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Kalai-Smorodinsky Bargaining Solution Equilibria

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Multicriteria games describe strategic interactions in which players, having more than one criterion to take into account, don't have an a-priori opinion on the rel- ative importance of all these criteria. Roemer (2005) introduces an organizational interpretation of the concept of equilibrium: each player can be viewed as running a bargaining game among criteria. In this paper, we analyze the bargaining problem within each player by considering the Kalai-Smorodinsky bargaining solution. We provide existence results for the so called *Kalai-Smorodinsky bargaining solution equilibria* for a general class of disagreement points which properly includes the one considered in Roemer (2005). Moreover we look at the *refinement power* of this equilibrium concept and show that it is an effective selection device even when combined with classical refinement concepts based on stability with respect to perturbations such as the the extension to multicriteria games of the Selten's (1975) trembling hand perfect equilibrium concept.

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Table of contents

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Abstract

Multicriteria games describe strategic interactions in which players , having more than one criterion to take into account, don't have an a-priori opinion on the relative importance of all these criteria. Roemer (2005) introduces an organizational interpretation of the concept of equilibrium: each player can be viewed as running a bargaining game among criteria. In this paper, we analyze the bargaining problem within each player by considering the Kalai-Smorodinsky bargaining solution. We provide existence results for the so called *Kalai-Smorodinsky bargaining solution equilibria* for a general class of disagreement points which properly includes the one considered in Roemer (2005). Moreover we look at the *refinement power* of this equilibrium concept and show that it is an effective selection device even when combined with classical refinement concepts based on stability with respect to perturbations such as the the extension to multicriteria games of the Selten's (1975) trembling hand perfect equilibrium concept.

1 Introduction

Multicriteria games describe strategic interactions in which players' payoff are vector-valued functions, representing players' multiple goals; in other words, agents, having more than one criterion to take into account, don't have an a-priori opinion on the relative importance of all these criteria. Different extensions of the classical concept of Nash equilibrium have been adopted for multicriteria games; the concepts of weak Pareto-Nash and Pareto-Nash equilibrium, as introduced in Shapley (1959), play a fundamental role and satisfy existence theorems under classical assumptions. Since in multicriteria games multiplicity of the equilibria arises even more drastically with respect to the standard scalar case, some contributions have also been made to generalize refinement concepts for Nash equilibria to the multicriteria games (see Puerto and Fernandez (1995) or Borm, van Megen and Tijs (1999) for perfect equilibria, Yang and Yu (2002) for essential equilibria).

Another approach is considered in Roemer (2005) where the author shows that, on the one hand, in applications it will often be the case that in a multicriteria game each player is an organization whose members have different goals and where the set of members sharing the same goal is called a faction (organizations might be political parties, firms, or trade unions). On the other hand, each player can be regarded as an organization whose factions are represented by the payoff's components. Therefore he introduces an organizational interpretation of the concept of equilibrium: each player/organization can be viewed as running a bargaining game among criteria/internal factions" in which the disagreement point corresponds to a fixed and exogenously given status quo strategy of the player. In particular, the bargaining problem within each player/organization is solved by considering the weighted Nash bargaining solution (Nash (1950)) and it has been shown that every Pareto-Nash equilibrium can been regarded as a weighted Nash bargaining solution equilibrium for a suitable choice of the weights. In this paper, we analyze the bargaining problem within each player/organization by considering the Kalai-Smorodinsky bargaining solution (Kalai-Smorodinsky (1975)). We provide existence results for the so called *Kalai-Smorodinsky bargaining solution equilibria* for a general class of disagreement points which properly includes the one considered in Roemer (2005) and the one *called minimal expectation disagreement point* defined in Roth (1977). Moreover, since in multicriteria games multiplicity of equilibria arises even more drastically with respect to the standard scalar case, we look at the *refinement power* of the Kalai-Smorodinsky bargaining solution equilibrium and show that, differently from the weighted Nash bargaining solution equilibrium, it is an effective selection device. Finally we show that it is possible to combine the Kalai-Smorodinsky bargaining solution equilibrium with the refinements based on trembles; more precisely we consider the extension to multicriteria games of Selten's (1975) trembling hand perfect equilibria as defined in Borm, van Megen and Tijs (1999). We show that the intersection between Kalai-Smorodinsky bargaining solution equilibria and perfect equilibria is not empty and that it provides a sharper selection device for weak Pareto-Nash equilibria.

2 Multicriteria Games

Multicriteria games describe interactions in which players' payoff are vector-valued functions; which means that players, having more then one criterion to take into account, don't have an a-priori opinion on the relative importance of all their criteria. In this paper we will consider games of the form:

$$\Gamma = \{I; X_1, \dots, X_n; J_1, \dots, J_n\}$$

where $I = \{1, \dots, n\}$ is the finite players' set; for every player i, the strategy set X_i is a subset of $\mathbb{R}^{l(i)}$ and the payoff is a vector-valued function $J_i : X \to \mathbb{R}^{r(i)}$, where $X = \prod_{j=1}^{n} X_j$ and $J_i = \left(J_i^h\right)_{h=1}^{r(i)}$; denote also $X_{-i} = \prod_{j \neq i} X_j$.

In case the players act non-cooperatively, different extensions of the classical concept of Nash equilibrium have been adopted; however, the concepts of weak Pareto-Nash and Pareto-Nash equilibrium, as introduced in Shapley (1959), play a fundamental role (see Wang (1993) for more general existence theorems and Morgan (2004) for variational stability, well-posedness and for an extensive list of references). We recall here some classical definitions and notations:

Definition 2.1: Given $x_{-i} \in X_{-i}$, the strategy $\widehat{x}_i \in X_i$ is said to be *strongly (Pareto) dominated* by the strategy $\overline{x}_i \in X_i$ if the vector $J_i(\widehat{x}_i, x_{-i})$ is strongly (Pareto) dominated by the vector $J_i(\overline{x}_i, x_{-i})$, that is

$$J_i(\overline{x}_i, x_{-i}) - J_i(\widehat{x}_i, x_{-i}) \in int\mathbb{R}_+^{r(i)}.$$

While, the strategy $\widehat{x}_i \in X_i$ is said to be *(Pareto) dominated* by the strategy $\overline{x}_i \in X_i$ if the vector $J_i(\widehat{x}_i, x_{-i})$ is (Pareto) dominated by the vector $J_i(\overline{x}_i, x_{-i})$, that is

$$J_i(\overline{x}_i, x_{-i}) - J_i(\widehat{x}_i, x_{-i}) \in \mathbb{R}_+^{r(i)} \setminus \{0\}.$$

Let $J_i(X_i, x_{-i}) = \{J_i(x_i, x_{-i}) \mid x_i \in X_i\}$, a vector y_i is a *weak Pareto point* in $J_i(X_i, x_{-i})$ if it is not strongly dominated by any other vector in $J_i(X_i, x_{-i})$, i.e. $\nexists z_i \in J_i(X_i, x_{-i})$ such that $z_i - y_i \in int\mathbb{R}_+^{r(i)}$. A vector y_i is a *Pareto point* in $J_i(X_i, x_{-i})$ if it is not dominated by any other vector in $J_i(X_i, x_{-i})$, i.e. $\nexists z_i \in J_i(X_i, x_{-i})$ such that $z_i - y_i \in \mathbb{R}_+^{r(i)} \setminus \{0\}$.

For every player i, let $\mathcal{W}_i : X_{-i} \rightsquigarrow \mathbb{R}^{r(i)}$ be the set-valued map where

$$\mathcal{W}_i(x_{-i}) \text{ is the set of all weak Pareto points in } J_i(X_i, x_{-i}) \quad \text{for all } x_{-i} \in X_{-i}. \tag{1}$$

and $\mathcal{P}_i : X_{-i} \rightsquigarrow \mathbb{R}^{r(i)}$ be the set-valued map where

$$\mathcal{P}_i(x_{-i}) \text{ is the set of all Pareto points in } J_i(X_i, x_{-i}) \quad \text{for all } x_{-i} \in X_{-i}. \tag{2}$$

Finally, for every player i and for every $x_{-i} \in X_{-i}$, a strategy $\overline{x}_i$ is a *weak-Pareto solution* for the vector-valued function $J_i(\cdot, x_{-i})$ in X_i if

$$\overline{x}_i \in \operatorname*{Arg\,wmax}_{x_i \in X_i} J_i(x_i, x_{-i}) = \{x_i \in X_i \mid J_i(x_i, x_{-i}) \in \mathcal{W}_i(x_{-i})\} \tag{3}$$

and a strategy $\overline{x}_i$ is a *Pareto solution* for the vector-valued function $J_i(\cdot, x_{-i})$ in X_i if

$$\overline{x}_i \in \operatorname*{Arg\,max}_{x_i \in X_i} J_i(x_i, x_{-i}) = \{x_i \in X_i \mid J_i(x_i, x_{-i}) \in \mathcal{P}_i(x_{-i})\}. \tag{4}$$

Note that

$$\mathcal{P}_i(x_{-i}) \subseteq \mathcal{W}_i(x_{-i}) \text{ and } \operatorname*{Arg\,max}_{x_i \in X_i} J_i(x_i, x_{-i}) \subseteq \operatorname*{Arg\,wmax}_{x_i \in X_i} J_i(x_i, x_{-i})$$

Definition 2.2: (Shapley (1959)). A strategy profile $x \in X$ is a *weak Pareto-Nash equilibrium* if, for every player i, x_i is a weak-Pareto solution for the vector-valued function $J_i(\cdot, x_{-i})$ in X_i; while $x \in X$ is a *Pareto-Nash equilibrium* if, for every player i, x_i is a Pareto solution for the vector-valued function $J_i(\cdot, x_{-i})$ in X_i.

