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Social Networks: Equilibrium Selection and Friendliness

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Abstract

Given their importance in determining the outcome of many economic interactions, different models have been proposed to determine how social networks form and which structures are stable. In Bala and Goyal (2000), the one-sided link formation model has been considered, which is based on a noncooperative game of network formation. They found out that the empty networks, the wheel in the one-way flow of benefits case and the center sponsored star in the two-way flow case play a fundamental role, since they are strict Nash equilibria of the corresponding games for certain classes of payoff functions. In this paper, firstly we prove that all these network structures are in weakly dominated strategies whenever there are no strict Nash equilibria. Then, we exhibit a more accurate selection device between these network architectures by considering 'altruistic behavior' refinements. Such refinements, that we investigate here in the framework of finite strategy sets games, have been introduced by the authors in previous papers.

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1 Introduction

The role of social-relationship network structures has been studied in many economic situations and their importance in determining the outcome of many economic interactions has been documented also in empirical work. Therefore, different models have been proposed to determine how these networks form and which structures are stable. The basic assumption behind models of network formation is that establishing and maintaining connections with other individuals is costly. As a consequence, individuals limit the number and the intensity of their connections and then network structures develop from agents' comparison of disutility (costs) versus benefits of connection.

In previous papers, the two-sided link formation, i.e. a situation in which a link between two people requires that both of them make some investments, has been investigated and the notion of pairwise stability has been considered and analyzed in a cooperative theoretic game framework (see Jackson and Wolinsky (1986) and Dutta and Mutuswami (1987)).

Successively, in Bala and Goyal (2000), the authors studied the one-sided link formation, analyzed via a notion of individual stability based on a simple game of network formation where each player simultaneously selects a list of the other players with whom he wishes to be linked. Individual stability then corresponds to a (pure strategy) Nash equilibrium of this game. In their paper, both the one-way and two-way flow of benefits (which correspond to directed and not directed graphs) are considered, and agents are supposed to be symmetric and maximizing real valued payoff functions depending on two variables: the number of people (directly or indirectly) accessed and the number of links the agent forms himself. Moreover payoffs are assumed to be strictly increasing in the first variable and strictly decreasing in the second one.
The first important result in Bala and Goyal is that a *Nash network* is either empty or connected, i.e., there is a path between any couple of players. However, it turns out that there is a great number of Nash network so, as stated in Bala and Goyal, *"multiplicity of equilibria motivates an examination of a stronger equilibrium concept"*. They focus on the concept of strict Nash equilibrium which is characterized by the uniqueness of the best reply correspondences in equilibrium. They find out that in the one-way flow model the unique not empty strict Nash network is the *wheel*, while, in the two-way flow model the unique not empty strict Nash network is the *center-sponsored star*.
However, in the Theory of Refinements of Nash Equilibria, it is well known that the strict Nash equilibrium concept might be too restrictive. Indeed, in the network formation models, strict Nash exist only for a certain class of payoff functions. In this paper we extend the analysis of the equilibria in the cases when strict Nash equilibria do not necessarily exist. A first approach is to consider the Admissibility Property (i.e. each equilibrium strategy is weakly undominated) because it is a necessary condition for many other classical refinement concepts based on stability with respect to perturbations: *perfect equilibria* (Selten (1975)), *proper equilibria* (Myerson (1978)), *essential equilibria* (Wu and Jiang (1962)), *regular equilibria* (Harsanyi (1973) and Ritzberger (1994)) ect... .
A different approach is to consider refinement concepts based on altruistic behavior of the players as done by the authors in previous papers. In fact, in De Marco and Mor-

gan (2007, 2008) the possibility of altruistic behavior of the players has been considered and used as a refinement criterion; in particular, two different refinement concepts have been introduced. *Friendliness equilibria* (De Marco and Morgan (2007)) are based on a property of robustness of the equilibrium with respect to a particular class of deviations: a player is supposed to move away from the equilibrium even only to guarantee a better payoff to the others and feasible deviations are unilateral and only towards Nash equilibria, that is *valid* deviations in the sense that there are no incentives to deviate from the deviation (see Bernheim et al. (1987) for valid deviations of coalitions of players). Recall also that *fiendly behavior* has been defined and used by Rusinowska (2002) for equilibrium selection in some 2-players bargaining models. On the other hand, the concept of *slightly altruistic equilibrium* (De Marco and Morgan (2008)) is based on a stability property with respect to trembles which capture an idea of reciprocal altruism: each player cares only about himself but his choice corresponds to the limit of choices he would have done in equilibrium if he had cared about the others, provided the others had done the same. In general, a slightly altruistic equilibrium is not necessarily a friendliness equilibrium and viceversa. However, sufficient conditions on the payoffs of the game guarantee that every slightly altruistic equilibrium is a friendliness equilibrium. Moreover, it is possible to enforce the robustness property of friendliness equilibria to obtain strategy profiles which are also slightly altruistic equilibria.

Aim of this paper is to analyze the equilibria of the one-way flow and the two-way flow models in light of the Admissibility Property and of the refinements concepts based on altruistic behavior. Since these games have finite strategy sets, firstly we look at the connections between frindliness and slightly altruistic equilibria in this context: it turns out that friendliness equilibria are slightly altruistic equilibria while the converse statement still doesn't hold. Then we look at the Bala and Goyal models and find out that empty networks, wheel and center sponsored star do not satisfy the Admissibility Property whenever they are not strict Nash equilibria, so that it seems hard to obtain a sharper and self-enforcing selection mechanism between these concepts in such a case. The previous arguments suggest to take into account the altruistic behavior refinements as a reasonable selection device between the network architectures quoted above. Indeed, it turns out that this latter selection device is effective: the wheel and the center sponsored star are friendliness and slightly altruistic equilibria in every case in which they are Nash equilibria, while the empty network is not of this kind in the two-way flow model. Moreover, the fact that the empty network is a friendliness and a slightly altruistic equilibrium in the one-way flow model is not surprising since it is an *efficient* Nash network architecture for the same class of payoff functions and, in De Marco and Morgan (2007), it has been shown that friendliness and efficiency are related even in general normal form games (in terms of not empty intersection between the corresponding set of equilibria). Nevertheless, in the two-way flow model, the empty network is an inefficient Nash network for a large class of payoff functions.

2 Equilibrium Selection and Altruistic Behavior in Normal Form Games

2.1 Preliminaries

Let $\Gamma = \{I; S_1, \dots, S_n; f_1, \dots, f_n\}$ be a n-player game where $I = \{1, \dots, n\}$ is the set of players, the strategy set S_i of Player i is a subset of $\mathbb{R}^{k(i)}$ and $f_i : S \to \mathbb{R}$ is the payoff function of Player i, with $S = \prod_{i \in I} S_i$. Let $\mathcal{E}$ be the set of Nash equilibria (see Nash (1950, 1951) of the game Γ; that is, a point $s^* \in S$ belongs to E if, for every Player i, $f_i(s_i^*, s_{-i}^*) \geq f_i(s_i, s_{-i}^*)$ for all $s_i \in S_i$, where (s_i, s_{-i}^*) denotes the vector $(s_1^*, \dots, s_{i-1}^*, s_i, s_{i+1}^*, \dots, s_N^*)$. Recall also that a strategy profile $s^* \in S$ is said to be a *strict Nash equilibrium* if, for every player i, $f_i(s_i^*, s_{-i}^*) > f_i(s_i, s_{-i}^*)$ for all $s_i \in S_i \setminus \{s_i^*\}$.

