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Stable Sets and Public Projects

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Maria Gabriella Graziano [*], **Maria Romaniello** [**]

Abstract
We introduce von Neumann-Morgenstern solution concepts in market models involving the choice of a public project. We show that vN-M stable sets, suitably defined in connection to public goods provision, are consistent with results from bargaining via cartels. We find as necessary the assumption that stability is defined with respect to blocking procedures in which coalitions do not necessarily pay for the whole realization of the project, but only for a fraction of it and that costs are distributed uniformly in each corner of the market. Under this assumption, we obtain large games solutions by the finite ones via embedding procedures. Going further in the investigation of stable solutions, we define stable sets following the "sophisticated" approach suggested by Harsanyi (see [15] and [17]), proving that a σ-sophisticated stable set corresponds to the solution in the associated payoff space.

JEL Classification: D51, D60, H41, C70.

Keywords: Public project, σ-core, von Neumann-Morgenstern σ-stable set, σ-sophisticated stable sets.

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1 Introduction

In this paper we try to explain and to predict the endogenous cartelization of markets with public projects.

We consider large market games in which the agents decompose into finitely many disjoint groups, each of which holds a corner of the market (glove markets with public projects). In glove games or markets, groups of agents of the same type have corners of different commodities. This very simple but basic situation represents a significant example in which the structure of the market seems to offer a strong incentive leading to the organization in cartels or syndicates (we assume that contracts generating cartels can be legally enforced). Agents join forces in subcoalitions (cartels or syndicates): they act intermediately as representative players of a short side market. The result of bargaining is then implemented to the long side market.

A general equilibrium approach in exchange economies with public projects is not able to predict, even in this simple situation, the formation of cartels. Specifically, results from bargaining via cartels cannot be seen as consistent with the traditional solution concepts. This is the case of the Foley core or of the core notions assuming a contribution measure. In the short side market, cartels may accept distribution of profits under core allocations much more favorable with respect to results obtained without cartelization. Using suitable equivalences between core notions with a given contribution measure and linear cost-share equilibria (see for example [7]), we conjecture a similar result in the case of non-cooperative behavior, that is results emerging from the cooperation within cartels differ from what emerges if agents show price and cost-share taking behavior. We expect a similar behavior for other game theoretical notions like the Shapley value (see the discussion presented in [19]).

Consequently, we look at different solution concepts that are capable to assign profits towards the long side of the markets, suggesting its endogenous cartelization. We introduce von Neumann-Morgenstern solution concepts, suitably defined in connection to public goods provision, exhibiting a markedly different behavior. We find as necessary the assumption that stability is defined with respect to blocking procedures in which coalitions do not necessarily pay for the whole realization of the project, but only for a fraction of it and that costs are distributed uniformly in each corner of the market. Under this assumption, we obtain large games solutions by embedding procedures in finite games (compare [16]).

In the second part of the paper, we go further in the investigation of vN-M stable sets in exchange economies with public goods and a large number of traders of the same type: we shall define stable sets following the "sophisticated" approach suggested by Harsanyi proving that a σ-sophisticated stable set corresponds to the solution in the associated payoff space (see [15], [17]).

We assume throughout the paper that private decisions of economic agents are influenced by non-market variables that we call public environments or public projects: examples include the public goods provision (transport, health, education, international public goods like the global climate), the regulation of private economic activities (regulation of quality standard, safety of labor conditions, trade institutions), social rules (laws, property rights) among the others. They are common to all the agents and affect individual budget sets and private objectives.

It is worth noticing that in the investigated model, the concept of public project or environment has to be considered as a very broad notion, allowing many different problems to be treated within one common setting. Technical difficulties deriving from a little structure imposed on the set of public projects represent a minor cost given the level of generality of our framework. Hence, building on the model originally proposed in Mas-Colell ([18]), we allow the public project to range over an abstract set, a priori with no special structure, each project being characterized by a cost in terms of private goods. Differently by the case of Samuelsonian public goods, represented by classical Euclidean structures, the great generality of the adopted model does not impose any kind of homogeneity assumption on the personal perceptions that agents may have of the same public good. Moreover, it better describes public decision problems in which a choice must be given among a few projects and permits the discussion of non convexity in public sector decisions. For the interpretation of the Mas-Colell approach to public goods economies we refer to [6], [7], [8], [11], [13], [14], for the extension of the Mas-Colell model to allow multiple private goods to [6], [5],

[12], [2].

The first part of the paper focuses on interactions among agents of the same type. In particular, the model aims to recognize and to anticipate the endogenous formation of cartels from the predicting power of game theoretical cooperative solution concepts. The seminal paper [15] discusses this problem in the case of large pure exchange economies in connection with the von Neumann-Morgenstern stable sets. He argues that the coalitions predicted by this solution concept do indeed reflect cooperation within cartels. He does not explicitly construct vNM-stable sets. Recently [20], [21] classify all vNM-stable sets in the non-atomic context investigating further versions of cartelization of markets. In particular, [20] allows for the actual construction of vNM-stable sets showing that agents on the short side of the market benefit according to their initial holdings. The theoretical basis developed in this paper are the premise in order to follow the analogous construction in the case of public goods. For conditions ensuring the stability of core allocations in non-atomic glove market games we shall refer to [9].

2 The Economic Model

We consider an economy $\mathcal{E}_{\mathcal{C}}$ in which:

- the space of agents is represented by a probability space (I, Σ, μ), where $I = [0, 1]$ is the set of agents, Σ is the set of all measurable subsets of I and μ is a nonatomic measure;
- the private commodity space is represented by positive cone of $I\!R^m$, denoted by $I\!R^m_+$;
- an abstract set $\mathcal{Y}$ without any mathematical structure represents the set of *public projects*. The cost of any public project in terms of private goods is expressed by means of a vector-valued function $c : \mathcal{Y} \to I\!R^m_+$, called *cost function*.
- the weight that a coalition has in the realization of any public project is described by a *contribution measure*, that is a probability measure $\widehat{\sigma} : \Sigma \to [0, 1]$ which is absolutely continuous with respect to μ[1].

We assume there are only finitely many types n of agents on the market, which means that we refer to the decomposition $I = \bigcup_{i=1}^{n} I_i$, where $I_i = \left[\frac{i-1}{n}, \frac{i}{n}\right[$ if $i \neq n$ and $I_n = \left[\frac{n-1}{n}, 1\right]$. Agents in the same set I_i share the same initial endowment and the same preference. In particular, we assume, that, for any $i = 1, \ldots, n$ and for any $t \in I_i$:

(1) consumer t has an initial endowment $\omega_t > 0$ and the total initial endowment $\omega = \int_I \omega_t d\mu \gg c(y)$[2], $\forall y \in \mathcal{Y}$, to ensure that each private commodity is present on the market regardless to the cost of the realized project.

(2) the preference of consumer t is represented by a function $u_t : I\!R^m_+ \times \mathcal{Y} \to I\!R_+$. We say that, u_t is strictly monotone, continuous and quasi concave if, for any public project $y \in \mathcal{Y}$, the restriction $u_t(\cdot, y)$ is strictly monotone, continuous and quasi concave.

(3) the contribution measure is uniform on agents of the same type, in the sense that, for any coalition $\widehat{S}$ it results:

$$\widehat{\sigma}(\widehat{S}) = \sum_{i=1}^{n} \sigma(I_i) \frac{\mu(\widehat{S} \cap I_i)}{\mu(I_i)}$$

[1] We mean that $\widehat{\sigma}(I) = 1$ and for any coalition $E \in \Sigma$ such that $\mu(E) = 0$ it results $\widehat{\sigma}(E) = 0$.

