CSEF
Centre for Studies in Economics and Finance

WORKING PAPER NO. 261

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October 2010

This version May 2012

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Abstract
We use a mixed market model for analyzing economies with public projects in which the condition of perfect competition is violated. We discuss core-equivalence results in the general framework of non-Euclidean representation of the collective goods. We show that if large traders are similar to each other, then they lose their market power and hence the equivalence theorem can be restored. This is possible assuming a cost distribution function to fix the fraction that each large or small agent is expected to cover of the total cost of providing the project. We show that, for each given individual and coalitional contribution scheme, the resulting core is equivalent to the corresponding linear cost share equilibria. Finally, we investigate on weaker equivalences when the assumption that all large traders are of the same type is dropped. An analysis of mixed markets with public goods via atomless economies is provided, joint with an extension of Schmeidler and Vind results on the measure of blocking coalitions.

JEL Classification: D51, D60, H41.

Keywords: Mixed market, public project, cost share equilibrium, core, contribution measure.

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1 Introduction

The purpose of this paper is to use a mixed market model for analyzing economies in which the ideal state of perfect competition is violated and in which the choice of a public project is also involved. In particular, we aim to find conditions ensuring that the basic connections between cooperative and competitive equilibrium notions can be restored despite of the presence of large traders and collective goods.

One of the major focus of the literature in general equilibrium theory during the 1970s and the 1980s has been the study of relations existing between the solution concept of core and the set of competitive allocations. An allocation of resources for a market economy is stable with respect to coalition improvements (or a core allocation) if it has the property that no group of people is able to achieve a preferred outcome for its members using only its initial aggregate resources. Differently, a competitive, price-taking, equilibrium outcome is the result of the simultaneous optimization of individual targets under independent budget constraints.

The core-equivalence theorem shows the formal coincidence existing between Walrasian competitive allocations and the core: this would be interpreted by saying that market choices emerging from individual private behavior (according to the competitive scheme) are the only possible even if agents are free to cooperate within coalitions in order to improve their own welfare.

The price-taking behavior assumption that underlies the notion of competitive equilibrium, implicitly assumes a high number of traders, so that none of them is able to affect prevailing prices and aggregate demand. In particular, if the number of agents is large enough, none of them has monopoly power. One of the major contribution to the literature is the idea, due to Aumann, of formalizing perfect competition by means of nonatomic market economies and the aggregation with the formal tool of Lebesgue integral (see [2]). As a consequence of the presence of a continuum of small traders, any change in individual behavior is negligible with respect to the aggregate demand. Hence, nonatomic economies constitute the formal basis of the economic intuition that, in markets embodying perfect competition, traders, being small, are price-takers.

Starting from the seminal contribution of Aumann, several authors recognized that the competition in the real economic activity is far form being perfect. The reason could be identified in two different distortions, both being representable by means of the same formal model of market economies. First, it is often the case that some traders may be present on the market concentrating in their hands an initial ownership of commodities that is large when compared with the total market endowment. This scenario includes monopolistic or, more generally, oligopolistic markets. Second, even if the initial endowment is spread over a continuum of negligible traders, it can be the case that some traders join groups (or coalitions) and decide to act on the market only together. As consequence of such collusive processes (cartels, syndicates or other form of institutional agreements) the large number of small traders may be not effective and trade may take place only between large groups, violating the perfect competition scheme (see for example the discussion in [18]).

Recently, models of international cooperation agreements aiming at the provision of global public goods, like climate for example, have led to the researchers attention the study of interaction amongst few large traders, representative of rich countries, and many small traders, representative of poor countries (see [3], [19] among the others). In particular, they have stimulated the interest in the non-cooperative provision of public goods (to describe the independent behavior of states), as well as the consideration of cooperative solutions (due to the emergence of coordination throughout international agreements to overcome inefficiency of non-cooperative policies). Surprisingly, only very little attention has been paid to the analysis, under a general equilibrium approach, of market models involving large traders as well as public goods. We try to fill this gap introducing general mixed models with public goods, defining suitable competitive and core notions, establishing equivalence properties.

We use a mixed economy. Namely an economy with an atomless sector of consumers as an idealization of the many negligible agents and with atoms (i.e. subsets with strictly positive mass containing no proper subsets of strictly positive mass) to represent the few influential agents (compare [18]). In this setting traders are faced with the problem of the unanimously agreed choice of a public project to be realized. It is especially noteworthy that we do not limit the choice of a public good to a set with an euclidean structure. Instead, we allow public projects to be drawn from a set without any mathematical structure. In [14], in contrast with the classical Samuelsonian Euclidean scheme, the choice of a public project takes place over a set where no special mathematical structure is imposed. The absence of a linear structure on the set of public projects allows in particular the treatment of those public goods on which there is no reason to assume a commonly accepted order by traders. This is the case of public goods for which different individuals may have different perceptions and hence different rankings (see the discussion in [4], [5], [6], [9]). Moreover, if public projects are interpreted as public environments, i.e. collections of variables common to all the agents but determined outside the market mechanisms, we end up with a general framework incorporating many different economic problems. This is the interpretation of the Mas-Colell approach given by [11], [12], where non-market variables include legal systems (such as the assignment of property rights), tax and benefits systems, but also private goods provided by the public sector.

In exchange economies involving large traders and public goods one should not expect the core-equivalence theorem to hold. On one hand, the presence of collusive agreements prevents the formation of some coalitions and enlarges the core. On the other hand, well known counter-examples show the failure of the core equivalence theorem in the public goods context when the blocking mechanism is adapted from private goods economies (see [15], [16], [22]). These negative results emerge even in nonatomic market models and could be roughly reconducted to the assumption that the cost for the public good provision has a uniform distribution among the agents (that is tipical of the Lindahl competitive approach) combined with the Foley veto mechanism used to define the core (see [7]). Indeed, an enlargement of the number of agents in order to achieve perfect competition, implies that the per-capita cost decreases, making small coalitions weaker. According to the Foley veto mechanism, a blocking coalition is expected to produce the public project on its own, covering the whole cost necessary for the provision. Hence, small coalitions lose their market power and the core enlarges.

The notion of competitive equilibrium we adopt originates in [14] and in the subsequent papers by [4], [5], where the concept of linear cost share equilibria for atomless markets with public goods has been introduced. In a linear cost share equilibrium, everyone is faced with an equal provision, but consumers may pay a different contribution according to a given cost share function. Agents maximize utility over their own budget sets taking into account the share of the cost of the project. Moreover, they are able to maximize their welfare taking into account changes in the price of private commodities deriving from changes in the public project. Indeed, the assumption that prices may depend on collective projects makes the approach different from the one based on the notion of Lindahl equilibrium. From the point of view of cooperative notions, the veto mechanism defining the core differs from the classical one due to Foley. Following [5] and [10], given an individual cost share function, a measure is accordingly defined on the set of all coalitions to fix the contribution that each blocking coalition is expected to cover. For large traders, this contribution is obtained just multiplying the trader size by the individual cost share.

To obtain our equivalence results, we adapt conditions of [18, Theorem B]. We require that there are at least two large traders and moreover that they are all of the same type (i.e. they have the same initial endowment and the same utility function). It comes out that large traders exploit their market power (and consequently a competitive price-taking equilibrium can be sustained), provided that the same individual cost share is exogenously fixed for each of them[1]. Thus, given that large agents have the same characteristics, in order to restore perfect competition they are required to pay

[1]Notice that, since large traders are exogenously assumed in the model, contracts generating cartels should be considered as being legally enforced and then institutionally taxed.

according to the same individual cost. Exactly like in [18, Theorem B], since large traders may have a priori a different size even if they are of the same type, large traders may be required to pay a different contribution in order to block a given allocation of resources. That is the equivalence result holds true regardless of the sizes of contribution of large traders[2]. In particular, our analysis clarifies relations between large traders and large tax-payers in mixed models with public goods.

Last part of the paper is devoted to the investigation, from a different point of view, of equivalence results without imposing additional conditions on the atomic sector. We turn our attention to the Aubin approach to the core analysis, that has been proved to be useful in abstract mixed economies by [17]. The latter extends the notion of coalition and the ordinary veto since it is allowed a participation of the agents with a fraction of their endowments when forming a blocking coalition. We show that in order to guarantee that a price taking behavior emerges, the contribution to the realization of the project for a blocking coalition has to be defined taking into account the share of participation of each agent in the coalition itself. In particular, we prove that the Aubin σ-core coincides with the σ-core of the associated split market even when these two sets are not guaranteed to coincide with the set of linear cost share equilibria.

Our main results in the paper are proved interpreting the public goods economy with large traders as a nonatomic economy with public goods that derives from the original one by splitting each large trader in a continuum.

The paper is organized as follows. In Section 2, we present the mathematical model and the main equilibrium notions. In Section 3, we interpret mixed economies with public goods as atomless economies obtained by splitting each atom into a continuum of small traders of the same type. Section 4 is devoted to the equivalence result under the assumption that large traders are of the same type. In Section 5 we present some counterexamples, while in Section 6 we show weaker equivalence results.

2 The Model

2.1 The Setting

We consider an economy $\mathcal{E}$ with public goods modeling it by means of the following collection:

$$\mathcal{E} = \left\{(T, \mathcal{T}, \mu);\ (\mathcal{Y}, c);\ \mathbb{R}^{\ell}_{+};\ (u_t, e_t)_{t \in T}\right\}$$

where:

[2]Hence, as for private market economies, the σ-core correspondence in the mixed setup is not lower hemicontinuous (compare [8]).

1. $(T, \mathcal{T}, \mu)$ is a complete, finite measure space representing the space of agents.

2. $I\!R_+^\ell$ is the space of private commodities.

3. $\mathcal{Y}$ is an abstract set containing all public projects and c is a function $c : \mathcal{Y} \to I\!R_+^\ell$ which expresses the cost of any public project in terms of private goods.

4. $\left(u_t, e_t\right)_{t\in T}$ is the profile of agents' characteristics:

- an utility function

$$u_t : \; I\!R_+^\ell \times \mathcal{Y} \quad \to \quad I\!R$$

  representing preferences of agent t

- an initial endowment of physical resources $e(t) \in I\!R_+^\ell$ of t.

The reference to a measure space of agents deserves some comments and specifications. Though topological considerations over the set T of agents are by no means relevant for this paper, let us suppose that T is a second countable Hausdorff space. μ is a Borel measure on T and furnishes the *economic weight,* or *influence,* on the market of subsets, or groups, of agents. The σ-field $\mathcal{T}$ is just the μ-null completion of the Borel σ-field of T.

The agents' market influence is an issue of this paper and to deal with an abstract measure space of agents allows us to investigate simultaneously markets which are competitive and markets which are not. Indeed consider the decomposition

$$T = T_0^\mu \cup T_1^\mu$$

where

$$T_0^\mu = \{t \in T : \mu(\{t\}) = 0\}, \;\; T_1^\mu = \{t \in T : \mu(\{t\}) > 0\}$$

in the light of [1, Lemma 12.18].

Then: a μ such that T_0^μ is μ-null describes classical discrete economies while a μ ensuring that non agents may be influential, i.e. T_1^μ empty, gives an atomless measure space of agents namely, as it is well known, the mathematical framework for proper treatment of perfect competition. If both sets T_0^μ and T_1^μ are of positive μ-measure, beside a multitude T_0^μ of "*small*" uninfluential agents we have (at most) countably many "*large*" influential agents (monopolists, oligopolies, cartels, syndicates or institutional forms of collusive agreements) forming the set T_1^μ. In such a situation we say that we have a *mixed market*.

We shall neglect groups of agents that as such do not effect aggregate consumption. This parallels the measure theoretical custom of identifying two sets with μ-null symmetric difference or two functions when they are equal μ-a.e. With such a convention in mind we reserve the term *coalition* to subsets of agents of positive μ-measure and identify large agents with atoms of the space $(T, \mathcal{T}, \mu)$ ([1, Lemma 12.18]). We shall write $T = T_0 \cup T_1$ instead of $T = T_0^{\mu} \cup T_1^{\mu}$ and $T_1 = \{A_1, A_2, \dots\}$ if no ambiguity may occur.

2.2 Further Assumptions, Main Definitions and Inclusions

Throughout the paper we shall assume that

(A.1) the total endowment $e = \int_T e(t)\, d\mu \gg 0$ satisfies the inequality $e \gg c(y)$ for every $y \in \mathcal{Y}$

(A.2) for each public project $y \in \mathcal{Y}$, all agents have a measurable, continuous, concave and strictly monotone utility function $u_t(\cdot, y)$.

Assumption (A.1) asserts that no private commodity is totally absent from the market regardless the cost of the project that is going to be realized.

Definition 1. *A pair (f, y), where f is an integrable function from T to $I\!R^{\ell}_{+}$ and $y \in \mathcal{Y}$, is said to be an* **allocation**. *An allocation is said to be* **feasible** *if the whole part of the total initial endowment not used for covering the cost of the realized public project is redistributed among the agents, i.e.,*

$$\int_T f(t)\, d\mu + c(y) = \int_T e(t)\, d\mu.$$

Definition 2. *A* **cost distribution** *is function $\varrho : T \times \mathcal{Y} \to I\!R_+$ such that for each public project $y \in \mathcal{Y}$ the partial function $\varrho(\cdot, y)$ is integrable and $\displaystyle\int_T \varrho(t, y)\, d\mu = 1$. We denote by Φ_μ the class of all cost distributions for the economy $\mathcal{E}$.*

A cost distribution function describes how much each agent contributes (per unit of weight) to the establishment of the provision level of each public good (notice that, differently from Diamantaras-Gilles, we assume now that the individual contribution varies across public projects). If ϱ is the given cost distribution, then an "infinitesimal"group of agents containing the individual t is expected to contribute $\varrho(t, y)p(y)\, d\mu(t) \cdot c(y)$, where $p(y)$ is the market price vector for private goods.