Different interesting attempts have been made to generalize some refinement concepts for Nash equilibria to the above solution concepts (see Puerto and Fernandez (1995) or Borm, van Megen and Tijs (1999) for perfect equilibria, Yang and Yu (2002) for essential equilibria). Moreover, in De Marco and Morgan (2007) it has been constructed a refinement concept that takes into account the methodology of the scalarization which adds to the original problem new endogenous parameters that are typical of the vector-valued model.

The purpose of this paper is to construct a refinement concept that takes into account the bargaining problem within the objectives of each player regarded as factions within an organization.

3 Games à la Kalai-Smorodinsky

Roemer (2005) shows that, on the one hand, in many applications players are organizations whose members have different goals and where the set of members of each organization sharing the same goal is called a faction; on the other hand, he points out that each player in a multicriteria game can be regarded as an organization whose factions are represented by the payoff's components. Therefore he introduces an organizational interpretation of the concept of equilibrium by considering a bargaining game among criteria/internal factions in which the disagreement point corresponds to a fixed and exogenously given status quo strategy of the player. In particular, the bargaining problem within each player/organization is solved by considering the weighted Nash bargaining solution (Nash (1951)) and it has been shown that every Pareto-Nash equilibrium can been regarded as a weighted Nash bargaining solution equilibrium for a suitable choice of the weights. In this paper, we analyze the bargaining problem within each player/organization by considering the Kalai-Smorodinsky bargaining solution (Kalai-Smorodinsky (1975)) for a general class of disagreement points which properly includes the one considered in Roemer (2005). We consider the case where the vector payoffs have only two components, that is, for every $i \in I$, $J_i : X \to \mathbb{R}^2$, because the properties of the Kalai-Smorodinsky solution become different when the dimension is greater or equal than 3 and this case will be considered in another paper.

More precisely, fixed a strategy profile for his opponents x_{-i}, each player/organization faces a bargaining problem $(J_i(X_i, x_{-i}), \varphi_i(x_{-i}))$ where:

1) $J_i(X_i, x_{-i}) = \{J_i(x_i, x_{-i}) \mid x_i \in X_i\}$ is the set of alternatives.

2) $\varphi_i(x_{-i})$ is the disagreement point.

In order to develop the theory we will use the following:

Assumption 1: *For every player i, X_i is not empty, compact and convex, the function $J_i : X \to \mathbb{R}^2$ is continuous in X and $J_i^h(\cdot, x_{-i})$ is concave in X_i for all $x_{-i} \in X_{-i}$ and for $h = 1, 2$.*

Assumption 2: *For every player i and for every $x_{-i} \in X_{-i}$, the disagreement point is given by the image $\varphi_i(x_{-i})$ of a disagreement point function $\varphi_i : X_{-i} \to \mathbb{R}^2$ satisfying the following condition*

$$\forall x_{-i} \in X_{-i} \quad \exists y(x_{-i}) \in J_i(X_i, x_{-i}) \text{ such that } y(x_{-i}) - \varphi_i(x_{-i}) \in int\mathbb{R}^2_+. \tag{5}$$

The Kalai-Smorodinsky bargaining solution to this problem is constructed as follows: given opponents' profile x_{-i}, let $\alpha_i(x_{-i})$ be the ideal point of player i given opponents' profile x_{-i}, that is,

$$\alpha_i(x_{-i}) = \left(\alpha_i^h(x_{-i})\right)_{h=1,2} \quad \text{with} \quad \alpha_i^h(x_{-i}) = \max_{x_i \in X_i} J_i^h(x_i, x_{-i}).$$

Denote with $L(\varphi_i(x_{-i}), x_{-i}) \subset \mathbb{R}^2$ the line joining $\varphi_i(x_{-i})$ to $\alpha_i(x_{-i})$. In light of condition (5), $L(\varphi_i(x_{-i}), x_{-i})$ has positive slope so that the partial order given by the

(Pareto) dominance relation in $\mathbb{R}^2$ induces a total order on $L(\varphi_i(x_{-i}), x_{-i})$. Therefore, if $L(\varphi_i(x_{-i}), x_{-i}) \cap J_i(X_i, x_{-i}) \neq \emptyset$, then there exists a unique maximal element in $L(\varphi_i(x_{-i}), x_{-i}) \cap J_i(X_i, x_{-i})$.

In order to show that $L(\varphi_i(x_{-i}), x_{-i}) \cap J_i(X_i, x_{-i}) \neq \emptyset$, denote with $\mathcal{C}_i(x_{-i})$ the closed convex hull of $J_i(X_i, x_{-i})$ and with $\widetilde{\mathcal{P}}_i(x_{-i})$ the set of Pareto points in $\mathcal{C}_i(x_{-i})$; then

Lemma 3.1: *In the Assumptions 1, for every player i and every $x_{-i} \in X_{-i}$ it results that $\mathcal{P}_i(x_{-i}) = \widetilde{\mathcal{P}}_i(x_{-i})$.*

Proof. Fixed $i \in I$, assume that $y \in \widetilde{\mathcal{P}}_i(x_{-i}) \setminus \mathcal{P}_i(x_{-i})$. Then $y \notin J_i(X_i, x_{-i})$, in fact $y \in J_i(X_i, x_{-i}) \cap \widetilde{\mathcal{P}}_i(x_{-i})$ imply $y \in \mathcal{P}_i(x_{-i})$. Since $y \in \mathcal{C}_i(x_{-i})$, then there exist $x_{i,1}, \ldots, x_{i,m} \in X_i$ and $\lambda_1, \ldots, \lambda_m \geq 0$, with $\sum_{j=1}^m \lambda_j = 1$, such that $y = \sum_{j=1}^m \lambda_j J_i(x_{i,j}, x_{-i})$. From concavity of J_i^h, for $h = 1, 2$ it follows that

$$J_i^h\left(\sum_{j=1}^m \lambda_j x_{i,j}, x_{-i}\right) \geq \sum_{j=1}^m \lambda_j J_i^h(x_{i,j}, x_{-i}) = y^h \quad h = 1, 2.$$

Since $J_i\left(\sum_{j=1}^m \lambda_j x_{i,j}, x_{-i}\right) \in J_i(X_i, x_{-i})$ and $y \notin J_i(X_i, x_{-i})$, then at least one of the previous inequalities is strict, then

$$J_i\left(\sum_{j=1}^m \lambda_j x_{i,j}, x_{-i}\right) - y \in \mathbb{R}^2_+ \setminus \{0\}.$$

Therefore $y \notin \widetilde{\mathcal{P}}_i(x_{-i})$, which is a contradiction; hence $y \in \mathcal{P}_i(x_{-i})$ and $\widetilde{\mathcal{P}}_i(x_{-i}) \subseteq \mathcal{P}_i(x_{-i})$.

Now, fixed $i \in I$, assume that $y \in \mathcal{P}_i(x_{-i}) \setminus \widetilde{\mathcal{P}}_i(x_{-i})$; then $y \in \mathcal{C}(x_{-i})$ and there exists $\widetilde{y} \in \mathcal{C}(x_{-i})$ such that $\widetilde{y} - y \in \mathbb{R}^2_+ \setminus \{0\}$. Since $\widetilde{y} \in \mathcal{C}(x_{-i})$ then there exist $x_{i,1}, \ldots, x_{i,m} \in X_i$ and $\lambda_1, \ldots, \lambda_m \geq 0$, with $\sum_{j=1}^m \lambda_j = 1$, such that $\widetilde{y} = \sum_{j=1}^m \lambda_j J_i(x_{i,j}, x_{-i})$. Then, from concavity of each $J_i^h(\cdot, x_{-i})$, it follows that $J_i\left(\sum_{j=1}^m \lambda_j x_{i,j}, x_{-i}\right) - \widetilde{y} \in \mathbb{R}^2_+$ and therefore $J_i\left(\sum_{j=1}^m \lambda_j x_{i,j}, x_{-i}\right) - y \in \mathbb{R}^2_+$, so $y \notin \mathcal{P}(x_{-i})$ and we get a contradiction. Hence $y \in \widetilde{\mathcal{P}}_i(x_{-i})$ and $\mathcal{P}_i(x_{-i}) \subseteq \widetilde{\mathcal{P}}_i(x_{-i})$. □