2.1.1 Admissibility

Strict Nash equilibria survive to most of the refinement criteria based on stability with respect to perturbations since they are essential equilibria (see for example van Damme (1989)). However, the strict Nash equilibrium concept might be too restrictive, since in a wide class of games they do no exist, so other weaker selection devices might be taken into account such as Perfectness or Admissibility (which is a necessary condition for perfectness).

A strategy $s_i^* \in S_i$ is said to be *weakly dominated* by the strateqy $\overline{s}_i \in S_i$ (or $\overline{s}_i$ *weakly dominates* s_i^*) if

$$f_i(s_i^*, s_{-i}) \leq f_i(\overline{s}_i, s_{-i}) \quad \forall s_{-i} \in \prod_{j \neq i} S_j \quad \text{and} \quad \exists \widehat{s}_{-i} \in \prod_{j \neq i} S_j \quad \text{s.t.} \quad f_i(s_i^*, \widehat{s}_{-i}) < f_i(\overline{s}_i, \widehat{s}_{-i})$$

Moreover, s_i^* is said to be *weakly undominated* if there does not exist a strategy $\overline{s}_i \in S_i$ such that s_i^* is weakly dominated by $\overline{s}_i$.

Equilibria in weakly undominated strategies play an important role in the theory of refinements of Nash equilibria. In fact, they are related to properties of stability of the equilibrium with respect to particular classes of perturbations (for example perfect equilibria, proper equilibria or essential equilibria) so that the so called *Admissibility property* (i.e. every solution is in weakly undominated strategies) is an important selection device within the set of Nash equilibria. However, it has been shown that equilibria in weakly undominated strategies might be payoff inefficient in the set of Nash equilibria (i.e. they might be payoff Pareto dominated by a Nash equilibrium in weakly dominated strategies)(see Example 1.5.2 in van Damme (1989)).

2.2 Altruistic Behavior

Following De Marco and Morgan (2007, 2008), we now introduce the concepts of friendliness equilibrium and slightly altruistic equilibrium which intend to capture the possibility of altruistic behavior in normal games. Let $K_i : S_{-i} \rightrightarrows S_i$ be the set valued map defined by:

$$K_i(s_{-i}) = \{s_i \in S_i \mid (s_i, s_{-i}) \in \mathcal{E}\} \quad \text{for all } s_{-i} \in S_{-i}$$

where $\mathcal{E}$ is the set of Nash equilibria of the game and let $g_i : S \to \mathbb{R}$ be the function defined by:

$$g_i(s) = \sum_{j \in I \setminus \{i\}} f_j(s) \quad \text{for all } s \in S. \tag{1}$$

then,

DEFINITION 2.1: A Nash equilibrium s^* is said to be a *friendliness equilibrium* of the game Γ if, for every player i, the following *friendly behavior property* is satisfied:

$$(\mathcal{FB}): \qquad g_i(s_i^*, s_{-i}^*) \geq g_i(s_i, s_{-i}^*) \quad \text{for all } s_i \in K_i(s_{-i}^*).$$

Moreover, if we call *reversed game* associated to a game $\Gamma = \{I; S_1, \ldots, S_n; f_1, \ldots, f_n\}$ the pseudo-game (Debreu (1952))

$$\Gamma^R = \{I; S_1, \ldots, S_N; K_1, \ldots, K_n; g_1, \ldots, g_n\}.$$

then, by definition, it follows that a strategy profile $s^* \in S$ is a friendliness equilibrium of the game Γ if and only if s^* is a social Nash equilibrium (Debreu (1952)) of Γ^R.

DEFINITION 2.2: Let ε be a positive real number and, for each player i, let $h_{i,\varepsilon} : S \to \mathbb{R}$ be the function, called *ε-altruistic payoff*, defined by:

$$h_{i,\varepsilon}(s) = f_i(s) + \varepsilon \left[\sum_{j \in I \setminus \{i\}} f_j(s) \right] \qquad \text{for all } s \in S. \tag{2}$$

For every $\varepsilon > 0$, the game $\Gamma_\varepsilon = \{I; S_1, \ldots, S_n; h_{1,\varepsilon}, \ldots, h_{n,\varepsilon}\}$ is called the *ε-altruistic game* associated to Γ and E_ε denotes the set of its Nash equilibria.

Each $h_{i,\varepsilon}$ represents the utility function of Player i supposed to take into account the sum of the payoffs of the opponents with weight ε.
Therefore:

DEFINITION 2.3: A Nash equilibrium s^* of the game Γ is said to be a *slightly altruistic equilibrium* if there exist a sequence of positive real numbers $(\varepsilon_\nu)_{\nu \in \mathbb{N}}$ decreasing to 0 and a sequence of strategy profiles $(s_\nu^*)_{\nu \in \mathbb{N}} \subseteq S$, such that

i) s_ν^* is a Nash equilibrium of the ε_ν-altruistic game Γ_{ε_ν} associated to Γ, for every $\nu \in \mathbb{N}$.

ii) s_ν^* converges to s^* as $\nu \to \infty$, $\nu \in \mathbb{N}$.

Counterexamples have been given showing that in general a slightly altruistic equilibrium is not necessarily a friendliness equilibrium and viceversa. However, in Theorem 4.2 in De Marco and Morgan (2008) it has been shown that a condition of pseudomonotonicity of a particular operator associated to the game guarantees that every slightly altruistic equilibrium is a friendliness equilibrium. On the other hand, in De Marco and Morgan (2007) it has been shown that it is possible to enforce the robustness property of friendliness equilibria to obtain elements which are also slightly altruistic equilibria in the context of finite games in mixed strategies.

