\beta) \gamma^2 \tau_{\epsilon_2} \tau_u) + \tau_v} > 0.$$

Hence, a first solution  $a_1^*$  to the equation  $f(a_1) = 0$  belongs to the interval  $(0, \gamma\tau_{\epsilon_1})$ .

Next, since

$$f(a_2/\beta) = \frac{a_2 (\tau_2 + \tau_{\epsilon_1})}{\beta \gamma (\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})} > 0,$$

and  $\lim_{a_1 \rightarrow \infty} f(a_1) = -\infty$ , a further solution  $a_1^{**}$  to  $f(a_1) = 0$  belongs to the interval  $(a_2/\beta, \infty)$ . To see that these are the only two real solutions (i.e. that the remaining two roots must be complex), notice that the cubic equation  $f'(a_1) = 0$  has a unique real root (its discriminant is positive). Hence, the graph of  $f(a_1)$  changes slope only once (between  $a_1^*$  and  $a_1^{**}$ ). Rearranging the first period market clearing equation yields (20), where  $\alpha_{P1} \equiv \alpha_{E1} (1 + (\beta\rho - 1) \tau_1 / (\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t}))$ , and  $\rho = a_1 / (\gamma\tau_{\epsilon_1})$ . According to the latter of (20), over-reliance on public information occurs whenever  $\alpha_{P1} < \alpha_{E1} \Leftrightarrow a_1 < (\gamma\tau_{\epsilon_1} / \beta)$ . which in the first equilibrium is always satisfied since we have  $a_1^* < \gamma\tau_{\epsilon_1} < \gamma\tau_{\epsilon_1} / \beta$ . Next, insufficient reliance on public information arises if and only if  $\alpha_{P1} > \alpha_{E1} \Leftrightarrow a_1 > (\gamma\tau_{\epsilon_1} / \beta)$ , which in the second equilibrium is



again always satisfied since  $a_1^{**} > a_2/\beta > \gamma\tau_{\epsilon_1}/\beta$ . Using the definition of  $\alpha_{P1}$  and rearranging the pricing equation yields:  $p_1 = \lambda_1 z_1 + (1 - \lambda_1 a_1)\bar{v}$ , where

$$\lambda_1 = \frac{\alpha_{P1}}{a_1} + \frac{(1 - \alpha_{P1})a_1\tau_u}{\tau_1}.$$

Finally, note that for  $f(a_1) \equiv a_1\lambda_2(\tau_2 + \tau_{\epsilon_1}) - \gamma\Delta a_2\tau_u\tau_{\epsilon_1} = 0$  to have a real solution it must be the case that  $\lambda_2$  and  $\Delta a_2$  have the same sign. In the high trading intensity equilibrium  $a_1^{**} > a_2/\beta$ , and  $\Delta a_2 < 0$ . Therefore,  $\lambda_2(a_1^{**}) < 0$ .

QED

*Proof of corollary 6*

Imposing  $\beta = 0$  in (43) one can see that the linear equilibria of the model are given by the solutions to the following cubic equation  $f(a_1) \equiv a_1\lambda_2(\tau_2 + \tau_{\epsilon_1}) - \gamma a_2\tau_u\tau_{\epsilon_1} = 0$ . Now, it is immediate to check that  $f(0) < 0$ ,  $f(\gamma\tau_{\epsilon_1}) > 0$ , and that

$$f'(a_1) = \frac{(1 + \gamma a_2\tau_u)(2a_1^2\tau_u\tau_{\epsilon_2} + (\tau_2 + \tau_{\epsilon_1})(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t}))}{\gamma(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})^2} > 0,$$

showing that the equilibrium is unique.