Lemma 3.2: *In the Assumptions 1 and 2, for every player i and every $x_{-i} \in X_{-i}$ it results that $L(\varphi_i(x_{-i}), x_{-i}) \cap J_i(X_i, x_{-i}) \neq \emptyset$.*

Proof. The ideal point of $\mathcal{C}_i(x_{-i})$ coincides with $\alpha_i(x_{-i})$, therefore, following the proof of the main Theorem in Kalai-Smorodinsky (1975) (p. 516) it results that $\mathcal{C}_i(x_{-i}) \cap L(\varphi_i(x_{-i}), x_{-i}) \neq \emptyset$. In fact, there exist $(\alpha_i^1(x_{-i}), \beta_i^2)$ and $(\beta_i^1, \alpha_i^2(x_{-i}))$ in $\widetilde{\mathcal{P}}_i(x_{-i})$ and the line connecting these points have negative slope; it can be checked that $(\alpha_i^1(x_{-i}), \beta_i^2)$ and $(\beta_i^1, \alpha_i^2(x_{-i}))$ are separated by the line $L(\varphi_i(x_{-i}), x_{-i})$. Therefore $L(\varphi_i(x_{-i}), x_{-i})$ intersects the segment connecting these two points and this intersection belongs to $\mathcal{C}_i(x_{-i}) \cap L(\varphi_i(x_{-i}), x_{-i})$. Following previous arguments, $L(\varphi_i(x_{-i}), x_{-i})$ has positive slope so that the partial order given by the Pareto dominance relation in $\mathbb{R}^2$ induces a total order on $L(\varphi_i(x_{-i}), x_{-i})$. Therefore, since $\mathcal{C}_i(x_{-i}) \cap L(\varphi_i(x_{-i}), x_{-i}) \neq \emptyset$, then there exists a unique maximal element $K_i = (K_i^1, K_i^2)$ with respect to Pareto dominance in

$L(\varphi_i(x_{-i}), x_{-i}) \cap \mathcal{C}_i(x_{-i})$. It follows that $K_i \in \widetilde{\mathcal{P}}_i(x_{-i})$. In fact suppose there exists $z_i = (z_i^1, z_i^2) \in \mathcal{C}_i(x_{-i})$ such that $z_i - K_i \in \mathbb{R}^2_+ \setminus \{0\}$. Since $z_i^1 \leq \alpha_i^1(x_{-i})$ and $z_i^2 \leq \alpha_i^2(x_{-i})$ with at least one of the two inequalities strict, then $L(\varphi_i(x_{-i}), x_{-i})$ intersects either the segment connecting z_i and $(\alpha_i^1(x_{-i}), \beta_i^2)$ or the segment connecting z_i and $(\beta_i^1, \alpha_i^2(x_{-i}))$. In both the cases the intersection dominates K_i with respect to Pareto dominance in $L(\varphi_i(x_{-i}), x_{-i}) \cap \mathcal{C}_i(x_{-i})$, but this is a contradiction and $K_i \in \widetilde{\mathcal{P}}_i(x_{-i})$.

From Lemma 3.1 it follows that $\widetilde{\mathcal{P}}_i(x_{-i}) = \mathcal{P}_i(x_{-i})$ and hence

$$J_i(X_i, x_{-i}) \cap L(\varphi_i(x_{-i}), x_{-i}) \supseteq \mathcal{P}_i(x_{-i}) \cap L(\varphi_i(x_{-i}), x_{-i}) = \widetilde{\mathcal{P}}_i(x_{-i}) \cap L(\varphi_i(x_{-i}), x_{-i}) \neq \emptyset.$$

□

So, in the Assumptions 1 and 2, the following definition is well posed:

Definition 3.3 (Kalai Smorodinsky (1975)): The Kalai-Smorodinsky solution (KS-s) $\kappa_i(\varphi_i(x_{-i}), x_{-i})$ to the bargaining problem $(J_i(X_i, x_{-i}), \varphi_i(x_{-i}))$ is the maximal element in $J_i(X_i, x_{-i})$ on the line joining $\varphi_i(x_{-i})$ to $\alpha_i(x_{-i})$.

Lemma 3.4: *In the Assumptions 1 and 2, for every player i and every $x_{-i} \in X_{-i}$ it results that $\kappa_i(\varphi_i(x_{-i}), x_{-i}) \in \mathcal{P}(x_{-i})$.*

Proof. Suppose there exists $z_i = (z_i^1, z_i^2) \in J_i(X_i, x_{-i})$ such that $z_i - \kappa_i(\varphi_i(x_{-i}), x_{-i}) \in \mathbb{R}^2_+ \setminus \{0\}$. Since $z_i^1 \leq \alpha_i^1(x_{-i})$ and $z_i^2 \leq \alpha_i^2(x_{-i})$ with at least one of the two inequalities strict, then $L(\varphi_i(x_{-i}), x_{-i})$ intersects either the segment connecting z_i and $(\alpha_i^1(x_{-i}), \beta_i^2)$ or the segment connecting z_i and $(\beta_i^1, \alpha_i^2(x_{-i}))$. In both the cases the intersection dominates $\kappa_i(\varphi_i(x_{-i}), x_{-i})$ with respect to Pareto dominance in $L(\varphi_i(x_{-i}), x_{-i}) \cap J_i(X_i, x_{-i})$, but this is a contradiction and $\kappa_i(\varphi_i(x_{-i}), x_{-i}) \in \mathcal{P}_i(x_{-i})$. □

Definition 3.5: A strategy profile $x^* \in X$ is said to be a *Kalai-Smorodinsky solution equilibrium (*KS-s *equilibrium)* with disagreement point functions $\varphi = (\varphi_1, \ldots, \varphi_n)$ if

$$\kappa_i(\varphi_i(x^*_{-i}), x^*_{-i}) = J_i(x^*_i, x^*_{-i}) \quad \forall i \in I. \tag{6}$$

The set of all KS-s equilibrium with disagreement point functions φ is denoted with $\mathcal{K}(\varphi)$.

Leontief Preferences

Now we introduce the game with the Leontief preferences deriving from the bargaining problems $(J_i(X_i, x_{-i}), \varphi_i(x_{-i}))$ for $i = 1, \ldots, n$ and we characterize KS-s equilibria in terms of equilibria of this game. For every player i and for every disagreement point function φ_i, let $f_i(\varphi_i(\cdot), \cdot) : X \to \mathbb{R}$ be the function defined by:

$$f_i(\varphi_i(x_{-i}), x) = \min \left\{ \frac{J_i^1(x_i, x_{-i}) - \varphi_i^1(x_{-i})}{\alpha_i^1(x_{-i}) - \varphi_i^1(x_{-i})}, \frac{J_i^2(x_i, x_{-i}) - \varphi_i^2(x_{-i})}{\alpha_i^2(x_{-i}) - \varphi_i^2(x_{-i})} \right\}. \tag{7}$$

So, we can consider the game between organizations with Leontief preferences and with disagreement point functions φ

$$\Gamma^O(\varphi) = \{I; X_1, \ldots, X_n; f_1(\varphi_1(\cdot), \cdot), \ldots, f_n(\varphi_n(\cdot), \cdot)\}. \tag{8}$$

DEFINITION 3.6: A strategy profile $x^* \in X$ is said to be a *bargaining solution equilibrium with Leontief preferences* and with disagreement point functions φ if it is a Nash equilibrium of the normal form game $\Gamma^O(\varphi)$. That is

$$f_i(\varphi_i(x^*_{-i}), x^*_i, x^*_{-i}) = \max_{x_i \in X_i} f_i(\varphi_i(x^*_{-i}), x_i, x^*_{-i}) \qquad \forall i \in I.$$

Therefore, the following characterization holds:

PROPOSITION 3.7: *In the Assumptions 1 and 2, it results that, for every $i \in I$,*

$$\kappa_i(\varphi_i(x_{-i}), x_{-i}) = J_i(\widetilde{x}_i, x_{-i}) \iff \widetilde{x}_i \in \arg\max_{x_i \in X_i} f_i(\varphi_i(x_{-i}), x_i, x_{-i}) \quad \forall i \in I.$$

Therefore x^* is a KS-S *equilibrium if and only if x^* is a bargaining solution equilibrium with Leontief preferences and with disagreement point functions φ.*

Proof. Fix $i \in I$ and let $\kappa_i(\varphi_i(x_{-i}), x_{-i}) = J_i(\widetilde{x}_i, x_{-i})$. By definition it follows that

$$\frac{J^1_i(\widetilde{x}_i, x_{-i}) - \varphi^1_i(x_{-i})}{\alpha^1_i(x_{-i}) - \varphi^1_i(x_{-i})} = \frac{J^2_i(\widetilde{x}_i, x_{-i}) - \varphi^2_i(x_{-i})}{\alpha^2_i(x_{-i}) - \varphi^2_i(x_{-i})}. \tag{9}$$