2.3 Finite strategy sets

Since the games of network formation that we will investigate below are games with finite strategy sets, it will be useful to investigate the connections between the two concepts in this particular case:

PROPOSITION 2.4: *Let S_i be finite sets (endowed with the discrete topology). If s^* is a slightly altruistic equilibrium then it is a friendliness equilibrium.*

Proof. Let s^* be a slightly altruistic equilibrium. Then there exist a sequence $(\varepsilon_\nu)_{\nu\in\mathbb{N}} \subset \mathbb{R}_+ \setminus \{0\}$ decreasing to 0 and a sequence of Nash equilibria, $(s_\nu)_{\nu\in\mathbb{N}} \subseteq S$, of Γ_{ε_ν} such that $s_\nu \longrightarrow s^*$ as $\nu \longrightarrow \infty$.
Since S is a finite set, there exists $\overline{\nu} > 0$ such that, for $\nu > \overline{\nu}$, s^* is a Nash equilibrium of Γ_{ε_ν}. Therefore, for $\nu > \overline{\nu}$ and for every player i,

$$f_i(s_i^*, s_{-i}^*) + \varepsilon_\nu \left[\sum_{j\in I\setminus\{i\}} f_j(s_i^*, s_{-i}^*) \right] \geq f_i(s_i', s_{-i}^*) + \varepsilon_\nu \left[\sum_{j\in I\setminus\{i\}} f_j(s_i', s_{-i}^*) \right] \quad \forall s_i' \in S_i$$

thus, since $\varepsilon_\nu > 0$, if $f_i(s_i, s_{-i}^*) = f_i(s_i^*, s_{-i}^*)$ it follows that

$$g_i(s_i^*, s_{-i}^*) = \sum_{j\in I\setminus\{i\}} f_j(s_i^*, s_{-i}^*) \geq \sum_{j\in I\setminus\{i\}} f_j(s_i, s_{-i}^*) = g_i(s_i, s_{-i}^*)$$

Since $f_i(s_i, s_{-i}^*) = f_i(s_i^*, s_{-i}^*)$ for every $s_i \in K_i(s_{-i}^*)$, then $g_i(s_i^*, s_{-i}^*) \geq g_i(s_i, s_{-i}^*)$ for all $s_i \in K_i(s_{-i}^*)$. Hence s^* is a friendliness equilibrium. □

The converse statement of the previous proposition does not hold even in the finite strategy set case as shown in the following example:

EXAMPLE 2.5: Consider the following 2 player game:

	L	R
T	1,1	-1,0
B	1,2	0,3

It is easy to check that (T, L) and (B, R) are the Nash equilibria in pure strategies, and they are, by definition, friendliness equilibria. Since the ε-altruistic game is

	L	R
T	$1+\varepsilon, 1+\varepsilon$	$-1, -\varepsilon$
B	$1+2\varepsilon, 2+\varepsilon$	$3\varepsilon, 3$

and (B,R) is the unique equilibrium in this game, then (B,R) is the unique slightly altruistic equilibrium.

However, a slight modification of Proposition (4.1) in De Marco and Morgan (2007) gives sufficient conditions for slight altruistic equilibria:

PROPOSITION 2.6: *Let S_i be finite sets (endowed with the discrete topology) and s^* be a Nash equilibrium of Γ such that, for every player i, the following property, called strong friendly behavior, is satisfied:*

$$(\mathcal{SFB}): \qquad g_i(s_i^*, s_{-i}^*) \geq g_i(s_i, s_{-i}^*) \qquad \text{for all } s_i \in BR_i(s_{-i}^*)$$

where $BR_i(s_{-i}^*)$ is the set of the best replies in Γ of Player i to his opponents' strategy profile s_{-i}^* and g_i is defined by equation (1). Then, s^* is a slightly altruistic equilibrium.

For the sake of completeness we report a short proof which, however, could be deduced from the proof of Proposition (4.1) in De Marco and Morgan (2007).

Proof of Proposition (2.6). Let s^* be a Nash equilibrium such that the property $(\mathcal{SFB})$ is satisfied for every player $i \in I$. Then, for every player $i \in I$ and for every pure strategy $\widehat{s}_i \in BR_i(s_{-i}^*)$ it follows from $(\mathcal{SFB})$ that

$$f_i(s_i^*, s_{-i}^*) + \varepsilon \left[\sum_{j \in I \setminus \{i\}} f_j(s_i^*, s_{-i}^*) \right] \geq f_i(\widehat{s}_i, s_{-i}^*) + \varepsilon \left[\sum_{j \in I \setminus \{i\}} f_j(\widehat{s}, s_{-i}^*) \right]$$

for every $\varepsilon > 0$.
Denote with $\Psi_i = \{\overline{s}_i \in S_i \mid \overline{s}_i \notin BR_i(s_{-i}^*)\}$. Then, for every $\overline{s}_i \in \Psi_i$, $f_i(s_i^*, s_{-i}^*) - f_i(\overline{s}_i, s_{-i}^*) > 0$ and there exists $\varepsilon_i(\overline{s}_i) > 0$ such that

$$f_i(s_i^*, s_{-i}^*) - f_i(\overline{s}_i, s_{-i}^*) > \varepsilon \left[\sum_{j \in I \setminus \{i\}} \left[f_j(\overline{s}_i, s_{-i}^*) - f_j(s_i^*, s_{-i}^*) \right] \right]$$

for all $0 < \varepsilon \leq \varepsilon_i(\overline{s}_i)$. Let $\varepsilon_i = \min_{\overline{s}_i \in \Psi_i} \varepsilon_i(\overline{s}_i)$, then

$$f_i(s_i^*, s_{-i}^*) - f_i(\overline{s}_i, s_{-i}^*) > \varepsilon \left[\sum_{j \in I \setminus \{i\}} \left[f_j(\overline{s}_i, s_{-i}^*) - f_j(s_i^*, s_{-i}^*) \right] \right] \quad \forall \overline{s}_i \in S_i$$

for all $0 < \varepsilon \leq \varepsilon_i$.
Therefore, if $\delta = \min_i \varepsilon_i$, s^* is a Nash equilibrium of the ε-altruistic game Γ_ε, for every $0 < \varepsilon \leq \delta$ and then s^* is a slightly altruistic equilibrium. □

3 The Social Network Formation Game: Previous Results

Following Bala and Goyal (2000), we consider one-sided link formation networks. Let $I = \{1, \dots, n\}$, with $n \geq 3$, be the set of agents, where each agent is assumed to be a source of benefits for the others. Then each agent can improve his utility connecting with the others incurring in some cost.

A strategy for a player i is a $n-1$ dimensional vector

$$x_i = (x_{i,1}, \dots, x_{i,i-1}, x_{i,i+1}, \dots, x_{i,n})$$

with $x_{i,j} \in \{0,1\}$, where $x_{i,j} = 1$ if i establishes a link with j and $x_{i,j} = 0$ otherwise, and we denote with X_i the strategy set of Player i and $X = X_1 \times \cdots \times X_n$.
A link between i and j can allow for either one-way or two-way flow of benefits. In the two-way flow of benefits $x_{i,j} = 1$ allows both i and j to access each other's benefit, while in the one-way flow $x_{i,j} = 1$ allows only Player i to access Player j's benefit.