QED

*Proof of proposition 4*

To prove the claim we check what happens to  $\rho = a_1(\tau_\delta + \tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})/(\gamma\tau_\delta\tau_{\epsilon_1})$  as  $\tau_\delta \rightarrow 0$ . Notice that

$$\begin{aligned} \lim_{\tau_\delta \rightarrow 0} a_1 &= \lim_{\tau_\delta \rightarrow 0} \frac{\gamma^2\tau_\delta\tau_{\epsilon_1}\tau_u\Delta a_2(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})}{(\tau_2 + \tau_{\epsilon_1})(\tau_\delta(1 + \gamma\Delta a_2\tau_u) + (\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t}))} \\ &= 0, \end{aligned}$$

and  $\lim_{\tau_\delta \rightarrow 0} a_2 = \lim_{\tau_\delta \rightarrow 0} \gamma\tau_\delta(\tau_{\epsilon_1} + \tau_{\epsilon_2})/(\tau_\delta + (\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})) = 0$ . Then,

$$\begin{aligned} \lim_{\tau_\delta \rightarrow 0} \rho &= \lim_{\tau_\delta \rightarrow 0} \frac{\gamma\tau_u\Delta a_2(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})}{\tau_2 + \tau_{\epsilon_1}} \\ &= 0, \end{aligned}$$

since, as argued above, in the limit  $a_1 = a_2 = 0$ . By continuity of  $\rho$  as a function of  $(a_1, a_2)$ , there must then exist an interval  $(0, \tau_\delta^*)$ , such that for all  $\tau_\delta \in (0, \tau_\delta^*)$  we have  $\rho < 1$ .

QED

## Appendix B

In this appendix we generalize the pricing formula obtained in section 4.1 to the  $N \geq 2$ -period case. To fix notation, let  $\bar{E}_n[Y] = \int_0^1 E[Y|\tilde{s}_{in}, p^n] di$ , and  $\text{Var}_n[Y] = \text{Var}[Y|\tilde{s}_{in}, p^n]$ . In the  $N$ -th period the market clearing equation reads as follows:

$$\gamma \frac{\bar{E}_N[v] - p_N}{\text{Var}_N[v]} + \theta_N = 0.$$

Therefore, the price of the asset in period  $N$  is given by

$$p_N = \bar{E}_N[v] + \frac{\text{Var}_N[v]}{\gamma} \theta_N, \quad (44)$$

In period  $N - 1$ , optimality and market clearing require that

$$\gamma \frac{\bar{E}_{N-1}[p_N] - p_{N-1}}{\text{Var}_{N-1}[p_N]} + \theta_{N-1} = 0,$$

and using (44) the corresponding market clearing price is given by:

$$\begin{aligned} p_{N-1} &= \bar{E}_{N-1}[p_N] + \frac{\text{Var}_{N-1}[p_N]}{\gamma} \theta_{N-1} \\ &= \bar{E}_{N-1} \left[ \bar{E}_N[v] + \frac{\text{Var}_N[v]}{\gamma} \beta \theta_{N-1} \right] + \frac{\text{Var}_{N-1}[p_N]}{\gamma} \theta_{N-1}. \end{aligned}$$

Iterating this procedure, and using the fact that

$$\theta_n = \beta \left( \beta^{n-1} \theta_1 + \sum_{t=1}^{n-1} \beta^{n+1-t} u_t \right) + u_n,$$

recursive substitution yields

$$\begin{aligned} p_n &= \\ &\bar{E}_n \left[ \bar{E}_{n+1} \left[ \dots \bar{E}_{N-1} \left[ \bar{E}_N[v] + \frac{\text{Var}_N[v]}{\gamma} \beta^{N-n} \theta_n \right] + \frac{\text{Var}_{N-1}[p_N]}{\gamma} \beta^{N-(n+1)} \theta_n \dots \right] + \frac{\text{Var}_{n+1}[p_{n+2}]}{\gamma} \beta \theta_n \right] \\ &\quad + \frac{\text{Var}_n[p_{n+1}]}{\gamma} \theta_n, \end{aligned}$$

for  $1 \leq n \leq N$ .