Assume there exists $\overline{x}_i$ such that $f_i(\varphi_i(x_{-i}), \overline{x}_i, x_{-i}) > f_i(\varphi_i(x_{-i}), \widetilde{x}_i, x_{-i})$, then

$$\frac{J^1_i(\overline{x}_i,x_{-i})-\varphi^1_i(x_{-i})}{\alpha^1_i(x_{-i})-\varphi^1_i(x_{-i})} > \frac{J^1_i(\widetilde{x}_i,x_{-i})-\varphi^1_i(x_{-i})}{\alpha^1_i(x_{-i})-\varphi^1_i(x_{-i})}, \quad \text{and} \quad \frac{J^2_i(\overline{x}_i,x_{-i})-\varphi^2_i(x_{-i})}{\alpha^2_i(x_{-i})-\varphi^2_i(x_{-i})} > \frac{J^2_i(\widetilde{x}_i,x_{-i})-\varphi^2_i(x_{-i})}{\alpha^2_i(x_{-i})-\varphi^2_i(x_{-i})}$$

From Assumption 2 it follows that $\alpha^1_i(x_{-i}) - \varphi^1_i(x_{-i}) > 0$ and $\alpha^2_i(x_{-i}) - \varphi^2_i(x_{-i}) > 0$; therefore

$$J^1_i(\overline{x}_i, x_{-i}) > J^1_i(\widetilde{x}_i, x_{-i}) \quad \text{and} \quad J^2_i(\overline{x}_i, x_{-i}) > J^2_i(\widetilde{x}_i, x_{-i})$$

which is a contradiction since from Lemma 3.4 it follows that $\kappa_i(\varphi_i(x_{-i}), x_{-i}) \in \mathcal{P}(x_{-i})$. Hence

$$\widetilde{x}_i \in \arg\max_{x_i \in X_i} f_i(\varphi_i(x_{-i}), x_i, x_{-i}). \tag{10}$$

Conversely, let $\widetilde{x}$ satisfy (10). By definition of KS-S, we have

$$\frac{\kappa^1_i(\varphi_i(x_{-i}), x_{-i}) - \varphi^1_i(x_{-i})}{\alpha^1_i(x_{-i}) - \varphi^1_i(x_{-i})} = \frac{\kappa^2_i(\varphi_i(x_{-i}), x_{-i}) - \varphi^2_i(x_{-i})}{\alpha^2_i(x_{-i}) - \varphi^2_i(x_{-i})}.$$

Assume that $\kappa_i(\varphi_i(x_{-i}), x_{-i}) \neq J_i(\widetilde{x}_i, x_{-i})$, then

$$\frac{\kappa^1_i(\varphi_i(x_{-i}),x_{-i})-\varphi^1_i(x_{-i})}{\alpha^1_i(x_{-i})-\varphi^1_i(x_{-i})} \leq \frac{J^1_i(\widetilde{x}_i,x_{-i})-\varphi^1_i(x_{-i})}{\alpha^1_i(x_{-i})-\varphi^1_i(x_{-i})}, \quad \text{and} \quad \frac{\kappa^2_i(\varphi_i(x_{-i}),x_{-i})-\varphi^2_i(x_{-i})}{\alpha^2_i(x_{-i})-\varphi^2_i(x_{-i})} \leq \frac{J^2_i(\widetilde{x}_i,x_{-i})-\varphi^2_i(x_{-i})}{\alpha^2_i(x_{-i})-\varphi^2_i(x_{-i})} \tag{11}$$

with at least one of the two inequalities strict. From Assumption 2 it follows $\alpha^1_i(x_{-i}) - \varphi^1_i(x_{-i}) > 0$ and $\alpha^2_i(x_{-i}) - \varphi^2_i(x_{-i}) > 0$; then (11) implies that

$$\kappa^1_i(\varphi_i(x_{-i}), x_{-i}) \leq J^1_i(\widetilde{x}_i, x_{-i}) \quad \text{and} \quad \kappa^2_i(\varphi_i(x_{-i}), x_{-i}) \leq J^2_i(\widetilde{x}_i, x_{-i})$$

with at least one of the two inequalities strict, but this is a contradiction since from Lemma 3.4 it follows that $\kappa_i(\varphi_i(x_{-i}), x_{-i}) \in \mathcal{P}(x_{-i})$. Then $\kappa_i(\varphi_i(x_{-i}), x_{-i}) = J_i(\widetilde{x}_i, x_{-i})$ and the assertion follows. □

REMARK 3.8: It can be easily deduced that the *only if part* of the previous Proposition can be immediately generalized to the case where $J_i : X \to \mathbb{R}^n$ with $n \geq 3$. While, the used proof of the other implication requires the dimension 2 of the image space for J_i.

Existence of KS-s equilibria

THEOREM 3.9: *In the Assumptions 1 and 2 hold true, if, for every player i, each component φ_i^h is continuous in X_{-i}, then there exists a* KS-S *equilibrium with disagreement point functions φ.*

Proof. We prove existence of equilibria with Leontief preferences and with disagreement point functions φ and then the assertion follows from Proposition 3.7. In fact, for every player i and from the Berge's maximum theorem each function $\alpha_i^h(\cdot)$ is continuous in X_{-i} for every $h = 1, \dots, r(i)$. Since the function each φ_i^h is continuous then the function defined by

$$F_i^h(x) = \frac{J_i^h(x_i, x_{-i}) - \varphi_i^h(x_{-i})}{\alpha_i^h(x_{-i}) - \varphi_i^h(x_{-i})},$$

is also continuous in X since, in light of the assumptions, $\alpha_i^h(x_{-i}) - \varphi_i^h(x_{-i}) \neq 0$ for all x_{-i} in X_{-i}. Moreover, from the concavity of $J_i^h(\cdot, x_{-i})$ for every x_{-i} it follows the concavity of $F_i^h(\cdot, x_{-i})$ for every x_{-i}. The min function of concave and continuous functions is concave and continuous so it is $F_i(\cdot, x_{-i})$. Therefore the game $\Gamma^O(\varphi)$ defined in (8) satisfies the Nash equilibrium existence theorem and there exists a KS-S equilibrium disagreement point functions φ. □

REMARK 3.10: From the proof of the previous theorem, the existence result for equilibria in the game with Leontief preferences efined in (8) can be obviously generalized to the case where $J_i : X \to \mathbb{R}^n$ with $n \geq 3$. While, in order to obtain also the existence of KS-S equilibria in this case, we would need additional conditions; for instance, the generalization of Proposition 3.7 would imply existence of KS-S equilibria when $n \geq 3$.

REMARK 3.11: If one of the players, say player h, has a unique criterion/faction then, setting $\kappa_h(\varphi_h(x_{-h}), x_{-h}) = \max_{x_h \in X_h} J_h(x_h, x_{-h})$, we can adapt the definition of KS-S equilibrium. In this case all the results contained in this paper hold true.

4 Models of disagreement point functions

In this section we analyze some examples by choosing explicit formulas for the disagreement point functions. The first example is the one considered in Roemer (2005) where the disagreement point function of each player i is given by an exogenous strategy of player i called *status quo strategy.* Then we consider the Roth's (1977) idea of *minimal expectation disagreement point* and we define two other different models of disagreement point functions.

4.1 Status quo strategy

As in Roemer (2005), for every player i we consider the disagreement point function $\delta_i : X_{-i} \to \mathbb{R}^2$ defined by

$$\delta_i(x_{-i}) = J_i(d_i, x_{-i}) \quad \text{for } h = 1, 2$$

where d_i is an exogenously given strategy of player i called *status quo strategy.* Then

DEFINITION 4.1: A strategy profile $x^* \in X$ is said to be a KS-S *equilibrium with status quo strategies* d if

$$\kappa_i(\delta_i(x^*_{-i}), x^*_{-i}) = J_i(x^*_i, x^*_{-i}) \quad \forall i \in I.$$

If the payoff function $x \to f_i(\delta_i(x_{-i}), x)$ of player i is obtained replacing φ_i with δ_i in (7) then we consider the game

$$\Gamma^O(\delta) = \{I; X_1, \dots, X_n; f_1(\delta_1(\cdot), \cdot), \dots, f_n(\delta_n(\cdot), \cdot)\}.$$

and as a direct application of Lemma 3.7 we get

COROLLARY 4.2: *A strategy profile $x^* \in X$ is a* KS-S *equilibrium with with status quo strategies d if and only if x^* is a Nash equilibrium of the normal form game $\Gamma^O(\delta)$.*

As a direct application of Theorem 3.9 we obtain:

PROPOSITION 4.3: *If the Assumptions 1 and 2 hold for the bargaining problems $(J_i(X_i, x_{-i}), \delta_i(x_{-i}))$ with $i = 1, \dots, n$, then there exists a* KS-S *equilibrium with status quo strategies d.*

4.2 Minimal expectations

In the previous section we constructed the KS-S equilibrium with fixed status quo strategy. However, in some situations the disagreement point is given endogenously; in particular, here we consider the Roth's idea of minimal expectations.