3.1 One-way flow

In the one-way flow model, a strategy profile x depicts one and only one directed network. We say there is *one-way path* from i to j if there exists a subset $\{j_1, \dots, j_m\} \subseteq I$ such that $i = j_1$, $j = j_m$ and $x_{j_{k-1},j_k} = 1$ for all $k = 2, \dots, m$. If there is a one-way path between player i and j then i is said to be *one-way connected* with j. For every player i the payoff $\chi : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is a function that associates to (q_i, l_i) the term $\chi(q_i, l_i)$ where q_i is the number of players with whom Player i is directly or indirectly one-way connected (i included) and l_i is the number of players $j \neq i$ such that $x_{i,j} = 1$. It is assumed that χ is strictly increasing in the first variable and strictly decreasing in the second one.
Obviously, q_i and l_i depend on the network formed and hence they are functions of the strategy profile, therefore setting $\chi(q_i(x_1, \dots, x_n), l_i(x_1, \dots, x_n)) = \zeta_i(x_1, \dots, x_n)$, it is possible to consider the following game of network formation:

$$\Gamma^O = \{I; \{X_i\}_i; \{\zeta_i\}_i\}.$$

3.1.1 Nash Networks

Proposition 3.1: [Bala and Goyal (2000)]. *If x is a Nash equilibrium of Γ^O then the corresponding network is either empty or satisfies the following:*

i) Every couple of players (i, j) is one way connected.

ii) Whenever a link $x_{i,j} = 1$ is replaced with $x'_{i,j} = 0$ then, there exist at least two players who are not longer one-way connected.

In the one-way flow model we have a great variety of Nash networks, however, in this case the wheel and the empty network have an important role. A network x is said to be a *wheel* if there exists a permutation $\delta : I \to I$ such that $\delta(I) = \{j_1, \dots, j_n\}$ and

$$x_{j_1,j_2} = x_{j_2,j_3} = \cdots = x_{j_{n-1},j_n} = x_{j_n,j_1} = 1;$$

and there aren't other links while the *empty network* is obviously the network x such that $x_{i,j} = 0$ for all $i \in I$ and $j \neq i$.

Proposition 3.2: [Bala and Goyal (2000)]. *Let x be a strict Nash equilibrium of Γ^O, then the corresponding network is either empty or a wheel. Moreover,*

a) if $\chi(w+1,w) > \chi(1,0)$ for some $w \in \{1,\ldots,n-1\}$ then the wheel x is the unique strict Nash equilibrium.

b) if $\chi(w+1,w) < \chi(1,0)$ for all $w \in \{1,\ldots,n-1\}$ and $\chi(n,1) > \chi(1,0)$ then the empty network and the wheel are both strict Nash equilibria.

c) if $\chi(w+1,w) < \chi(1,0)$ for all $w \in \{1,\ldots,n-1\}$ and $\chi(n,1) < \chi(1,0)$ then the empty network is the unique strict Nash equilibrium.

3.1.2 Efficient Networks

Let $\Phi : X \to \mathbb{R}$ be the function defined by $\Phi(x) = \sum_{i\in I} \zeta_i(x)$ for all $x \in X$, then a network $\overline{x}$ is said to be *one-way efficient* if

$$\Phi(\overline{x}) \geq \Phi(x) \quad \forall x \in X$$

Proposition 3.3: [Bala and Goyal (2000)].

i) If $\chi(n,1) > \chi(1,0)$ then the wheel is the unique one-way efficient network.

ii) If $\chi(n,1) < \chi(1,0)$ then the empty network is the unique one-way efficient network.

3.2 Two-way flow

In the two-way flow model, a strategy profile x depicts one and only one undirected network. Let $\mu(x_{i,j}) = \max\{x_{i,j}, x_{j,i}\}$, then, we say there exists a *two-way path* between i and j if there exists a subset $\{j_1,\ldots,j_m\} \subseteq I$ such that $i = j_1$, $j = j_m$ and $\mu(x_{j_{k-1},j_k}) = 1$ for all $k = 2,\ldots,m$ (in this case i and j are also said to be *two-way connected*). For every player i, the payoff is the function $\psi : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ which associates to (z_i, l_i) the term $\psi(z_i, l_i)$ where z_i is the number of players with whom Player i is (directly or indirectly) two-way connected (i included) and l_i is the number of players $j \neq i$ such that $x_{i,j} = 1$. It is assumed that ψ is strictly increasing in the first variable and strictly decreasing in the second one.
Obviously, z_i and l_i depend on the network formed and hence they are functions of the strategy profile, therefore setting $\psi(z_i(x_1,\ldots,x_n), l_i(x_1,\ldots,x_n)) = \gamma_i(x_1,\ldots,x_n)$, it is possible to consider the following game of network formation:

$$\Gamma^T = \{I; \{X_i\}_i; \{\gamma_i\}_i\}$$

3.2.1 Nash Networks

In Bala and Goyal it has been proved that

Proposition 3.4: [Bala and Goyal (2000)]. *If x is a Nash equilibrium of Γ^T then the corresponding network is either empty or satisfies the following:*

i) Every couple of players (i,j) is two-way connected.

ii) There does not exist a cycle, i.e. a subset of players $\{j_1,\dots,j_q\} \subseteq I$ such that

$$\mu(x_{j_1,j_2}) = \cdots = \mu(x_{j_{q-1},j_q}) = \mu(x_{j_q,j_1}) = 1.$$

The previous result shows that a great variety of networks can be implemented by Nash equilibria of the corresponding game, however some network structures play a predominant role: a network x is said to be a *center-sponsored star* if there exists $i \in I$ such that $x_{i,j} = 1$ for all j and $x_{j,h} = 0$ for all $j \neq i$ and for all h; a network x is said to be the empty network if $x_{i,j} = 0$ for all $i, j \in I$ with $i \neq j$.

PROPOSITION 3.5: [Bala and Goyal (2000)]. *Let x be a strict Nash equilibrium x of Γ^T, then x is either a center-sponsored star or the empty network. If x is a center-sponsored star then it is a strict Nash equilibrium if and only if $\psi(n, n-1) > \psi(w+1, w)$ for all $w \in \{0,\dots,n-2\}$. If x is the empty network then it is a strict Nash equilibrium if and only if $\psi(1,0) > \psi(w+1,w)$ for all $w \in \{1,\dots,n-1\}$.*

3.2.2 Efficient Networks

Let $\Psi : X \to \mathbb{R}$ be the function defined by $\Psi(x) = \sum_{i\in I} \gamma_i(x)$ for all $x \in X$, then a network $\overline{x}$ is said to be *two-way efficient* if

$$\Psi(\overline{x}) \geq \Psi(x) \quad \forall x \in X$$

PROPOSITION 3.6: [Bala and Goyal (2000)] *Given a two-way efficient network then every set of two-way connected players (two-way component) does not have a cycle (i.e. it is minimal). Moreover, if $\chi(w+1, h+1) > \chi(w,h)$ for all $h \in \{0,1,\dots,n-2\}$ and $w \in \{h+1,\dots,n-1\}$, every couple of players is two-way connected.*

4 The Social Network Formation Game: New Results

4.1 One-way flow

4.1.1 The wheel

PROPOSITION 4.1: *A wheel is a Nash equilibrium of Γ^O if and only if $\chi(n,1) \geq \chi(1,0)$. Moreover, a wheel is a strict Nash equilibrium if and only if $\chi(n,1) > \chi(1,0)$.*

