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Giuseppe De Marco[*] and Jacqueline Morgan[**]

Abstract

The famous Harsanyi's (1973) Theorem states that *generically* a finite game has an odd number of Nash equilibria in mixed strategies. In this paper, we show that for finite multicriteria games (games with vector-valued payoffs) this kind of result does not hold. In particular, we show, by examples, that it is possible to find balls in the space of games such that every game in this set has uncountably many equilibria so that uncountable sets of equilibria are not *nongeneric* in multicriteria games. Moreover, we point out that, surprisingly, all the equilibria of the games cor- responding to the center of these balls are essential, that is, they are stable with respect to every possible perturbation on the data of the game. However, if we consider the *scalarization stable equilibrium* concept (introduced in De Marco and Morgan (2007) and which is based on the scalarization technique for multicriteria games), then we show that it provides an effective selection device for the equilibria of the games corresponding to the centers of the balls. This means that the scalarization stable equilibrium concept can provide a sharper selection device with respect to the other classical refinement concepts in multicriteria games.

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1 Introduction

Harsanyi (1973) proved that *generically* a finite game has an odd number of Nash equilibria in mixed strategies and that they are all regular. Regularity of an equilibrium implies that the graph of the Nash equilibrium correspondence, that is, the set-valued map defined in the space of games having the same strategy sets that associates to every game the set of its Nash equilibria, is the graph of a continuous function in a neighborhood of such equilibrium considered as a point in the graph (see also Ritzberger (1994) or van Damme (1989)). Obviously, this property implies stability of the equilibrium with respect to every possible perturbation on the data of the game.

However, when an equilibrium is not regular it might be "unstable" with respect to perturbations on the strategies or on the payoffs and therefore refinements of the Nash equilibrium concept have been introduced in order to select equilibria stable with respect to particular classes of perturbation. Just to quote a few, we recall perfectness (Selten 1975), properness (Myerson (1978), essentiality (Jiang and Wu (1965)), strategic stability (Kohlberg and Mertens (1986)). In particular, essentiality is the property of an equilibrium to be stable with respect to every possible perturbation on the payoffs of a game.

In this paper, we show by examples that, when the payoffs of the game are vector-valued (multicriteria games), the previous considerations about the non genericity of infinite sets of equilibria do not hold anymore. Recall that multicriteria games describe strategic interactions in which players have different goals whose relative values cannot be ascertained a-priori, for example, representing a single individual with multiple objectives or an organization of individuals which have to jointly take a single decision and where each of the criteria corresponds to the concerns of a different faction of the organization. Different extensions of the classical concept of Nash equilibrium have been adopted for multicriteria games; the concepts of weak Pareto-Nash and Pareto-Nash equilibrium, as introduced in Shapley (1959), play a fundamental role and satisfy existence theorems under classical assumptions. We consider these concepts of equilibrium and find balls in the space of games such that every element in this set has uncountably many equilibria so that uncountable sets of equilibria are not *nongeneric* in multicriteria games. Moreover, we show that, surprisingly, all the equilibria of the games corresponding to the centers of these balls are essential, that is, they are stable with respect to every possible perturbation on the data of the game. More precisely, we present two examples which independently satisfy these properties; however, the first example is much easier from the mathematical point of view (just two strategies for each player and only one vector-valued payoff) but it is much less interesting from the game theoretic point of view since it involves a trivial best reply correspondence of the first player which coincides with his strategy set for every strategy of the second player. Therefore, the first example shows that the results hold in the simplest class of multicriteria games and the second example (in which the first player has three strategies instead of two) emphasizes that the results coming out from the first example do not depend on the *degenerate* behavior of the best reply correspondences and may arise in nontrivial games.

In the final part of the paper, we notice that, even though essentiality is usually a very sharp selection device, another kind of stability requirement (namely the *scalarization-stable equilibrium* concept defined in De Marco and Morgan (2007)) provides an effective selection device for the weak Pareto-Nash equilibria of the games in both the examples. The scalarization-stable equilibrium concept is based on the following scalarization technique for finite multicriteria games studied in Shapley (1959): every weak Pareto-Nash equilibrium is a Nash equilibrium of a scalar game in which the real-valued payoff is obtained by weighting the components of the vector-valued payoff with weights in the simplex (called *trade-off game*), and conversely. In other words, when an equilibrium x of a multicriteria game is played, the choice of a strategy x_i by a player i as a best reply to his opponents' strategy profile x_{-i} implicitly implies that Player i is using particular total order relations to implement x_i as a maximum point and Player i is assuming the others playing x_{-i} and then, he is implicitly assuming the others using particular total order relations. Therefore scalarization-stable equilibria are obtained by perturbing the weights in the scalarization and by requiring a lower semicontinuity-like stability in the equilibria, in order to capture the stability of the equilibrium with respect to perturbations on the total order relations of every player and with respect to perturbations on the expectations of each player about others' total order relations. As a final remark, the examples presented in this paper show also that essentiality in a multicriteria game differs from essentiality in the corresponding trade-off games, since there exists (at least) an essential (weak Pareto-Nash) equilibrium of the multicriteria game such that, for every possible scalarization, it is not an essential (Nash) equilibrium of the corresponding trade-off game. This latter result is in line with an analogous result obtained for perfect equilibria in Borm, van Megen and Tijs (1999).

The paper is organized as follows: Section 2 presents the basic definitions of equilibrium in multicriteria games and introduces the genericity problem in finite games. In Section 3, the two examples are analyzed. Section 4 presents the scalarization stable equilibrium concept and its application to the examples.

2 Finite Multicriteria Normal Form Games

2.1 Equilibria

Multicriteria games describe interactions in which players' payoff are vector-valued functions; which means that players, having more than one criterion to take into account, don't have an a-priori opinion on the relative importance of all their criteria. Given a n-player finite game $\Omega = \{I; \Psi_1, \ldots, \Psi_n; H_1, \ldots, H_n\}$ where $\Psi_i = \{\psi_i^1, \ldots, \psi_i^{k(i)}\}$ is the (finite) pure strategy set of player i, $\Psi = \prod_{i \in I} \Psi_i$ and $H_i : \Psi \to \mathbb{R}^{r(i)}$ is the vector-valued payoff function of player i, then in this section $\Gamma = \{I; X_1, \ldots, X_n; J_1, \ldots, J_n\}$ denotes the mixed extension of Ω. Therefore, each strategy $x_i \in X_i$ is a vector $x_i = (x_i(\psi_i))_{\psi_i \in \Psi_i} \in \mathbb{R}_+^{k(i)}$ such that $\sum_{\psi_i \in \Psi_i} x_i(\psi_i) = 1$ and the expected payoff function $J_i : X \to \mathbb{R}^{r(i)}$, with $X = \prod_{i=1}^n$, is defined by:

$$J_i^h(x) = \sum_{\psi\in\Psi} \left[\prod_{i\in I} x_i(\psi_i)\right] H_i^h(\psi) \quad \text{for all } x\in X \quad \text{and for all } h=1,\dots,r(i).$$