Definition 3. *A* **contribution measure** *is a function $\sigma : \mathcal{T} \times \mathcal{Y} \to [0, 1]$ such that for each project $y \in \mathcal{Y}$ the partial function $\sigma(\cdot, y) \in M_\mu$, the collection of all probability measures on $\mathcal{T}$ which are absolutely continuous with respect to μ. The symbol M_μ will also denote the collection of all contribution measures of $\mathcal{E}$.*

Contribution measures fix the share of the total cost of the public project provision for which each coalition is responsible. Again this share for a coalition varies across projects. In the blocking mechanism a coalition S must guarantee the cost $\sigma(S,z)c(z)$ in order to block the allocation (f,y) by means of a different allocation (g,z).
The natural correspondence that we expect to hold between cost distribution functions and contribution measures follows. If $\varrho \in \Phi_\mu$ then the associated $\sigma_\varrho \in M_\mu$ is given by

$$\sigma_\varrho(S,y) = \int_S \varrho(t,y)\, d\mu \qquad \text{for all } S \in \mathcal{T}, \text{for all } y \in \mathcal{Y}. \tag{1}$$

Conversely, if $\sigma \in M_\mu$ then the associated ϱ_σ is the Radon-Nikodym derivative of $\sigma(\cdot,y)$ with respect to μ.

Though, from the analytical point of view, we merely register that μ-atoms and σ-atoms (σ is, for short, $\sigma(\cdot,y)$ for a given contribution measure) do not necessarily coincide [3], the market influential-uninfluential decomposition of the space of agents really effects a similar decomposition with respect to the provision necessary to the realization of a public project.

Next to the decomposition of T into the sets T_0^μ and T_1^μ we also have the analogous decomposition $T = T_0^\sigma \cup T_1^\sigma$ and the absolute continuity of σ with respect to μ tells that $T_1^\sigma \subseteq T_1^\mu$ and that

$$\sigma(\{t\},y) = \varrho_\sigma(t,y)\,\mu(\{t\})$$

holds for any t of T_1^μ and μ-a.e. on T_0^μ. Therefore

$$\begin{cases} T_1^\sigma = T_1^\mu \\ T_0^\sigma = T_0^\mu \end{cases} \Leftrightarrow \ \varrho_\sigma(t,y) > 0,\ \forall t \in T_1^\mu.$$

In other words, any contribution measure σ induces a decomposition of T in large taxpayers and small or negligible taxpayers [4]. The former are

[3] A μ-atom S may well not be a σ-atom, indeed just take $\sigma(\cdot) = \mu(\cdot \setminus S)$. A σ-atom is not necessarily a μ-atom; nonetheless if μ is atomless then so is σ. The converse is false and σ may be atomless even if μ does have atoms. To see all this, first consider $T = \{1,2,3\}$ and μ giving weight one third on each individual. Take the cost distribution function equal to zero on the first agent then equal to one. The coalition S made of the first two agents is a σ-atom which is not a μ-atom. To understand that an atomless μ determines an atomless σ, note that an atom S with respect to σ contains a point of T with positive σ-measure and therefore with positive μ-measure. Finally, if $T = [0,\frac{1}{2}] \cup \{1\}$, the measure μ is the Lebesgue measure on $[0,\frac{1}{2}]$ and gives weight one half to agent 1 and the cost distribution function is constantly one on $[0,\frac{1}{2}]$ and zero on agent 1, then σ is atomless though μ is not.

[4] We can imagine that, similarly to the distinction between large and small traders - according to the fact that a large trader is an agent whose consumption affects the aggregate total consumption - there is a distinction between large and small contributors. A contributor is then large if a change in its contribution affects the total provision for the realization of the public project, otherwise he is small.

necessarily agents with market power while the latter are not necessarily small traders. Unless we assume that market power must necessarily entail positive contribution to the public project, in which case we can perfectly identify large traders with large taxpayers and small traders with small taxpayers. This paper does not assume such hypothesis and therefore with an arbitrary cost distribution we may well have that large traders do not contribute to public projects.

Denote by Δ the simplex of $I\!R^{\ell}_{+}$, that is $\Delta = \{p \in R^{\ell}_{+} : \sum_{h=1}^{\ell} p^{\ell} = 1\}$.

Definition 4. *A feasible allocation (f, y) is said to be a* **linear cost share equilibrium** *for the economy $\mathcal{E}$ if there exist a price system $p : \mathcal{Y} \to \Delta$ and a cost distribution ϱ such that, for almost all $t \in T$, $(f(t), y)$ maximizes the utility function u_t on the budget set*

$$\left\{(h, z) \in I\!R^{\ell}_{+} \times \mathcal{Y} : p(z) \cdot h + \varrho(t, z)p(z) \cdot c(z) \leq p(z) \cdot e(t)\right\}.$$

Let ϱ be a cost distribution function. The set of linear cost share equilibria whose corresponding cost distribution function is ϱ will be denoted by $LCE_{\varrho}(\mathcal{E})$. Thus the set of all linear cost share equilibria of $\mathcal{E}$, $LCE(\mathcal{E})$, is given by

$$LCE(\mathcal{E}) = \bigcup_{\varrho \in \Phi_{\mu}} LCE_{\varrho}(\mathcal{E}).$$

If $\varrho(t, y) = \textit{constant}$ for almost all $t \in T$ and $y \in \mathcal{Y}$, then we end up into **equal cost share equilibria**.

Definition 5. *Let (f, y), (g, z) be two feasible allocations. (f, y) is said to be* **dominated** *by (g, z) if $u_t(g(t), z) > u_t(f(t), y)$ for almost all $t \in T$. A feasible, non-dominated allocation is said to be* **Pareto-optimal**.

Definition 6. *Given a contribution measure σ, we say that a coalition $S \in \mathcal{T}$* **σ-blocks** *an allocation (f, y) if there exists a public project $z \in \mathcal{Y}$ and an integrable assignment of private goods $g : S \to I\!R^{\ell}_{+}$ such that*

$$\begin{array}{ll} (i) & u_t(g(t), z) > u_t(f(t), y) \quad \textit{for almost all } t \in S, \\ (ii) & \displaystyle\int_S g(t)\, d\mu \, + \sigma(S, z)c(z) = \int_S e(t)\, d\mu. \end{array}$$

We denote by $C_{\sigma}(\mathcal{E})$ the set of σ-core allocations, namely feasible allocations that cannot be σ-blocked by any coalition. When the contribution measure $\sigma(\cdot, y)$ coincide with μ for any $y \in \mathcal{Y}$, then the corresponding core $C_{\mu}(\mathcal{E})$ is called the **proportional core**.

It is clear that any σ-core allocation is Pareto optimal. One just need to put S equal to T and notice that $\sigma(T, \cdot) = 1$.

In [4] and [5] a definition of core is introduced according to a veto mechanism of standard (Foley) type. It does not require a contribution measure, as reported below.

Definition 7. *A coalition $S \in \mathcal{T}$* **blocks** *an allocation (f, y) if there exists a public project $z \in \mathcal{Y}$ and an integrable assignment of private goods $g : S \to I\!R^{\ell}_{+}$ such that*

$$\begin{array}{ll} (i) & u_t(g(t), z) > u_t(f(t), y) \quad \textit{for almost all } t \in S, \\ (ii) & \displaystyle\int_S g(t)\, d\mu + c(z) = \int_S e(t)\, d\mu. \end{array}$$

We denote by $C(\mathcal{E})$ the set of core allocations, i.e. feasible allocations that cannot be blocked by any coalition. Notice that in the veto mechanism a blocking coalition when dissenting from a given allocation, is required to cover the whole cost of the alternative public project. It is easy to show that $C_\sigma(\mathcal{E}) \subseteq C(\mathcal{E})$ for any σ, for any contribution measure σ.

Proposition 1. *For any contribution measure σ, we have $C_\sigma(\mathcal{E}) \subseteq C(\mathcal{E})$.*

PROOF: Let σ be a contribution measure and (f, y) be a σ-core allocation. Assume, on the contrary, that there exist a coalition S and (g, z) such that

$$\begin{array}{ll} (i) & u_t(g(t), z) > u_t(f(t), y) \quad \text{for almost all } t \in S, \\ (ii) & \displaystyle\int_S g(t)\, d\mu + c(z) = \int_S e(t)\, d\mu. \end{array}$$

Since for any coalition S and public good z, $\sigma(S, z) \leq 1$ and $c(z) \in I\!R^{\ell}_{+}$, then

$$\int_S g(t)\, d\mu + \sigma(S, z)c(z) \leq \int_S g(t)\, d\mu + c(z) = \int_S e(t)\, d\mu.$$

Define $\epsilon = \int_S e(t)\, d\mu - \int_S g(t)\, d\mu - \sigma(S, z)c(z)$ and notice that $\epsilon \geq 0$ and S σ-blocks (f, y) via $\left(g + \frac{\epsilon}{\mu(S)}, z\right)$, which is a contradiction. $\square$

The next example illustrates that the above inclusion may be strict.

Example 1. *There exist a mixed market and a contribution measure σ such that a core allocation need not be a σ-core allocation.*

PROOF: Consider an economy with only two atoms A and B and μ such that $\mu(A) = \mu(B) = \frac{1}{2}$. Fix $e(A) = (3, 0)$, while $e(B) = (0, 2)$. Assume there are two private goods and two public projects $\mathcal{Y} = \{y, z\}$ such that $c(y) = (\frac{1}{2}, 0)$ and $c(z) = (0, \frac{1}{2})$. Moreover, suppose that agents have the same utility functions defined as follows:

$$\begin{aligned} u_t(f(t), y) &= f_1(t) + f_2(t) \\ u_t(f(t), z) &= \frac{1}{3} f_1(t) + f_2(t). \end{aligned}$$

The allocation (f, y), where $f(t) = (1,1)$ for $t = A, B,$ belongs to the core.
Notice that (f, y) is feasible, indeed

$$\begin{aligned}
f_1(A)\mu(A) + f_1(B)\mu(B) + c_1(y) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} &= \frac{3}{2} = \frac{3}{2} + 0 = e_1(A)\mu(A) + e_1(B)\mu(B) \\
f_2(A)\mu(A) + f_2(B)\mu(B) + c_2(y) = \frac{1}{2} + \frac{1}{2} + 0 &= 1 = 0 + \frac{2}{2} = e_2(A)\mu(A) + e_2(B)\mu(B).
\end{aligned}$$

Assume that (f, y) is not a core allocation. Then there exist a coalition S and a distribution of private goods g such that

$$\begin{array}{lll}
\text{I CASE}: & (i) & u_t(g(t), y) = g_1(t) + g_2(t) > 2 = f_1(t) + f_2(t) = u_t(f(t), y) \quad \text{for all } t \in S \\
& (ii) & \displaystyle\int_S g(t)\, d\mu + c(y) = \int_S e(t)\, d\mu, \\
\text{or II CASE}: & (i) & u_t(g(t), z) = \dfrac{1}{3} g_1(t) + g_2(t) > 2 = f_1(t) + f_2(t) = u_t(f(t), y) \quad \text{for all } t \in S \\
& (ii) & \displaystyle\int_S g(t)\, d\mu + c(z) = \int_S e(t)\, d\mu.
\end{array}$$

Three subcases may occur according to the three possible coalitions $S = \{A\}, \{B\}, T$.

If $S = \{A\}$, then

$$\begin{aligned}
\text{I CASE}: g_1(A) + g_2(A) &> 2 \\
\frac{1}{2} g_1(A) + \frac{1}{2} &= \frac{3}{2} \qquad \text{that is } g_1(A) = 2 \\
\frac{1}{2} g_2(A) + 0 &= 0 \qquad \text{that is } g_2(A) = 0.
\end{aligned}$$

This means that $2 < g_1(A) + g_2(A) = 2$, which is a contradiction. Similarly,

$$\begin{aligned}
\text{II CASE}: \frac{1}{3} g_1(A) + g_2(A) &> 2 \\
\frac{1}{2} g_1(A) + 0 &= \frac{3}{2} \\
\frac{1}{2} g_2(A) + \frac{1}{2} &= 0 \qquad \text{which is a contradiction.}
\end{aligned}$$

If $S = \{B\}$, then

$$\begin{aligned}
\text{I CASE}: g_1(B) + g_2(B) &> 2 \\
\frac{1}{2} g_2(B) + 0 &= 1 \\
\frac{1}{2} g_1(B) + \frac{1}{2} &= 0 \qquad \text{which is a contradiction.} \\
\text{II CASE}: \frac{1}{3} g_1(B) + g_2(B) &> 2 \\
\frac{1}{2} g_1(B) + 0 &= 0 \qquad \text{that is } g_1(B) = 0 \\
\frac{1}{2} g_2(B) + \frac{1}{2} &= 1 \qquad \text{that is } g_2(B) = 1.
\end{aligned}$$

This means that $2 < g_1(B) + g_2(B) = 1$, which is a contradiction.

If $S = T$, we have the following.

$$
\begin{aligned}
\text{I CASE}: g_1(A) + g_2(A) &> 2 \\
g_1(B) + g_2(B) &> 2 \\
\frac{1}{2} g_1(A) + \frac{1}{2} g_1(B) + \frac{1}{2} &= \frac{3}{2} \\
\frac{1}{2} g_2(A) + \frac{1}{2} g_2(B) + 0 &= 1.
\end{aligned}
$$

Summing up, we get that $4 < [g_1(A) + g_2(A)] + [g_1(B) + g_2(B)] = 2 + 2 = 4$, which is a contradiction. Similarly,

$$
\begin{aligned}
\text{II CASE}: \frac{1}{3} g_1(A) + g_2(A) &> 2 \\
\frac{1}{3} g_1(B) + g_2(B) &> 2 \\
\frac{1}{2} g_1(A) + \frac{1}{2} g_1(B) + 0 &= \frac{3}{2} \qquad \text{and hence } \frac{1}{3} g_1(A) + \frac{1}{3} g_1(B) = 1 \\
\frac{1}{2} g_2(A) + \frac{1}{2} g_2(B) + \frac{1}{2} &= 1.
\end{aligned}
$$

Summing up, we get that $4 < [\frac{1}{3} g_1(A) + g_2(A)] + [\frac{1}{3} g_1(B) + g_2(B)] = 1 + 1 = 2$, which is a contradiction.