[2] We follow the standard notation according to which for two vectors $x \equiv (x_1, \ldots x_m)$ and $z \equiv (z_1, \ldots z_m)$ of $I\!R^m_+$ $x \gg z$ means that $x_i > z_i$, for each $i = 1, \ldots m$.

A *feasible allocation* for the economy $\mathcal{E}_{\mathcal{C}}$ is a couple (f, y) where $f : I \to I\!R^m_+$ is a μ-integrable function and $y \in \mathcal{Y}$, such that

$$\int_I f d\mu + \widehat{c}(y) \leq \int_I \omega d\mu.$$

A feasible allocation (f, y) is called *symmetric* if it assigns identical bundles to agents of the same type, that is $f(t) = x_i$ for any $i = 1, \ldots, n$ and for any $t \in I_i$.

Now we can consider a finite economy $\mathcal{E}$ which has the same data of $\mathcal{E}_{\mathcal{C}}$, but consists of n agents. For this economy a contribution measure is an additive function σ defined on the set of all the agents $N = \{1, 2, \ldots, n\}$ such that $\sigma(\emptyset) = 0$ and $\sigma(N) = 1$. Obviously, we have that

if σ is a contribution measure for $\mathcal{E}$ then $\widehat{\sigma} : \Sigma \to [0, 1]$ defined as

$$\widehat{\sigma}(E) = \sum_{i=1}^{n} \sigma(\{i\}) \frac{\mu(E \cap I_i)}{\mu(I_i)}, \text{ for all } E \in \Sigma \tag{1}$$

is a contribution measure for $\mathcal{E}_{\mathcal{C}}$;

if $\widehat{\sigma}$ is a contribution measure for $\mathcal{E}_{\mathcal{C}}$ then $\sigma : N \to [0, 1]$ defined as

$$\sigma(S) = \sum_{i \in S} \widehat{\sigma}(I_i), \text{ for all } S \subseteq N \tag{2}$$

is a contribution measure for $\mathcal{E}$.

An assignment $(x_1, \ldots, x_n, y) \in I\!R^m_+ \times \mathcal{Y}$ is a *feasible allocation* in $\mathcal{E}$ if

$$\sum_{i=1}^{n} x_i + c(y) \leq \sum_{i=1}^{n} \omega_i.$$

Observe that an allocation $(x_1, \ldots, x_n, y)$ in $\mathcal{E}$ can be interpreted as a symmetric allocation (f, y) in $\mathcal{E}_{\mathcal{C}}$, where f is the function defined as $f(t) = x_i$, if $t \in I_i$. Reciprocally, an allocation (f, y) in $\mathcal{E}_{\mathcal{C}}$ can be interpreted as an allocation $(x_1, \ldots, x_n, y)$ in $\mathcal{E}$, with $x_i = \dfrac{1}{\mu(I_i)} \displaystyle\int_{I_i} f d\mu$.

Consider the economy $\mathcal{E}_{\mathcal{C}}$ (or equivalently $\mathcal{E}$) and assume that the sets

$$M_i = \left\{ k \mid \omega_i^k > 0 \right\} \qquad \text{for all } \ 1 \leq i \leq n \tag{3}$$

are disjoint, that is each commodity is initially owned by only one type of trader.

We assume the existence of a distinguished project, denoted by 0, such that $c(0) = 0$. It is to be interpreted as the "status quo", i.e., as the situation from which a change is being contemplated.

Definition 2.1 *A feasible allocation (f, y) (or $(x_1, \ldots, x_n, y)$) is a individually rational allocation if $u_t(f(t), y) \geq u_t(\omega(t), 0)$ for almost all $t \in I$ (or $u_i(x_i, y) \geq u_i(\omega_i, 0)$ for all $i = 1, \ldots, n$).*

Definition 2.2 *Given two allocations (f, y) and (g, z), a coalition $\widehat{S}$ with nonnull measure and a contribution measure $\widehat{\sigma}$, we say that (f, y) $\widehat{\sigma}$-dominates (g, z) on $\widehat{S}$ if*

$$\int_{\widehat{S}} g d\mu + \widehat{\sigma}(\widehat{S}) \widehat{c}(z) \leq \int_{\widehat{S}} \omega d\mu \qquad \textit{and}$$

$$u_t(f(t), y) > u_t(g(t), z) \textit{ for almost all } t \in \widehat{S}.$$

We say that (f, y) $\widehat{\sigma}$-dominates (g, z), if there exists a coalition $\widehat{S}$ with nonnull measure such that (f, y) $\widehat{\sigma}$-dominates (g, z) on $\widehat{S}$.

Analogously in the finite economy $\mathcal{E}$, if σ is a contribution measure, we say that the allocation $(x_1,\ldots,x_n,y)$ σ-dominates $(g_1,\ldots,g_n,z)$ on a nonempty coalition S if

$$\sum_{i\in S} g_i + \sigma(S)c(z) \le \sum_{i\in S}\omega_i \quad \text{and}$$

$$u_i(x_i,y) > u_i(g_i,z) \text{ for all } i\in S.$$

We say that $(x_1,\ldots,x_n,y)$ σ-dominates (g,z) if there exists a nonempty coalition S such that $(x_1,\ldots,x_n,y)$ σ-dominates $(g_1,\ldots,g_n,z)$ on S.

Definition 2.3 *Given a contribution measure $\widehat{\sigma}$, a solution of $\mathcal{E}_{\mathcal{C}}$ is a set $\mathcal{A}$ of individually rational allocations such that:*

- $\mathcal{A}$ is internally consistent, that is no two elements of $\mathcal{A}$ $\widehat{\sigma}$-dominate each other,

- $\mathcal{A}$ is extradominative, that is each individually rational allocation not in $\mathcal{A}$ is $\widehat{\sigma}$-dominated by some element of $\mathcal{A}$.

Analogously we can define a *solution* for the finite economy $\mathcal{E}$.

A *permutation* in $\mathcal{E}_{\mathcal{C}}$ is a one-to-one μ-measure preserving function π from I to I, measurable in both directions, such that for all $t\in I$, πt and t belong to I_i for the same i.

A set $\mathcal{A}$ of allocations is *symmetric* if for each permutation π and each $(f,y)\in\mathcal{A}$, the allocation $(\pi f,y)$, defined as $(\pi f)(t)=f(\pi t)$, is also in $\mathcal{A}$.