Recall that, for every player i, let $\mathcal{W}_i : X_{-i} \rightsquigarrow \mathbb{R}^2$ be the set-valued map where, for all $x_{-i} \in X_{-i}$, $\mathcal{W}_i(x_{-i})$ is the set of all weak Pareto points in $J_i(X_i, x_{-i})$. The minimum expectation point $m_i(x_{-i}) \in \mathbb{R}^2$ is defined by

$$m_i^h(x_{-i}) = \min_{y_i \in \mathcal{W}_i(x_{-i})} y_i^h \qquad \text{for } h = 1, 2.$$

If we consider the function $m_i : X_{-i} \to \mathbb{R}^2$ as the disagreement point function of player i, we obtain the following

DEFINITION 4.4: A strategy profile $x^* \in X$ is said to be a KS-S *equilibrium with minimal expectations* if

$$\kappa_i(m_i(x^*_{-i}), x^*_{-i}) = J_i(x^*_i, x^*_{-i}) \quad \forall i \in I.$$

If the payoff function $x \to f_i(m_i(x_{-i}), x)$ of player i is obtained replacing φ_i with m_i in (7) then we consider the game

$$\Gamma^O(m) = \{I; X_1, \dots, X_n; f_1(m_1(\cdot), \cdot), \dots, f_n(m_n(\cdot), \cdot)\}$$

and as a direct application of Lemma 3.7 we get

COROLLARY 4.5: *A strategy profile $x^* \in X$ is a* KS-S *equilibrium with minimal expectations m if and only if x^* is a Nash equilibrium of the normal form game $\Gamma^O(m)$.*

Recall that $\mathcal{P}_i : X_{-i} \rightsquigarrow \mathbb{R}^2$ the set-valued map where, for all $x_{-i} \in X_{-i}$, $\mathcal{P}_i(x_{-i})$ is the set of all weak Pareto points in $J_i(X_i, x_{-i})$ Then we have

PROPOSITION 4.6: *If the Assumptions 1 and 2 hold for the bargaining problems* $(J_i(X_i, x_{-i}), m_i(x_{-i}))$ *with* $i = 1, \dots, n$ *an if, for every player* i*, there exists a convex cone* $K_i \subseteq int\mathbb{R}^2_-$ *such that*

$$J_i(X_i, x_{-i}) \subset \mathcal{P}_i(x_{-i}) + K_i \quad \forall x_{-i} \in X_{-i}, \tag{12}$$

then there exists a KS-S *equilibrium with minimal expectations.*

Proof. From the assumptions, the set-valued map $\mathcal{W}_i : X_{-i} \rightsquigarrow \mathbb{R}^2$ is continuous on X_{-i} (see, for example, Theorems 2.2 and 2.3 in Loridan, Morgan and Raucci (1999) and references therein). Then each m_i^h is continuous on X_{-i}. Moreover, since m_i satisfies condition (5), the hypothesis of Theorem 3.9 hold true and hence the game admits at least a KS-S equilibrium with minimal expectations. □

REMARK 4.7: Property (12), called *Strong Domination Property* and defined in Bednarczuk (1994), guarantees also that $\mathcal{W}_i(x_{-i}) = \mathcal{P}_i(x_{-i})$ for all $x_{-i} \in X_{-i}$.

Strong minimal expectation

Here we propose a slight modification of the previous model where the minimal expectations are taken over a larger set, that is, the set of all the values of the payoff function of the player, for every given strategy profile of his opponents. It turns out also that in this case the existence is obtained under relaxed assumptions.

For every player and every $x_{-i} \in X_{-i}$, let $\mu_i(x_{-i}) \in \mathbb{R}^2$ be defined by

$$\mu_i^h(x_{-i}) = \min_{y_i \in J_i(X_i, x_{-i})} y_i^h \quad \text{for } h = 1, 2$$

If we consider the function $\mu_i : X_{-i} \to \mathbb{R}^2$ as the disagreement point function of player i, then

DEFINITION 4.8: A strategy profile $x^* \in X$ is said to be a KS-S *equilibrium with strong minimal expectations* if

$$\kappa_i(\mu_i(x^*_{-i}), x^*_{-i}) = J_i(x^*_i, x^*_{-i}) \quad \forall i \in I.$$

If the payoff function $x \to f_i(\mu_i(x_{-i}), x)$ of player i is obtained replacing φ_i with μ_i in (7) then we consider the game

$$\Gamma^O(\mu) = \{I; X_1, \dots, X_n; f_1(\mu_1(\cdot), \cdot), \dots, f_n(\mu_n(\cdot), \cdot)\}$$

and, as a direct application of Lemma 3.7, we get

COROLLARY 4.9: *A strategy profile* $x^* \in X$ *is a* KS-S *equilibrium with strong minimal expectations* m *if and only if* x^* *is a Nash equilibrium of the normal form game* $\Gamma^O(\mu)$.

Proposition 4.10: *If the Assumptions 1 and 2 hold for the bargaining problems* $(J_i(X_i, x_{-i}), \mu_i(x_{-i}))$ *with* $i = 1, \ldots, n$, *then there exists a* KS-s *equilibrium with strong minimal expectations.*

Proof. The set-valued map $J_i(X_i, \cdot) : X_{-i} \rightsquigarrow \mathbb{R}^2$ is obviously continuous on X_{-i}. Then each μ_i^h is continuous on X_{-i}. Moreover, since μ_i satisfies condition (5), then the hypothesis of Theorem 3.9 hold true and hence the game admits at least a KS-s equilibrium with strong minimal expectations. □

5 Stability

In this section we focus on stability of the KS-s equilibrium concept with respect to perturbations on the data. More precisely, given the game Γ, we consider a sequence of perturbed games $(\Gamma_\nu)_{\nu \in \mathbb{N}}$ with

$$\Gamma_\nu = \{I; X_{1,\nu}, \ldots, X_{n,\nu}; J_{1,\nu}, \ldots, J_{n,\nu}\} \quad \forall \nu \in \mathbb{N}.$$

We investigate conditions of convergence of the data of the game which guarantee the convergence of KS-s equilibria of perturbed games to a KS-s equilibrium of the original game.

Theorem 5.1: *Given the multicriteria game* Γ *and a disagreement point function* φ *satisfying (5), assume that, for every player* i:

i) $(X_{i,\nu})_{\nu \in \mathbb{N}}$ *is a sequence of sets converging to* X_i *in the sense of Painlevé-Kuratowski, that is*

$$\operatorname*{Lim\,sup}_{\nu \to \infty} X_{i,\nu} \subseteq X_i \subseteq \operatorname*{Lim\,inf}_{\nu \to \infty} X_{i,\nu} \qquad \textit{where} \tag{13}$$

$$\operatorname*{Lim\,inf}_{\nu \to \infty} X_{i,\nu} = \left\{ x_i \in \mathbb{R}^{m(i)} \mid \forall \varepsilon > 0 \; \exists \overline{\nu} \; s.t. \; for \nu \geq \overline{\nu} \; B_i(x_i, \varepsilon) \cap X_{i,\nu} \neq \emptyset \right\}, \tag{14}$$

$$\operatorname*{Lim\,sup}_{\nu \to \infty} X_{i,\nu} = \left\{ x_i \in \mathbb{R}^{m(i)} \mid \forall \varepsilon > 0 \; \forall \overline{\nu} \in \mathbb{N} \; \exists \nu \geq \overline{\nu} \; s.t. \; B_i(x_i, \varepsilon) \cap X_{i,\nu} \neq \emptyset \right\}. \tag{15}$$

ii) $(J_{i,\nu})_{\nu \in \mathbb{N}}$ *is a sequence of functions from* X_ν *to* $\mathbb{R}^2$ *for every* $\nu \in \mathbb{N}$, *such that* $J_{i,\nu}$ *continuously converges to* J_i, *i.e., for every* $x \in X$ *and for every sequence* $(x_\nu)_{\nu \in \mathbb{N}}$ *converging to* x, *with* $x_\nu \in X_{i,\nu}$ *for every* $\nu \in \mathbb{N}$, *it follows that*

$$\lim_{\nu \to \infty} J_{i,\nu}(x_\nu) = J_i(x).$$

iii) $(\varphi_{i,\nu})_{\nu \in \mathbb{N}}$*is a sequence of functions from* $X_{-i,\nu}$ *to* $\mathbb{R}$ *such that:*

a) $\forall x_{-i} \in X_{-i,\nu} \quad \exists y_\nu(x_{-i}) \in J_{i,\nu}(X_{i,\nu}, x_{-i})$ such that $y_\nu(x_{-i}) - \varphi_{i,\nu}(x_{-i}) \in int\mathbb{R}^2_+$;

b) $(\varphi_{i,\nu})_{\nu \in \mathbb{N}}$ *continuously converges to* φ_i, *i.e., for every* $x_{-i} \in X_{-i}$ *and for every sequence* $(x_{-i,\nu})_{\nu \in \mathbb{N}}$ *converging to* x_{-i}, *with* $x_{-i,\nu} \in X_{-i,\nu}$ *for every* $\nu \in \mathbb{N}$, *it follows that*

$$\lim_{\nu \to \infty} \varphi_{i,\nu}(x_{-i,\nu}) = \varphi_i(x_{-i}).$$

If x^*_ν is a KS-S *equilibrium of the game Γ_ν with disagreement point function φ_ν for every $\nu \in \mathbb{N}$ and the sequence $(x^*_\nu)_{\nu\in\mathbb{N}}$ converges to $x^* \in X$, then x^* is a* KS-S *equilibrium of the game Γ with disagreement point function φ.*

Proof. We prove the result for equilibria with Leontief preferences and with disagreement point functions φ and then the assertion follows from Proposition 3.7.