In case the players act non-cooperatively, different extensions of the classical concept of Nash equilibrium have been adopted; however, the concepts of weak Pareto-Nash and Pareto-Nash equilibrium, as introduced in Shapley (1959), play a fundamental role (see Wang (1993) for more general existence theorems and Morgan (2005) for variational stability, well-posedness and for an extensive list of references). We recall here some classical definitions and notations:

DEFINITION 2.1: Given $x_{-i}\in X_{-i}$, the strategy $\widehat{x}_i \in X_i$ is said to be *strongly (Pareto) dominated* by the strategy $\overline{x}_i\in X_i$ if the vector $J_i\left(\widehat{x}_i, x_{-i}\right)$ is strongly (Pareto) dominated by the vector $J_i\left(\overline{x}_i, x_{-i}\right)$, that is

$$J_i\left(\overline{x}_i, x_{-i}\right) - J_i\left(\widehat{x}_i, x_{-i}\right) \in int\mathbb{R}_+^{r(i)}.$$

While, the strategy $\widehat{x}_i\in X_i$ is said to be *(Pareto) dominated* by the strategy $\overline{x}_i\in X_i$ if the vector $J_i\left(\widehat{x}_i, x_{-i}\right)$ is (Pareto) dominated by the vector $J_i\left(\overline{x}_i, x_{-i}\right)$, that is

$$J_i\left(\overline{x}_i, x_{-i}\right) - J_i\left(\widehat{x}_i, x_{-i}\right) \in \mathbb{R}_+^{r(i)} \setminus \{0\}.$$

Let $J_i(X_i, x_{-i}) = \{J_i(x_i, x_{-i}) \mid x_i\in X_i\}$, a vector y_i is a *weak Pareto point* in $J_i(X_i, x_{-i})$ if it is not strongly dominated by any other vector in $J_i(X_i, x_{-i})$, i.e. $\nexists z_i\in J_i(X_i,x_{-i})$ such that $z_i - y_i \in int\mathbb{R}_+^{r(i)}$. A vector y_i is a *Pareto point* in $J_i(X_i, x_{-i})$ if it is not dominated by any other vector in $J_i(X_i, x_{-i})$, i.e. $\nexists z_i \in J_i(X_i, x_{-i})$ such that $z_i - y_i \in \mathbb{R}_+^{r(i)}\setminus\{0\}$.

For every player i, let $\mathcal{W}_i : X_{-i} \rightsquigarrow \mathbb{R}^{r(i)}$ be the set-valued map where

$$\mathcal{W}_i(x_{-i}) \text{ is the set of all weak Pareto points in } J_i(X_i, x_{-i}) \quad \text{for all } x_{-i}\in X_{-i}. \tag{1}$$

and $\mathcal{P}_i : X_{-i} \rightsquigarrow \mathbb{R}^{r(i)}$ be the set-valued map where

$$\mathcal{P}_i(x_{-i}) \text{ is the set of all Pareto points in } J_i(X_i, x_{-i}) \quad \text{for all } x_{-i}\in X_{-i}. \tag{2}$$

Finally, for every player i and for every $x_{-i}\in X_{-i}$, a strategy $\overline{x}_i$ is a *weak Pareto solution* for the vector-valued function $J_i\left(\cdot, x_{-i}\right)$ in X_i if

$$\overline{x}_i \in \operatorname*{Arg\,wmax}_{x_i\in X_i} J_i(x_i, x_{-i}) = \{x_i\in X_i \mid J_i(x_i, x_{-i}) \in \mathcal{W}_i(x_{-i})\} \tag{3}$$

and a strategy $\overline{x}_i$ is a *Pareto solution* for the vector-valued function $J_i\left(\cdot, x_{-i}\right)$ in X_i if

$$\overline{x}_i \in \operatorname*{Arg\,max}_{x_i\in X_i} J_i(x_i, x_{-i}) = \{x_i\in X_i \mid J_i(x_i, x_{-i}) \in \mathcal{P}_i(x_{-i})\}. \tag{4}$$

Note that

$$\mathcal{P}_i(x_{-i}) \subseteq \mathcal{W}_i(x_{-i}) \text{ and } \operatorname*{Arg\,max}_{x_i\in X_i} J_i(x_i, x_{-i}) \subseteq \operatorname*{Arg\,wmax}_{x_i\in X_i} J_i(x_i, x_{-i})$$

Definition 2.2: (Shapley (1959)). A strategy profile $x \in X$ is a *weak Pareto-Nash equilibrium* if, for every player i, x_i is a weak-Pareto solution for the vector-valued function $J_i(\cdot, x_{-i})$ in X_i; while $x \in X$ is a *Pareto-Nash equilibrium* if, for every player i, x_i is a Pareto solution for the vector-valued function $J_i(\cdot, x_{-i})$ in X_i.

Moreover, we recall that different interesting attempts have been made to generalize some refinement concepts for Nash equilibria to the above solution concepts (see Puerto and Fernandez (1995) or Borm, van Megen and Tijs (1999) for perfect equilibria, Yang and Yu (2002) for essential equilibria).

2.2 Genericity

Scalar Games

In order to illustrate the genericity problem for multicriteria games, we first recall the basic genericity arguments for the class of scalar games. Let $P = \{I; \Psi_1, \ldots, \Psi_n; v_1, \ldots, v_n\}$ denote a finite game, where $v_i : \Psi \to \mathbb{R}$ is the payoff function of player i. In this case, we denote with $G = \{I; X_1, \ldots, X_n; f_1, \ldots, f_n\}$ its mixed extension where the expected payoff function $f_i : X \to \mathbb{R}$ is defined by: $f_i(x) = \sum_{\psi \in \Psi} \left[\prod_{i \in I} x_i(\psi_i)\right] v_i(\psi)$ for all $x \in X$. Let $|\Psi| = K$ denote the cardinality of the set of all pure strategy profiles, then every payoff function $v_i : \Psi \to \mathbb{R}$ has finite range, in particular $y_i = (v_i(\psi))_{\psi \in \Psi}$ is a K-dimensional vector for every player i. Then, it is possible to identify the mixed extension G of the game P with the point $y = (y_1, \ldots, y_n) \in \mathbb{R}^{nK}$. Therefore, denoting with $\mathcal{G}(X_1, \ldots, X_n)$ the set of n-player finite games with mixed strategy sets $(X_1, \ldots, X_n)$, there is a one to one correspondence between $\mathbb{R}^{nK}$ and $\mathcal{G}(X_1, \ldots, X_n)$. Then, one can define a distance, denoted by $d(G', G'')$, between the games G' and G'' using the classical Euclidean distance between the corresponding vectors in $\mathbb{R}^{nK}$. Following this approach, it has been proved (see Harsanyi (1973) or Ritzberger (1994)) that almost all games in $\mathbb{R}^{nK}$ have an odd number of Nash equilibria and that they are all regular. More precisely, the graph of the Nash equilibrium correspondence $\mathcal{N} : \mathcal{G}(X_1, \ldots, X_n) \rightsquigarrow X$ is given by the union of an odd number of graphs of continuous functions outside a residual in $\mathcal{G}(X_1, \ldots, X_n)$. We will show in the next section that this kind of characterization does not hold in the case of multicriteria games.