Once we have established that (f, y) belongs to the core, define the following cost distribution function $\varrho(A, \cdot) = \frac{1}{2}$ and $\varrho(B, \cdot) = \frac{3}{2}$. Notice that the associated contribution measure σ_ϱ is such that $\sigma_\varrho(A, \cdot) = \frac{1}{4}$ and $\sigma_\varrho(B, \cdot) = \frac{3}{4}$. Moreover, $S = \{A\}$ σ_ϱ-blocks (f, y) via (g, y), where $g(A) = \left(\frac{11}{4}, 0\right)$. Indeed

$$
\begin{aligned}
u_A(g(A), y) = g_1(A) + g_2(A) &= \frac{11}{4} > 2 = u_A(f(A), y) \\
g_1(A)\mu(A) + \sigma_\varrho(A, y) c_1(y) = \frac{11}{8} + \frac{1}{8} &= \frac{12}{8} = \frac{3}{2} = e_1(A)\mu(A) \\
g_2(A)\mu(A) + \sigma_\varrho(A, y) c_2(y) = 0 &= e_2(A)\mu(A).
\end{aligned}
$$

This means that $(f, y) \in C(\mathcal{E}) \setminus C_{\sigma_\varrho}(\mathcal{E})$. $\square$

Not surprisingly, any linear cost share equilibrium of $\mathcal{E}$ with cost distribution function ϱ is a σ_ϱ-core allocation.

Proposition 2. *Let (f, y) be a linear cost share equilibrium in $\mathcal{E}$ with cost distribution function ϱ and let σ_ϱ be the corresponding contribution measure. Then, (f, y) is a $\sigma-$core allocation and hence it is Pareto optimal.*

PROOF: Let (f, y) be a linear cost share equilibrium in $\mathcal{E}$ with cost distribution function ϱ and assume, on the contrary, that (f, y) is not a σ_ϱ-core allocation. This means that there exist a coalition $S \in \mathcal{T}$ and an allocation (g, z) such that

$$\begin{array}{ll} (i) & u_t(g(t), z) > u_t(f(t), y) \quad \text{for almost all } t \in S, \text{ and} \\ (ii) & \displaystyle\int_S g(t)\, d\mu + \sigma_\varrho(S, z)c(z) = \int_S e(t)\, d\mu. \end{array}$$

Since (f, y) is a linear cost share equilibrium, by condition (i), there exists a price system $p : \mathcal{Y} \to \Delta$ such that for almost all $t \in S$,

$$p(z) \cdot g(t) + \varrho(t, z)p(z) \cdot c(z) > p(z) \cdot e(t).$$

This implies that

$$p(z) \cdot \int_S g(t)\, d\mu + \int_S \varrho(t, z)\, d\mu \; p(z) \cdot c(z) > p(z) \cdot \int_S e(t)\, d\mu,$$

that is, by (1),

$$p(z) \cdot \int_S g(t)\, d\mu + \sigma_\varrho(S, z)p(z) \cdot c(z) > p(z) \cdot \int_S e(t)\, d\mu,$$

which contradicts (ii). □

It is useful to remark that in suitable situations a σ-core allocation assigns equivalent bundles to agents that are of the same type (i.e. they have the same economic characteristics: the same initial endowment and the same utility function) and are also the same as taxpayers. This observation is made precise in the next proposition.

Proposition 3. *Let (f, y) be a σ_ϱ-core allocation. Suppose A is a coalition such that $T \setminus A$ does not contain any large trader. Assume further that for $t, s \in A$ one has*

$$u_t(\cdot, y) = u_s(\cdot, y), \;\; e(t) = e(s) \;\; \textit{and} \;\; \varrho(t, y) = \varrho(s, y).$$

Then, μ-almost all individuals of A get the same utility under (f, y) namely $u_t(f(t), y)$ is μ-a. e. constant on A.

PROOF: Let us fix for simplicity $u_t(\cdot, y) = u(\cdot)$, $\varrho(t, y) = \varrho$ when $t \in A$ and $\sigma(\cdot) = \sigma_\varrho(\cdot, y)$. Let g be the average value of f over A: $\quad g = \dfrac{1}{\mu(A)} \displaystyle\int_A f\, d\mu.$

Consider the two sets $C = \{t \in A : u(g) > u(f(t))\}$ and $D = \{t \in A : u(g) < u(f(t))\}$. If we show that none of them is of positive measure, then we are done.

Claim 1. $\mu(C) = 0$.

Suppose not and take $\alpha = \frac{\mu(C)}{\mu(A)}$. From the continuity of u we derive that there exists $\epsilon \in]0,1[$ such that the set $B = \{t \in C : u(\epsilon g) > u(f(t))\}$ is of positive measure.

Since $T \setminus A$ is atomless, Liapunov convexity theorem applies to the measure

$$\eta(\cdot) = \left(\int_{(\cdot)} f \, d\mu, \int_{(\cdot)} e \, d\mu, \sigma(\cdot) \right)$$

defined over members of $\mathcal{T}$ contained in $T \setminus A$. Take therefore $S \subseteq T \setminus A$ with $\eta(S) = \alpha \eta(T \setminus A)$. Without loss of generality we assume that $\mu(S) > 0$[5] and prove that the coalition $R = S \cup C$ blocks (f, y), against the fact that the latter is a σ-core allocation.

Consider the assignment $h(\cdot)$ defined to be equal to ϵg on B, equal to g on $C \setminus B$ and equal to $f(\cdot) + (1-\epsilon) g \frac{\mu(B)}{\mu(S)}$ on S. By monotonicity of u:

$$u_t(h(t), y) > u_t(f(t), y), \text{ for a. e. } t \text{ in } R.$$

The equality

$$\int_R h \, d\mu + \sigma(R) c(y) = \int_R e \, d\mu$$

will complete the argument. Now:

$$\int_R h \, d\mu + \sigma(R) c(y) =$$

$$\int_S f \, d\mu + (1-\epsilon) g \mu(B) + \epsilon g \mu(B) + g \mu(C \setminus B) + c(y) \alpha \sigma(T \setminus A) + c(y) \varrho \mu(C) =$$

$$\alpha \int_{T \setminus A} f \, d\mu + g \mu(C) + c(y)[\alpha \sigma(T \setminus A) + \varrho \mu(C)] =$$

$$\alpha \int_{T \setminus A} f \, d\mu + \alpha \int_A f \, d\mu + c(y) \alpha [\sigma(T \setminus A) + \sigma(A)] = \alpha \int_T e \, d\mu.$$

but, since e is constant over A,

$$\alpha \int_T e \, d\mu = \alpha \int_A e \, d\mu + \alpha \int_{T \setminus A} e \, d\mu = \alpha \int_A e \, d\mu + \int_S e \, d\mu =$$

$$\int_C e \, d\mu + \int_S e \, d\mu = \int_R e \, d\mu.$$

Claim 2. $\mu(D) = 0$.

[5] If not, it is trivial that coalition C already blocks (f, y) via (g, y).

Suppose not and take $\alpha = \frac{\mu(D)}{\mu(A)}$. Without loss of generality suppose $\alpha < 1$ or, what is the same, $\mu(A \setminus D) > 0$. Let a and b the average values of f over D and $A \setminus D$ respectively. By definition of D and concavity of u,we have

$$u(a) \geq \frac{1}{\mu(D)} \int_D u(f)\, d\mu > u(g).$$

Similarly $u(b) \geq u(g)$. Since $g = \alpha a + (1-\alpha)b$, we get $u(g) = u(\alpha a + (1-\alpha)b) > u(g)$ (concavity again) which is a contradiction. $\square$

3 Interpretation of mixed market via atomless economies

In this section, we associate to a mixed economy an atomless economy obtained by splitting each atom into a continuum of small traders of the same type. The aim is to use the interpretation of the original economy as **embedded into** an atomless one for deriving a corresponding interpretation of core and competitive allocations.

Remind that large traders have been indicated by $A_1, A_2, \ldots$ and they form the subset T_1 of the set T of agents. Correspondingly, we split the interval $[\mu(T_0), \mu(T)]$, to be denoted by T_1^*, as follows. T_1^* is the disjoint union of the intervals A_i^* defined as

$$A_1^* := [\mu(T_0), \mu(T_0) + \mu(A_1)[, ...,$$

$$A_i^* := [\mu(T_0) + \mu\{A_1, ..., A_{i-1}\}, \mu(T_0) + \mu\{A_1, ..., A_i\}[, ...$$

Now, given the economy

$$\mathcal{E} = \left\{(T, \mathcal{T}, \mu);\ (\mathcal{Y}, c);\ I\!R_+^\ell;\ (u_t, e_t)_{t \in T}\right\}$$

the associated atomless economy

$$\mathcal{E}^* = \left\{(T^*, \mathcal{T}^*, \mu^*);\ (\mathcal{Y}, c);\ I\!R_+^\ell;\ (u_t, e_t)_{t \in T^*}\right\}$$

is defined as follows.
The measure space $(T^*, \mathcal{T}^*, \mu^*)$ of agents is the direct sum of $(T_0, \mathcal{T}_0, \mu)$ and the interval T_1^* endowed with the σ-algebra of its Lebesgue measurable subsets and with the Lebesgue measure.

The set $\mathcal{Y}$ of public projects as well as the cost function c are unchanged.

Finally, the profile $(u_t, e_t)_{t \in T^*}$ of agents' characteristics extends the original profile of members of T by assuming that u_t and e_t, for $t \in A_i^*$, coincide with u_{A_i} and e_{A_i}. We shall say that A_i^* is the split of the atom A_i.

It is worth to fix some notation.

Given a real or vector-valued function $\varphi \in L_1(\mu^*)$, call φ_{A_i} the average of φ over the interval A_i^*, i.e. $\dfrac{1}{\mu^*(A_i^*)} \displaystyle\int_{A_i^*} \varphi \, d\mu^*$.

Define the element $\widetilde{\varphi} \in L_1(\mu)$ by

$$\widetilde{\varphi}(t) = \begin{cases} \varphi(t), & \text{for } t \in T_0 \\ \varphi_{A_i}, & \text{for } t = A_i. \end{cases}$$

Reciprocally, for a given real or vector-valued function $f \in L_1(\mu)$ define over T^* the function $f^* \in L_1(\mu^*)$ by

$$f^*(t) = \begin{cases} f(t), & \text{if } t \in T_0 \\ f(A_i), & \text{if } t \in A_i^*. \end{cases}$$

It is easily checked that for any $f \in L_1(\mu)$,

$$\|f^*\|_{L_1(\mu^*)} = \|f\|_{L_1(\mu)} \text{ and } (f^*)^\sim = f$$

while for any φ,

$$\|\widetilde{\varphi}\|_{L_1(\mu)} \leq \|\varphi\|_{L_1(\mu^*)}$$

and clearly for nonnegative f, φ

$$\int_T f \, d\mu = \int_{T^*} f^* \, d\mu^*, \quad \int_{T^*} \varphi \, d\mu^* = \int_T \widetilde{\varphi} \, d\mu \ .$$

So $L_1(\mu)$ is embedded, by means of the $f \mapsto f^*$ mapping, into $L_1(\mu^*)$ being isomorphic to the subspace $L_1^s(\mu^*)$ of the latter consisting of functions φ which are μ^*-a.e. constant over any interval A_i^*.

On the other hand, the mapping $\varphi \mapsto \widetilde{\varphi}$ maps continuously $L_1(\mu^*)$ onto $L_1(\mu)$.

At this point an allocation (f, y) in the economy $\mathcal{E}$ can be interpreted as an allocation (f^*, y) in $\mathcal{E}^*$ preserving the total consumption of private goods.

Reciprocally, an allocation (φ, y) in $\mathcal{E}^*$ can be projected as an allocation $(\widetilde{\varphi}, y)$ in $\mathcal{E}$ again preserving the total consumption of private goods.

Since in $\mathcal{E}^*$ the initial endowment is just e^*, we can observe that under operators $*$ and $\sim$, feasibility is preserved in the sense described in the next proposition.

Proposition 4. *Let (f, y) a feasible allocation of the mixed market $\mathcal{E}$, then the associated allocation (f^*, y) is feasible for $\mathcal{E}^*$. Reciprocally, if (φ, y) is a feasible allocation of the atomless economy $\mathcal{E}^*$, the associated allocation $(\widetilde{\varphi}, y)$ is feasible for $\mathcal{E}$.*

We also register explicitly the useful fact that follows.

Fact 1. *Feasible allocations of the original economy $\mathcal{E}$ are in a one-to-one correspondence with those allocations of $\mathcal{E}^*$ which are (μ^*-) constant over any set A_i^*.*

A further step is to point explicitly out the embedding, made possible by $*$ and $\sim$ operators, of $(\mathcal{T}, \mu)$ into $(\mathcal{T}^*, \mu^*)$. Let us denote by $*(\mathcal{T})$ the sub-sigma-algebra of $\mathcal{T}^*$ made of those sets X such that for any index i the intersection $X \cap A_i^*$ is either empty or coincides with A_i^*. If we restrict the operator $*$ to characteristic functions, the equation

$$(\chi_S)^* = \chi_{S^*}$$

defines, for any $S \subseteq T$, the set

$$S^* = (S \cap T_0) \cup \bigcup_{\{i: A_i \in S\}} A_i^*$$

belonging to $*(\mathcal{T})$. Notice that if we restrict $\sim$ to characteristic functions, we do not necessarily get another characteristic function, therefore set for any $X \in \mathcal{T}^*$

$$\widetilde{X} := (X \cap T_0) \cup \{A_i : i \in I_X\}$$

where $I_X := \{i : X \cap A_i^* = A_i^*\}$.