Definition 2.4 *A symmetric solution of $\mathcal{E}_{\mathcal{C}}$ is a solution that is a symmetric set.*

Recalling that a feasible allocation (f,y) of $\mathcal{E}_{\mathcal{C}}$ is said to be symmetric if $f(t)=x_i$ for almost all $t\in I_i$ and for all $i=1,\ldots,n$, we can prove that

Lemma 2.5 *Let (f,y) and (g,z) be symmetric allocations in $\mathcal{E}_{\mathcal{C}}$ with $f(t)=x_i$ and $g(t)=g_i$ for almost all $t\in I_i$. If (f,y) $\widehat{\sigma}$-dominates (g,z) in $\mathcal{E}_{\mathcal{C}}$, then $(x_1,\ldots,x_n,y)$ σ-dominates $(g_1,\ldots,g_n,z)$ in $\mathcal{E}$.*

PROOF: Since (f,y) $\widehat{\sigma}$-dominates (g,z), there exists a coalition $\widehat{S}$ with $\mu(\widehat{S})>0$, such that

$$\int_{\widehat{S}} f\,d\mu + \widehat{\sigma}(\widehat{S})\widehat{c}(y) \le \int_{\widehat{S}}\omega\,d\mu, \text{ and}$$

$$u_t(f(t),y) > u_t(g(t),z) \text{ for almost all } t\in\widehat{S}.$$

Let $S=\left\{i \mid \widehat{S}\cap I_i\neq\emptyset\right\}$, then S is not empty and $u_i(x_i,y)>u_i(g_i,z)$ for all $i\in S$. Moreover, since $\widehat{\sigma}(\widehat{S})=\sum_{i\in S}\sigma(\{i\})\frac{\mu(\widehat{S}\cap I_i)}{\mu(I_i)}$, we have

$$\sum_{i\in S}\left(x_i\mu(\widehat{S}\cap I_i)+\sigma(\{i\})\frac{\mu(\widehat{S}\cap I_i)}{\mu(I_i)}\,\frac{c(y)}{n}\right)\le\sum_{i\in S}\omega_i\mu(\widehat{S}\cap I_i)$$

that is

$$\sum_{i\in S}\mu(\widehat{S}\cap I_i)\left(x_i+\frac{\sigma(\{i\})}{\mu(I_i)}\,\frac{c(y)}{n}\right)\le\sum_{i\in S}\omega_i\mu(\widehat{S}\cap I_i)$$

so, for any $k=1,\ldots,m$

$$\sum_{i\in S}\mu(\widehat{S}\cap I_i)\left(x_i^k+\frac{\sigma(\{i\})}{\mu(I_i)}\,\frac{c^k(y)}{n}\right)\le\sum_{i\in S}\omega_i^k\mu(\widehat{S}\cap I_i).$$

If $k \notin \cup_{i\in S} M_i$, then $\sum_{i\in S} \omega_i^k \mu(\widehat{S} \cap I_i) = 0$ and so $x_i^k + \frac{\sigma(\{i\})}{\mu(I_i)} \frac{c^k(y)}{n} = 0$ for all $i \in S$. Otherwise $\sum_{i\in S} \omega_i^k = \sum_{i=1}^n \omega_i^k$, and, being $(x_1, \ldots, x_n, y)$ a feasible allocation, it results

$$\sum_{i\in S} [x_i^k + \sigma(\{i\}) c^k(y)] \leq \sum_{i=1}^n x_i^k + \sigma(\{i\}) c^k(y) \leq \sum_{i=1}^n \omega_i^k = \sum_{i\in S} \omega_i^k .$$

So $\sum_{i\in S} x_i + \sigma(S) c(y) \leq \sum_{i\in S} \omega_i$, and $(x_1, \ldots, x_n, y)$ σ-dominates $(g_1, \ldots, g_n, z)$ in $\mathcal{E}$. □

Lemma 2.6 *Let (f, y) be an i.r. allocation such that the set*

$$S_i = \left\{ t \in I_i \mid f(t) \neq \xi_i = \frac{1}{\mu(I_i)} \int_{I_i} f d\mu \right\}$$

has nonnull measure. Then there exists $\eta_i \leq \xi_i$ such that the set

$$U_i = \{ t \in I_i \mid u_i(f(t), y) < u_i(\eta_i, y) \}$$

has nonnull measure and $u_i(\eta_i, y) > u_i(\omega(t), 0)$ on a subset of I_i having nonnull measure.

PROOF: Since (f, y) is an individually rational feasible allocation we have

$$\int_I f d\mu + \widehat{c}(y) \leq \int_I \omega d\mu$$

$$u_t(f(t), y) \geq u_t(\omega(t), 0) \text{ for almost all } t \in I.$$

Let us consider the set

$$V_i = \{ t \in I_i \mid u_i(f(t), y) < u_i(\xi_i, y) \} .$$

If $\mu(V_i) = 0$, then $u_i(f(t), y) \geq u_i(\xi_i, y)$ for almost all $t \in I_i$. The set

$$C = \left\{ l \in I\!R_+^m \mid u_i(l, y) \geq u_i(\xi_i, y) \text{ with } l \neq \xi_i \right\}$$

is not empty and convex and so $\dfrac{1}{\mu(S_i)} \displaystyle\int_{S_i} f d\mu \in C$. Since

$$\mu(I_i)\xi_i = \int_{I_i} f d\mu = \int_{S_i} f d\mu + \int_{I_i \setminus S_i} f d\mu = \int_{S_i} f d\mu + \xi_i \mu(I_i \setminus S_i)$$

then

$$\xi_i \left[\mu(I_i) - \mu(I_i \setminus S_i) \right] = \int_{S_i} f d\mu$$

and so $\frac{1}{\mu(S_i)} \int_{S_i} f d\mu = \xi_i \in C$, hence a contradiction. Then $\mu(V_i) > 0$.
If $\xi_i = 0$ then $f(t) = 0$ almost everywhere on I_i and $\mu(S_i) = 0$, so $\xi_i \geq 0$ and for almost a component k it results $\xi_i^k > 0$. Since utility functions are continuous, we can find $\varepsilon > 0$ such that if $\eta_i = \xi_i - \varepsilon \mathbf{e_k}$, then the set

$$U_i = \{ t \in I_i \mid u_i(f(t), y) < u_i(\eta_i, y) \}$$

has nonnull measure and $u_i(\eta_i, y) > u_i(f(t), y) \geq u_i(\omega(t), 0)$ for all $t \in U_i$.

□

Lemma 2.7 *Let (f, y) be an allocation in $\mathcal{E}_\mathcal{C}$, let $S \subset I$ and $0 \leq \alpha \leq 1$. Then there exists $S_\alpha \subset S$ such that*

$$\mu(S_\alpha) = \alpha \mu(S) \quad \widehat{\sigma}(S_\alpha) = \alpha \widehat{\sigma}(S) \text{ and } \quad \int_{S_\alpha} f d\mu = \alpha \int_S f d\mu$$

PROOF: The proof follows from the Lyapunov convexity theorem. □

Theorem 2.8 *Let $\mathcal{B}$ be a solution of the economy $\mathcal{E}$. If $\mathcal{A}$ denotes the set of symmetric allocations of the economy $\mathcal{E}_{\mathcal{C}}$ associated to $\mathcal{B}$, then $\mathcal{A}$ is a symmetric solution of $\mathcal{E}_{\mathcal{C}}$.*

PROOF: The set $\mathcal{A}$ is symmetric by construction, in light of lemma (2.5), is internally consistent and its elements are obviously individually rational. We have to prove the extradominance.
Let (f, y) be an individually rational allocation not in $\mathcal{A}$.