Proof. Let x be a wheel (i.e. there exists a permutation $\delta : I \to I$ such that $\delta(I) = \{j_1,\dots,j_n\}$ and $x_{j_1,j_2} = x_{j_2,j_3} = \cdots = x_{j_{n-1},j_n} = x_{j_n,j_1} = 1$ and there aren't other links). It follows that for every player the payoff in equilibrium is equal to $\chi(n,1)$. Given a player i, for every strategy $\widehat{x}_i \neq x_i$ different from the strategy "no links" x_i^* it follows that

$$q_i(\widehat{x}_i, x_{-i}) < n \quad \text{or} \quad l_i(\widehat{x}_i, x_{-i}) > 1 \quad \text{or both.}$$

Since

$$\forall w = 2, \ldots, n-1 \quad \chi(n,1) > \chi(w,1) \qquad \text{and} \quad \chi(w,1) > \chi(w,d) \quad \forall d = 2, \ldots, w-1 \quad (3)$$

then, $\{x_i, x_i^*\} = BR_i(x_{-i})$ if and only if $\chi(n,1) = \chi(1,0)$ and $\{x_i\} = BR_i(x_{-i})$ if and only if $\chi(n,1) > \chi(1,0)$. So x is a Nash equilibrium if and only if $\chi(n,1) \geq \chi(1,0)$ and x is a strict Nash equilibrium if and only if $\chi(n,1) > \chi(1,0)$. □

PROPOSITION 4.2: *If a wheel is a Nash equilibrium but not a strict Nash equilibrium of* Γ^O *then it is an equilibrium in weakly dominated strategies.*

Proof. From the assumptions and Proposition 4.1, it follows that $\chi(n,1) = \chi(1,0)$. Let x be a wheel and i be such that $x_{i,j} = 1$ and $x_{i,k} = 0$ $k \neq i,\ k \neq j$. We claim that the strategy x_i^* in which Player i establishes no links weakly dominates x_i. In fact $\zeta_i(x_i^*, x_{-i}) = \chi(q_i(x_i^*, x_{-i}), 0) = \chi(1,0)$. Being $\zeta_i(x_i, x_{-i}) = \chi(q_i(x_i, x_{-i}), 1) = \chi(n,1)$, it follows that $\zeta_i(x_i^*, x_{-i}) = \zeta_i(x_i, x_{-i})$.
Moreover, for every other strategy profile $\overline{x}_{-i}$ of his opponents, we have that $\zeta_i(x_i^*, \overline{x}_{-i}) = \chi(1,0)$ while $\zeta_i(x_i, \overline{x}_{-i}) = \chi(w+1,1)$ if Player j is directly one-way connected with $w-1 \leq n-2$ players different from Player i in the network $(x_i, \overline{x}_{-i})$. From strict monotonicity $\chi(w+1,1) \leq \chi(n,1)$ with the inequality strict as $w+1 < n$. Since there exists a strategy profile $\widehat{x}_{-i}$ such that Player j is directly one-way connected with $h < n-2$ players in the network $(x_i, \widehat{x}_{-i})$, it follows that

$$\zeta_i(x_i^*, \overline{x}_{-i}) \geq \zeta_i(x_i, \overline{x}_{-i})\ \forall \overline{x}_{-i} \quad \text{and} \quad \exists \widehat{x}_{-i} \text{ s.t. } \zeta_i(x_i^*, \widehat{x}_{-i}) > \zeta_i(x_i, \widehat{x}_{-i})$$

which means that x_i^* weakly dominates x_i. □

PROPOSITION 4.3: *If* $\chi(n,1) \geq \chi(1,0)$ *then every wheel is a friendliness equilibrium and a slightly altruistic equilibrium of* Γ^O.

Proof. Let x be a wheel (i.e. there exists a permutation $\delta : I \to I$ such that $\delta(I) = \{j_1, \ldots, j_n\}$ and $x_{j_1,j_2} = x_{j_2,j_3} = \cdots = x_{j_{n-1},j_n} = x_{j_n,j_1} = 1$, and there are no other links). If x is a strict Nash equilibrium then it is obviously a friendliness equilibrium. Suppose that x is not a strict Nash, that is, $\chi(n,1) = \chi(1,0)$. So $BR_i(x_{-i}) = \{x_i, x_i^*\}$, where x_i^* is the strategy in which Player i establishes no links. Note that, for every other player j, $q_j(x_i^*, x_{-i}) < q_j(x_i, x_{-i}) = n$ so that

$$\zeta_j(x_i^*, x_{-i}) = \chi(q_j(x_i^*, x_{-i}), 1) < \chi(q_j(x_i, x_{-i}), 1) = \chi(n,1) = \zeta_j(x_i, x_{-i})$$

So $\sum_{j \neq i} \zeta_j(x_i, x_{-i}) > \sum_{j \neq i} \zeta_j(x_i^*, x_{-i})$. Since $K_i(x_{-i}) \subseteq BR_i(x_{-i})$ x is a friendliness equilibrium and in light of Proposition 2.6 a slightly altruistic equilibrium. □

4.1.2 The empty network

PROPOSITION 4.4: *The empty network is a Nash equilibrium of* Γ^O *if and only if*

$$\chi(w+1,w) \leq \chi(1,0) \quad \forall w \in \{1,\dots,n-1\}. \tag{4}$$

Moreover, the empty network is a strict Nash equilibrium if and only if the inequalities are strict.

Proof. Let x be the empty network. Given a player i, for every strategy $\widehat{x}_i \neq x_i$, it follows that

$$q_i(\widehat{x}_i, x_{-i}) > 1 \quad \text{and} \quad l_i(\widehat{x}_i, x_{-i}) = q_i(\widehat{x}_i, x_{-i}) - 1$$

So, for every strategy $\widehat{x}_i \neq x_i$, let $q_i(\widehat{x}_i, x_{-i}) = w+1$, then

$$\zeta_i(\widehat{x}_i, x_{-i}) = \chi(q_i(\widehat{x}_i, x_{-i}), l_i(\widehat{x}_i, x_{-i})) = \chi(w+, w) \quad \text{with}$$

and then x is a Nash equilibrium if and only if

$$\chi(w+1,w) \leq \chi(1,0) = \zeta_i(x_i, x_{-i}) \quad \forall w \in \{1,\dots,n-1\}$$

while x is a strict Nash equilibrium if and only if the inequalities are strict. □

PROPOSITION 4.5: *If the empty network is a Nash equilibrium but not a strict Nash equilibrium of* Γ^O *then it is a Nash equilibrium in weakly dominated strategies.*

Proof. From the assumptions and Proposition 4.4, it follows that $\chi(w+1,w) \leq \chi(1,0)$ for all $w \in \{1,\dots,n-1\}$ with at least one equality. Let $\overline{w}$ be such that $\chi(\overline{w}+1,\overline{w}) = \chi(1,0)$, x be the empty network and $\overline{x}_i$ be a strategy for Player i in which Player i establishes $\overline{w}$ links. We claim that $\overline{x}_i$ weakly dominates x_i. In fact, it follows that

$$\zeta_i(\overline{x}_i, x_{-i}) = \chi(\overline{w}+1,\overline{w}) = \chi(1,0) = \zeta_i(x_i, x_{-i})$$