One can note the following trivial facts:

$$(S^*)^\sim = S, \quad [\cup_k S_k]^* = \cup_k S_k^*, \; \mu^*(S^*) = \mu(S) \text{ for } k \in I\!N, \, S, S_k \in \mathcal{T}$$

$$(\chi_X)^\sim = \chi_{\widetilde{X}}, \quad (\widetilde{X})^* = X, \quad [\cup_k X_k]^\sim = \cup_k \widetilde{X}_k, \quad \text{for } \; k \in I\!N, X, X_k \in *(\mathcal{T})$$

to say that

Fact 2. *$(\mathcal{T}, \mu)$ can be identified with $(*(\mathcal{T}), \mu^*)$ by means of the mappings, each inverse of the other, $S \mapsto S^*$ and $X \mapsto \widetilde{X}$.*

and

Fact 3. *The set of probabilities M_μ can be identified with the set of probabilities ν defined on $*(\mathcal{T})$ which are absolutely continuous with respect to μ^* by means of the two mappings*

$$\sigma \mapsto *(\sigma), \quad \textit{where } \; *(\sigma)(X) := \sigma(\widetilde{X}), \; \textit{for } X \in *(\mathcal{T})$$

and

$$\nu \mapsto \widetilde{\nu} \quad \textit{where } \widetilde{\nu}(S) := \nu(S^*), \; \textit{for } S \in \mathcal{T}$$

each other inverses.

Cost distribution functions, as well as contribution measures, for $\mathcal{E}$ and for $\mathcal{E}^*$ can also be similarly connected, completing in this way the embedding. Let us see first cost distribution functions.

Let ϱ be a cost distribution function for $\mathcal{E}$. The associated function ϱ^*, naturally defined by

$$\varrho^*(\cdot, y) = [\varrho(\cdot, y)]^*, \quad \forall y \in \mathcal{Y}$$

is a cost distribution function in the atomless economy $\mathcal{E}^*$.

On the other hand, if γ is a cost distribution function for $\mathcal{E}^*$ then the associated function $\widetilde{\gamma}$ defined by

$$\widetilde{\gamma}(\cdot, y) = [\gamma(\cdot, y)]^{\sim}, \quad \forall y \in \mathcal{Y}$$

is a cost distribution function for the original mixed market $\mathcal{E}$.

Fact 4. *Cost distribution functions of the original economy $\mathcal{E}$ are in a one-to-one correspondence with those cost distribution functions of $\mathcal{E}^*$ which are, for any public project, (μ^*-) constant over any set A_i^*.*

Moving to contribution measure, let us introduce a proper subset $M_{\mu^*}^s$ of M_{μ^*} [6] we shall made use of.

Measures $M_{\mu^*}^s$ are those $\nu \in M_{\mu^*}$ for which

$$X \in \mathcal{T}^*, \, X \subseteq A_n^* \Rightarrow \nu(X) = \frac{\mu^*(X \cap A_n^*)}{\mu^*(A_n^*)} \nu(A_n^*).$$

We shall say that measures ν of M_{μ^*} belonging to $M_{\mu^*}^s$ are splitted. For a given $\sigma \in M_\mu$, define

$$\sigma^*(X) = \sigma(X \cap T_0) + \sum_{n \in \mathbb{N}} \frac{\mu^*(A_n^* \cap X)}{\mu^*(A_n^*)} \, \sigma(\{A_n\}), \text{ for } X \in \mathcal{T}^*. \tag{2}$$

Due to Fact 3 σ^* can be interpreted as the unique splitted extension of σ to $\mathcal{T}^*$. Naturally σ^* [7] is also μ^*-absolutely continuous. Reciprocally, for any $\nu \in M_{\mu^*}$ the probability $\widetilde{\nu}$ [8] defined in Fact 3 may be seen as the restriction of ν from $\mathcal{T}^*$ to $\mathcal{T}$.

Straightforward calculations show that

$$[\widetilde{\nu}]^* = \nu, \text{ for any } \nu \in M_{\mu^*}^s$$

[6] the collection of probability measures on $\mathcal{T}^*$ that are absolutely continuous with respect to μ^*

[7] Obviously $[*(\sigma)]^*$ is the same as σ^*.

[8] Notice that for any $S \in \mathcal{T}$ we have

$$\widetilde{\nu}(S) = \nu(S^*) = \nu(S \cap T_0) + \sum_{\{i : A_i \in S\}} \nu(A_i^*).$$

and

$$\text{if } \nu(\cdot) = \int_{(\cdot)} \varphi \, d\mu^*, \text{ then } \widetilde{\nu}(\cdot) = \int_{(\cdot)} \widetilde{\varphi} \, d\mu.$$

$$[\sigma^*]^{\sim} = \sigma, \text{ for any } \sigma \in M_\mu$$

and

$$\text{if } \sigma(\cdot) = \int_{(\cdot)} \varrho \, d\mu, \text{ then } \sigma^*(\cdot) = \int_{(\cdot)} \varrho^* \, d\mu^*.$$

Then we can register what follows.

Fact 5. *M_μ is in a one-to-one correspondence with $M^s_{\mu^*}$ being maps*

$$\sigma \in M_\mu \mapsto \sigma^* \in M^s_{\mu^*}$$

and

$$\nu \in M^s_{\mu^*} \mapsto \widetilde{\nu} \in M_\mu$$

the each other inverses.

Now, let σ be a contribution measure for $\mathcal{E}$. The associated function σ^*, naturally defined by

$$\sigma^*(\cdot, y) = [\sigma(\cdot, y)]^*, \quad \forall y \in \mathcal{Y}$$

is a contribution measure in the atomless economy $\mathcal{E}^*$.

On the other hand, if ν is a contribution measure for $\mathcal{E}^*$ then the associated function $\widetilde{\nu}$ defined by

$$\widetilde{\nu}(\cdot, y) = [\nu(\cdot, y)]^{\sim}, \quad \forall y \in \mathcal{Y}$$

is a contribution measure for the original mixed market $\mathcal{E}$.

Fact 6. *Contribution measures of the original economy $\mathcal{E}$ are in a one-to-one correspondence with those contribution measures of $\mathcal{E}^*$ which are splitted, for any public project.*

Fact 7. *If in the original economy $\mathcal{E}$, the contribution measure σ is associated to the cost distribution function ϱ, then in $\mathcal{E}^*$ the contribution measure σ^* and the cost distribution function ϱ^* are associated too.*

Reciprocally, if in the atomless economy $\mathcal{E}^*$, the contribution measure ν is associated to the cost distribution function φ, then in the original economy $\mathcal{E}$ the contribution measure $\widetilde{\nu}$ and the cost distribution function $\widetilde{\varphi}$ are associated too.

As a consequence of proposition 3, we have that

Proposition 5. *Let σ be a contribution measure for the economy $\mathcal{E}$. If the allocation (φ, y) of the economy $\mathcal{E}^*$ is a σ^*-core allocation, then the allocation $(\widetilde{\varphi}, y)$ is a σ-core allocation for $\mathcal{E}$* [9].

PROOF: Suppose, on the contrary, that $(\widetilde{\varphi}, y)$ is not a σ-core allocation and let S be a coalition that blocks it so that for suitable g and z we have $u_t(g(t), z) > u_t(\widetilde{\varphi}(t), y)$ for almost all $t \in S$ and the equality

$$\int_S g\, d\mu + \sigma(S, z)c(z) = \int_S e\, d\mu$$

holds. We shall see that S^* blocks (φ, y) via g^* and z. Indeed,

$$\int_{S^*} g^*\, d\mu^* = \int_S g\, d\mu, \qquad \int_{S^*} e^*\, d\mu^* = \int_S e\, d\mu$$

and $\sigma^*(S^*) = (\sigma^*)^{\sim}(S) = \sigma(S)$. Moreover $u_t(g^*(t), z) > u_t(\varphi(t), y)$ for μ^*-all $t \in S^*$. To check this consider $t \in A_i^*$ with $A_i \in S$:

$$u_t(g^*(t), z) = u_{A_i}(g(A_i), z) > u_{A_i}(\varphi_{A_i}, y) \geq \frac{1}{\mu^*(A_i^*)} \int_{A_i^*} u_{A_i}(\varphi(t), y)\, d\mu^*.$$

Now apply proposition 3 to $(T^*, \mathcal{T}^*, \mu^*)$, (φ, y) and for $A = A_i^*$. It follows that $u_{A_i}(\varphi(t), y)$ is μ^*-a. e. constant over the set A_i^* and the assertion follows. $\square$

Reversing the above proposition will be a task of the next session. Concerning linear cost share equilibria we remark what follows.

Proposition 6. *Let ϱ be a cost distribution function for the economy $\mathcal{E}$. Then the following implications hold true.*

$$\begin{array}{ll} (i) & (f, y) \in LCE_{\varrho}(\mathcal{E}) \Rightarrow (f^*, y) \in LCE_{\varrho^*}(\mathcal{E}^*) \\ (ii) & (\varphi, y) \in LCE_{\varrho^*}(\mathcal{E}^*) \Rightarrow (\widetilde{\varphi}, y) \in LCE_{\varrho}(\mathcal{E}). \end{array}$$

Consequently, for the given cost distribution function ϱ, linear cost share equilibria of the original economy $\mathcal{E}$ are (via $f \mapsto f^*$) in a one-to-one correspondence with those of $\mathcal{E}^*$, with respect to ϱ^*, which are μ^*-a. e. constant over any set A_i^*.

---

[9] In particular one has that

$$(f^*, y) \in C_{\sigma^*}(\mathcal{E}^*) \Rightarrow (f, y) \in C_{\sigma}(\mathcal{E}).$$

Note that the same proof shows that, given any contribution measure ν for the economy $\mathcal{E}^*$,

$$(f^*, y) \in C_{\nu}(\mathcal{E}^*) \Rightarrow (f, y) \in C_{\widetilde{\nu}}(\mathcal{E})$$

while it does not guarantee that $(\varphi, y) \in C_{\nu}(\mathcal{E}^*) \Rightarrow (\widetilde{\varphi}, y) \in C_{\widetilde{\nu}}(\mathcal{E}^*)$.

PROOF: (i) is trivial.
(ii) Take the price system p that ensures a. e. on T^*

$$(+) \qquad u_t(g,z) > u_t(\varphi(t),y) \Rightarrow p(z)\cdot g + \varrho^*(t,z)p(z)\cdot c(z) > p(z)\cdot e^*(t).$$

Now for $t \in T$ assume $u_t(g,z) > u_t(\widetilde{\varphi}(t),y)$. If $t \in T_0$, because of (+), $p(z)\cdot g + \varrho(t,z)p(z)\cdot c(z) > p(z)\cdot e(t)$. If $t = A_i$ we have, because of concavity of utility functions, $u_{A_i}(g,z) > u_{A_i}(\varphi_{A_i},y) \geq \dfrac{1}{\mu^*(A_i^*)}\displaystyle\int_{A_i^*} u_{A_i}(\varphi(t),y)\,d\mu^*$ and on a subset of A_i^* of positive μ^*-measure, $u_{A_i}(g,z) > u_{A_i}(\varphi(t),y)$. So, for a $t \in A_i^*$, we have $u_{A_i}(g,z) > u_{A_i}(\varphi(t),y)$ to be rewritten $u_t(g,z) > u_t(\varphi(t),y)$. From (+), we get $p(z)\cdot g + \varrho^*(t,z)p(z)\cdot c(z) > p(z)\cdot e^*(t) \Leftrightarrow p(z)\cdot g + \varrho(A_i,z)p(z)\cdot c(z) > p(z)\cdot e(A_i)$.

□

4 Equivalence Theorem

This section focuses on the equivalence between σ-core allocations and linear cost share equilibria with respect to the cost distribution function ϱ_σ associated to σ. In order to prove the equivalence theorem the following hypotheses are needed.

(A.3) Compensation condition For every agent $t \in T$, every bundle $v \in \mathbb{R}_+^\ell$, and all public projects $y, z \in \mathcal{Y}$, there exists a bundle $g \in \mathbb{R}_+^\ell$ such that $u_t(g,z) \geq u_t(v,y)$.

Setting

$$P(t,y,z,v) := \{g \in \mathbb{R}_+^\ell \,:\, u_t(g,z) \geq u_t(v,y)\}$$

and

$$P^s(t,y,z,v) := \{g \in \mathbb{R}_+^\ell \,:\, u_t(g,z) > u_t(v,y)\}$$

the above condition just says that $P(t,y,z,v) \neq \emptyset$.

(A.4) Second essentially condition For any agent $t \in T$, for all $y, z \in \mathcal{Y}$ and $v \in \mathbb{R}_+^\ell$ the inequality $u_t(0,z) \leq u_t(v,y)$ holds, i.e. $P(t,z,y,0) = \mathbb{R}_+^\ell$.

Given an allocation (f,y) and a public project z, set

$$\Psi(t,z) := P^s(t,y,z,f(t)) = \{g \in \mathbb{R}_+^\ell \,:\, u_t(g,z) > u_t(f(t),y)\}.$$

Observe that, because of monotonicity of utilities, under (A.3) Ψ is never empty[10].

(A.5) Integrable preferred bundles For any allocation (f, y) and for all $z \in \mathcal{Y}$, there exists g such that (g, z) is feasible and moreover μ-a. e. on T one has

$$v \in I\!R^{\ell}_{+},\ u_t(v, z) > u_t(f(t), y) \Rightarrow u_t(g(t), z) \geq u_t(f(t), y).$$

Of course the above implication can be rewritten as

$$P^s(t, y, z, f(t)) \neq \emptyset \Rightarrow g(t) \in P(t, y, z, f(t))$$

leading to the interpretation[11] of g as an integrable selection of $P(t, y, z, f(t))$. A selection that certainly exists under the simultaneous validity of both (A.3) and (A.5).

Our main aim is to prove that when atoms are of the same type, it generates such intense competition among them that we can consider a mixed economy equivalent to an atomless one, and hence the Equivalence Theorem can be restored. First notice that the following results hold true.