For every player i and every $\nu \in \mathbb{N}$, consider the functions $\alpha^h_{i,\nu} : X_{-i,\nu} \to \mathbb{R}$, for $h = 1, 2$, defined by:

$$\alpha^h_{i,\nu}(x_{-i}) = \max_{x_i \in X_{i,\nu}} J^h_{i,\nu}(x_i, x_{-i}) \quad \forall x_{-i} \in X_{-i,\nu}$$

From Berge's Theorem it follows that the sequence $(\alpha^h_{i,\nu})_\nu$ continuously converges to α^h_i for every $h \in \{1, 2\}$. Therefore, for every player i and every $h = 1, 2$, it follows that the sequence of functions $(F^h_{i,\nu})_{\nu\in\mathbb{N}}$ defined by

$$F^h_{i,\nu}(x) = \frac{J^h_{i,\nu}(x) - \varphi^h_{i,\nu}(x_{-i})}{\alpha^h_{i,\nu}(x_{-i}) - \varphi^h_{i,\nu}(x_{-i})}$$

continuously converges to the function F^h_i defined by

$$F^h_i(x) = \frac{J^h_i(x) - \varphi^h_i(x_{-i})}{\alpha^h_i(x_{-i}) - \varphi^h_i(x_{-i})}.$$

Again, in light of the Berge's Theorem, the sequence of functions $(F_{i,\nu})_\nu$ defined by $F_{i,\nu}(x) = \min\left\{F^1_{i,\nu}(x), F^2_{i,\nu}(x)\right\}$ continuously converges to F_i defined by $F_i(x) = \min\left\{F^1_i(x), F^2_i(x)\right\}$. If $(x^*_\nu)_\nu$ is a sequence of KS-S equilibria of Γ_ν with disagreement point functions φ_ν converging to x^*, then, for every ν and every player i, it follows that $x^*_{i,\nu} \in \arg\max_{z_i \in X_{i,\nu}} F_{i,\nu}(z_i, x^*_{-i,\nu})$. Therefore, in light of the Berge's Theorem , it follows that $x^*_i \in \arg\max_{z_i \in X_i} F_i(z_i, x^*_{-i})$. Hence x^* is a KS-S equilibrium and the assertion follows.

□

6 Perfectness

In this section we refine the KS-S equilibrium concept by considering the perfectness approach in Selten (1975), in the context of games in mixed strategies with a finite number of pure strategies. Selten's idea is to consider the possibility that agents make mistakes playing their equilibrium strategies. When such mistakes occur, it may happen that equilibria are not stable, therefore, the concept of trembling hand perfect equilibrium for normal form games is defined by a limit process and it is based on the idea that players coordinate their choices on a Nash equilibrium which is stable with respect to mistakes in the choice of their equilibrium strategies. More precisely, if an equilibrium is not perfect then it is unstable with respect to every Selten's perturbation on the strategies. Different interesting contributions have been provided to generalize this solution concepts to the multicriteria case (see Puerto and Fernandez (1995) or Borm, van Megen and Tijs (1999)); in this section we propose to combine perfectness with the game between organizations approach of Roemer. It turns out that not only it is possible to combine the KS-S equilibria with perfect equilibria, but also that this selection device is sharper than perfectness and KS-S equilibrium concept.

Given a n-player finite game $\Omega = \{I; \Phi_1, \dots, \Phi_n; H_1, \dots, H_n\}$ where $\Phi_i = \{\varphi_i^1, \dots, \varphi_i^{k(i)}\}$ is the (finite) pure strategy set of player i, $\Phi = \prod_{i \in I} \Phi_i$ and $H_i : \Phi \to \mathbb{R}^2$ is the vector-valued payoff function of player i, then in this section $\Gamma = \{I; X_1, \dots, X_n; J_1, \dots, J_n\}$ denotes the mixed extension of Ω. Therefore, each strategy $x_i \in X_i$ is a vector $x_i = (x_i(\varphi_i))_{\varphi_i \in \Phi_i} \in \mathbb{R}_+^{k(i)}$ such that $\sum_{\varphi_i \in \Phi_i} x_i(\varphi_i) = 1$ and the expected payoff function $J_i : X \to \mathbb{R}^2$ is defined by:

$$J_i^h(x) = \sum_{\varphi \in \Phi} \left[\prod_{i \in I} x_i(\varphi_i) \right] H_i^h(\varphi) \quad \text{for all } x \in X \quad \text{for all } h = 1, 2$$

We recall the following definition

Definition 6.1 (Selten (1975)): Let Ω be a finite game and Γ its mixed extension. For every player i, let $\eta_i : \Phi_i \to]0, 1[$ be a function satisfying

$$\sum_{\varphi_i \in \Phi_i} \eta_i(\varphi_i) < 1$$

Let $\eta = (\eta_1, \dots, \eta_n)$ and $X_{i,\eta} = \{x_i \in X_i \mid x_i(\varphi_i) \geq \eta_i(\varphi_i) \quad \forall \varphi_i \in \Phi_i\}$. The game $(\Gamma, \eta) = \{I; X_{1,\eta}, \dots, X_{n,\eta}; J_1, \dots, J_n\}$ will be called *Selten's perturbed game.*

The natural extension to multicriteria games of the trembling hand perfect equilibrium concept (Puerto and Fernandez (1995) or Borm, van Megen and Tijs (1999)) is the following:

Definition 6.2: Let Ω be a finite game and Γ its mixed extension. A weak Pareto-Nash equilibrium x of Γ is a *trembling hand perfect equilibrium* of Γ if there exist a sequence of perturbed games $\{(\Gamma, \eta_\nu)\}_{\nu \in \mathbb{N}}$ and a sequence of strategy profiles $\{x_\nu\}_{\nu \in \mathbb{N}}$ such that:

i) for all $\nu \in \mathbb{N}$, x_ν is a weak Pareto-Nash equilibrium of (Γ, η_ν)

ii) $\lim_{\nu \to \infty} x_\nu = x, \qquad \lim_{\nu \to \infty} \eta_\nu = 0$

So, the perfectness approach for KS-S equilibria reads naturally as follows

Definition 6.3: Let Ω be a finite game and Γ its mixed extension. A strategy profile $x \in X$ is said to be a *perfect* KS-S *equilibrium* of Γ with disagreement point functions φ satisfying (5) if there exist a sequence of perturbed games $\{(\Gamma, \eta_\nu)\}_{\nu \in \mathbb{N}}$, a sequence of strategy profiles $\{x_\nu\}_{\nu \in \mathbb{N}}$ and, for every player i, a sequence of disagreement point functions $\{\varphi_{i,\nu}\}_{\nu \in \mathbb{N}}$ from X_{-i,η_ν} to $\mathbb{R}$ satisfying *(iii)*,a)) in Theorem (5.1), such that

i) x_ν is a KS-S equilibrium of (Γ, η_ν) with disagreement point function φ_ν, for all $\nu \in \mathbb{N}$;

ii) (α): $\lim_{\nu\to\infty} x_\nu = x$;
(β): $\lim_{\nu\to\infty} \eta^\nu = 0$,
(γ): $\{\varphi_{i,\nu}\}_{\nu\in\mathbb{N}}$ continuously converges to φ_i for every $i \in I$.

THEOREM 6.4: *Let Ω be a finite game, Γ be its mixed extension and φ be a profile of disagreement point functions satisfying condition (5). Then there exists a perfect* KS-S *equilibrium with disagreement point functions φ.*

Proof. For every Selten's perturbation η, each set $X_{i,\eta}$ is compact and contained in the simplex X_i. Therefore, given a sequence of perturbations $\{\eta_\nu\}_{\nu\in\mathbb{N}}$, it follows that

$$J_i(X_{i,\eta_\nu}, x_{-i}) \subseteq J_i(X_i, x_{-i}) \quad \forall x_{-i} \in X_{-i},\ \forall \nu \in \mathbb{N}$$

holds for every player i. For every $x_{-i} \in X_{-i}$, $J_i(X_i, x_{-i})$ is compact, then

$$\varphi_i(x_{-i}) + \mathbb{R}^2_+ \supset J_i(X_i, x_{-i}) \supseteq J_i(X_{i,\eta_\nu}, x_{-i}) \quad \forall \nu \in \mathbb{N}$$

So, for every player i and for every ν, the disagreement point function from X_{-i,η_ν} to $\mathbb{R}$ satisfies (*iii)*,a)) in Theorem 5.1, for all ν, and (5). Let $\{x_\nu\}_{\nu\in\mathbb{N}}$ be a sequence of KS-S equilibria with disagreement point functions φ for the games (Γ, η_ν); since the sequence is compact then it admits a subsequence converging to $x^* \in X$. Then x^* is, by definition, perfect KS-S equilibrium of Γ with disagreement point functions φ. □

REMARK 6.5: From the final part of the previous proof, we can deduce that, if a constant function φ satisfies (5) and (*iii)*,a)) in Theorem (5.1) along a sequence of Selten's perturbed games, then there exists a perfect KS-S equilibrium of Γ with disagreement point functions φ.