Moreover, for any other strategy profile $\widehat{x}_{-i}$, it follows that strategy $\overline{x}_i$ guarantees to Player i $q_i(\overline{x}_i, \widehat{x}_{-i}) = \overline{w}+k \geq \overline{w}+1$ with $k \geq 1$. Therefore

$$\zeta_i(\overline{x}_i, \widehat{x}_{-i}) = \chi(\overline{w}+k,\overline{w}) \geq \chi(\overline{w}+1,\overline{w})$$

with $k \geq 1$, with the inequality strict as $k > 1$ and

$$\chi(\overline{w}+1,\overline{w}) = \chi(1,0) = \zeta_i(x_i, \widehat{x}_{-i}).$$

Let Player s be such that $\overline{x}_{i,s} = 1$. Obviously, there exists a strategy profile $\widehat{x}_{-i}$ in which Player s is one-way connected with a Player t such that $x^*_{i,t} = 0$, hence $k > 1$ and

$$\zeta_i(x^*_i, \widehat{x}_{-i}) = \chi(\overline{w}+k,\overline{w}) > \chi(1,0) = \zeta_i(x_i, \widehat{x}_{-i}).$$

Therefore x^*_i weakly dominates x_i. □

PROPOSITION 4.6: *If*

$$\chi(w+1,w) \leq \chi(1,0) \quad \forall w \in \{1,\dots,n-1\} \tag{5}$$

then, the empty network is a friendliness equilibrium and a slightly altruistic equilibrium of Γ^O*.*

Proof. Let x be the empty network. If x is a strict Nash equilibrium then it is a friendliness equilibrium. Assume that x is not a strict Nash, then, from the assumptions and Proposition 4.4, it follows that there exists $1 < \overline{w} < n-1$ such that $\chi(\overline{w}+1,\overline{w}) = \chi(1,0)$. For every other strategy $\overline{x}_i$ of Player i in which he establishes $\overline{w}$ direct links we have $\zeta_i(\overline{x}_i, x_{-i}) = \chi(\overline{w}+1,\overline{w})$, so $\overline{x}_i \in BR_i(x_{-i})$ if and only if $\chi(\overline{w}+1,\overline{w}) = \chi(1,0) = \zeta_i(x_i, x_{-i})$. In light of the previous arguments, there exists a strategy $\overline{x}_i \neq x_i$ with $\overline{x}_i \in BR_i(x_{-i})$ Since in the new network $(\overline{x}_i, x_{-i})$ only Player i is one-way connected with other players then, for every other player j, the payoff is given by

$$\zeta_j(\overline{x}_i, x_{-i}) = \chi(1,0) = \zeta_j(x_i, x_{-i})$$

which implies that

$$\sum_{j\neq i} \zeta_j(x_i, x_{-i}) = \sum_{j\neq i} \zeta_j(\overline{x}_i, x_{-i}) \quad \forall \overline{x}_i \in BR_i(x_{-i})$$

Then x is a friendliness equilibrium and in light of Proposition 2.6 it is a slightly altruistic equilibrium. □

4.2 Two-way flow

As done for the one-way flow case, now we investigate the admissibility property and the altruistic behavior properties for the wheel and the empty network.

4.2.1 The center-sponsored star

PROPOSITION 4.7: *A center-sponsored star x is a Nash equilibrium of Γ^T if and only if*

$$\psi(n,n-1) \geq \psi(w+1,w) \quad \text{for all } w \in \{0,\dots,n-2\}.$$

In particular, x is a strict Nash equilibrium if and only if all the inequalities are strict.

Proof. Let i be the center of the star then $\gamma_i(x_i, x_{-i}) = \psi(n, n-1)$. In every other strategy $\widehat{x}_i \neq x_i$ Player i establishes $\widehat{w} < n-1$ direct links so that $\gamma_i(\widehat{x}_i, x_{-i}) = \psi(\widehat{w}+1, \widehat{w}) \leq \psi(n, n-1)$. For every player $j \neq i$, $\gamma_j(x_j, x_{-j}) = \psi(n, 0)$. Moreover, in every other strategy $\widehat{x}_j \neq x_j$ Player j establishes $0 < \widehat{w} < n-1$ direct links so that $\gamma_j(\widehat{x}_j, x_{-j}) = \psi(n, \widehat{w}) < \psi(n, 0)$. Then, from the assumptions, it obviously follows the assertion. □

PROPOSITION 4.8: *If x is a center-sponsored star and it is not a strict Nash equilibrium of Γ^T, then x is an equilibrium in weakly dominated strategies.*

Proof. From the assumptions it follows that $\psi(n, n-1) \geq \psi(w+1, w)$ for all $w \in \{0, \ldots, n\text{-}2\}$ and there exists $\overline{w}$ such that $\psi(n, n-1) = \psi(\overline{w}+1, \overline{w})$. Let x be a center-sponsored star and i be the center of the star so that Player i's payoff is $\psi(n, n-1)$.
Consider another strategy $\overline{x}_i$ for Player i in which he establishes only $\overline{w}$ links such that $\psi(n, n-1) = \psi(\overline{w}+1, \overline{w})$. We claim that $\overline{x}_i$ weakly dominates strategy x_i. In fact, first suppose that every player $j \neq i$ is playing his equilibrium strategy ($x_{j,k} = 0\ \forall k \neq j$), then Player i's payoff is

$$\gamma_i(\overline{x}_i, x_{-i}) = \psi(z_i(\overline{x}_i, x_{-i}), l_i(\overline{x}_i, x_{-i})) = \psi(\overline{w}+1, \overline{w}).$$

Then, suppose that a player $j \neq i$ plays a different strategy x'_j. From $x'_j \neq x_j$, it follows that $x'_{j,k} = 1$ for at least another player k. For the sake of simplicity assume first that k is unique (the other cases follow similarly). If $k = i$ then payoff of Player i does not change. Consider $k \neq i$. If strategy $\overline{x}_i$ is such that $\overline{x}_{i,j} = 1$ or $\overline{x}_{i,k} = 1$ (but not both of them) then, payoff of Player i is $\psi(\overline{w}+2, \overline{w})$, with $\psi(\overline{w}+2, \overline{w}) > \psi(\overline{w}+1, \overline{w}) = \psi(n, n-1)$. In the other cases, the payoff of Player i remains $\psi(\overline{w}+1, \overline{w})$. Summarizing, for every strategy profile of his opponents x^*_{-i} we get

$$\gamma_i(x_i, x^*_{-i}) = \psi(z_i(x_i, x^*_{-i}), l_i(x_i, x^*_{-i})) = \psi(n, n-1)$$

while

$$\gamma_i(\overline{x}_i, x^*_{-i}) = \psi(z_i(\overline{x}_i, x^*_{-i}), l_i(\overline{x}_i, x^*_{-i})) = \psi(\overline{w}+h, \overline{w}) \quad \text{with } h \geq 1.$$

Hence

$$\gamma_i(x_i, x^*_{-i}) = \psi(n, n-1) = \psi(\overline{w}+1, \overline{w}) \leq \psi(\overline{w}+h, \overline{w}) = \gamma_i(\overline{x}_i, x^*_{-i}) \quad \text{for all } x^*_{-i} \tag{6}$$

with the inequality strict as $h > 1$. Note also that there exists a strategy profile $\widehat{x}_{-i}$ such that the inequality in (6) is strict. In fact, there obviously exist players $j \neq i$ and $s \neq i$ such that $\overline{x}_{i,j} = 1$ and $\overline{x}_{i,s} = 0$, so if $\widehat{x}_{-i}$ is such that $\widehat{x}_{j,s} = 1$ or $\widehat{x}_{s,j} = 1$ then, $\gamma_i(\overline{x}_i, \widehat{x}_{-i}) = \psi(w+h, w)$ with $h > 1$, because, in this case, a link with Player j guarantees to Player i an indirect two-way connection with Player s.