Moreover, we recall that

Definition 2.3: (Wu and Jiang (1962)). An equilibrium in mixed strategies x^* of G is said to be an *essential equilibrium* for G if for every $\eta > 0$ there exists $\delta > 0$ such that for every game G' with $d(G, G') < \delta$ there exists an equilibrium x' with $d(x^*, x') < \eta$.

Multicriteria games

If $\Omega = \{I; \Psi_1, \ldots, \Psi_n; H_1, \ldots, H_n\}$ denotes a n-player finite game in pure strategies and $\Gamma = \{I; X_1, \ldots, X_n; J_1, \ldots, J_n\}$ denotes the mixed extension of Ω. For every player i, $w_i = (J_i(\psi))_{\psi \in \Psi}$ is a $r(i)K$-dimensional vector for every player i. Then, it is possible to identify the mixed extension Γ of the game Ω with the point $w = (w_1, \ldots, w_n) \in \mathbb{R}^{(\sum_{i \in I} r(i))K}$. Therefore, denoting with $\mathcal{MG}(X_1, \ldots, X_n)$ the set of n-player finite multicriteria games with mixed strategy sets $(X_1, \ldots, X_n)$ and payoff dimensions $r(i)$ with $i = 1, \ldots, n$, there is a one to one correspondence between $\mathbb{R}^{(\sum_{i \in I} r(i))K}$ and $\mathcal{MG}(X_1, \ldots, X_n)$. Then, analogously to the scalar case, one can define a distance, denoted by $d(\Gamma', \Gamma'')$, between the games Γ' and Γ'' using the classical Euclidean distance between the corresponding vectors in $\mathbb{R}^{(\sum_{i \in I} r(i))K}$. Then, we can define the weak Pareto-Nash equilibrium correspondence $\mathcal{WPN} : \mathcal{MG}(X_1, \ldots, X_n) \rightsquigarrow X$. We show below, by an example, that the images of the set-valued map of weak Pareto-Nash equilibria have uncountably many elements for every point in a particular open ball in $\mathcal{MG}(X_1, \ldots, X_n)$ and that $\mathcal{WPN}$ is continuous in the sense of Painlevé-Kuratowski in the center of the ball. This implies that every weak Pareto-Nash equilibrium of this game is essential (see Yang and Yu (2002) for the natural extension to multicriteria games of the essential equilibrium concept). We recall that a set-valued map $K : \mathcal{MG}(X_1, \ldots, X_n) \rightsquigarrow X$ is continuous (in the sense of Painlevé-Kuratowski) in $w \in \mathcal{MG}(X_1, \ldots, X_n)$ if and only if K is lower semicontinuous and closed in w (see for example Aubin and Frankowska (1990) or Border (1985)).

3 The Examples

EXAMPLE 3.1: We consider the following multicriteria game Γ_1 in which Player 1 has two criteria, selects rows and has two strategies, Player 2 has one criterion, selects columns and has two strategies. The payoffs are given as follows

	L	R
T	(1,0)	(1,0)
B	(0,1)	(0,1)

Payoffs of Player 1

	L	R
T	1	0
B	0	1

Payoffs of Player 2

We consider mixed strategies and we denote with $p = Prob(T)$ and $q = Prob(L)$. With an abuse of notation, we denote with X_i the set of strategies of player i defined as follows $X_1 = [0, 1]$ and $X_2 = [0, 1]$, since there is a one to one correspondence between the set of mixed strategies of player i and X_i.

For every $q \in [0, 1]$, $J_1(T, q) = (1, 0)$, $J_1(B, q) = (0, 1)$ so the set $\mathcal{W}_1(q)$ of weak Pareto points in $J_1(X_1, q)$ coincides with $J_1(X_1, q)$. Then, the best reply correspondence of Player 1 is given by:

$$\operatorname*{Arg\,wmax}_{p \in X_1} J_1(p, q) = X_1 \quad \forall q \in [0, 1]$$

and the best reply correspondence of Player 2 is given by

$$\operatorname*{arg\,max}_{q \in X_2} J_2(p, q) = \begin{cases} \{1\} & \text{if } p > 1/2 \\ X_2 & \text{if } p = 1/2 \\ \{0\} & \text{if } p < 1/2 \end{cases} .$$

Denoted with $\mathcal{V}_1 = \{(p, 0) \mid p \in [0, 1/2[\}$, $\mathcal{V}_2 = \{(1/2, q) \mid q \in [0, 1]\}$, $\mathcal{V}_3 = \{(p, 1) \mid p \in]1/2, 1]\}$, the set of weak Pareto-Nash equilibria $\mathcal{WPN} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$. Consider now a perturbation of the previous game:

	L	R
T	$(1+\varepsilon_{1,1},\varepsilon_{1,2})$	$(1+\varepsilon_{2,1}\ ,\ \varepsilon_{2,2})$
B	$(\varepsilon_{3,1},1+\varepsilon_{3,2})$	$(\varepsilon_{4,1},\ 1+\varepsilon_{4,2})$

Payoffs of Player 1

	L	R
T	$1+\delta_1$	δ_2
B	δ_3	$1+\delta_4$

Payoffs of Player 2

Denoted with $g^\delta(p) = p(2+\delta_1-\delta_2-\delta_3+\delta_4)-(1+\delta_4-\delta_3)$, the payoff function of Player 2 takes the following form

$$J_2^\delta(p,q) = g^\delta(p)q + h^\delta(p).$$

Let δ_i, with $i = 1,\ldots,4$, be sufficiently small, there exists a unique element p^δ in $[0,1]$ such that $g^\delta(p^\delta) = 0$. Therefore the best reply correspondence of Player 2 is given by

$$\arg\max_{q\in X_2} J_2^\delta(p,q) = \begin{cases} \{1\} & \text{if } p > p^\delta \\ X_1 & \text{if } p = p^\delta \\ \{0\} & \text{if } p < p^\delta \end{cases};$$