If (f, y) is symmetric, then $f(t) = x_i$ for almost all $t \in I_i$ for all $i = 1, \ldots, n$. Clearly, the allocation $(x_1, \ldots, x_n, y)$ is not in $\mathcal{B}$, and so there exists an allocation $(g_1, \ldots, g_n, z) \in \mathcal{B}$ and a not empty coalition S such that

$$\sum_{i \in S} g_i + \sigma(S)c(z) \leq \sum_{i \in S} \omega_i \ , \quad \text{and}$$

$$u_i(g_i, z) > u_i(x_i, y) \ \forall i \in S.$$

Let $\widehat{S} = \cup_{i \in S} I_i$ and $g(t) = g_i$ if $t \in I_i$, then $(g, z) \in \mathcal{A}$ and $\widehat{\sigma}$-dominates (f, y) since, dividing by n the previous inequalities, we obtain

$$\int_{\widehat{S}} g d\mu + \widehat{\sigma}(\widehat{S})\widehat{c}(z) \leq \int_{\widehat{S}} \omega d\mu \ , \quad \text{and}$$

$$u_t(g(t), z) > u_t(f(t), y) \ \text{ for almost all } t \in \widehat{S}.$$

If (f, y) is not symmetric, then for all $i \in S = \{ j \in \{1, \ldots, n\} \mid \mu(S_j) > 0 \}$, the set

$$S_i = \left\{ t \in I_i \mid f(t) \neq \xi_i = \frac{1}{\mu(I_i)} \int_{I_i} f d\mu \right\}$$

has nonnull measure, and so from lemma (2.6) there exists $\eta_i \leq \xi_i$ such that the set

$$U_i = \{ t \in I_i \mid u_i(f(t), y) < u_i(\eta_i, y) \}$$

has nonnull measure and $u_i(\eta_i, y) \geq u_i(\omega(t), 0)$ on U_i.
Let $\delta = \sum_{i \in S} \xi_i - \eta_i \geq 0$, and consider the allocation $(\eta_1, \ldots, \eta_n, y)$ with $\eta_i = \xi_i + \frac{1}{n - |S|}\delta$ if $i \notin S$, then $(\eta_1, \ldots, \eta_n, y)$ is i.r. and

$$\sum_{i=1}^{n} \eta_i = \sum_{i \in S} \eta_i + \sum_{i \notin S} \eta_i = \sum_{i \in S} \eta_i + \sum_{i \notin S} \xi_i + \frac{1}{n - |S|}\delta =$$

$$= \sum_{i \in S} \eta_i + \sum_{i \notin S} \xi_i + \frac{n - |S|}{n - |S|} \sum_{i \in S} \xi_i - \eta_i \leq \sum_{i=1}^{n} \omega_i - c(y)$$

Let be $U_i = \begin{cases} U_i & \text{if } i \in S \\ I_i & \text{if } i \notin S \end{cases}$, $\alpha = min_{1 \leq i \leq n} \mu(U_i)$ and $V_i \subseteq U_i$ with $\mu(V_i) = \alpha$.

If $(\eta_1, \ldots, \eta_n, y) \in \mathcal{B}$, then the allocation (g, y) with $g(t) = g_i$ if $t \in I_i$ belongs to $\mathcal{A}$ and, from the previous inequalities, since

$$\widehat{\sigma}(\cup_{i=1}^{n} V_i) = \sum_{i=1}^{n} \sigma(\{i\}) \frac{\mu(V_i \cap I_i)}{\mu(I_i)} = \sum_{i=1}^{n} \sigma(\{i\}) \alpha n,$$

it follows that (g, z) $\widehat{\sigma}$-dominates (f, y) on the coalition $\cup_{i=1}^{n} V_i$.

If $(\eta_1, \ldots, \eta_n, y) \notin \mathcal{B}$, then, being $\mathcal{B}$ a solution, there exists an allocation $(h_1, \ldots, h_n, z)$ in $\mathcal{B}$ and a not empty coalition H, such that,

$$\sum_{i \in H} h_i + \sigma(H)c(z) \leq \sum_{i \in H} \omega_i, \ \text{and}$$

$$u_i(h_i, z) > u_i(\eta_i, y) \text{ for any } i \in H.$$

Then the allocation (h, z) with $h(t) = h_i$ if $t \in I_i$ belongs to $\mathcal{A}$ and $\widehat{\sigma}$-dominates (f, y) on the coalition $\widehat{H} = \cup_{i\in H} I_i$ since $\widehat{\sigma}(\widehat{H}) = \sigma(H)$.
In any case the property of extradominance is proved and $\mathcal{A}$ is a solution. □

To prove the converse of theorem (2.8) we need the following lemma which does not make use of assumption (3).

Lemma 2.9 *Let $\mathcal{A}$ a symmetric solution of the economy $\mathcal{E}_{\mathcal{C}}$. Then every allocation (f, y) in $\mathcal{A}$ is symmetric.*

PROOF: Suppose $(f, y) \in \mathcal{A}$ is not symmetric, then there is some i_0 such that the set

$$\left\{ t \in I_{i_0} \mid f(t) \neq \varepsilon_{i_0} = \frac{1}{\mu(I_{i_0})} \int_{I_{i_0}} f d\mu \right\}$$

is nonnull. Using lemma (2.6) for this i_0, we get $\eta_{i_0} \leq \varepsilon_{i_0}$ such that the set

$$\widehat{S}_{i_0} = \{t \in I_{i_0} \mid u_t(f(t), y) < u_t(\eta_{i_0}, y)\}$$

has a nonnull measure and $u_t(\eta_{i_0}, y) \geq u_t(\omega_{i_0}, 0)$ for almost all $t \in S_{i_0}$.
Let us define $\delta = \varepsilon_{i_0} - \eta_{i_0} \geq 0$, $\frac{\alpha}{n} = \mu(\widehat{S}_{i_0})$ and

$$g(t) = \begin{cases} f(t) + \frac{\delta}{n-1} & \forall t \in I_i \text{ with } i \neq i_0 \\ \eta_{i_0} & \forall t \in I_{i_0} \end{cases}.$$

Using lemma (2.7), for all $i \neq i_0$ we find a set $\widehat{S}_i \subset I_i$ such that $\mu(\widehat{S}_i) = \alpha\mu(I_i)$ and $\int_{\widehat{S}_i} g d\mu = \alpha \int_{I_i} g d\mu$. Then the allocation (g, y) $\widehat{\sigma}$-dominates (f, y) on the coalition $S = \cup_{i=1}^n \widehat{S}_i$. In fact, obviously, $u_t(g(t), y) > u_t(f(t), y)$ for almost all $t \in S$. Moreover, since $\widehat{\sigma}(S) \leq 1$, it results

$$\int_S g d\mu = \int_{\cup_{i\neq i_0} \widehat{S}_i} g d\mu + \int_{\widehat{S}_{i_0}} g d\mu = \alpha \int_{\cup_{i\neq i_0} I_i} g d\mu + \eta_{i_0} \frac{\alpha}{n} =$$

$$= \alpha \int_{\cup_{i\neq i_0} I_i} f d\mu + \alpha \frac{\varepsilon_{i_0} - \eta_{i_0}}{n-1} \frac{n-1}{n} + \eta_{i_0} \frac{\alpha}{n} = \alpha \int_I f d\mu \leq$$

$$\leq \alpha \int_I \omega d\mu - \widehat{c}(y) \leq \int_{\cup \widehat{S}_i} \omega d\mu - \widehat{\sigma}(S)\widehat{c}(y)$$

Hence (g, y) is not a member of $\mathcal{A}$, and there exists some $(h, z) \in \mathcal{A}$ which $\widehat{\sigma}$-dominates (g, y) on a nonnull coalition U. Without loss of generality (we may choose U so that $\mu(U)$ arbitrarily small), we assume that

$$\mu(U \cap I_{i_0}) \leq \mu(\widehat{S}_{i_0})$$

If $\mu(U \cap I_{i_0}) = 0$, then (h, z) $\widehat{\sigma}$-dominates (f, y) via U. If $\mu(U \cap I_{i_0}) > 0$, let $V_{i_0} \subset \widehat{S}_{i_0}$ such that $\mu(U \cap I_{i_0}) = \mu(V_{i_0})$, and define π to be a permutation interchanging V_{i_0} with $U \cap I_{i_0}$, and being otherwise the identity. Then $\pi(f, y) = (\pi f, y)$ is also in $\mathcal{A}$, since $\mathcal{A}$ is symmetric, and obviously (h, z) $\widehat{\sigma}$-dominates $\pi(f, y)$ via U. In any case we get domination between two members of $\mathcal{A}$, hence a contradiction since $\mathcal{A}$ is internally consistent. □