Therefore, there exists a strategy $\overline{x}_i$ such that

$$\gamma_i(x_i, x^*_{-i}) \leq \gamma_i(\overline{x}_i, x^*_{-i})$$

for all x^*_{-i} with at least one inequality strict. □

Proposition 4.9: *Assume $\psi(n, n-1) \geq \psi(w+1, w)$ for all $w \in \{0, \ldots, n-2\}$. If x is a center-sponsored star then it is a slightly altruistic equilibrium and a friendliness equilibrium of Γ^T.*

Proof. If x is a center-sponsored star then, from the assumptions, it follows that it is an equilibrium of Γ^T. Let i be the center of the star; by construction, for $j \neq i$, $x_{j,k} = 0$ for all $k \neq j$. A strategy $\overline{x}_i$ of Player i belongs to the set of best replies $BR_i(x_{-i})$ if and only if in the strategy $\overline{x}_i$ player i establishes $\overline{w}$ links for a number $\overline{w} \in \{0, \ldots, n-2\}$ such that $\psi(n, n-1) = \psi(\overline{w}+1, \overline{w})$, in fact, in this case payoff of Player i is

$$\gamma_i(\overline{x}_i, x_{-i}) = \psi(z_i(\overline{x}_i, x_{-i}), l_i(\overline{x}_i, x_{-i})) = \psi(\overline{w}+1, \overline{w}).$$

Moreover, the strategy profile $(\overline{x}_i, x_{-i})$ gives payoff $\psi(\overline{w}+1, 0)$ to the players two-way connected with Player i and $\psi(1, 0)$ to the other players. Since $\overline{w}+1 < n$ implies $\psi(\overline{w}+1, 0) < \psi(\overline{n}, 0)$, x_i maximizes $\sum_{j \neq i} \gamma_j(\cdot, x_{-i})$ in the set $BR_i(x_{-i})$. Then, since $K_i(x_{-i}) \subseteq BR_i(x_{-i})$, from Proposition 2.6 we get the assertion. □

4.2.2 The empty network

PROPOSITION 4.10: *The empty network x is a Nash equilibrium of Γ^T if and only if*

$$\psi(1, 0) \geq \psi(w+1, w) \quad \text{for all } w \in \{1, \ldots, n-1\}.$$

Moreover, the empty network x is a strict Nash equilibrium if and only if all the inequalities are strict.

Proof. Since x is the empty network then, for every player i, $\gamma_i(x_i, x_i) = \psi(1, 0)$. In every other strategy $\overline{x}_i \neq x_i$ Player i establishes $\overline{w} > 0$ direct links so that $\gamma_i(\overline{x}_i, x_i) = \psi(\overline{w}+1, \overline{w})$. Then, from the assumptions, it obviously follows the assertion. □

PROPOSITION 4.11: *If the empty network x is a Nash equilibrium but not a strict Nash equilibrium of Γ^T then, x is an equilibrium in weakly dominated strategies.*

Proof. From the assumptions it follows that $\psi(1, 0) \geq \psi(w+1, w)$ for all $w \in \{0, \ldots, n-1\}$ and there exists $\overline{w} \in \{1, \ldots, n-1\}$ such that $\psi(1, 0) = \psi(\overline{w}+1, \overline{w})$. From the assumptions and the previous proposition the empty network x is a Nash but not a strict Nash equilibrium. Given a player i, consider another strategy $\overline{x}_i$ in which Player i establishes $\overline{w}$ links such that $\psi(1, 0) = \psi(\overline{w}+1, \overline{w})$. Then $\overline{x}_i$ weakly dominates x_i. In fact, following the proof of Proposition 4.8, there exists a strategy profile $\widehat{x}_{-i}$ for players $j \neq i$ such that $z_i(\overline{x}_i, \widehat{x}_{-i}) > \overline{w}+1$. Being ψ strictly increasing in the first variable then

$$\gamma_i(\overline{x}_i, \widehat{x}_{-i}) = \psi(z_i(\overline{x}_i, \widehat{x}_{-i}), \overline{w}) > \psi(\overline{w}+1, \overline{w}) = \psi(1, 0).$$

Moreover, every other strategy profile x^*_{-i} is such that $z_i(\overline{x}_i, x^*_{-i}) \geq \overline{w}+1$ therefore

$$\gamma_i(\overline{x}_i, x^*_{-i}) = \psi(z_i(\overline{x}_i, x^*_{-i}), \overline{w}) \geq \psi(\overline{w}+1, \overline{w}) = \psi(1, 0)$$

and $\overline{x}_i$ weakly dominates x_i. □

PROPOSITION 4.12: *If the empty network x is a Nash equilibrium and $K_i(x_{-i}) \neq \{x_i\}$ then x is not a friendliness equilibrium nor a slightly altruistic equilibrium of Γ^T.*

Proof. From the assumptions it follows that $\psi(1,0) \geq \psi(w+1,w)$ for all $w \in \{1,\dots,n-1\}$. Moreover $K_i(x_{-i}) \neq \{x_i\}$ implies that the empty network x is not a strict Nash then, there exists $\overline{w} \in \{1,\dots,n-1\}$ such that $\psi(1,0) = \psi(\overline{w}+1,\overline{w})$. A strategy $\overline{x}_i$ belongs to $K_i(x_{-i})$ only if in $\overline{x}_i$ Player i establishes $\overline{w}$ links such that $\psi(\overline{w}+1,\overline{w}) = \psi(1,0)$. From the assumptions, there exists a Nash equilibrium $(\overline{x}_i, x_{-i})$, then

$$\gamma_i(\overline{x}_i, x_{-i}) = \psi(\overline{w}+1,\overline{w}) = \psi(1,0) = \gamma_i(x_i, x_{-i}), \quad \text{with } \overline{w} \in \{1,\dots,n-1\}$$

Moreover, for every player j two-way connected with i in the network $(\overline{w}_i, x_{-i})$ the payoffs are

$$\gamma_j(\overline{x}_i, x_{-i}) = \psi(\overline{w}+1,0) > \psi(1,0) = \gamma_j(x_i, x_{-i})$$

while, for the other players, payoffs are

$$\gamma_j(\overline{x}_i, x_{-i}) = \psi(1,0) = \gamma_j(x_i, x_{-i}).$$

It easily follows that

$$\sum_{j \neq i} \gamma_j(\overline{x}_i, x_{-i}) > \sum_{j \neq i} \gamma_j(x_i, x_{-i})$$

so x is not a friendliness equilibrium. Finally, from Proposition (2.4), x is not a slightly altruistic equilibrium. □