Theorem 1. *There exists an atomless economy $\mathcal{E}^*$ with public goods with preferences represented by strictly monotone and quasi-concave utility functions for which the set $LCE(\mathcal{E}^*)$ of linear cost share equilibria is non empty.*

PROOF: See [5, Theorem 1]. □

Theorem 2. *Let $\mathcal{E}$ be an atomless economy with public goods (i.e. $T_1 = \emptyset$ and $T = T_0$) satisfying the assumptions $(A.1) - -(A.4)$. Let $\sigma \in M_\mu$ be a contribution measure and let ϱ_σ be the corresponding cost distribution function. Then $C_\sigma(\mathcal{E}) = LCE_{\varrho_\sigma}$.*

PROOF: Let (f, y) be a σ-core allocation. Define for any $z \in \mathcal{Y}$ and $t \in T$,

$$\begin{aligned} F(t, z) &= \{\Psi(t, z) - e(t)\} \cup \{-\varrho_\sigma(t, z)c(z)\} \\ F(z) &= \int_T F(t, z)\, d\mu(t) + c(z). \end{aligned}$$

It is easy to show that $0 \in F(z)$ for any $z \in \mathcal{Y}$. Indeed, since for all t, $-\varrho_\sigma(t, z)c(z) \in F(t, z)$ then

$$\int_T [-\varrho_\sigma(t, z)c(z)]d\mu + c(z) = -c(z) + c(z) = 0 \in F(z).$$

Moreover the following holds true:

[10] Indeed, the compensation condition implies that for all t and z, there exists $g \in R^{\ell}_{+}$ such that $u_t(g, z) \geq u_t(f(t), y)$. Then, define $\tilde{g} = g + 1_\ell$, where $1_\ell = (1, \ldots, 1) \in R^{\ell}_{+}$. From monotonicity it follows that $u_t(\tilde{g}, z) > u_t(f(t), y)$, that is $\tilde{g} \in \Psi(t, z)$.

[11] If we consider selection of a multifunction a function whose values belong to the nonempty values of the multifunction.

Claim 3. *$F(z) \cap Int(I\!R^\ell_-) = \emptyset$ for all $z \in \mathcal{Y}$.*

Suppose, on the contrary, that there exist $z \in \mathcal{Y}$ and $h \in F(z) \cap Int(I\!R^\ell_-)$. Take, for all $t \in T$, a suitable $g(t) \in F(t,z)$ such that $g(\cdot)$ is integrable and $h = \int_T g(t)\, d\mu + c(z)$. Define

$$S = \{t \in T \,:\, g(t) = -\varrho_\sigma(t,z)c(z)\}\,,$$

and observe that if $t \in T \setminus S$, then $g(t) \in \Psi(t,z) - \{e(t)\}$, that is there exists $\tilde{g}(t)$ such that $u_t(\tilde{g}(t),z) > u_t(f(t),y)$ and $g(t) = \tilde{g}(t) - e(t)$. It is easy to check that $\mu(T \setminus S) > 0$. For if not, then $g(t) = -\varrho_\sigma(t,z)c(z)$ for almost all $t \in T$, and hence $h = \int_T [-\varrho_\sigma(t,z)c(z)]d\mu + c(z) = 0$, which is a contradiction since $h \in Int I\!R^\ell_-$. Furthermore,

$$\begin{aligned} 0 > h &= \int_T g(t)d\mu + c(z) = \int_S [-\varrho_\sigma(t,z)c(z)]d\mu + \int_{T\setminus S} [\tilde{g}(t) - e(t)]d\mu + c(z) \\ &= -\sigma(S,z)c(z) + \int_{T\setminus S} \tilde{g}(t)d\mu - \int_{T\setminus S} e(t)td\mu + \sigma(T,z)c(z) \\ &= \int_{T\setminus S} \tilde{g}(t)d\mu + \sigma(T\setminus S,z)c(z) - \int_{T\setminus S} e(t)d\mu. \end{aligned}$$

Hence $u_t(\tilde{g}(t),z) > u_t(f(t),y)$ for almost all $t \in T \setminus S$ and

$$\int_{T\setminus S} \tilde{g}(t)d\mu + \sigma(T\setminus S,z)c(z) < \int_{T\setminus S} e(t)d\mu.$$

It is easy to show that $T \setminus S$ σ-blocks (f,y) via $(\bar{g},z)$. This completes the proof the Claim 3 since it contradicts the hypothesis that (f,y) is a σ-core allocation.

From Liapunov convexity theorem, it follows that for all $z \in \mathcal{Y}$, the set $F(z)$ is convex. Therefore, by Minkowski's separating hyperplane theorem, for each $z \in \mathcal{Y}$ there exists $p(z) \in \Delta$ such that $p(z) \cdot F(z) \geq 0$. Due to the fact that $0 \in F(z)$, this implies that $\inf\{p(z) \cdot h \,|\, h \in F(z)\} = 0$ that is

$$\inf \left\{ p(z) \cdot \int_T F(t,z)d\mu + p(z) \cdot c(z) \right\} = 0. \tag{3}$$

Since for all $t \in T$, $-\varrho_\sigma(t,z)c(z) \in F(t,z)$, it follows that

$$\inf p(z) \cdot F(t,z) \leq -\varrho_\sigma(t,z)p(z) \cdot c(z). \tag{4}$$

Therefore from (3) and (4) it follows that for almost all[12] $t \in T$

$$\inf p(z) \cdot F(t,z) + \varrho_\sigma(t,z)p(z) \cdot c(z) = 0.$$

[12] Let $n(t)$ be $\inf p(z) \cdot F(t,z) + \varrho_\sigma(t,z)p(z) \cdot c(z)$. Condition (3) implies that $\int_T n(t)d\mu = 0$, while (4) implies that $n(t) \leq 0$ for all $t \in T$. Hence $n(t) = 0$ for almost all $t \in T$ (see Claim 2 in [5] and Proposition 6 in [13] p.63).

This means that for almost all $t \in T$, for all $z \in \mathcal{Y}$ and $v \in I\!R^{\ell}_{+}$, we have that if $u_t(v,z) > u_t(f(t),y)$, then $p(z)\cdot v + \varrho_\sigma(t,z)p(z)\cdot c(z) \geq p(z)\cdot e(t)$.

Moreover for almost all $t \in T$, $p(y)\cdot f(t) + \varrho_\sigma(t,y)p(y)\cdot c(y) = p(y)\cdot e(t)$.

Indeed, by continuity, $p(y)\cdot f(t) + \varrho_\sigma(t,y)p(y)\cdot c(y) \geq p(y)\cdot e(t)$ for almost all $t \in T$. Assume that there exists a coalition $S \in \mathcal{T}$ such that for almost all $t \in S$, $p(y)\cdot f(t) + \varrho_\sigma(t,y)p(y)\cdot c(y) > p(y)\cdot e(t)$ and for almost all $t \in T\setminus S$, $p(y)\cdot f(t) + \varrho_\sigma(t,y)p(y)\cdot c(y) = p(y)\cdot e(t)$. Thus,

$$p(y)\cdot\left[\int_T f(t)\,d\mu + c(y) - \int_T e(t)\,d\mu\right] = p(y)\cdot\left[\int_S f(t)\,d\mu + \sigma(S,y)c(y) - \int_S e(t)\,d\mu\right] +$$

$$+p(y)\cdot\left[\int_{T\setminus S} f(t)\,d\mu + \sigma(T\setminus S,y)c(y) - \int_{T\setminus S} e(t)\,d\mu\right] > 0,$$

which contradicts the feasibility of (f,y). Therefore, for almost all $t \in T$, $p(y)\cdot f(t) + \varrho_\sigma(t,y)p(y)\cdot c(y) = p(y)\cdot e(t)$.

Define $B = \{t \in T \,:\, \varrho_\sigma(t,z)p(z)\cdot c(z) < p(z)\cdot e(t)\}$ and notice that $\mu(B) > 0$, since otherwise $p(z)\cdot c(z) \geq p(z)\cdot \int_T e(t)d\mu$ and, since $p(z) > 0$, we contradict the assumption that for any $z \in \mathcal{Y}$, $c(z) \ll \int_T e(t)d\mu$. Moreover, for all z and $t \in B$, all $v \in I\!R^{\ell}_{+}$, if $u_t(v,z) > u_t(f(t),y)$, then $p(z)\cdot v + \varrho_\sigma(t,z)p(z)\cdot c(z) \geq p(z)\cdot e(t) > \varrho_\sigma(t,z)p(z)\cdot c(z)$, and hence $p(z)\cdot v > 0$. Since utility functions are continuous, there exists $\epsilon \in (0,1)$ such that if $u_t(v,z) > u_t(f(t),y)$, then $u_t(\epsilon v,z) > u_t(f(t),y)$. Hence,

$$p(z)\cdot e(t) \leq p(z)\cdot \epsilon v + \varrho_\sigma(t,z)p(z)\cdot c(z) < p(z)\cdot v + \varrho_\sigma(t,z)p(z)\cdot c(z).$$

Thus for all $t \in B$ and $z \in \mathcal{Y}$ if $u_t(v,z) > u_t(f(t),y)$, then $p(z)\cdot v + \varrho_\sigma(t,z)p(z)\cdot c(z) > p(z)\cdot e(t)$. This implies that for any $t \in B$ and $z \in \mathcal{Y}$, there exists $\bar{f}$ such that $(\bar{f}(t),z)$ maximizes $u_t(\cdot,z)$ in[13]

$$B_t(p,z) = \{g \in I\!R^{\ell}_{+} \,:\, p(z)\cdot g + \varrho_\sigma(t,z)p(z)\cdot c(z) \leq p(z)\cdot e(t)\}.$$

Claim 4. *For any $z \in \mathcal{Y}$, $p(z) \gg 0$.*

Let t be in B and assume on the contrary that there exists $z \in \mathcal{Y}$ such that $H(z) := \{k \in \{1,\ldots,\ell\} \,:\, p^k(z) = 0\}$ is non empty. Define

$$\tilde{g} = \begin{cases} \bar{f}^k & \text{if } k \notin H(z) \\ \bar{f}^k + \delta & \text{otherwise,} \end{cases}$$

[13]Notice that for all $t \in B$, $B_t(p,z)$ is non empty. Indeed for all $t \in B$, $\epsilon(t) = p(z)\cdot e(t) - \varrho_\sigma(t,z)p(z)\cdot c(z) > 0$ and $p(z) > 0$, then put $g = \left(\frac{\epsilon(t)}{\sum_{k=1}^{\ell} p^k(z)},\ldots,\frac{\epsilon(t)}{\sum_{k=1}^{\ell} p^k(z)}\right) \in I\!R^{\ell}_{+}$ and notice that $g \in B_t(p,z) \neq \emptyset$.

where $\delta > 0$ and notice that $u_t(\tilde{g}(t), z) > u_t(\bar{f}(t), z)$ and

$$\begin{aligned} p(z) \cdot \tilde{g} + \varrho_\sigma(t, z) p(z) \cdot c(z) &= \sum_{k \notin H(z)} p^k(z) \bar{f}^k(t) + \varrho_\sigma(t, z) p(z) \cdot c(z) \\ &= p(z) \cdot \bar{f}(t) + \varrho_\sigma(t, z) p(z) \cdot c(z) \leq p(z) \cdot e(t). \end{aligned}$$

This is a contradiction since $\bar{f}$ is maximal. Therefore for any $z \in \mathcal{Y}$, $p(z) \gg 0$. We already know that for all $t \in T$ if $u_t(g, z) > u_t(f(t), y)$, then $p(z) \cdot g + \varrho_\sigma(t, z) p(z) \cdot c(z) \geq p(z) \cdot e(t)$. We just need to show that $p(z) \cdot g + \varrho_\sigma(t, z) p(z) \cdot c(z) > p(z) \cdot e(t)$. Assume, on the contrary that for some $t \in T$, $p(z) \cdot g + \varrho_\sigma(t, z) p(z) \cdot c(z) = p(z) \cdot e(t)$, thus by continuity it follows that there exists $\epsilon \in (0, 1)$ such that $u_t(\epsilon g, z) > u_t(f(t), y)$. Moreover, since $p(z) \gg 0$ and assumption $(A.4)$ holds, then

$$p(z) \cdot e(t) \leq \epsilon p(z) \cdot g + \varrho_\sigma p(z) \cdot c(z) < p(z) \cdot g + \varrho_\sigma(t, z) p(z) \cdot c(z) = p(z) \cdot e(t),$$

which is a contradiction. Therefore for all $t \in T$, if $u_t(g, z) > u_t(f(t), y)$, then $p(z) \cdot g + \varrho_\sigma(t, z) p(z) \cdot c(z) > p(z) \cdot e(t)$, that is (f, y) is a linear cost share equilibrium. □

The next result reverses proposition 5

Proposition 7. *Let $\mathcal{E}$ be a mixed market with public goods that satisfies the hypotheses $(A.1) - -(A.5)$. Let $\sigma \in M_\mu$ be a contribution measure and let ϱ_σ be the corresponding cost distribution function for $\mathcal{E}$. Moreover, assume that there are at least two atoms and that all atoms are of the same type and also the same as taxpayers under ϱ_σ. Then*

$$(f, y) \in C_\sigma(\mathcal{E}) \Rightarrow (f^*, y) \in C_{\sigma^*}(\mathcal{E}^*).$$

To prove the above proposition some lemmas are useful. First we give a straightforward extension of Schmeidler's theorem to economies with public projects.

Lemma 1. *Let $\mathcal{E}$ be an atomless economy with public goods (i.e. $T_1 = \emptyset$ and $T = T_0$). Let σ be a contribution measure for $\mathcal{E}$ and S be a coalition which σ-blocks the allocation (f, y) via (g, z). Suppose that $\mu(S \cap P) \geq \alpha > 0$ for a coalition P. Then, there exists a coalition $R \subseteq S$ that σ-blocks (f, y) via (g, z) and with $\mu(R \cap P) = \alpha$.*

PROOF: If S σ-blocks (f, y) via (g, z), so that:

$$\begin{array}{ll} (i) & u_t(g(t), z) > u_t(f(t), y) \quad \text{for almost all } t \in S \\ (ii) & \displaystyle\int_S g(t)\, d\mu + \sigma(S, z) c(z) = \int_S e(t)\, d\mu, \end{array}$$

define the vector measure ν on the subsets of S which are in $\mathcal{T}$ by setting

$$\nu(\cdot) = \left(\int_{(\cdot)} e(t)\, d\mu\, ; \int_{(\cdot)} g(t)\, d\mu\, ;\, \mu(\cdot \cap P);\, \sigma(\cdot, z) \right).$$

Since $\frac{\alpha}{\mu(S \cap P)} \leq 1$, then, by Liapunov Theorem, there exists a coalition $R \subseteq S$ such that

$$\nu(R) = \frac{\alpha}{\mu(S \cap P)} \nu(S).$$

The coalition R σ-blocks (f, y) via (g, z) and it is such that $\mu(R \cap P) = \alpha$. $\square$

Now we extend Vind's theorem to economies with public projects.