Theorem 2.10 *Let $\mathcal{A}$ be a symmetric solution of $\mathcal{E}_{\mathcal{C}}$. If $\mathcal{B}$ denotes the set of allocations of the economy $\mathcal{E}$ associated to $\mathcal{A}$, then $\mathcal{A}$ is a solution of $\mathcal{E}$.*

PROOF: The set $\mathcal{B}$ is well defined by lemma (2.9). Its internal consistency and extradominance follow easily from the same properties of $\mathcal{A}$. □

Remark 2.11 *We can give the stability definitions above using the classical concept of dominance without involving contribution measure, that is an allocation (f, y) dominates (g, z) if there exists a nonnull coalition S such that*

$$\int_S f d\mu + \widehat{c}(y) \leq \int_S \omega d\mu, \text{ and}$$

$$u_t(f(t), y) > u_t(g(t), z) \text{ for almost all } t \in S.$$

results hold also for this case.

We just point out that σ-stability and stability without contribution measure are independent concepts. Indeed it is easy to show that

if the set $\mathcal{A}$ is σ-internally stable, then $\mathcal{A}$ is also internally stable;

if the set $\mathcal{A}$ is externally stable, then $\mathcal{A}$ is also σ-externally stable.

Our interest in σ-stability is motivated by the fact that the σ-dominance concept ensures core equivalence results which do not hold using the classical concept of dominance.

The next results refer to a contribution measure σ which is not necessarily uniform on agents of the same type.

Proposition 2.12 *Let σ be a contribution measure and A_σ a stable set with respect to σ. Then the allocations contained in A_σ are Pareto optimal.*

PROOF: Assume that the allocation $(f, y) \in A_\sigma$ is not Pareto optimal. Then there exists a feasible allocation (g, z) such that $u_t(g(t), z) > u_t(f(t), y)$ for almost all $t \in I$. If $(g, z) \in A_\sigma$, then we contradict the internal σ-stability of A_σ, then $(g, z) \notin A_\sigma$. If (g, z) is individually rational, then we can find an allocation $(h, s) \in A_\sigma$ which σ-dominates (g, z) and so also (f, y), a contradiction. Hence (g, z) is not individually rational, that is there exists a coalition S with nonnull measure such that $u_t(\omega(t), 0) > u_t(g(t), z) > u_t(f(t), y)$ for almost all $t \in S$, a contradiction that completes the proof. □

Proposition 2.13 *Let σ be a contribution measure and A_σ a stable set with respect to σ. Then the σ-core is contained in A_σ.*

PROOF: Let (f, y) be an allocation contained in the σ-core. If (f, y) is not individually rational, then we can find a coalition S with nonnull measure such that $u_t(\omega(t), 0) > u_t(f(t), y)$ for almost all $t \in S$ and, being $c(0) = 0$, $\int_S \omega d\mu + \sigma(S) \cdot 0 = \int_S \omega d\mu$. So $(\omega, 0)$ σ-dominates (f, y) on S, a contradiction. Then (f, y) is individually rational. If $(f, y) \notin A_\sigma$, we can find an allocation $(g, z) \in A_\sigma$ which σ-dominates (f, y), a contradiction since (f, y) is in the σ-core. Hence $(f, y) \in A_\sigma$. □

Let us consider the economy $\mathcal{E}_\mathcal{C}$ and denote by P the set of all Pareto optimal allocations which are symmetric and individually rational. If A_σ is a σ-solution (with σ not necessarily uniform on agents), then in light of Lemma 1.9 and Proposition 2.12, $A_\sigma \subseteq P$. Let us assume that

(h1) the set $\mathcal{Y}$ of public projects is compact,

(h2) the σ-dominance is defined with respect to those coalition S such that $\mu(S \cap I_i) > 0$ for every $i = 1, \cdots, n$.

Then our assumptions ensure that P is the unique solution with respect to σ.

We have to prove that P is internally and externally σ-stable.

Proposition 2.14 *Under the assumption (h1), the set P is σ-internally stable.*

PROOF: If P is not σ-internally stable, then we can find (f,y), $(g,z)\in P$ and a coalition S with $\mu(S\cap I_i)>0$ for every $i=1,\cdots,n$, such that

$$u_t(f(t),y)>u_t(g(t),z)\quad \text{for almost all } t\in S$$

$$\int_S f d\mu+\sigma(S)c(z)\leq \int_S \omega d\mu.$$

Being the allocations in P symmetric, it results $f(t)\sim f_i$ and $g(t)\sim g_i$ for almost all $t\in I_i$ and for every $i=1,\cdots,n$. So, being $u_i(f_i,y)>u_i(g_i,z)$ for every $i=1,\cdots,n$, and since (f,y) is feasible we obtain a contradiction since (g,z) is a Pareto optimal allocation. □

Proposition 2.15 *Under the assumption (h2), the set P is σ-externally stable.*

PROOF: Let us consider an individually rational allocation (f,y) which is not in P. Then or

(1) (f,y) is symmetric but not Pareto optimal or

(2) (f,y) is a nonsymmetric Pareto optimal allocation.

In both cases we have to find an allocation in P which σ-dominates (f,y).
If (1) holds, then, there exists an allocation (g,z) (that we can assume to be a Pareto optimal by compactness of $\mathcal{Y}$), such that

$$u_t(g(t),z)>u_t(f(t),y)\quad \text{for almost all } t\in I$$

$$\int_I g d\mu+c(z)\leq \int_I \omega d\mu.$$

So (g,z) is individually rational and if it is also symmetric, we can conclude that $(g,z)\in P$. Otherwise, as in the case (2), (g,z) is a nonsymmetric Pareto optimal allocation. Consequently, to complete the proof, we have to show the statement if the case (2) holds.
For every $j=1,\cdots,n$, let us define the set

$$S_j=\left\{t\in I_j\ :\ u_t(\widetilde{f}_j,y)>u_t(f(t),y)\text{ with } \widetilde{f}_j=\frac{1}{\mu(I_j)}\int_{I_j} f d\mu\right\}$$

We claim that there exists at least one index j such that $\mu(S_j)>0$. If $\mu(S_j)=0$, then it results $u_t(f(t),y)\geq u_t(\widetilde{f}_j,y)$ for almost all $t\in I_j$. Then,

$u_t(f(t),y)=u_t(\widetilde{f}_j,y)$ for almost all $t\in I_j$, or

$\mu(A_j)>0$, with $A_j=\left\{t\in I_j\ :\ u_j(f(t),y)>u_j(\widetilde{f}_j,y)\right\}$.