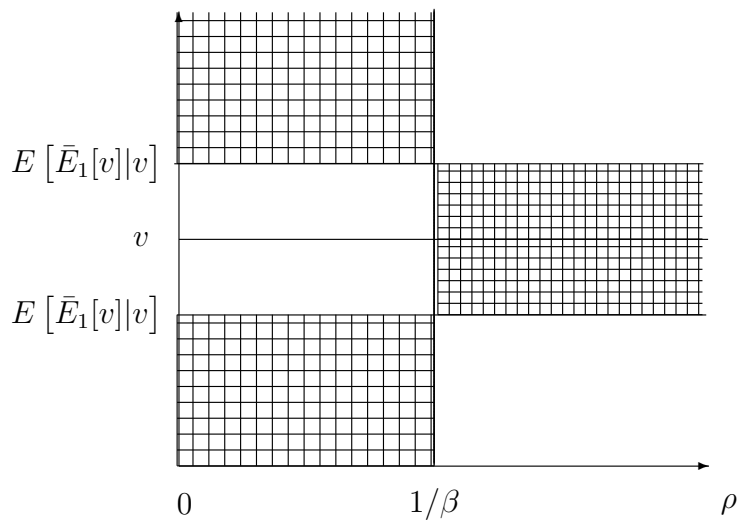


Figure 1: If  $\rho < 1/\beta$  ( $\rho > 1/\beta$ )  $E[p_1|v]$  lays in the thickly (thinly) meshed area, and over- (under-) reliance on public information occurs. Note that since  $E[\bar{E}_1[v]|v] = \alpha_{E1}v + (1 - \alpha_{E1})E[E[v|z_1]|v]$ ,  $E[\bar{E}_1[v]|v] \geq v \Leftrightarrow \bar{v} \geq v$ .

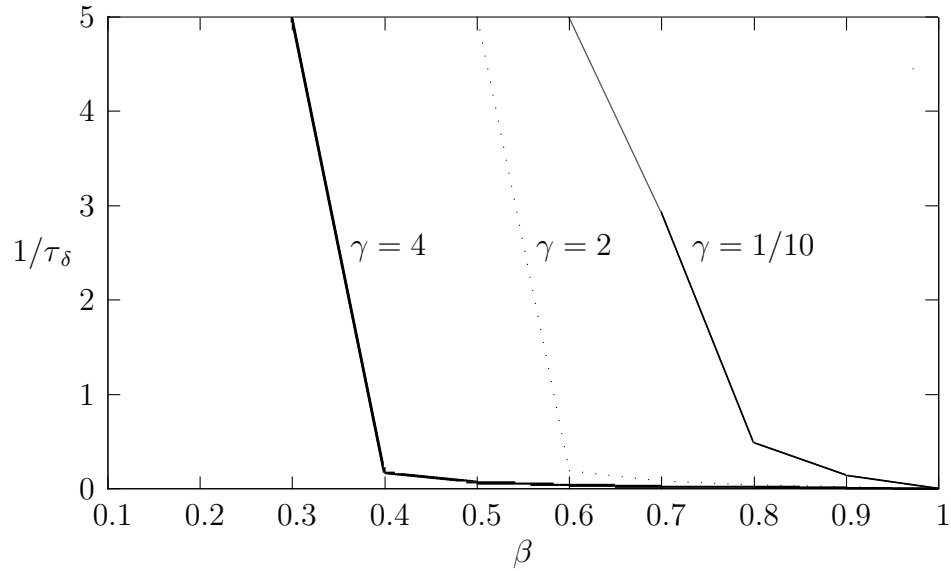


Figure 2: The figure plots the set  $\Omega \equiv \{(\beta, 1/\tau_\delta) \in [0, 1] \times \mathfrak{R}_+ \mid \beta\rho(\beta) = 1\}$  for  $\gamma \in \{1/10, 2, 4\}$  (parameters' values:  $\tau_v = \tau_u = \tau_{\epsilon_n} = 1$ ). Values of  $(\beta, 1/\tau_\delta)$  that fall below (above) this set identify market conditions leading to over (under) reliance on public information. As  $\gamma$  increases, traders speculate more and more aggressively on their first period information. This increases the relevance of informed trades in the aggregate order-flow, widening the parameter space for which under reliance on public information occurs.