Lemma 2. *Let $\mathcal{E}$ be an atomless economy with public goods which satisfies the hypotheses $(A.1) - -(A.5)$. Given a contribution measure σ, if (f, y) is not a σ-core allocation, then for any $0 < \alpha < \mu(T)$ there exists a coalition $S \in \mathcal{T}$ with $\mu(S) = \alpha$ which σ-blocks (f, y).*

PROOF: Since (f, y) is not a σ-core allocation, there exist a coalition S and an allocation $(\overline{g}, z)$ such that

$$(i) \qquad u_t(\overline{g}(t), z) > u_t(f(t), y) \quad \text{for almost all } t \in S$$
$$(ii) \qquad \int_S \overline{g}(t)\, d\mu + \sigma(S, z)c(z) = \int_S e(t)\, d\mu.$$

If $\alpha \in]0, \mu(S)]$, the previous lemma applies. Therefore let us suppose $\alpha > \mu(S)$.

Since utilities are continuous, there exist $\epsilon > 0$ and $B \subseteq S$, with $\mu(B) > 0$ such that $u_t(\epsilon \overline{g}(t), z) > u_t(f(t), y)$ for almost all $t \in B$. Define $g(t) = \epsilon \overline{g}(t)\chi_B + \overline{g}(t)\chi_{S \setminus B}$, and notice that $u_t(g(t), z) > u_t(f(t), y)$ for almost all $t \in S$. Moreover, thanks to the second essentially condition,

$$\int_S g(t)\, d\mu + \sigma(S, z)c(z) - \int_S e(t)\, d\mu = -\gamma < 0.$$

As we have seen, the joint action of assumptions (A.3) and (A.5) leads to the existence of a distribution of private goods $\bar{h} : T \to I\!R^{\ell}_{+}$ such that $(\bar{h}, z)$ is feasible and $u_t(\bar{h}(t), z) \geq u_t(f(t), y)$ for all $t \in T$.

For any $\delta \in (0, 1)$, concavity assumption guarantees that for almost all $t \in S$, $u_t(\delta g(t) + (1 - \delta)\bar{h}(t), z) > u_t(f(t), y)$. Define the vector measure

$$\lambda(I) = \left(\int_I \bar{h}(t)\, d\mu,\, \int_I e(t)\, d\mu,\, \mu(I),\, \sigma(I, z) \right),$$

on subsets I of $T \setminus S$ which are members of $\mathcal{T}$. Since $\delta \in (0, 1)$, there exists $S_\delta \subseteq T \setminus S$ such that $\lambda(S_\delta) = (1 - \delta)\lambda(T \setminus S)$. Define the following distribution of private goods:

$$\hat{g}(t) = \begin{cases} \delta g(t) + (1-\delta)\bar{h}(t) & \text{if } t \in S \\ \bar{h}(t) + \frac{\delta\,\gamma}{\mu(S_\delta)} & \text{if } t \in S_\delta \end{cases}$$

By monotonicity, $u_t(\hat{g}(t), z) > u_t(f(t), y)$ for almost all $t \in S \cup S_\delta$. Moreover

$$\int_{S\cup S_\delta} \hat{g}(t)\, d\mu + \sigma(S\cup S_\delta, z)c(z) - \int_{S\cup S_\delta} e(t)\, d\mu =$$

$$= \delta \int_S g(t)\, d\mu + (1-\delta)\int_S \bar{h}(t)\, d\mu + \int_{S_\delta} \bar{h}(t)\, d\mu + \delta\gamma + [\sigma(S,z) + \sigma(S_\delta, z)]c(z) +$$

$$- \int_S e(t)\, d\mu - \int_{S_\delta} e(t)\, d\mu =$$

$$= -\delta\gamma + (1-\delta)\sigma(S,z)c(z) - (1-\delta)\int_S e(t)\, d\mu + (1-\delta)\int_S \bar{h}(t)\, d\mu +$$

$$+(1-\delta)\int_{T\setminus S} \bar{h}(t)\, d\mu + \delta\,\gamma + (1-\delta)\sigma(T\setminus S, z)c(z) - (1-\delta)\int_{T\setminus S} e(t)\, d\mu =$$

$$= (1-\delta)\sigma(T,z)c(z) + (1-\delta)\int_T \bar{h}(t)\, d\mu - (1-\delta)\int_T e(t)\, d\mu =$$

$$= (1-\delta)\left[\int_T \bar{h}(t)\, d\mu + c(z) - \int_T e(t)\, d\mu\right] \leq 0.$$

Notice that $\mu(S \cup S_\delta) = \mu(S) + (1-\delta)\mu(T \setminus S)$.

Let $\epsilon = \int_{S\cup S_\delta} e(t)\, d\mu - \int_{S\cup S_\delta} \hat{g}(t)\, d\mu - \sigma(S\cup S_\delta, z)c(z) \geq 0$ and define

$$k(t) = \hat{g}(t) + \frac{\epsilon}{\mu(S\cup S_\delta)}.$$

By arbitrariness of δ, we have constructed a coalition of arbitrary measure that σ-blocks (f, y) via (k, z). In particular we can set δ in such a way that $\mu(S \cup S_\delta) = \alpha$. □

The proof above shows that a slightly more general formulation can be given to the lemma.

Lemma 3. *Let $\mathcal{E}$ be an atomless economy with public goods which satisfies the hypotheses $(A.1), (A.2)$ and $(A.4)$. Let σ be a contribution measure for $\mathcal{E}$. Let (f, y) be an allocation such that, for any public project z, the correspondence $P(t, y, z, f(t))$ admits a selection $\bar{h}$ such that $(\bar{h}, z)$ is a feasible. If (f, y) is not a σ-core allocation, then for any $0 < \alpha < \mu(T)$ there exists a coalition $S \in \mathcal{T}$ with $\mu(S) = \alpha$ which σ-blocks (f, y).*

We give a counterexample to the validity of Vind's theorem for the σ-core of our model of cost sharing with a general cost function $\varrho(t, y)$.

Example 2. Consider an atomless economy with public goods with $T = [0, 1]$, two commodities, the set of public projects $\mathcal{Y} = \{y, z\}$, the initial endowment density uniform over agents and equal to $e_t \equiv (1, 1)$, the cost function defined by $c(y) = (\frac{1}{2}, \frac{1}{2})$ and $c(z) = (\frac{3}{4}, \frac{3}{4})$, utility functions defined by $u_t(h_1, h_2, y) = u_t(h_1, h_2, z) = h_1$ for each $t \in T$, the cost distribution function equal to $\varrho(t, z) = 0.4$ for each $t \in [0, \frac{1}{2}]$, $\varrho(t, z) = 2.4$ for each $t \in (\frac{1}{2}, \frac{3}{4}]$ and $\varrho(t, z) = 0.8$ for each $t \in (\frac{3}{4}, 1]$ while $\varrho(t, y) = 1$ for each $t \in T$. The allocation (f, y), defined by $f(t) = (\frac{1}{2}, \frac{1}{2})$ over T, is clearly feasible. It does not belong to the σ_ϱ-core since it is blocked by the coalition $[0, \frac{1}{2}]$ via the allocation (g, z) defined by $g(t) = (0.7, 0.7)$. Indeed, it results

$$u_t(g(t), z) = 0.7 > 0.5 = u_t(f(t), y), \qquad \text{for each } t \in [0, \frac{1}{2}]$$

and

$$\int_0^{\frac{1}{2}} g(t) d\mu + \int_0^{\frac{1}{2}} \varrho(t, z) c(z) d\mu = (0.5, 0.5) = \int_0^{\frac{1}{2}} e(t) d\mu.$$

Observe now that (f, y) cannot be blocked by a coalition of measure $\frac{7}{8}$. Indeed, assuming that a coalition S exists such that $\mu(S) = \frac{7}{8}$ and S σ-blocks (f, y) by means of a suitable allocation (g, z), then the following contradiction arises.

Denote by S_1, S_2 and S_3 the intersections of such coalition with $[0, \frac{1}{2}]$, $(\frac{1}{2}, \frac{3}{4}]$ and $(\frac{3}{4}, 1]$ respectively. Then $\mu(S_i) \geq \frac{1}{8}$ for $i = 1, 2, 3$. From

$$u_t(g(t), z) = g_1(t) > 0.5 = u_t(f(t), y), \qquad \text{for each } t \in S$$

and

$$\int_S g(t) d\mu + \int_S \varrho(t, z) c(z) d\mu = \int_S e(t) d\mu$$

it follows, along the first coordinate, that the equality becomes

$$\mu(S) = \int_S g_1(t) d\mu + \frac{3}{4} \int_S \varrho(t, z) d\mu$$

that means

$$0.2\mu(S_1) - 1.3\mu(S_2) - 0.1\mu(S_3) > 0$$

and, considering that S_2 and S_3 measures at least 1/8, we get that $\mu(S_1) > \frac{7}{8}$. Too much! On the other hand the allocation (f, y) can't be blocked by a coalition S under the same project y by construction. Hence the conclusion follows.

Observe also that in this example the existence of an allocation under the project z which is feasible and ensures at least the same levels of utilities given by (f, y) (i.e. $u_t(g(t), z) = (g(t))_1 \geq 0.5 = u_t(f(t), y)$ would imply, due to feasibility, the contradiction $1 \geq \frac{1}{2} + \frac{3}{4}$.

We are now ready to prove proposition 7. Then the equivalence between the σ-core and the set of linear cost share equilibria in a mixed market with public goods, stated below, is just an application of proposition 2, proposition 7, theorem 2 and proposition 6.

Theorem 3. *Let $\mathcal{E}$ be a mixed market with public goods that satisfies the hypotheses $(A.1)--(A.5)$. Let $\sigma \in M_\mu$ be a contribution measure and let ϱ_σ be the corresponding cost distribution function for $\mathcal{E}$. Moreover, assume that there are at least two atoms and that all atoms are of the same type and also the same as taxpayers under ϱ_σ. Then $C_\sigma(\mathcal{E}) = LCE_{\varrho_\sigma}(\mathcal{E})$.*

PROOF OF PROPOSITION 7: Let the allocation (f, y) be in the σ-core. Apply proposition 3 for $A = T_1 = \{A_1, A_2, ...\}$. Since there are at least two elements in T_1, if B is any of the atoms, then $\mu(T) > \mu(T_0) + \mu(\{B\})$ and $u_{A_i}(f(A_i), y) = u_B(f(B), y)$ for any i. Note also that uniqueness of type for atoms says that on T_1^* we have $u_t = u_B \ \forall t$.

An appeal to lemma 3, for the economy $\mathcal{E}^*$ under the assumption that $(f^*, y) \notin C_{\sigma^*}(\mathcal{E}^*)$ [14], says that a coalition $R \subseteq T^*$ exists such that $\mu^*(R) = \mu(T_0) + \mu(\{B\})$ and such that it σ^*-blocks (f^*, y) via an allocation (φ, z). Obviously $\mu^*(R \cap T_1^*) \geq \mu(\{B\})$ and lemma 1 gives us the possibility to assume that $\mu^*(R \cap T_1^*) = \mu(\{B\})$. Now we show that we violate the assumption that (f, y) is in the σ-core. Indeed, let S be the coalition $S = R \cap T_0 \cup \{B\}$ and g the distribution of private goods equal to φ on $R \cap T_0$ and in B equal to the average value of φ over $R \cap T_1^*$: $g(B) = \dfrac{1}{\mu(\{B\})} \displaystyle\int_{R \cap T_1^*} \varphi \, d\mu^*$.

On S we have $u_t(g(t), z) > u_t(f(t), y)$. This is clear for $t \in R \cap T_0$. For $t = B$ we have $u_B(g(B), z) \geq \dfrac{1}{\mu(\{B\})} \displaystyle\int_{R \cap T_1^*} u_B(\varphi(t), z) \, d\mu^*$ for concavity. Then for $t \in R \cap T_1^*$:
from $u_B(\varphi(t), z) = u_t(\varphi(t), z)$ we get

$$u_B(g(B), z) \geq \frac{1}{\mu(\{B\})} \int_{R \cap T_1^*} u_t(\varphi(t), z) \, d\mu^*,$$

from $u_t(\varphi(t), z) > u_t(f^*(t), y)$ we get

$$u_B(g(B), z) > \frac{1}{\mu(\{B\})} \int_{R \cap T_1^*} u_t(f^*(t), y) \, d\mu^*,$$

and from $u_t(f^*(t), y) = u_B(f(B), y)$ we get

$$u_B(g(B), z) > u_B(f(B), y)$$

[14] Assumptions (A.1), (A.2) and (A.4) are automatically inherited by $\mathcal{E}^*$. As of the existence of feasible selections of $P(t, y, z, f^*(t))$, note that because of (A.3) and (A.5), on T the correspondence $P(t, y, z, f(t))$ has a feasible selection, say (g, z). As a consequence (g^*, z) is a feasible allocation such that on T^* the function g^* is a selection of $P(t, y, z, f^*(t))$.

as declared.