Let be $B_j=I_j\setminus A_j$, and assume $\mu(B_j)=0$. Then $u_j(f(t),y)>u_j(\widetilde{f}_j,y)$ for almost all $t\in I_j$, and so, by applying the Jensen' s inequality, we obtain $u_j(\frac{1}{\mu(I_j)}\int_{I_j} fd\mu,y)=u_j(\widetilde{f}_j,y)>u_j(f(t),y)$ and a contradiction. Then it results $\mu(B_j)>0$.
Now, let us denote by

$$A=\frac{1}{\mu(A_j)}\int_{A_j} fd\mu\ \ ,\ \ B=\frac{1}{\mu(B_j)}\int_{B_j} fd\mu\ \text{ and }\ \alpha=\frac{\mu(A_j)}{\mu(I_j)}.$$

Then $u_j(A,y)>u_j(\widetilde{f}_j,y)$, $u_j(B,y)>u_j(\widetilde{f}_j,y)$ and $\alpha A+(1-\alpha)B=\widetilde{f}_j$. Since utility functions are convex, we have $u_j(\alpha A+(1-\alpha)B,y)>u_j(\widetilde{f}_j,y)$, a contradiction. Then $\mu(a_j)=0$ and the allocation (f,y) is symmetric, a contradiction. Hence, without loss of generality, we can assume

that $\mu(S_1) > 0$.
By continuity, it results

$$S_1 = \cup_{n\in I\!N} \left\{ t \in S_1 \ : \ u_1(\frac{n}{n+1}\widetilde{f}_1, y) > u_1(f(t), y) \right\}$$

and, without loss of generality, we can assume that the set

$$P_1 = \left\{ t \in S_1 \ : \ u_1(\frac{n_0}{n_0+1}\widetilde{f}_1, y) > u_1(f(t), y) \right\}$$

has nonnull measure.
For any $j = 2, \cdots, n$ let us consider the set

$$P_j = \left\{ t \in S_1 \ : \ u_j(\widetilde{f}_j, y) \geq u_j(f(t), y) \right\}.$$

If $\mu(P_j) = 0$, then $u_j(f(t), y) > u_j(\widetilde{f}_j, y)$ for almost all $t \in I_j$ and so $u_j(\widetilde{f}_j, y) > u_j(\widetilde{f}_j, y)$, a contradiction. Hence $\mu(P_j) > 0$ for every $j = 1, \ldots, n$.
Let us define $\varepsilon < min\left\{ \frac{\mu(P_j)}{\mu(I_j)} \ : \ j = 1, \cdots, m \right\}$, and consider

- the set $Q_j \subset P_j$ such that $\sigma(Q_j) = \varepsilon\sigma(P_j)$ and $\mu(Q_j) = \varepsilon\mu(P_j)$,
- the function $h(t) = \begin{cases} \frac{n_0}{n_0+1}\widetilde{f}_1 & \text{if } t \in Q_1 \\ \widetilde{f}_j + \frac{1}{(m-1)(n_0+1)}\frac{\mu(Q_1)}{\mu(Q_j)}\widetilde{f}_1 & \text{if } t \in Q_j \text{ and } j \neq 1 \\ \widetilde{f}_j & \text{if } t \in I_j \setminus Q_j. \end{cases}$

It results

$$\int_I hd\mu = \frac{n_0}{n_0+1}\widetilde{f}_1\mu(Q_1) + \sum_{j=2}^{n} \widetilde{f}_j\mu(Q_j) + \frac{1}{n_0+1}\widetilde{f}_1\mu(Q_1) + \sum_{j=1}^{n} \widetilde{f}_j\mu(I_j \setminus Q_j) =$$

$$= \sum_{j=1}^{n} \widetilde{f}_j\mu(Q_j) + \sum_{j=1}^{n} \widetilde{f}_j\mu(I_j \setminus Q_j) = \sum_{j=1}^{n} \int_{I_j} fd\mu = \int_I fd\mu,$$

so $\int_I hd\mu + c(y) \leq \int_I \omega d\mu$, and the allocation (h, y) is feasible. Moreover

$$\sigma(Q = \cup_{j=1}^{m} Q_j) = \sum_{j=1}^{n} \sigma(Q_j) = \sum_{j=1}^{n} \varepsilon\sigma(I_j) = \varepsilon\sigma(I) = \varepsilon$$

and also $\mu(Q) = \varepsilon$. Hence

$$\int_Q hd\mu = \frac{n_0}{n_0+1}\widetilde{f}_1\mu(Q_1) + \sum_{j=2}^{n} \widetilde{f}_j\mu(Q_j) + \frac{1}{n_0+1}\widetilde{f}_1\mu(Q_1) = \sum_{j=1}^{n} \widetilde{f}_j\varepsilon\mu(I_j) = \varepsilon\int_I fd\mu$$

and

$$\int_Q hd\mu + \sigma(Q)c(y) = \varepsilon\int_I fd\mu + \varepsilon c(y) \leq \varepsilon\int_I \omega d\mu = \int_Q \omega d\mu.$$

By construction the allocation (h, y) is strictly preferred to (f, y) on Q, so if (h, y) is not individually rational, then we have $u_t(\omega(t), 0) > u_t(\widetilde{f}_j, y)$ for almost all $t \in I_j$ and $j = 1, \cdots, n$, a contradiction. Now let us consider the function $\widetilde{h}$ defined as $\widetilde{h}_j(t) = \frac{1}{\mu(I_j)}\int_{I_j} hd\mu$ for all $t \in I_j$ and $j = 1, \cdots, n$. By definition of h and choosing suitable $\widetilde{\varepsilon}$ and a_j, we can write $\widetilde{h}_j = \widetilde{\varepsilon}\widetilde{f}_j + (1-\widetilde{\varepsilon})a_j$, so that the allocation $(\widetilde{h}, y)$ is strictly preferred to (f, y) almost everywhere on Q. Moreover $(\widetilde{h}, y)$ is obviously feasible on Q, individually rational and symmetric; so if $(\widetilde{h}, y)$ is also Pareto optimal the proof is complete. If $(\widetilde{h}, y)$ is not Pareto optimal, then the corresponding allocation in the finite economy $\mathcal{E}$,

$(\widetilde{h_1}, \cdots, \widetilde{h_n}, y)$ is not Pareto optimal, so there exists a Pareto optimal allocation $(l_1, \cdots, l_m, s)$ such that $\sum_{j=1}^{m} l_j + c(s) \leq \sum_{j=1}^{m} \omega_j$ and $u_j(l_j, s) > u_j(\widetilde{h_j}, y)$ for all $j = 1, \cdots, n$. The corresponding allocation (l, s) in the continuum economy $\mathcal{E}_{\mathcal{C}}$ is then symmetric, Pareto optimal and dominates (h, y), a contradiction. □

Using previous results we can conclude that

Proposition 2.16 *Under assumptions (h1) and (h2) the set P is the unique σ-stable set.*

3 σ-sophisticated stable sets in $\mathcal{E}$

Let us denote by A the set of feasible allocations of the economy $\mathcal{E}$ and by V the set of feasible payoffs that is

$$V = \{u(x, y) = (u_1(x_1, y), \ldots, u_n(x_n, y)) \mid (x_1, \cdots, x_n, y) \in A\}.$$

Moreover for a set $B \subseteq A$, let

$$u(B) \equiv \{u(x, y) \mid (x, y) = (x_1, \ldots, x_n, y) \in B\}.$$

Finally, for a coalition S, $(x_1, \cdots, x_n, y) \in A$ and $\xi \in V$, we denote by (x^S, y) and ξ^S the restrictions of $(x_1, \cdots, x_n, y)$ and ξ on S, respectively.
For a coalition S, the set of *S_σ-feasible allocations* is given by

$$A(S) = \left\{ \Big((x_i)_{i \in S}, y \Big) \in I\!\!R_+^{m|S|} \times \mathcal{Y} \mid \sum_{i \in S} x_i + \sigma(S) c(y) \leq \sum_{i \in S} \omega_i \right\},$$

and the set of *S_σ-feasible payoffs* is given by

$$V(S) = \left\{ \xi \in I\!\!R_+^{|S|} \mid \exists \Big((x_i)_{i \in S}, y \Big) \in A(S) \text{ such that } \xi_i = u_i(x_i, y) \right\}.$$