In order to calculate the best replies of Player 1, note that, for every $q \in [0,1]$,

$$\begin{aligned} J_1^\varepsilon(T,q) &= (1+q(\varepsilon_{1,1}-\varepsilon_{2,1})+\varepsilon_{2,1}, q(\varepsilon_{1,2}-\varepsilon_{2,2})+\varepsilon_{2,2}), \\ J_1^\varepsilon(B,q) &= (q(\varepsilon_{3,1}-\varepsilon_{4,1})+\varepsilon_{4,1}, 1+q(\varepsilon_{3,2}-\varepsilon_{4,2})+\varepsilon_{4,2}) \end{aligned}$$

It can be checked that, for $\varepsilon_{i,j}$ sufficiently small, the set $\mathcal{W}_1^\varepsilon(q)$ of weak Pareto points in $J_1^\varepsilon(X_1,q)$ coincides with $J_1^\varepsilon(X_1,q)$. Therefore

$$\operatorname*{Arg\,wmax}_{p\in X_1} J_1^\varepsilon(p,q) = X_1 \quad \forall q \in [0,1]$$

Denoted with $\mathcal{V}_1^\delta = \{(p,0) \mid p \in [0,p^\delta[\}$, $\mathcal{V}_2^\delta = \{(p^\delta,q) \mid q \in [0,1]\}$, $\mathcal{V}_3^\delta = \{(p,1) \mid p \in]p^\delta,1]\}$, the set of weak Pareto-Nash equilibria of the perturbed game is $\mathcal{WPN}^{\varepsilon,\delta} = \mathcal{V}_1^\delta \cup \mathcal{V}_2^\delta \cup \mathcal{V}_3^\delta$. Then $p^\delta \to 1/2$ as the vector $\delta \to 0$, so that $\mathcal{WPN}^{\varepsilon,\delta} \to \mathcal{WPN}$ in the sense of Painlevé-Kuratowski as $\varepsilon \to 0$ and $\delta \to 0$.

Example 3.2: We consider the following multicriteria game Γ_2 in which Player 1 has two criteria, selects rows and has three strategies, Player 2 has one criterion, selects columns and has two strategies and where the payoffs are given as follows

	L	R
T	(1,1)	(0,0)
M	(0,0)	(1, 1)
B	(3,-1)	(2,-1)

Payoffs of Player 1

	L	R
T	1	0
M	0	1
B	0	0

Payoffs of Player 2

We consider mixed strategies and we denote with $p_1 = Prob(T)$, $p_2 = Prob(M)$, $1-p_1-p_2 = Prob(B)$ and $q = Prob(L)$, $1-q = Prob(R)$. With an abuse of notation, we denote with X_i the set of strategies of player i defined as follows

$$X_1 = \{(p_1,p_2) \in \mathbb{R}^2 \mid p_1,p_2 \geq 0;\ p_1+p_2 \leq 1\}, \quad X_2 = \{q \in \mathbb{R} \mid 0 \leq q \leq 1\},$$

since there is a one to one correspondence between the set of mixed strategies of player i and X_i.

For every $q \in [0,1]$, $J_1(T,q) = (q,q)$, $J_1(M,q) = (1-q, 1-q)$ and $J_1(B,q) = (2+q, -1)$ so for every $q \in [0,1]$ the set $J_1(X_1,q)$ of the images of the vector-valued expected payoff of Player 1 is given by the convex hull of the points $(q,q), (1-q,1-q), (2+q,-1)$. Denote with $\gamma_1(q)$ the segment joining $J_1(T,q)$ to $J_1(M,q)$, with $\gamma_2(q)$ the segment joining $J_1(M,q)$ to $J_1(B,q)$ and with $\gamma_3(q)$ the segment joining $J_1(T,q)$ with $J_1(B,q)$. Then one can check that the set $\mathcal{W}_1(q)$ of weak Pareto points in $J_1(X_1,q)$ is given by the following:

$$\mathcal{W}_1(q) = \begin{cases} \gamma_2(q) & \text{if } q \in [0,1/2[\\ \gamma_1(q) \cup \gamma_2(q) \cup \gamma_3(q) = \gamma_2(q) = \gamma_3(q) & \text{if } q = 1/2 \\ \gamma_3(q) & \text{if } q \in]1/2,1] \end{cases}.$$

The best reply correspondence of Player 1 is given by:

$$\operatorname*{Arg\,wmax}_{(p_1,p_2)\in X_1} J_1(p_1,p_2,q) = \begin{cases} \{(p_1,p_2) \in X_1 \mid p_1 = 0\} & \text{if } q \in [0,1/2[\\ X_1 & \text{if } q = 1/2 \\ \{(p_1,p_2) \in X_1 \mid p_2 = 0\} & \text{if } q \in]1/2,1] \end{cases}$$

and the best reply correspondence of Player 2 is given by

$$\operatorname*{arg\,max}_{q\in X_2} J_2(p_1,p_2,q) = \begin{cases} \{1\} & \text{if } p_1 > p_2 \\ X_2 & \text{if } p_1 = p_2 \\ \{0\} & \text{if } p_1 < p_2 \end{cases}.$$

Denoted with

$$\begin{array}{l} \mathcal{P}_1 = \{(0,p_2,0) \mid p_2 \in]0,1]\} \\ \mathcal{P}_2 = \{(0,0,q) \mid q \in [0,1]\} \\ \mathcal{P}_3 = \{(p_1,p_2,1/2) \mid p_2 = p_1,\ p_1 \in]0,1/2]\} \\ \mathcal{P}_4 = \{(p_1,0,1) \mid p_1 \in]0,1]\} \end{array},$$

the set of weak Pareto-Nash equilibria $\mathcal{WPN}$ is

$$\mathcal{WPN} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4.$$

Consider now a perturbation of the previous game:

	L	R
T	$(1+\varepsilon_{1,1}, 1+\varepsilon_{1,2})$	$(\varepsilon_{2,1}, \varepsilon_{2,2})$
M	$(\varepsilon_{3,1}, \varepsilon_{3,2})$	$(1+\varepsilon_{4,1}, 1+\varepsilon_{4,2})$
B	$(3+\varepsilon_{5,1}, -1+\varepsilon_{5,2})$	$(2+\varepsilon_{6,1}, -1+\varepsilon_{6,2})$