For simplicity just denote by ϱ the cost distribution function associated to σ, we know that ϱ^* is then associated to σ^*. Since all atoms are the same as taxpayers, we have that $\varrho^*(t,\cdot) = \varrho(B,\cdot)$ for all $t \in T_1^*$. Hence, starting from equation $\displaystyle\int_R \varphi \, d\mu^* + \sigma^*(R,z)c(z) = \int_R e^* \, d\mu^*$, we get a contradiction by computing

$$
\begin{array}{rcl}
\displaystyle\int_S g(t)\, d\mu + \sigma(S,z)c(z) & = & \displaystyle\int_{R\cap T_0} \varphi(t)\, d\mu + \\
g(B)\mu(B) + \sigma(R\cap T_0, z)c(z) + \sigma(\{B\},z)c(z) & = & \displaystyle\int_{R\cap T_0} \varphi(t)\, d\mu^* + \int_{R\cap T_1^*} \varphi(t)\, d\mu^* \\
+\sigma^*(R\cap T_0,z)c(z) + \mu(\{B\})\, \varrho(B,z)c(z) & = & \displaystyle\int_R \varphi(t)\, d\mu^* \\
+\sigma^*(R\cap T_0,z)c(z) + \displaystyle\int_{R\cap T_1^*} \varrho^*(t,z)\, d\mu^* c(z) & = & \displaystyle\int_R \varphi(t)\, d\mu^* \\
+\sigma^*(R\cap T_0,z)c(z) + \sigma^*(R\cap T_1^*,z)c(z) & = & \displaystyle\int_R \varphi(t)\, d\mu^* + \sigma^*(R,z)c(z) = \\
\displaystyle\int_R e^*(t)\, d\mu^* & = & \displaystyle\int_{R\cap T_0} e(t)\, d\mu^* + \int_{R\cap T_1^*} e^*(t)\, d\mu^* \\
\displaystyle\int_{R\cap T_0} e(t)\, d\mu + \mu(\{B\})e(B) & = & \displaystyle\int_S e(t)\, d\mu.
\end{array}
$$

□

5 Some Counterexamples

Here, we underline the importance of hypotheses made on atoms in order to get the Equivalence Theorem in mixed markets with public projects. We present two examples. The first shows that if we assume the existence of an unique atom, then the Equivalence Theorem is not valid anymore. The second shows the failure of the Equivalence theorem in oligopolistic markets if we admits that atoms may be different from each other.

Example 3. Consider a mixed market with only one atom A with $\mu(\{A\}) = \frac{1}{2}$. For example, let the measure space of agents be the direct sum of the ordinary interval $T_0 = [0, \frac{1}{2}[$ and the point $A = \frac{1}{2}$ on which we concentrate a weight equal to one half. Assume two private goods and two public projects y, z with costs $c(y) = (1,1)$ and $c(z) = (0,1)$. Assume that $e(t) = (5,1)$ for $t < \frac{1}{2}$ and $e(A) = (1,5)$. The utility functions are the same for all agents, that is for all $t \in T, h \in \mathbb{R}^2_+$, we have $u_t(h,y) = \sqrt{h_1} + \sqrt{h_2}$ and $u_t(h,z) = \sqrt{h_1} + \sqrt{h_2} - 2$. Consider the cost distribution function $\varrho = 1$

constantly. The associated contribution measure σ_ϱ obviously coincide with μ independently on the public project y or z.

We first notice that the allocation (f, y) where $f(t) = (2, 2)$ for all $t \in T$ is the unique ϱ-linear cost share equilibrium. An equilibrium price system necessarily entails $p(y) = (\frac{1}{2}, \frac{1}{2})$.

Indeed, for agent A and public good y, we need to solve max $\sqrt{f_1(A)} + \sqrt{f_1(A)}$ subject to

$$p_1(y)f_1(A) + p_2(y)f_2(A) + p_1(y) + p_2(y) = p_1(y) + 5p_2(y).$$

The solution is $f_1(A) = \frac{4[p_2(y)]^2}{p_1(y)}$ and $f_2(A) = 4p_1(y)$.
For any $t \in T_0$ and y, we need to solve max $\sqrt{f_1(t)} + \sqrt{f_1(t)}$ subject to

$$p_1(y)f_1(t) + p_2(y)f_2(t) + p_1(y) + p_2(y) = 5p_1(y) + p_2(y).$$

The solution is $f_1(t) = 4p_2(y)$ and $f_2(t) = \frac{4[p_1(y)]^2}{p_2(y)}$.
Imposing the feasibility condition we get that $p(y) = (\frac{1}{2}, \frac{1}{2})$ and hence $f(t) = (2, 2)$ for all $t \in T$.

On the other hand, for the agent A and public good z, we have to solve max $\sqrt{f_1(A)} + \sqrt{f_1(A)} - 2$ subject to

$$p_1(z)f_1(A) + p_2(z)f_2(A) + p_2(z) = p_1(z) + 5p_2(z).$$

The solution is $f_1(A) = \frac{p_2(z)[p_1(z)+4p_2(z)]}{p_1(z)}$ and $f_2(A) = \frac{p_1(z)[p_1(z)+4p_2(z)]}{p_2(z)}$.
For any $t \in T_0$ and z, we need to solve max $\sqrt{f_1(t)} + \sqrt{f_1(t)} - 2$ subject to

$$p_1(z)f_1(t) + p_2(z)f_2(t) + p_2(z) = 5p_1(z) + p_2(z).$$

The solution is $f_1(t) = 5p_2(z)$ and $f_2(t) = \frac{5[p_1(z)]^2}{p_2(z)}$.
Imposing the feasibility condition we get that $p(z) = \left(\sqrt{6} - 2, 3 - \sqrt{6}\right)$ and hence $f(t) = \left(5(3 - \sqrt{6}), \frac{10(3-\sqrt{6})}{3}\right)$ for all $t \in T_0$ and $f(A) = \left(5\sqrt{6} - 9, \frac{10\sqrt{6}-18}{3}\right)$.

Notice that for all $t \in T_0$, $u_t((2, 2), y) > u_t\left(\left(5(3 - \sqrt{6}), \frac{10(3-\sqrt{6})}{3}\right), z\right)$ and $u_A((2, 2), y) > u_A\left(\left(5\sqrt{6} - 9, \frac{10\sqrt{6}-18}{3}\right)), z\right)$. Therefore, (f, y), with $f(t) = (2, 2)$ for all $t \in T$ is the unique ϱ-linear cost share equilibrium.

On the other hand the feasible allocation (h, y) with $h(t) = (1, 1)$ for all $t < \frac{1}{2}$ and $h(A) = (3, 3)$ is a μ-core allocation. Indeed, assume, on the contrary, that there exist a coalition S, a public project x and a distribution of private goods g such that

$$(i) \qquad u_t(g(t), x) > u_t(h(t), y) \qquad \text{for almost all } t \in S$$

$$(ii) \qquad \int_S g(t)\, d\mu + \mu(S)c(x) = \int_S e(t)\, d\mu.$$

The project x may be either y or z, in any case we get a contradiction. If $x = y$ we have

$$\begin{cases} \sqrt{g_1(t)} + \sqrt{g_2(t)} > 2 \text{ for almost all } t \in S,\ t < \frac{1}{2} \\ \sqrt{g_1(A)} + \sqrt{g_2(A)} > 2\sqrt{3} \text{ if } A \in S \\ \int_{S\setminus\{A\}} g_1(t)d\mu + \mu(S \cap \{A\})g_1(A) = 4\mu(S \setminus \{A\}) \\ \int_{S\setminus\{A\}} g_2(t)d\mu + \mu(S \cap \{A\})g_2(A) = 4\mu(S \cap \{A\}) \end{cases}$$

and see that both components $S \setminus \{A\}$ and $S \cap \{A\}$ of S must have positive measure. In particular the coalition S contains the agent A.

Since from $\sqrt{g_1(t)}+\sqrt{g_2(t)} > 2$ it follows that $g_1(t)+g_2(t) > 2$ and from $\sqrt{g_1(A)} + \sqrt{g_2(A)} > 2\sqrt{3}$ it follows that $g_1(A) + g_2(A) > 6$, by summing the last two conditions in the above system we get $\mu(S \setminus \{A\}) > \frac{1}{2}$, which is a contradiction.

If $x = z$ we have

$$\begin{cases} \sqrt{g_1(t)} + \sqrt{g_2(t)} - 2 > 2 \text{ for almost all } t \in S,\ t < \frac{1}{2} \\ \sqrt{g_1(A)} + \sqrt{g_2(A)} - 2 > 2\sqrt{3} \text{ if } A \in S \\ \int_{S\setminus\{A\}} g_1(t)d\mu + \mu(S \cap \{A\})g_1(A) = 5\mu(S \setminus \{A\}) + \mu(S \cap \{A\}) \\ \int_{S\setminus\{A\}} g_2(t)d\mu + \mu(S \cap \{A\})g_2(A) = 4\mu(S \cap \{A\}) \end{cases}$$

and again see that the coalition S contains the agent A.

Since from $\sqrt{g_1(t)} + \sqrt{g_2(t)} - 2 > 2$ it follows that $g_1(t) + g_2(t) > 8$, we have, by summing the last two conditions in the above system, that

$$5\mu(S\setminus\{A\})+\frac{5}{2} = \int_{S\setminus\{A\}} (g_1(t) + g_2(t))\, d\mu+\frac{g_1(A) + g_2(A)}{2} > 8\mu(S\setminus\{A\})+\frac{g_1(A) + g_2(A)}{2}$$

and therefore

$$5 > 6\mu(S \setminus \{A\}) + g_1(A) + g_2(A).$$

Now, if $g_1(A) \leq 1$, then $\sqrt{g_2(A)} > 1 + 2\sqrt{3}$, that is $g_2(A) > 13 + 4\sqrt{3}$, which is impossible.

If $g_1(A) > 1$, then $1 + g_2(A) < g_1(A) + g_2(A) < 5$, so $g_2(A) < 4$ and $\sqrt{g_1(A)} + \sqrt{g_2(A)} - 2 < \sqrt{g_1(A)}$. Finally $12 < g_1(A)$ which is also impossible.

Therefore, (h, y) is a μ-core allocation but it is not a ϱ-linear cost share equilibrium.

Example 4. Consider a mixed market with two different atoms B and A with $\mu(\{B\}) = \frac{1}{4}$ and $\mu(\{A\}) = \frac{1}{2}$ just by making the following modification to the previous example: $[\frac{1}{4}, \frac{1}{2}[$ is now an atom B (to be identified with $\frac{1}{4}$) of weight one quarter and T_0 is now the ordinary interval $[0, \frac{1}{4}[$. So agents are now points in $T = [0, \frac{1}{4}[\cup\{\frac{1}{4}, \frac{1}{2}\}$. All the rest is unchanged.

Since budget sets result the same as in example 3, again the allocation (f, y) where $f(t) = (2, 2)$ for all $t \in T$ is the unique ϱ-linear cost share equilibrium and an equilibrium price system necessarily entails $p(y) = (\frac{1}{2}, \frac{1}{2})$.

On the other hand the same feasible allocation (h, y) of example 3, i.e. with $h(t) = (1, 1)$ for all $t \in T_0 \cup B$ and $h(A) = (3, 3)$ is a μ-core allocation. For that assertion, the argument is exactly the same as in example 3.

Example 5. Consider a mixed market with two atoms A_1 and A_2, $T = A_1 \cup A_2$, with $\mu(A_1) = \mu(A_2) = \frac{1}{2}$, two private goods and two public projects $\mathcal{Y} = \{y, z\}$. Assume that $e(A) = (3, 0)$ and $e(A_2) = (0, 3)$. Notice that atoms A_1 and A_2 are not of the same type, since they have different initial endowment. The utility functions are defined as follows: $u_A(f, y) = u_B(f, z) = 3\sqrt{f_1(t)} + \sqrt{f_2(t)}$, $u_A(g, z) = \sqrt{g_1(t)} + \sqrt{g_2(t)}$ and $u_B(g, z) = log g_1(B) + g_2(B)$. Moreover, assume that $c(y) = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $c(z) = \left(\frac{1}{2}, 0\right)$. Consider the following distribution cost function $\varrho(A, \cdot) = \varrho(B, \cdot) = 1$ and notice that $\int_t \varrho(t, \cdot)\, d\mu = 1$. The associated contribution measure σ_ϱ is such that for each coalition S, $\sigma(S, \cdot) = \mu(S)$.

With the same arguments used in the previous example, it is easy to prove that the allocation (f, y) where $f(A) = \left(\frac{7}{4}, \frac{7}{4}\right)$ and $f(B) = \left(\frac{1}{4}, \frac{1}{4}\right)$ is the unique ϱ-linear cost share equilibrium whose price is given by $p(y) = (3p, p)$. On the other have the allocation (h, y) with $h(A) = h(B) = (1, 1)$ is a σ_ϱ-core allocation. Indeed, if on the contrary there exist a coalition S and a distribution of private goods g such that

$$\begin{array}{ll} (i) & u_t(g(t), y) > u_t(h(t), y) \qquad \text{for all } t \in S \\ (ii) & \displaystyle\int_S g_1(t)\, d\mu + \sigma(S, y) c_1(y) = \int_S e_1(t)\, d\mu \\ (iii) & \displaystyle\int_S g_2(t)\, d\mu + \sigma(S, y) c_2(y) = \int_S e_2(t)\, d\mu, \end{array}$$

or such that

$$\begin{array}{ll} (i) & u_t(g(t), z) > u_t(h(t), y) \qquad \text{for almost all } t \in S \\ (ii) & \displaystyle\int_S g_1(t)\, d\mu + \sigma(S, z) c_1(z) = \int_S e_1(t)\, d\mu \\ (iii) & \displaystyle\int_S g_2(t)\, d\mu + \sigma(S, z) c_2(z) = \int_S e_2(t)\, d\mu. \end{array}$$

This means that

$$\begin{cases} 3\sqrt{g_1(t)} + \sqrt{g_2(t)} > 4 \text{ for all } t \in S \\ \mu(S \cap A) g_1(A) + \mu(S \cap B) g_1(B) + \frac{1}{2}\mu(S \cap A) + \frac{1}{2}\mu(S \cap B) = 3\mu(S \cap A) \\ \mu(S \cap A) g_2(A) + \mu(S \cap B) g_2(B) + \frac{1}{2}\mu(S \cap A) + \frac{1}{2}\mu(S \cap B) = 3\mu(S \cap B) \end{cases}$$

or

$$\begin{cases} \sqrt{g_1(A)} + \sqrt{g_2(A)} > 4 \\ log g_1(B) + g_2(B) > 4 \\ \mu(S \cap A) g_1(A) + \mu(S \cap B) g_1(B) + \frac{1}{2}\mu(S \cap A) + \frac{1}{2}\mu(S \cap B) = 3\mu(S \cap A) \\ \mu(S \cap A) g_2(A) + \mu(S \cap B) g_2(B) = 3\mu(S \cap B) \end{cases}$$

First, notice that in both cases S cannot equal to A neither to B, thus S must be $T = A \cup B$. Assume that

$$\begin{cases} 3\sqrt{g_1(A)} + \sqrt{g_2(A)} > 4 \\ 3\sqrt{g_1(B)} + \sqrt{g_2(B)} > 4 \\ g_1(A) + g_1(B) = 2 \\ g_2(A) + g_2(B) = 2, \end{cases}$$

From the first two conditions it follows that

$$9[g_1(A) + g_1(B)] + [g_2(A) + g_2(B)] + 6\sqrt{g_1(A) g_2(A)} + 6\sqrt{g_1(B) g_2(B)} > 32.$$

From the last two condition it follows that $6\sqrt{(2-x)(2-y)} + 6\sqrt{xy} > 12$, where $x = g_1(B)$ and $y = g_2(B)$. Making the calculations it follows that $(x-y)^2 < 0$, which is a contradiction. On the other hand if we consider the second system

$$\begin{cases} \sqrt{g_1(A)} + \sqrt{g_2(A)} > 4 \\ log g_1(B) + g_2(B) > 4 \\ g_1(A) + g_1(B) = 2 \\ g_2(A) + g_2(B) = 3 \end{cases}$$

then if $g_1(A) > 1$, it follows that $g_1(B) < 1$, that is $log g_1(B) < 0$ and hence $g_2(B) > 4$, which implies that $g_2(A) + 4 < 3$, a contradiction. Otherwise, if $g_1(A) \leq 1$, then $g_2(A) > 9$ and hence $g_2(B) + 9 < 3$, which is impossible. This allows us to conclude that (h, y) is a $\sigma-$ core allocation, which is trivially not a ϱ-linear cost share equilibrium.