The conditions in the previous Proposition are satisfied even in simple cases as shown by the following:

EXAMPLE 4.13: Consider a three player two-sided social network in which the payoff function ψ is given by

$$\psi(1,0) = \psi(3,2) = \psi(2,1) = 3; \ \psi(2,0) = \psi(3,1) = 4; \ \psi(3,0) = 5. \tag{7}$$

Obviously, the empty network x is a Nash but not a strict Nash equilibrium. Let $\overline{x}_{i_1}$ be the strategy of Player i_1 in which he establishes a link with Player i_2 and a link with Player i_3. It follows that the payoff of Player i_1 is $\gamma_{i_1}(\overline{x}_{i_1}, x_{-i_1}) = \psi(3,2)$ so $\overline{x}_{i_1} \in BR_{i_1}(x_{-i_1})$. Moreover, the payoff of Player i_2 is $\gamma_{i_2}(x_{i_2}, \overline{x}_{i_1}, x_{i_3}) = \psi(3,0)$, while, if Player i_2 establishes direct links himself, his payoff is $\psi(3,2)$ or $\psi(3,1)$, with $\psi(3,2) < \psi(3,1) < \psi(3,0)$. The same arguments hold true for Player i_3 so that $(\overline{x}_{i_1}, x_{-i_1})$ is a Nash equilibrium and $\overline{x}_{i_1} \in K_{i_1}(x_{-i_1})$ with $\overline{x}_{i_1} \neq x_{i_1}$.

REMARK 4.14: When

$$\psi(n, n-1) < \psi(n,1)$$

then

$$\psi(1,0) > \psi(n,1) \implies \psi(1,0) > \psi(n,n-1)$$

Since in Bala and Goyal (pp. 1205), it has been shown that the empty network is a Nash efficient architecture if and only if $\psi(1,0) > \psi(n,1)$ then the empty network is an inefficient Nash network for a large class of payoff functions.

4.2.3 The periphery-sponsored star

Besides the notions of empty network and center-sponsored star another network plays and important role: a network x is said to be a *periphery-sponsored star* if there exists $i \in I$ such that, for all $j \neq i$, $x_{j,i} = 1$ and $x_{j,h} = 0$ for all $h \neq i$, while $x_{i,k} = 0$ for all $k \neq i$. It immediately follows that

PROPOSITION 4.15: *A periphery sponsored star x is a Nash equilibrium of Γ^T if and only if $\psi(n, 1) \geq \psi(1, 0)$.*

Proof. Let i be the center of the star then $\gamma_i(x_i, x_{-i}) = \psi(n, 0)$. In every other strategy $\overline{x}_i \neq x_i$ Player i establishes $0 < \overline{w} \leq n - 1$ direct links so that $\gamma_i(\overline{x}_i, x_{-i}) = \psi(n, \overline{w}) < \psi(n, 0)$. For every player $j \neq i$, $\gamma_j(x_j, x_{-j}) = \psi(n, 1)$. Moreover, in every other strategy $\overline{x}_j \neq x_j$ Player j establishes $\overline{w} \leq n-1$ direct links so that $\gamma_j(\overline{x}_j, x_{-j}) = \psi(\alpha, \overline{w})$. If $\overline{w} = 0$ then $\alpha = 1$ while $\overline{w} > 0$ implies $1 < \alpha \leq n$ and therefore, from the assumptions it follows that $\psi(\alpha, \overline{w}) \leq \psi(n, 1)$. So x is a Nash equilibrium. □

REMARK 4.16: Note that a periphery sponsored star x is not a strict Nash equilibrium even when $\psi(n, 1) > \psi(1, 0)$. In fact, let i be the center of the star, then for every player $j \neq i$ every other strategy $\widehat{x}_j$ in which Player j makes a unique link with a player k, with $k \neq i$, assures to Player j the same payoff $\psi(n, 1)$ being the $n - 1$ opponents two-way connected.

PROPOSITION 4.17: *If $\psi(n, 1) \geq \psi(1, 0)$ then the periphery sponsored star is a friendliness and a slightly altruistic equilibrium of Γ^T.*

Proof. Let x be a periphery sponsored star and i be the center of the star. In light of the assumptions and Remark 4.16 then, for every player $j \neq i$, every strategy $\overline{x}_j$ in which Player j makes a unique link with a player $k \neq i$ assures to Player j the same payoff $\psi(n, 1)$ of the equilibrium strategy x_j, being the $n-1$ opponents two-way connected. This implies that $\overline{x}_j \in BR_j(x_{-j})$. Note that, if $\overline{x}_j$ is defined as above, the payoff of every player k with $k \neq j$, $k \neq i$ is

$$\gamma_k(x_j, x_{-j}) = \gamma_k(\overline{x}_j, x_{-j}) = \psi(n, 1)$$

while

$$\gamma_i(x_j, x_{-j}) = \gamma_i(\overline{x}_j, x_{-j}) = \psi(n, 0)$$

Moreover, any other strategy in which Player j makes a $\beta \geq 2$ links gives to j payoff $\psi(\alpha, \beta) < \psi(n, 1)$ with $\alpha \leq n$.

Finally, let x_j^* be the no links strategy of Player j. Then $x_j^* \in BR_j(x_{-j})$ if and only if $\psi(n, 1) = \psi(1, 0)$. If this is the case, for every player k with $k \neq j$, $k \neq i$ we have $z_k(x_j^*, x_{-j}) = n - 1$, $l_k(x_j^*, x_{-j}) = 1$ and

$$\gamma_k(x_j^*, x_{-j}) = \psi(n - 1, 1) < \psi(n, 1) = \gamma_k(x_j, x_{-j}).$$

Moreover, $z_i(x_j^*, x_{-j}) = n - 1$, $l_i(x_j^*, x_{-j}) = 0$ and

$$\gamma_i(x_i^*, x_{-i}) = \psi(n - 1, 0) < \psi(n, 0) = \gamma_j(x_j, x_{-j}).$$

Summarizing,

$$\sum_{k\neq j} \gamma_k(x_j^*, x_{-j}) < \sum_{k\neq j} \gamma_k(x_j, x_{-j}) \quad \forall x_j^* \in BR_j(x_{-j})$$

and x is a friendliness equilibrium. Finally, from Proposition 2.6, x is a slightly altruistic equilibrium. □

REMARK 4.18: When

$$\psi(n, n-1) < \psi(n, 1)$$

then

$$\psi(1,0) \leq \psi(n, n-1) \implies \psi(1,0) \leq \psi(n,1).$$

So, the condition for the center sponsored star to be a Nash and a friendliness equilibrium is more restrictive than the condition for the periphery sponsored to be a Nash and a friendliness equilibrium.

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