Payoffs of Player 1

	L	R
T	$1+\delta_1$	δ_2
M	δ_3	$1+\delta_4$
B	δ_5	δ_6

Payoffs of Player 2

Denoted with

$$g^\delta(p_1,p_2) = p_1(1+\delta_1-\delta_2-\delta_5+\delta_6) - p_2(1-\delta_3+\delta_4+\delta_5-\delta_6) - (\delta_6-\delta_5),$$

the payoff function of Player 2 takes the following form

$$J_2^\delta(p_1,p_2,q) = g^\delta(p_1,p_2)q + h^\delta(p_1,p_2)$$

and the best replies of Player 2 are given by

$$\arg\max_{q\in X_2} J_2^\delta(p_1,p_2,q) = \begin{cases} \{1\} & \text{if } g^\delta(p_1,p_2) > 0 \\ X_2 & \text{if } g^\delta(p_1,p_2) = 0 \\ \{0\} & \text{if } g^\delta(p_1,p_2) < 0 \end{cases};$$

In order to calculate the best replies of Player 1, note that, for every $q \in [0,1]$,

$$\begin{aligned} J_1^\varepsilon(T,q) &= (q(1+\varepsilon_{1,1}-\varepsilon_{2,1})+\varepsilon_{2,1}, q(1+\varepsilon_{1,2}-\varepsilon_{2,2})+\varepsilon_{2,2}), \\ J_1^\varepsilon(M,q) &= (q(\varepsilon_{3,1}-\varepsilon_{4,1}-1)+1+\varepsilon_{4,1}, q(\varepsilon_{3,2}-\varepsilon_{4,2}-1)+1+\varepsilon_{4,2})) \\ J_1^\varepsilon(B,q) &= (q(1+\varepsilon_{5,1}-\varepsilon_{6,1})+2+\varepsilon_{6,1}, q(\varepsilon_{5,2}-\varepsilon_{6,2})-1+\varepsilon_{6,2}) \end{aligned}$$

For every $q \in [0,1]$, the set $J_1^\varepsilon(X_1,q)$ of the images of the vector-valued expected payoff of Player 1 is the convex hull of the points $J_1^\varepsilon(M,q)$, $J_1^\varepsilon(T,q)$, $J_1^\varepsilon(B,q)$. Denote with $\gamma_1^\varepsilon(q)$ the segment joining $J_1^\varepsilon(T,q)$ to $J_1^\varepsilon(M,q)$, with $\gamma_2^\varepsilon(q)$ the segment joining $J_1^\varepsilon(M,q)$ to $J_1^\varepsilon(B,q)$ and with $\gamma_3^\varepsilon(q)$ the segment joining $J_1^\varepsilon(T,q)$ with $J_1^\varepsilon(B,q)$. Denote also with

$$\underline{q}^\varepsilon = \frac{1+\varepsilon_{4,1}-\varepsilon_{2,1}}{2+\varepsilon_{1,1}-\varepsilon_{2,1}-\varepsilon_{3,1}+\varepsilon_{4,1}}$$

and

$$\overline{q}^\varepsilon = \frac{1+\varepsilon_{4,2}-\varepsilon_{2,2}}{2+\varepsilon_{1,2}-\varepsilon_{2,2}-\varepsilon_{3,2}+\varepsilon_{4,2}}.$$

where $\underline{q}^\varepsilon$ is such that the first components of $J_1^\varepsilon(T,\underline{q}^\varepsilon)$ and $J_1^\varepsilon(M,\underline{q}^\varepsilon)$ coincide and $\overline{q}^\varepsilon$ is such that the second components of $J_1^\varepsilon(T,\overline{q}^\varepsilon)$ and $J_1^\varepsilon(M,\overline{q}^\varepsilon)$ coincide.

Case 1: Assume that $\underline{q}^\varepsilon < \overline{q}^\varepsilon$. For $\varepsilon_{i,j}$ sufficiently small, if $q < \underline{q}^\varepsilon$ then $J_1^\varepsilon(M,q)$ strongly dominates $J_1^\varepsilon(T,q)$, if $\underline{q}^\varepsilon \le q \le \overline{q}^\varepsilon$ then $J_1^\varepsilon(M,q)$ and $J_1^\varepsilon(T,q)$ are not comparable, finally if $\overline{q}^\varepsilon < q$ then $J_1^\varepsilon(T,q)$ strongly dominates $J_1^\varepsilon(M,q)$. Moreover, there exists a point $\widehat{q}^\varepsilon \in [\underline{q}^\varepsilon, \overline{q}^\varepsilon]$ such that $J_1^\varepsilon(T,q)$, $J_1^\varepsilon(M,q)$, $J_1^\varepsilon(B,q)$ lie on the same line. A simple geometric analysis shows that even when $q \in [\underline{q}^\varepsilon, \widehat{q}^\varepsilon[$ the set $\gamma_1^\varepsilon(q) \setminus \{J_1^\varepsilon(M,q)\}$ is strongly Pareto dominated by $\gamma_2^\varepsilon(q)$ and therefore it is easy to check that the set $\mathcal{W}_1^\varepsilon(q)$ of weak Pareto points in $J_1^\varepsilon(X_1,q)$ is given by the following:

$$\mathcal{W}_1^\varepsilon(q) = \begin{cases} \gamma_2^\varepsilon(q) & \text{if } q \in [0,\widehat{q}^\varepsilon[\\ \gamma_2^\varepsilon(q) = \gamma_1^\varepsilon(q) \cup \gamma_3^\varepsilon(q) & \text{if } q \in q = \widehat{q}^\varepsilon \\ \gamma_3^\varepsilon(q) & \text{if } q \in]\widehat{q}^\varepsilon, 1] \end{cases}.$$

The best reply correspondence of Player 1 is given by:

$$\operatorname*{Arg\,wmax}_{(p_1,p_2)\in X_1} J_1^\varepsilon(p_1,p_2,q) = \begin{cases} \{(p_1,p_2) \in X_1 \mid p_1 = 0\} & \text{if } q \in [0,\widehat{q}^\varepsilon[\\ X_1 & \text{if } q = \widehat{q}^\varepsilon \\ \{(p_1,p_2) \in X_1 \mid p_2 = 0\} & \text{if } q \in]\widehat{q}^\varepsilon, 1] \end{cases};$$

Let $\mathcal{P}_3^{\varepsilon,\delta}$ be the subset of $X_1 \times X_2$ such that

$$\mathcal{P}_3^{\varepsilon,\delta} = \{(p_1,p_2,\widehat{q}^\varepsilon) \mid g^\delta(p_1,p_2) = 0\}.$$

Then, the set of weak Pareto-Nash equilibria $\mathcal{WPN}^{\varepsilon,\delta}$ in the perturbed game is

$$\mathcal{WPN}^{\varepsilon,\delta} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3^{\varepsilon,\delta} \cup \mathcal{P}_4.$$

As the vector $\varepsilon \to 0$, then $\underline{q}^\varepsilon \to 1/2$, $\widehat{q}^\varepsilon \to 1/2$, $\overline{q}^\varepsilon \to 1/2$ so that, as $\varepsilon \to 0$ and $\delta \to 0$, $\mathcal{WPN}^{\varepsilon,\delta} \to \mathcal{WPN}$ in the sense of Painlevé-Kuratowski.