6 A weaker equivalence result

We have proved that in a mixed market with more than two atoms which are all of the same type, the equivalence between the core and the set of linear cost share equilibria can be restored. This interesting result fails in an economy with only one atom or with "different" large traders. In particular, Theorem 3 does not provide a characterization of linear cost share equilibria for finite economies whose traders have different characteristics. However, it is possible to characterized the set of linear cost share equilibria with a weaker cooperative notion involving generalized coalitions. To this aim, let us recall the definition of Aubin core in an economy with public projects introduced by [10].

Definition 8. *Define the set*

$$\mathcal{A} = \{\gamma : T \to [0,1] : \ \gamma \textit{ is simple and measurable}\}.$$

Any element γ of $\mathcal{A}$ such that $\mu(supp\,\gamma) = \mu(\{t \in T : \ \gamma(t) > 0\}) > 0$ is said to be a **generalized coalition**.

The contribution of a generalized coalition γ to the realization of a public project is defined by

$$\tilde{\sigma}(\gamma, \cdot) = \int_T \gamma(t)\, d\sigma = \int_T \gamma(t)\varrho(t, \cdot)\, d\mu. \tag{5}$$

Notice that since $\gamma(t) = 0$ for all $t \notin supp\,\gamma$, (5) can be replaced by

$$\tilde{\sigma}(\gamma, \cdot) = \int_{supp\,\gamma} \gamma(t)\, d\sigma = \int_{supp\,\gamma} \gamma(t)\varrho(t, \cdot)\, d\mu.$$

Definition 9. *Given a contribution measure σ, a generalized coalition γ σ-blocks an allocation (f, y) if there exist a public project z and a distribution of private goods $g : T \to I\!R^\ell_+$ such that*

$$\begin{array}{ll} (i) & u_t(g(t), z) > u_t(f(t), y) \quad \textit{for almost all } t \in supp\,\gamma \\ (ii) & \displaystyle\int_T \gamma(t) g(t)\, d\mu + \tilde{\sigma}(\gamma, z) c(z) = \int_T \gamma(t) e(t)\, d\mu. \end{array}$$

A feasible allocation (f, y) is said to be an **Aubin σ-core** *allocation if it cannot be σ-blocked by any generalized coalition. According to [10], we denote by $C^A_\sigma(\mathcal{E})$ the Aubin σ-core.*

It is easy to see that any Aubin σ-core allocation is in the σ-core. In the case of atomless economies the reverse inclusion can be proved.

Proposition 8. *Let $\mathcal{E}$ be an atomless economy with public goods which satisfies the hypotheses $(A.1) - -(A.4)$. Let σ be a contribution measure for $\mathcal{E}$. Then the Aubin σ-core $C^A_\sigma(\mathcal{E})$ of $\mathcal{E}$ coincides with the σ-core $C_\sigma(\mathcal{E})$.*

PROOF: We have already observed that the inclusion $C^A_\sigma(\mathcal{E}) \subseteq C_\sigma(\mathcal{E})$ holds true. Consider now an allocation (f, y) in the σ-core and assume by contradiction that it does not belong to $C^A_\sigma(\mathcal{E})$. Then there exists a generalized coalition $\gamma \in \mathcal{A}$ and an assignment (g, z) such that

$$\begin{array}{ll} (i) & u_t(g(t), z) > u_t(f(t), y) \quad \textit{for almost all } t \in supp\,\gamma \\ (ii) & \displaystyle\int_T \gamma(t) g(t)\, d\mu + \tilde{\sigma}(\gamma, z) c(z) = \int_T \gamma(t) e(t)\, d\mu. \end{array}$$

Since γ is a simple function, there exists the measurable sets $S_i \subseteq T$ and the real numbers $\gamma_i \in [0, 1]$ such that

$$0 = \int_T \gamma(t) \left(g(t) + \varrho(t, z) c(z) - e(t)\right)\, d\mu = \sum \int_{S_i} \gamma_i \left(g(t) + \varrho(t, z) c(z) - e(t)\right)\, d\mu$$

and consequently, applying the Lyapunov convexity Theorem to the nonatomic vector measures $\int_{S_i} \gamma_i \left(g(t) + \varrho(t,z)c(z) - e(t)\right) \, d\mu$ when $S_i \subseteq supp\gamma$, we get the existence of coalitions $\widehat{S}_i \subseteq supp\gamma$ such that

$$0 = \sum \int_{\widehat{S}_i} \left(g(t) + \varrho(t,z)c(z) - e(t)\right) \, d\mu.$$

By means of the coalition $\widehat{S} = \bigcup_i \widehat{S}_i$ we infer a contradiction. □

In the case of a general mixed market with public goods, we can show that the Aubin σ-core $C^A_\sigma(\mathcal{E})$ can be identified with the σ-core of the associated nonatomic economy $\mathcal{E}^*$, $C_{\sigma^*}(\mathcal{E}^*)$, without additional assumptions on the large traders.

Proposition 9. *Let $\mathcal{E}$ be a mixed market with public goods which satisfies assumptions $(A.1) - -(A.4)$. Let $\sigma \in M_\mu$ be a contribution measure. The following statements hold true.*

(i) *If the allocation (f, y) of the economy $\mathcal{E}$ is an Aubin σ-core allocation, then the allocation (f^*, y) is a σ^*-core allocation for $\mathcal{E}^*$;*

(ii) *if the allocation (φ, y) of the economy $\mathcal{E}^*$ is a σ^*-core allocation, then the allocation $(\widetilde{\varphi}, y)$ is an Aubin σ-core allocation for $\mathcal{E}$.*

PROOF: Let (f, y) be an Aubin σ-core allocation. Let us prove that the corresponding allocation (f^*, y) in the associated atomless economy is in the σ^*-core, where σ^* is defined by (2). Assume on the contrary that there exist a coalition $S^* \in \mathcal{T}^*$ and an allocation (g^*, z) such that

$$\begin{array}{ll} (i) & u_t(g^*(t), z) > u_t(f^*(t), y) \quad \text{for almost all } t \in S^* \\ (ii) & \displaystyle\int_{S^*} g^*(t) \, d\mu^* + \sigma^*(S^*, z)c(z) = \int_{S^*} e(t) \, d\mu^*. \end{array}$$

Define

$$\begin{aligned} S_i^* &= S^* \cap A_i^*, \\ I &= \{i \in I\!N \, : \, \mu^*(S_i^*) > 0\}, \\ S &= (S^* \cap T_0) \cup \left(\bigcup_{i \in I} A_i \right), \\ g(t) &= g^*(t) \quad \text{for all } t \in S^* \cap T_0, \\ g(A_i) &= \frac{1}{\mu^*(S_i^*)} \int_{S_i^*} g^*(t) \, d\mu^* \quad \text{for all } i \in I, \end{aligned}$$

where A_i^* denotes the split atom of A_i.

First notice that from the above definitions and from (i) it follows that for almost all $t \in S$, $u_t(g(t), z) > u_t(f(t), y)$. Indeed, in the atomless part $S^* \cap T_0$ there is no change, while for each $i \in I$ and $t \in S_i^*$

$$
\begin{aligned}
u_{A_i}(g(A_i), z) &= u_t(g(A_i), z) = u_t\left(\frac{1}{\mu^*(S_i^*)}\int_{S_i^*} g^*(t)\, d\mu^*, z\right) \geq \\
\frac{1}{\mu^*(S_i^*)}\int_{S_i^*} u_t(g^*(t), z)\, d\mu^* &> \frac{1}{\mu^*(S_i^*)}\int_{S_i^*} u_t(f^*(t), z)\, d\mu^* = \\
\frac{1}{\mu^*(S_i^*)}\int_{S_i^*} u_{A_i}(f(A_i), z)\, d\mu^* &= u_{A_i}(f(A_i), y).
\end{aligned}
$$

Moreover, define

$$
\begin{aligned}
\gamma(t) &= 1 \qquad \text{for all } t \in S^* \cap T_0 \\
\gamma(A_i) &= \frac{\mu^*(S_i^*)}{\mu(A_i)} \quad \text{for all } i \in I,
\end{aligned}
$$

and notice that $supp\, \gamma = S$. Furthermore,

$$
\int_S \gamma(t) g(t)\, d\mu + \tilde{\sigma}(\gamma, z) c(z) - \int_S \gamma(t) e(t)\, d\mu =
$$

$$
\int_{S^*\cap T_0} \gamma(t) g^*(t)\, d\mu^* + \sum_{i\in I} \gamma(A_i) g(A_i) \mu(A_i) + \int_S \gamma(t) \varrho(t, z)\, d\mu c(z) - \int_{S^*\cap T_0} \gamma(t) e(t)\, d\mu^* +
$$

$$
-\sum_{i\in I} \gamma(A_i) e(A_i) \mu(A_i) = \int_{S^*\cap T_0} g^*(t)\, d\mu^* + \sum_{i\in I} \frac{\mu^*(S_i^*)}{\mu(A_i)} \frac{1}{\mu^*(S_i^*)} \int_{S_i^*} g^*(t)\, d\mu^* \mu(A_i) +
$$

$$
+\int_{S^*\cap T_0} \varrho(t, z)\, d\mu^* c(z) + \sum_{i\in I} \varrho(A_i, z) \mu(A_i) \frac{\mu^*(S_i^*)}{\mu(A_i)} c(z) - \int_{S^*\cap T_0} e(t)\, d\mu^* - \sum_{i\in I} \frac{\mu^*(S_i^*)}{\mu(A_i)} e(A_i) \mu(A_i) =
$$

$$
\int_{S^*} g^*(t)\, d\mu^* + \int_{S^*\cap T_0} \varrho(t, z)\, d\mu^* c(z) + \sum_{i\in I} \int_{S_i^*} \varrho^*(t, z)\, d\mu^* c(z) - \int_{S^*} e(t)\, d\mu^* =
$$

$$
\int_{S^*} g^*(t)\, d\mu^* + \sigma^*(S^*, z) c(z) - \int_{S^*} e(t)\, d\mu^* = 0.
$$

This means that (f, y) is σ-blocked by the generalized coalition γ defined above via the allocation (g, z), which is a contradiction. Therefore the associated allocation (f^*, y) is in the σ^*-core of the atomless economy $\mathcal{E}^*$.

Suppose now that (φ, y) is a σ^*-core allocation of the economy $\mathcal{E}^*$ and assume by contradiction that $(\widetilde{\varphi}, y)$ is not in the Aubin σ-core. Let γ be a generalized coalition that blocks it so that for suitable g and z we have $u_t(g(t), z) > u_t(\widetilde{\varphi}(t), y)$ for almost all $t \in S = supp\gamma$ and the equality

$$\int_S \gamma(t) g \, d\mu + \tilde{\sigma}(S,z) c(z) = \int_S \gamma(t) e \, d\mu$$

holds.

Let us define the generalized coalition γ^* on $\mathcal{E}^*$ by $\gamma = \gamma^*$ on $S \cap T_o$, $\gamma^*(t) = \gamma(A_i)$, for each $t \in A_i^*$ such that $A_i \in S$. Similarly to the proof of Proposition 5 we shall see that γ^* blocks (φ, y) via g^* and z, since $\tilde{\sigma}(S,z) = \tilde{\sigma^*}(S^*, z)$. Then the assertion follows from Proposition 8. □

The next proposition relates linear cost share equilibria to Aubin σ-core allocations.

Proposition 10. *Let (f, y) be a linear cost share equilibrium in $\mathcal{E}$ with cost distribution function ϱ and let σ_ϱ be the corresponding contribution measure. Then, (f, y) belongs to the Aubin σ_ϱ-core of $\mathcal{E}$.*

PROOF: See [10, Proposition 3.7].

Notice that Proposition 2 is a direct consequence of Proposition 6.

We are now ready to prove the equivalence between the Aubin σ-core and the set of linear cost share equilibria in a mixed market without any additional requirements on large traders.

Theorem 4. *Let $\mathcal{E}$ be a mixed market with public goods which satisfies assumptions $(A.1) - -(A.4)$. Let $\sigma \in M_\mu$ be a contribution measure and let ϱ_σ be the corresponding Radon-Nikodym derivative. Then*

$$C_\sigma^A(\mathcal{E}) = LCE_{\varrho_\sigma}(\mathcal{E}).$$

PROOF: The proof follows from Proposition , Proposition 8 and Theorem 2. □

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