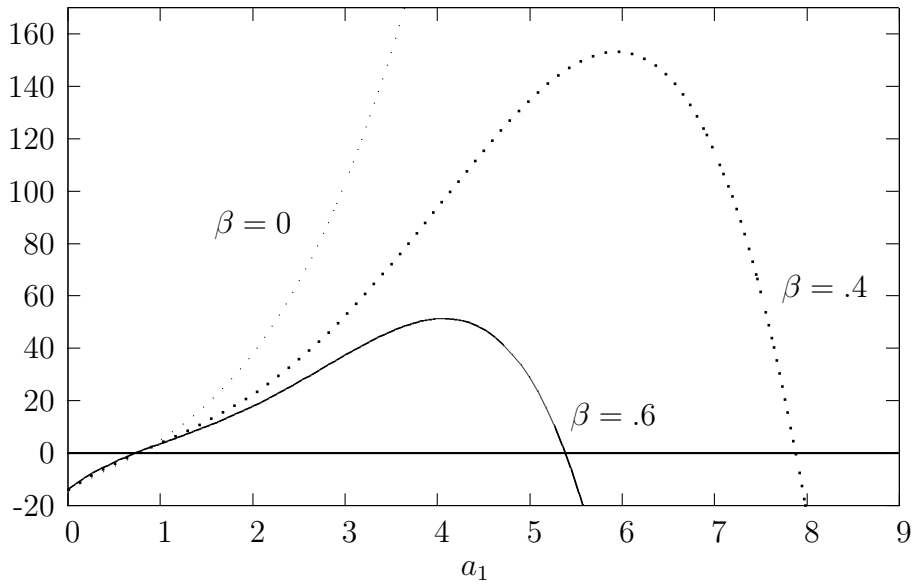


Figure 3: The continuous (heavily, lightly dotted) curve graphs the equation that determines the equilibria when  $\beta = .6$  ( $\beta = .4$ ,  $\beta = 0$ ). Since  $a_1^* < \gamma\tau_{\epsilon_1}/\beta < a_1^{**}$ , as  $\beta \rightarrow 0$ ,  $\gamma\tau_{\epsilon_1}/\beta \rightarrow \infty$ , the high trading intensity equilibrium disappears and only the low trading intensity equilibrium survives (parameter values:  $\tau_v = \tau_u = \tau_{\epsilon_n} = \gamma = 1$  and  $\beta \in \{0, .4, .6\}$ ).

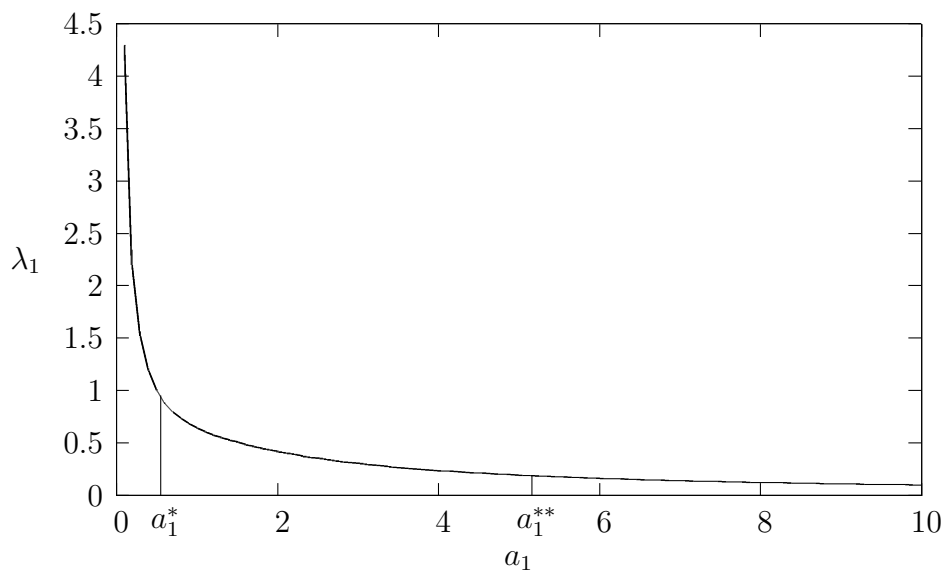


Figure 4: Differently from the market with long-term traders, with short-term traders a over- (under-) reliance on public information occurs in a thin (thick) market (parameter values:  $\tau_v = \tau_u = \tau_{\epsilon_n} = \gamma = 1$  and  $\beta = .6$ ;  $a_1^* = 0.732$ ,  $a_1^{**} = 5.393$ ).