Case 2: Assume that $\overline{q}^\varepsilon < \underline{q}^\varepsilon$. For $\varepsilon_{i,j}$ sufficiently small, if $q < \overline{q}^\varepsilon$ then $J_1^\varepsilon(M,q)$ strongly dominates $J_1^\varepsilon(T,q)$, if $\overline{q}^\varepsilon \le q \le \underline{q}^\varepsilon$ then $J_1^\varepsilon(M,q)$ and $J_1^\varepsilon(T,q)$ are not comparable, finally if $\underline{q}^\varepsilon < q$ then $J_1^\varepsilon(T,q)$ strongly dominates $J_1^\varepsilon(M,q)$. Also in this case there exists a point $\widehat{q}^\varepsilon \in [\overline{q}^\varepsilon, \underline{q}^\varepsilon]$ such that $J_1^\varepsilon(T,q)$, $J_1^\varepsilon(M,q)$, $J_1^\varepsilon(B,q)$ lie on the same line; however, differently from the previous case, here when $q \in [\overline{q}^\varepsilon, \widehat{q}^\varepsilon[$ the set $\gamma_1^\varepsilon(q) \setminus \{J_1^\varepsilon(M,q)\}$ is not strongly Pareto dominated by $\gamma_2^\varepsilon(q)$ and therefore the set $\mathcal{W}_1^\varepsilon(q)$ of weak Pareto points in $J_1^\varepsilon(X_1,q)$ is given by the following:

$$\mathcal{W}_1^\varepsilon(q) = \begin{cases} \gamma_2^\varepsilon(q) & \text{if } q \in [0, \overline{q}^\varepsilon[\\ \gamma_1^\varepsilon(q) \cup \gamma_2^\varepsilon(q) & \text{if } q \in [\overline{q}^\varepsilon, \widehat{q}^\varepsilon[\\ \gamma_1^\varepsilon(q) \cup \gamma_2^\varepsilon(q) = \gamma_3^\varepsilon(q) & \text{if } q \in q = \widehat{q}^\varepsilon \\ \gamma_3^\varepsilon(q) & \text{if } q \in]\widehat{q}^\varepsilon, 1] \end{cases}$$

then, the best reply correspondence of Player 1 is given by:

$$\operatorname*{Arg\,wmax}_{(p_1,p_2)\in X_1} J_1^\varepsilon(p_1,p_2,q) =$$

$$\begin{cases} \{(p_1,p_2) \in X_1 \mid p_1 = 0\} & \text{if } q \in [0, \overline{q}^\varepsilon[\\ \{(p_1,p_2) \in X_1 \mid p_1 = 0\} \cup \{(p_1,p_2) \in X_1 \mid p_1 + p_2 = 1\} & \text{if } q \in [\overline{q}^\varepsilon, \widehat{q}^\varepsilon[\\ X_1 & \text{if } q = \widehat{q}^\varepsilon \\ \{(p_1,p_2) \in X_1 \mid p_2 = 0\} & \text{if } q \in]\widehat{q}^\varepsilon, 1] \end{cases}.$$

Let $\overline{\mathcal{P}}_5^{\varepsilon,\delta}$ be the subset of $X_1 \times X_2$ such that

$$\overline{\mathcal{P}}_5^{\varepsilon,\delta} = \{(p_1,p_2,q) \mid g^\delta(p_1,p_2) = 0, p_1 + p_2 = 1, q \in [\overline{q}^\varepsilon, \widehat{q}^\varepsilon]\} \ .$$

In this case, the set of weak Pareto-Nash equilibria $\mathcal{WPN}^{\varepsilon,\delta}$ in the perturbed game is

$$\mathcal{WPN}^{\varepsilon,\delta} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3^{\varepsilon,\delta} \cup \mathcal{P}_4 \cup \overline{\mathcal{P}}_5^{\varepsilon,\delta}.$$

Since as $\varepsilon \to 0$ we get $\underline{q}^\varepsilon \to 1/2$, $\widehat{q}^\varepsilon \to 1/2$, $\overline{q}^\varepsilon \to 1/2$, then also in this case $\mathcal{WPN}^{\varepsilon,\delta} \to \mathcal{WPN}$, as $\varepsilon \to 0$ and $\delta \to 0$.

Case 3: Assume that $\overline{q}^\varepsilon = \underline{q}^\varepsilon = \widehat{q}^\varepsilon$; then the best reply of Player 1 is

$$\operatorname*{Arg\,wmax}_{(p_1,p_2)\in X_1} J_1^\varepsilon(p_1,p_2,q) = \begin{cases} \{(p_1,p_2) \in X_1 \mid p_1 = 0\} & \text{if } q \in [0, \widehat{q}^\varepsilon[\\ X_1 & \text{if } q = \widehat{q}^\varepsilon \\ \{(p_1,p_2) \in X_1 \mid p_2 = 0\} & \text{if } q \in]\widehat{q}^\varepsilon, 1] \end{cases};$$

then the set of weak Pareto-Nash equilibria $\mathcal{WPN}^{\varepsilon,\delta}$ in the perturbed game is

$$\mathcal{WPN}^{\varepsilon,\delta} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3^{\varepsilon,\delta} \cup \mathcal{P}_4.$$

and, also in this case, $\mathcal{WPN}^{\varepsilon,\delta} \to \mathcal{WPN}$, as $\varepsilon \to 0$ and $\delta \to 0$.

Summarizing, in each example, we found a game Γ_h ($h = 1, 2$) and an open neighborhood (in the space of multicriteria games having the same set of strategies and the same number of criteria) of Γ_h such that every game in this set have uncountably many weak Pareto-Nash equilibria. Moreover, the equilibria of each Γ_h are all essential, that is "stable" with respect to every perturbation on the data of the game. It could also be checked that all the equilibria of each Γ_h are trembling hand perfect (see Borm, van Megen and Tijs (1999) for the definition of perfect in multicriteria games).