Proposition 3.1 *Let C^P denote the payoff σ-core, that is the set of all payoffs in V that are undominated. Then*

$$C^P = \{u(x, y) \mid (x, y) = (x_1, \cdots, x_n, y) \in \sigma\text{-}core\}$$

PROOF: Assume by contradiction that there exists $\xi \in C^P$ and $(x, y) = (x_1, \cdots, x_n, y) \in A$ such that $\xi = u(x, y)$ and (x, y) not belonging to the σ-core. Then there exists a not empty coalition S and $\Big((x_i)_{i \in S}, y \Big) \in A(S)$ such that

$$\sum_{i \in S} g_i + \sigma(S) c(z) \leq \sum_{i \in S} \omega_i \text{ , and}$$

$$u_i(g_i, z) > u_i(x_i, y) \text{ for all } i \in S.$$

But then, $(u_i(g_i, z))_{i \in S}$ dominates ξ on S, contradicting $\xi \in C^P$.
Conversely, let $(x, y) = (x_1, \cdots, x_n, y)$ belong to the σ-core and assume $u(x, y) \notin C^P$. Then there exists $\eta \in V$ that dominates $u(x, y)$, that is there exists a not empty coalition S and $\Big((x_i)_{i \in S}, y \Big)$ belonging to $A(S)$ such that $\eta_i = u_i(g_i, z) > u_i(x_i, y)$ for all $i \in S$. Then $\Big((x_i)_{i \in S}, y \Big)$ dominates (x, y) on S, hence a contradiction. So the proposition is proved. □

Since for every coalition S we have $V(S) = u(A(S))$, it seems plausible to expect that the analogous of proposition 3.1 holds for σ-stable sets, namely, that is if H^A is an allocation σ-stable set then $u(H^A)$ is a payoff σ-stable set and conversely if H^P is a payoff σ-stable set then $u^{-1}(H^P)$ is an allocation σ-stable set. However, as shown in [17] for the case of economies without public projects and contribution measures, this result is not true. Because of such reasoning, in line with [15], we suggest the following modification of the notion of σ-stability.

Definition 3.2 *For ξ and η in V, we say that η indirectly σ-dominates ξ and write $\eta \succ\succ \xi$, if there exists a sequence of feasible payoff vectors and coalitions $\{\{\xi^\nu\}_{\nu=0}^m, \{S^\nu\}_{\nu=1}^m\}$ such that $\xi^0 = \xi$, $\xi^m = \eta$ and for $j = 1, \ldots, m$ and for all $i \in S^j$ the following three conditions hold:*

$$\xi^{j,S_j} \in V(S^j), \qquad \xi_i^{j-1} < \xi_i^j, \qquad \xi_i^{j-1} < \xi_i^m, \tag{4}$$

where ξ^{j,S_j} represents the restriction to coalition S_j of the feasible payoff ξ^j.
A set of payoffs $H^P \subseteq V$ is a payoff σ-sophisticated stable set if

$$\xi \in V \setminus H^P \iff \text{there exists } \eta \in H^P \text{ such that } \eta \succ\succ \xi.$$

Extending the definition to the allocations space, we have

Definition 3.3 *For $(x_1, \cdots, x_n, y)$ and $(g_1, \ldots, g_n, z)$ in A, we say that $(g_1, \ldots, g_n, z)$ indirectly σ-dominates $(x_1, \cdots, x_n, y)$ and write $(g, z) \succ\succ (f, y)$, if there exists a sequence of feasible allocations and coalitions $\{\{(x^\nu, y^\nu)\}_{\nu=0}^m, \{S^\nu\}_{\nu=1}^m\}$ such that $(x^0, y^0) = (x, y)$, $(x^m, y^m) = (g, z)$ and for $j = 1, \ldots, m$ and for all $i \in S^j$ the following three conditions hold:*

$$(x^{j,S_j}, y^j) \in A(S^j), \quad u_i(x_i^{j-1}, y^{j-1}) < u_i(x_i^j, y^j), \quad u_i(x_i^{j-1}, y^{j-1}) < u_i(x_i^m, y^m), \tag{5}$$

where (x^{j,S_j}, y) represents the restriction to coalition S_j of the feasible allocation (x^j, y).

A set of allocations $H^A \subseteq A$ is an allocation σ-sophisticated stable set if

$$(x_1, \cdots, x_n, y) \in A \setminus H^A \iff \text{there exists } (g_1, \ldots, g_n, z) \in H^A \text{ s.t. } (g, z) \succ\succ (x, y).$$

An important feature of the σ-sophisticated stable sets is that every element in such a set is individually rational.

Proposition 3.4 *Assume that the set of linear cost share equilibria is not empty. If $(g_1, \ldots, g_n, z)$ belongs to the allocations σ-sophisticated stable set H^A, then $(g_1, \ldots, g_n, z)$ is individually rational. If ξ belongs to the payoffs σ-sophisticated stable set H^P, then ξ is individually rational.*

PROOF: Let $(g, z) = (g_1, \ldots, g_n, z) \in A$ be not individually rational, that is there exists $k \in I$ such that $u_k(g_k, z) < u_k(\omega_k, 0)$. Since $u_k(\cdot, 0)$ is continuous and strictly monotone, and $\omega_k \neq 0$, there exists $h_k < \omega_k$ such that

$$u_k(g_k, z) < u_k(h_k, 0) < u_k(\omega_k, 0).$$

Moreover, being $c(0) = 0$, we have that the allocation $(h, 0)$ with $h_i = 0$ if $i \neq k$ σ-dominates (g, z) on the coalition $\{k\}$. Let $(x, y) = (x_1, \ldots, x_n)$ be a linear cost share equilibrium, then (x, y) is individually rational. If not, we can find a $k \in I$ such that, denoted with $p(\cdot)$ the price system corresponding to (x, y), it results

$$p(0) \cdot \omega_k = p(0) \cdot \omega_k + p(0) \cdot \varphi(k) c(0) > p(0) \cdot \omega_k,$$

hence a contradiction.

So $(h, 0)$ is σ-dominated by (x, y) using the grand coalition.

So $(x, y) \succ\succ (g, z)$. Since (x, y) in the σ-Aubin core, which is contained in the σ-core, external stabilities implies $(x, y) \in H^A$, so $(g, z) \notin H^A$.
An analogous argument used in the utility space shows that if $\xi \in V$ is not individually rational then $\xi \notin H^P$. □

Lemma 3.5 *Let η and ξ in V, and let $\eta = u(x,y)$ be individually rational. If $\eta \succ\succ \xi$, then there exists $\overline{\eta} = u(\overline{x}, y)$ such that $\overline{\eta} \succ\succ \xi$ and $\overline{\eta}_i < \eta_i$.*

PROOF: Since $\eta \succ\succ \xi$ there exists a sequence of feasible payoff vectors and coalitions $\{\{\xi^\nu\}_{\nu=0}^m, \{S^\nu\}_{\nu=1}^m\}$ such that $\xi^0 = \xi$, $\xi^m = \eta$ and for $j = 1, \ldots, m$ and for all $i \in S^j$ conditions (4) hold. Being $\omega_i \neq 0$, $\eta_i = u_i(x_i, y) \geq u_i(\omega_i, 0)$ for all i and $u_i(\cdot, y)$ continuous and strictly monotone, we can choose $\alpha \in (0,1)$ such that $\left\{\{(\xi^\nu)_{\nu=0}^{m-1}, u(\alpha x, y)\right\}, \{S^\nu\}_{\nu=1}^m\}$ satisfies conditions (4). So the lemma is proved with $\overline{\eta} = u(\alpha x, y)$. □