PROPOSITION 6.6: *Let Ω be a finite game and Γ its mixed extension. Then, every perfect* KS-S *equilibrium with disagreement point functions φ of Γ is a perfect Pareto-Nash equilibrium of Γ and a* KS-S *equilibrium with disagreement point functions φ of Γ .*

Proof. Assume x^* is a perfect KS-S equilibrium of Γ with disagreement point functions φ. Then there exists a sequence $\{x_\nu\}_{\nu\in\mathbb{N}}$ converging to x such that x_ν is a KS-S equilibrium of (Γ, η_ν) with disagreement point function φ_ν, for all $\nu \in \mathbb{N}$ and *ii)* in Definition 6.3 are satisfied. In light of the definition of the KS-S, x_ν is a weak Pareto-Nash equilibrium of (Γ, η_ν) for all $\nu \in \mathbb{N}$, therefore x is a trembling hand perfect equilibrium of Γ. Moreover, from Theorem 5.1 it also follows that x is a KS-S equilibrium of Γ with disagreement point functions φ. Hence the assertion follows. □

7 An example

Now we show with an example that not only KS-S equilibria refine Pareto-Nash equilibria but also that perfect KS-S equilibria refine perfect Pareto-Nash equilibria.

Consider the following two player game.

	L	R
T	1,0	0,0
M	0,0	0, 1
B	-1,2	-1,0

payoffs of Player 1

	L	R
T	1	0
M	1	1
B	0	0

payoffs of Player 2

We consider mixed strategies and we denote with $p_1 = Prob(T)$, $p_2 = Prob(M)$, $1 - p_1 - p_2 = Prob(B)$ and $q = Prob(L)$, $1 - q = Prob(R)$. Denote with X_i the set of mixed strategies of player i, i.e.,

$$X_1 = \{(p_1, p_2) \in \mathbb{R}^2 \mid p_1, p_2 \geq 0;\ p_1 + p_2 \leq 1\}, \quad X_2 = \{q \in \mathbb{R} \mid 0 \leq q \leq 1\}.$$

Note that, for every $q \in [0, 1]$, $J_1(T, q) = (q, 0)$, $J_1(M, q) = (0, 1 - q)$ and $J_1(B, q) = (-1, 2q)$ so for every $q \in [0, 1]$ the set $J_1(X_1, q)$ of the images of the vector-valued expected payoff of Player 1 is given by the convex hull of the points $(q, 0), (0, 1 - q), (-1, 2q)$. Denote with $\gamma_1(q)$ the segment joining $J_1(T, q)$ to $J_1(M, q)$, with $\gamma_2(q)$ the segment joining $J_1(M, q)$ to $J_1(B, q)$ and with $\gamma_3(q)$ the segment joining $J_1(T, q)$ with $J_1(B, q)$, i.e.

$$\begin{aligned}
\gamma_1(q) &= \big\{\big(sq, (1-s)(1-q)\big),\ \forall s \in [0, 1]\big\},\\
\gamma_2(q) &= \big\{\big(-s, 2qs + (1-s)(1-q)\big),\ \forall s \in [0, 1]\big\}\\
\gamma_3(q) &= \big\{\big(-s + (1-s)q, 2qs\big),\ \forall s \in [0, 1]\big\}
\end{aligned}$$

It is easy to check that the set $\mathcal{W}_1(q)$ of weak Pareto points in $J_1(X_1, q)$ is given by the following:

$$\mathcal{W}_1(q) = \begin{cases} \gamma_1(q) & \text{if } q \in [0, 1/3[\\ \gamma_1(q) \cup \gamma_2(q) & \text{if } q \in [1/3, 1/\sqrt{3}[\\ \gamma_3(q) & \text{if } q \in [1/\sqrt{3}, 1] \end{cases}$$

where for $q = 1/\sqrt{3}$ it results that $\gamma_1(q) \cup \gamma_2(q) = \gamma_3(q)$; then, the best reply correspondence of Player 1 is given by:

$$\operatorname*{Arg\,wmax}_{(p_1,p_2)\in X_1} J_1(p_1, p_2, q) = \begin{cases} \{(p_1, p_2) \in X_1 \mid p_1 + p_2 = 1\} & \text{if } q \in [0, 1/3[\\ \{(p_1, p_2) \in X_1 \mid p_1 + p_2 = 1\} \cup \{0\} \times [0, 1] & \text{if } q \in [1/3, 1/\sqrt{3}[\\ X_1 & \text{if } q = 1/\sqrt{3} \\ [0, 1] \times \{0\} & \text{if } q \in]1/\sqrt{3}, 1] \end{cases};$$

the best reply correspondence of Player 2 is given by

$$\operatorname*{arg\,max}_{q\in[0,1]} J_2(p_1, p_2, q) = \begin{cases} q = 1 & \text{if } p_1 > 0 \\ q \in [0, 1] & \text{if } p_1 = 0 \end{cases}$$

Denoted with

$$\begin{aligned}
\mathcal{P}_1 &= \{(0, 1, q) \mid q \in [0, 1/3[\}\\
\mathcal{P}_2 &= \{(0, p_2, q) \mid p_2 \in [0, 1],\ q \in [1/3, 1/\sqrt{3}]\}\\
\mathcal{P}_3 &= \{(0, 0, q) \mid q \in [1/\sqrt{3}, 1[\}\\
\mathcal{P}_4 &= \{(p_1, 0, 1) \mid p_1 \in [0, 1]\}.
\end{aligned}$$

Then the set of weak Pareto-Nash equilibria WPN is

$$WPN = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4.$$

In order to find the KS-S equilibria, observe that the ideal point function $\alpha_1(q)$ of Player 1 is

$$\alpha_1(q) = \begin{cases} (q, 1-q) & \text{if } q \in [0, 1/3[\\ (q, 2q) & \text{if } q \in [1/3, 1] \end{cases}.$$

Consider the strong minimal expectation disagreement point function of Player 1 (which in this case coincides with the minimal expectation disagreement point function), that is,

$$\varphi_1(q) = (-1, 0) \quad \forall q \in [0, 1]. \tag{16}$$

Therefore, for every $q \in [0, 1]$, the KS-S $\kappa_i(q)$ to $J_1(X_1, q)$ is the maximal element (with respect to Pareto dominance) in $J_1(X_1, q)$ on the line $l(q)$ joining $\varphi_1(q)$ to $\alpha_1(q)$, where it can be checked that

$$l(q) = \begin{cases} \{(z_1, z_2) \in \mathbb{R}^2 \mid l_1(q) = (1-q)z_1 - (q+1)z_2 + 1 - q = 0\} & \text{if } q \in [0, 1/3[\\ \{(z_1, z_2) \in \mathbb{R}^2 \mid l_2(q) = 2qz_1 - (q+1)z_2 + 2q = 0\} & \text{if } q \in [1/3, 1] \end{cases}.$$

Therefore it can be checked that the KS-S $\kappa_1(q)$ is given by:

$$\kappa_1(q) = \begin{cases} \gamma_1(q) \cap l_1(q) & \text{if } q \in [0, 1/3[\\ \gamma_1(q) \cap l_2(q) & \text{if } q \in]1/3, -1+\sqrt{2}[\\ \gamma_2(q) \cap l_2(q) & \text{if } q \in [-1+\sqrt{2}, 1/\sqrt{3}[\\ \gamma_3(q) \cap l_2(q) & \text{if } q \in [1/\sqrt{3}, 1] \end{cases}.$$

where for $q = -1 + \sqrt{2}$ the line $l_2(q)$ passes through the point $(0, 1-q)$. It can be calculated that

$$\kappa_1(q) = \begin{cases} \left(\frac{q^2}{1+2q}, \frac{(1-q^2)}{1+2q}\right) & \text{if } q \in [0, 1/3[\\ \left(\frac{q(1-2q-q^2)}{q^2+1}, \frac{2q(1-q^2)}{q^2+1}\right) & \text{if } q \in [1/3, -1+\sqrt{2}[\\ \left(\frac{q^2+2q-1}{1-4q-3q^2}, \frac{-4q^2}{1-4q-3q^2}\right) & \text{if } q \in [-1+\sqrt{2}, 1/\sqrt{3}[\\ \left(\frac{q-1}{2}, q\right) & \text{if } q \in]1/\sqrt{3}, 1] \end{cases}.$$