Note that the first example is much easier from the mathematical point of view but it is much less interesting from the game theoretic point of view because in the game Γ_1 the best reply correspondence of the first player coincide with his strategy set for every strategy of the second player. Therefore, it shows that the results hold in the simplest class of multicriteria games. The second example shows that the results coming out from the first example do not depend on the *degenerate* behavior of the best reply correspondences and may arise in nontrivial games.

4 Scalarization stable equilibria

As already mentioned in the Introduction, in this section we show that is possible to refine the set of weak Pareto-Nash equilibria of the games in the previous section by considering a non-classical property of stability with respect to perturbations, even if all these equilibria are essential. Moreover, since this result holds in both the examples, then it does not depend on the particular form of the best reply correspondences in the first example. More precisely, we consider a property of stability which deals with the scalarization technique which is a peculiarity of the vector-valued payoffs case.

Given a system of weights $\lambda = (\lambda_1, \ldots, \lambda_n)$, where each λ_i is a $r(i)$-dimensional vector $(\lambda_i^1, \ldots, \lambda_i^{r(i)})$ in the $(r(i)$-1)-dimensional simplex $\Delta(r(i))$, it is possible to consider the scalar game

$$\Gamma(\lambda) = \{I; X_1, \ldots, X_n; \lambda_1 J_1, \ldots, \lambda_n J_n\}$$

called *trade-off game*, where for every player i, the payoff function is defined as $\lambda_i J_i(x) = \sum_{k=1}^{r(i)} \lambda_i^k J_i^k(x)$ for every $x \in X$. In Shapley (1959), it has been proved that:

PROPOSITION 4.1: *Let Γ be the mixed extension of a finite multicriteria game. The strategy profile x is a weak Pareto-Nash equilibrium (resp. Pareto-Nash equilibrium) for Γ if and only if there exists a system of weights $\lambda \in \Delta(r(1)) \times \cdots \times \Delta(r(n))$ (resp. $\lambda \in$ relint $(\Delta(r(1)) \times \cdots \times \Delta(r(n)))$) such that x is a Nash equilibrium (Nash (1950), (1951)) of the trade-off game $\Gamma(\lambda)$.*

To obtain a refinement concept which captures the idea of stability shown in the previous example, we give the following:

DEFINITION 4.2 (De Marco and Morgan (2007)): Let Γ be a multicriteria game and x' be a weak Pareto-Nash equilibrium of Γ. Then, x' is said to be a *scalarization-stable* equilibrium (s-stable equilibrium for short) if there exists $\lambda' \in \prod_{i=1}^{n} \Delta(r(i))$ such that:

i) x' is a Nash equilibrium of $\Gamma(\lambda')$

ii) for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $\lambda \in \prod_{i=1}^{n} \Delta(r(i))$ verifying $d(\lambda, \lambda') < \delta$ there exists $x \in E(\lambda)$ such that $d(x, x') < \varepsilon$.

EXAMPLE 4.3: Consider the game Γ_1 in the Example 3.1. Let $\lambda \in [0,1]$, with an abuse of notation denote with $\lambda_1 = (\lambda_1^1, \lambda_1^2) = (\lambda, 1-\lambda)$. Then, we consider the trade-off games $\Gamma_1(\lambda)$ with $\lambda \in [0,1]$:

Play. 1, Play. 2	L	R
T	λ,1	λ,0
B	1-λ,0	1-λ,1

The best reply correspondences of the Player 1 in the trade off games are given by:

$$\arg\max_{p \in X_1} \lambda_1 J_1(p,q) = \begin{cases} \{0\} & \forall q \in [0,1] \quad if\ \lambda \in [0, 1/2[\\ X_1 & \forall q \in [0,1] \quad if\ \lambda = 1/2 \\ \{0\} & \forall q \in [0,1] \quad if\ \lambda \in]1/2, 1] \end{cases}$$

The set valued function $E_1(\cdot)$ which associates to every λ the set of Nash equilibria of the trade off game $\Gamma_1(\lambda)$ is given by:

$$E_1(\lambda) = \begin{cases} \{(0,0)\} & if\ \lambda \in [0, 1/2[\\ \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3 & if\ \lambda = 1/2 \\ \{(1,1)\} & if\ \lambda \in]1/2, 1] \end{cases}$$

The equilibrium correspondence $E_1(\cdot)$ is not lower semicontinuous in $\lambda = 1/2$. Since $(0,0)$ and $(1,1)$ are the only equilibria which belong to $E_1(\lambda)$ for some $\lambda \neq 1/2$ then they are the only scalarization stable equilibria, that is, $E^s = \{(0,0), (1,1)\}$.

EXAMPLE 4.4: Consider the game Γ_2 in the Example 3.2. Let $\lambda \in [0,1]$, with an abuse of notation denote with $\lambda_1 = (\lambda_1^1, \lambda_1^2) = (\lambda, 1-\lambda)$. Then, we consider the following trade-off games $\Gamma_2(\lambda)$ with $\lambda \in [0,1]$:

Play. 1, Play. 2	L	R
T	1,1	0,0
M	0,0	1,1
B	4λ-1,0	3λ-1,0

For every $\lambda \in [0,1]$ and for every $q \in [0,1]$, let $J_1(T,q)$, $J_1((M,q)$, $J_1((B,q),\lambda)$ be the expected payoff of Player 1 when he plays respectively T, M or B.

Let $m(q) = \max\{J_1(T,q), J_1(M,q)\}$ for all $q \in [0,1]$ then

$$m(q) = \begin{cases} 1-q & \text{if } q \in [0, 1/2] \\ q & \text{if } q \in]1/2, 1] \end{cases}$$

then, being $\lambda_1 J_1(B,q) = q\lambda + 3\lambda - 1$, we get

$$\begin{cases} i)\ \nexists q \in [0,1] \quad \text{such that } \lambda_1 J_1(B,q) \geq m(q) & if \lambda \in [0, 3/7[\\ ii)\ \lambda_1 J_1(B,q) \geq m(q)\ \forall q \in \left[\frac{2-3\lambda}{1+\lambda}, \frac{3\lambda-1}{1-\lambda}\right] & if \lambda \in [3/7, 1/2] \\ iii)\ \lambda_1 J_1(B,q) \geq m(q)\ \forall q \in \left[\frac{2-3\lambda}{1+\lambda}, 1\right] & if \lambda \in]1/2, 2/3] \\ iv)\ \lambda_1 J_1(B,q) \geq m(q)\ \forall q \in [0,1] & if \lambda \in]2/3, 1] \end{cases}$$

Denote with $\alpha(\lambda) = \frac{2-3\lambda}{1+\lambda}$ and $\beta(\lambda) = \frac{3\lambda-1}{1-\lambda}$. Then, as $\alpha(\lambda), \beta(\lambda) \in [0,1]$, $\lambda_1 J_1(B,q)$ intersects $J_1(M,q)$ for $q = \alpha(\lambda)$, and $\lambda_1 J_1(B,q)$ intersects $J_1(T,q)$ for $q = \beta(\lambda)$, that is

$$\lambda_1 J_1(B,\alpha(\lambda)) = J_1(M,\alpha(\lambda)) \qquad \lambda_1 J_1(B,\beta(\lambda)) = J_1(T,\beta(\lambda))$$

Moreover, since $\alpha(\cdot)$ and $\beta(\cdot)$ are increasing in the interval $[0,1]$ and $\alpha(3/7) = \beta(3/7) = 1/2$, $\beta(1/2) = 1$ and $\alpha(2/3) = 1$, the best reply of the Player 1 in the trade off game is:

$$\arg\max_{(p_1,p_2)\in X_1} \lambda_1 J_1(p_1,p_2,q) =$$

$$\begin{cases} \begin{cases} \{(0,1)\} & if\ q \in [0,1/2[\\ \{(p_1,p_2) \in X_1 \mid p_1 + p_2 = 1\} & if\ q = 1/2 \\ \{(1,0)\} & if\ q \in]1/2,1] \end{cases} & if\ \lambda \in [0,3/7[\\ \\ \begin{cases} \{(0,1)\} & if\ q \in [0,1/2[\\ X_1 & if\ q = 1/2 \\ \{(1,0)\} & if\ q \in]1/2,1] \end{cases} & if\ \lambda = 3/7 \\ \\ \begin{cases} \{(0,1)\} & if\ q \in \left[0, \frac{2-3\lambda}{1+\lambda}\right[\\ \{(p_1,p_2) \in X_1 \mid p_1 = 0\} & if\ q = \frac{2-3\lambda}{1+\lambda} \\ \{(0,0)\} & if\ q \in \left]\frac{2-3\lambda}{1+\lambda}, \frac{3\lambda-1}{1-\lambda}\right[\\ \{(p_1,p_2) \in X_1 \mid p_2 = 0\} & if\ q = \frac{3\lambda-1}{1-\lambda} \\ \{(1,0)\} & if\ q \in \left]\frac{3\lambda-1}{1-\lambda}, 1\right] \end{cases} & if\ \lambda \in]3/7,1/2] \\ \\ \begin{cases} \{(0,1)\} & if\ q \in \left[0, \frac{2-3\lambda}{1+\lambda}\right[\\ \{(p_1,p_2) \in X_1 \mid p_1 = 0\} & if\ q = \frac{2-3\lambda}{1+\lambda} \\ \{(0,0)\} & if\ q \in \left]\frac{2-3\lambda}{1+\lambda} 1\right] \end{cases} & if\ \lambda \in]1/2,2/3] \\ \\ \{(0,0)\} \quad if\ q \in [0,1] & if\ \lambda \in]2/3,1] \end{cases}$$

The set valued function $E_2(\cdot)$ which associates to every λ the set of Nash equilibria of the trade off game $\Gamma_2(\lambda)$ is given by:

$$E_2(\lambda) = \begin{cases} \{(0,1,0), (1/2,1/2,1/2), (1,0,1)\} & if\ \lambda \in [0,3/7[\\ \\ \{(0,1,0), (1,0,1)\} \cup \{(p_1,p_2,1/2)\ |p_2 = p_1,\ p_1 \in]0,1/2]\} & if\ \lambda = 3/7 \\ \\ \{(0,1,0), (1,0,1)\} \cup \left\{(0,0,q) \mid q \in \left[\frac{2-3\lambda}{1+\lambda}, \frac{3\lambda-1}{1-\lambda}\right]\right\} & if\ \lambda \in]3/7,1/2[\\ \\ \{(0,1,0\} \cup \left\{(0,0,q) \mid q \in \left[\frac{2-3\lambda}{1+\lambda}, 1\right]\right\} \cup \{(p_1,0,1) \mid p_1 \in]0,1]\} & if\ \lambda = 1/2 \\ \\ \{(0,1,0\} \cup \left\{(0,0,q) \mid q \in \left[\frac{2-3\lambda}{1+\lambda}, 1\right]\right\} & if\ \lambda \in]1/2,2/3[\\ \\ \{(0,0,q) \mid q \in [0,1]\} \cup \{(0,p_2,0) \mid p_2 \in]0,1]\} & if\ \lambda = 2/3 \\ \\ \{(0,0,q) \mid q \in [0,1]\} & if\ \lambda \in]2/3,1] \end{cases}$$

It follows that the equilibrium correspondence $E_2(\cdot)$ is not lower semicontinuous in $\lambda = 3/7$, $\lambda = 1/2$ and $\lambda = 2/3$.

Since the set of equilibria $\{(p_1, p_2, 1/2) \mid p_2 = p_1,\ p_1 \in]0, 1/2]\}$ belongs only to $E_2(3/7)$, $\{(p_1, 0, 0) \mid p_1 \in]0, 1]\}$ belongs only to $E_2(1/2)$ and $\{(0, p_2, 0) \mid p_2 \in]0, 1]\}$ belongs only to $E_2(2/3)$, then they are not scalarization stable equilibria and the set of scalarization stable equilibria is a proper subset of $\mathcal{WPN}$, i.e..

$$E^s = \mathcal{P}_2 \cup \{(0, 1, 0), (1/2, 1/2, 1/2), (1, 0, 1)\}.$$

Remark 4.5: In De Marco and Morgan (2007) it has been shown that if x is an essential equilibrium of the trade-off game $\Gamma(\lambda)$ (derived from the multicriteria game Γ) for some $\lambda \in \Delta(r(1)) \times \cdots \times \Delta(r(n))$ then it is a scalarization stable equilibrium for the multicriteria game Γ. On the other hand, the previous example shows that there exists (at least) an essential (weak Pareto-Nash) equilibrium for Γ which is not a scalarization stable equilibrium for Γ. Then, it follows that essentiality in a multicriteria game Γ differs from essentiality in the trade off games $\Gamma(\lambda)$ derived from Γ, since there exists (at least) an essential (weak Pareto-Nash) equilibrium for Γ which is not an essential equilibrium of the trade-off game $\Gamma(\lambda)$ for every $\lambda \in \Delta(r(1)) \times \cdots \times \Delta(r(n))$.

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