Therefore, denoted with $BR_1^\kappa(q)$ the best reply correspondence of Player 1 when he runs a KS-S within his criteria then, denoted with v^t the transpose of the row vector v,

$$(p_1^\kappa(q), p_2^\kappa(q)) \in BR_1^\kappa(q) \text{ if and only if } p_2^\kappa(q), p_2^\kappa(q) \geq 0,\ p_1^\kappa(q) + p_2^\kappa(q) \leq 1, \text{ and}$$

$$\begin{cases} i)\ p_1^\kappa(q)J_1(T,q)^t + p_2^\kappa(q)J_1(M,q)^t = \left(\frac{q^2}{1+2q}, \frac{(1-q^2)}{1+2q}\right)^t & \text{if } q \in [0, 1/3[\\ ii)\ p_1^\kappa(q)J_1(T,q)^t + p_2^\kappa(q)J_1(M,q)^t = \left(\frac{q(1-2q-q^2)}{q^2+1}, \frac{2q(1-q^2)}{q^2+1}\right)^t & \text{if } q \in [1/3, -1+\sqrt{2}[\\ iii)\ p_2^\kappa(q)J_1(M,q)^t + (1-p_2^\kappa(q))J_1(B,q)^t = \left(\frac{q^2+2q-1}{1-4q-3q^2}, \frac{-4q^2}{1-4q-3q^2}\right)^t & \text{if } q \in [-1+\sqrt{2}, 1/\sqrt{3}] \\ iv)\ p_1^\kappa(q)J_1(T,q)^t + p_2^\kappa(q)J_1(M,q)^t + & \\ +(1-p_1^\kappa(q)-p_2^\kappa(q))J_1(B,q)^t = \left(\frac{q^2+2q-1}{1-4q-3q^2}, \frac{-4q^2}{1-4q-3q^2}\right)^t & \text{if } q = 1/\sqrt{3} \\ v)\ p_1^\kappa(q)J_1(T,q)^t + (1-p_1^\kappa(q))J_1(B,q)^t = \left(\frac{q-1}{2}, q\right)^t & \text{if } q \in]1/\sqrt{3}, 1] \end{cases}.$$

Therefore $i)$, $ii)$, $iii)$ $iv)$, $v)$ are linear systems which can be rewritten as

$$i) \begin{cases} p_1^\kappa(q)q = \frac{q^2}{1+2q} \\ p_2^\kappa(q)(1-q) = \frac{(1-q^2)}{1+2q} \end{cases}, \; ii) \begin{cases} p_1^\kappa(q)q = \frac{q(1-2q-q^2)}{q^2+1} \\ p_2^\kappa(q)(1-q) = \frac{2q(1-q^2)}{q^2+1} \end{cases}$$

$$iii) \begin{cases} -(1-p_2^\kappa(q)) = \frac{q^2+2q-1}{1-4q-3q^2} \\ p_2^\kappa(q)(1-q) + 2q(1-p_2^\kappa(q)) = \frac{-4q^2}{1-4q-3q^2} \end{cases},$$

$$iv) \begin{cases} \frac{1}{\sqrt{3}}p_1^\kappa(q) - (1-p_1^\kappa(q)-p_2^\kappa(q)) = \frac{1-\sqrt{3}}{2\sqrt{3}} \\ p_2^\kappa(q)\frac{\sqrt{3}-1}{\sqrt{3}} + \frac{2}{\sqrt{3}}(1-p_1^\kappa(q)-p_2^\kappa(q)) = \frac{1}{\sqrt{3}} \end{cases}, \; v) \begin{cases} p_1^\kappa(q)q - (1-p_1^\kappa(q)) = \frac{q-1}{2} \\ 2q(1-p_1^\kappa(q)) = q \end{cases}.$$

Therefore, solving the systems we get,

$$BR_1^\kappa(q) = \begin{cases} \left(\frac{q}{1+2q}, \frac{(1+q)}{1+2q}\right) & \text{if } q \in [0, 1/3[\\ \left(\frac{1-2q-q^2}{q^2+1}, \frac{2q(1+q)}{q^2+1}\right) & \text{if } q \in [1/3, -1+\sqrt{2}[\\ \left(0, \frac{-2q(q+1)}{1-4q-3q^2}\right) & \text{if } q \in [-1+\sqrt{2}, 1/\sqrt{3}[\\ (p_1, p_2) \text{ s.t. } 2p_1 + \left(\frac{2\sqrt{3}}{\sqrt{3}+1}\right)p_2 = 1, \; p_1 \in [0, 1/2] & \text{if } q = 1/\sqrt{3} \\ (1/2, 0) & \text{if } q \in]1/\sqrt{3}, 1] \end{cases}.$$

Finally, the set $\mathcal{K}(\mu)$ of KS-S equilibria is

$$\mathcal{K}(\mu) = \{(0,1,0)\} \cup \left\{ \left(0, \frac{-2q(q+1)}{1-4q-3q^2}, q\right) \mid q \in [-1+\sqrt{2}, 1/\sqrt{3}] \right\} \cup \left\{ \left(\frac{1}{2}, 0, 1\right) \right\}.$$

The set of the multicriteria trembling hand perfect equilibria E^T of this game is $E^T = \{(p_1, 0, 1) \mid p_1 \in [0,1]\}$. In fact consider a Selten's perturbation η on the set of strategy profiles, that is a pair of functions $\eta = (\eta_1, \eta_2)$, where $\eta_1 : \{T, M, B\} \to]0,1[$ and $\eta_2 : \{L, R\} \to]0,1[$ such that:

$$\eta_1(T) + \eta_1(M) + \eta_1(B) < 1 \qquad \eta_2(L) + \eta_2(R) < 1.$$

Let $(\Gamma, \eta) = \{2; X_{1,\eta}, X_{2,\eta}; J_1, J_2\}$ be the corresponding perturbed multicriteria game, where

$$X_{1,\eta} = \{(p_1, p_2) \in X_1 \mid \eta_1(T) \leq p_1 \leq 1-(\eta_1(M)+\eta_1(B)), \; \eta_1(M) \leq p_2 \leq 1-(\eta_1(T)+\eta_1(B))\}$$

$$X_{2,\eta} = \{q \in X_2 \mid \eta_2(L) \leq q \leq 1-\eta_2(R)\}.$$

The best reply correspondences in the perturbed games are:

$$(p_1, p_2) \in \underset{(p_1,p_2)\in X_{1,\eta}}{\text{Arg wmax}} \, J_1(p_1, p_2, q) \text{ if and only if}$$

$$\begin{cases} i)\{(p_1,p_2) \mid p_1 \geq \eta_1(T), p_2 \geq \eta_1(M),\ p_1+p_2 = 1-\eta_1(B)\} & \text{if } q \in \eta_2(L), 1/3[\\ ii)\{(p_1,p_2) \mid p_1 \geq \eta_1(T), p_2 \geq \eta_1(M),\ p_1+p_2 = 1-\eta_1(B)\} & \text{and} \\ \{(\eta_1(T),p_2) \mid p_2 \in [\eta_1(T), 1-\eta_1(T)-\eta_1(B)]\} & \text{if } q \in [1/3, 1/\sqrt{3}[\\ iii)\{(p_1,p_2) \mid p_1 \geq \eta_1(T), p_2 \geq \eta_1(M),\ p_1+p_2 \leq 1-\eta_1(B)\} & \text{if } q = 1/\sqrt{3} \\ iv)\{(p_1,\eta_1(M)) \mid p_1 \in [\eta_1(T), 1-\eta_1(M)-\eta_1(B)]\} & \text{if } q \in]1/\sqrt{3}, 1-\eta_2(R)] \end{cases} ;$$

the best reply correspondence of Player 2 is given by

$$q \in \arg\max q \in X_{2,\eta} J_2(p_1,p_2,q) \iff \{\ q = 1-\eta_2(R) \quad \text{for all } (p_1,p_2) \in X_{1,\eta}$$

Then, the set WPE_η of the weak Pareto-Nash equilibria of (Γ, η) is:

$$WPE_\eta = \{(p_1, \eta_1(M), 1-\eta_2(R)) \mid p_1 \in [\eta_1(T), 1-\eta_1(M)-\eta_1(B)]\}$$

Therefore, the set E^T of the multicriteria trembling hand perfect equilibria of this game is

$$E^T = WPE \cap \operatorname*{Lim\,sup}_{\eta\to 0} WPE_\eta = \{(p_1, 0, 1) \mid p_1 \in [0,1]\}$$

In light of Theorem (6.4) and Proposition (6.6), the set $\mathcal{PK}(\mu)$ of perfect KS-S equilibria satisfies

$$\emptyset \neq \mathcal{PK}(\mu) \subseteq \mathcal{K}(\mu) \cap E^T = \left\{ \left(\frac{1}{2}, 0, 1 \right) \right\}.$$

Therefore there exists a unique perfect KS-S equilibrium with the disagreement point function defined in (16).

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