Lemma 3.6 *Let $(g_1, \ldots, g_n, z)$ and $(x_1, \ldots, x_n, y)$ in A, $\xi = u(g, z)$ and assume that $\eta = u(x, y)$ is individually rational. Then, $\eta \succ\succ \xi$ if and only if $(x, y) \succ\succ (g, z)$.*

PROOF: Assume that $\eta \succ\succ \xi$, by lemma 3.5 there exists $\overline{\eta} = u(\overline{x}, y)$ such that $\overline{\eta} \succ\succ \xi$ and $\overline{\eta}_i < \xi_i$. That is, there exists a sequence of feasible payoff vectors and coalitions $\{\{\xi^\nu\}_{\nu=0}^m, \{S^\nu\}_{\nu=1}^m\}$ such that $\xi^0 = \xi$, $\xi^m = \overline{\eta}$ and for $j = 1, \ldots, m$ and for all $i \in S^j$ conditions (4) hold. So, for every $j = 1, \ldots, m-1$ there exists a feasible allocation (x^j, y^j) such that $(x^{j,S_j}, y^j) \in A(S^j)$ and $\xi_i^j = u_i(x_i^j, y^j)$. We now consider the sequence $\left\{(x^j, y^j)\right\}_{j=0}^{m+1}$ of allocations in A with $(x^0, y^0) = (g, z)$, $(x^m, y^m) = (\overline{x}, y)$ and $(x^{m+1}, y^{m+1}) = (x, y)$. Obviously $\left\{\{(x^\nu, y^\nu\}_{\nu=0}^{m+1}, \{(S^\nu)_{\nu=1}^m, I\}\right\}$ satisfies conditions (5), thus $(x, y) \succ\succ (g, z)$. Conversely, assume that $(x, y) \succ\succ (g, z)$. Then there exists a sequence of feasible allocations and coalitions $\{\{(x^\nu, y^\nu)\}_{\nu=0}^m, \{S^\nu\}_{\nu=1}^m\}$ such that $(x^0, y^0) = (g, z)$, $(x^m, y^m) = (x, y)$ and for $j = 1, \ldots, m$ and for all $i \in S^j$ conditions (5) hold. It is easy to check that, by using the sequence $\{\{\xi^\nu\}_{\nu=0}^m, \{S^\nu\}_{\nu=1}^m\}$ with $\xi^j = u(x^j, y^j)$, we can conclude that $\eta \succ\succ \xi$. □

Theorem 3.7 *If H^A is an allocation σ-sophisticated stable set then $u(H^A)$ is a payoff σ-sophisticated stable set and conversely, if H^P is a payoff σ- sophisticated stable set then $u^{-1}(H^P)$ is an allocation σ-sophisticated stable set.*

PROOF: Let H^A an allocation σ-sophisticated stable set, we need to prove that $\xi \in V \setminus u(H^A)$ if and only if there exists η in $u(H^A)$ such that $\eta \succ\succ \xi$. Let $\xi \in V \setminus u(H^A)$ and assume that $\xi = u(x, y)$ with $(x, y) = (x_1, \ldots, x_n, y) \in A$. Then $(x, y) \notin H^A$ and, since H^A is a σ-sophisticated stable set, there exists $(g, z) = (g_1, \ldots, g_n, z) \in H^A$ such that $(g, z) \succ\succ (x, y)$. Then $\eta = u(g, z) \in u(H^A)$ indirectly dominates ξ. Conversely assume by contradiction that there exists ξ and η in $u(H^A)$ such that $\eta \succ\succ \xi$. Then there exists $(x, y) = (x_1, \ldots, x_n, y)$ and $(g, z) = (g_1, \ldots, g_n, z)$ in H^A such that $\xi = u(x, y)$ and $\eta = u(g, z)$. By lemma 3.6 and proposition 3.4, it follows that $(g, z) \succ\succ (x, y)$ which contradicts the internal stability of H^A. An analogous argument proves the property in the utility space. □

4 σ-sophisticated stable sets in $\mathcal{E}_\mathcal{C}$

It is the aim of this section to extend the previous results to the continuum economy $\mathcal{E}_\mathcal{C}$. Recall that, if we consider the associated economies $\mathcal{E}_\mathcal{C}$ and $\mathcal{E}$, then conditions (1) and (2) define the natural relation existing between the contribution measures $\widehat{\sigma}$ and σ.

Let us denote by $A_\mathcal{C}$ the set of feasible allocations of the economy $\mathcal{E}_\mathcal{C}$ and by $V_\mathcal{C}$ the set of feasible payoffs that is

$$V = \{\xi : I \to I\!R_+ \mid \xi(t) = u_t(f(t), y) \forall t \in I \text{ with } (f, y) \in A_\mathcal{C}\}.$$

Moreover for a set $B \subseteq A_\mathcal{C}$, let

$$u(B) \equiv \{\xi : I \to I\!R_+ \mid \xi(t) = u_t(f(t), y) \forall t \in I \text{ with } (f, y) \in B\}.$$

Finally, for a coalition S, $(f, y) \in A_\mathcal{C}$ and $\xi \in V_\mathcal{C}$, we denote by (f^S, y) and ξ^S the restrictions of (f, y) and ξ on S, respectively.

For a coalition S, the set of S_σ*-feasible allocations* is given by

$$A_{\mathcal{C}}(S) = \left\{(f,y) \text{ with } f \text{ intergable function and } y \in \mathcal{Y} \mid \int_S f d\mu + \sigma(S)c(y) \leq \int_S \omega d\mu \right\},$$

and the set of S_σ*-feasible payoffs* is given by

$$V_{\mathcal{C}}(S) = \{\xi : S \to I\!R_+ \mid \exists (f,y) \in A_{\mathcal{C}}(S) \text{ such that } \xi(t) = u_t(f(t),y) \; \forall t \in S\}.$$

Proposition 4.1 *Let $C^P_{\mathcal{C}}$ denote the payoff σ-core, that is the set of all payoffs in $V_{\mathcal{C}}$ that are undominated. Then*

$$C^P_{\mathcal{C}} = \{\xi : I \to I\!R_+ \mid \xi(t) = u_t(f(t),y) \; \forall t \in I \text{ and } (f,y) \in \sigma\text{-core}\}$$

PROOF: Assume by contradiction that there exists $\xi \in C^P_{\mathcal{C}}$ and $(f,y) \in A_{\mathcal{C}}$ such that $\xi(t) = u_t(f(t),y)$ and (f,y) does not belong to the σ-core. Then there exist a coalition S with nonnull measure, a μ-integrable function $g : S \to I\!R^m_+$, and a public project z such that

$$\int_S g d\mu + \sigma(S)c(z) \leq \int_S \omega d\mu \quad \text{, and}$$

$$u_t(g(t),z) > u_t(f(t),y) \text{ for almost all } t \in S.$$

But then, $\eta(t) = u_t(g(t),z)$ dominates ξ on S, contradicting $\xi \in C^P_{\mathcal{C}}$.
Conversely, let (f,y) belong to the σ-core and assume that the function $\xi : I \to I\!R_+$ defined as $\xi(t) = u_t(f(t),y) \notin C^P_{\mathcal{C}}$. Then there exists $\eta \in V_{\mathcal{C}}$ that dominates ξ, that is there exists a coalition S with nonnull measure and (g,z) belonging to $A_{\mathcal{C}}(S)$ such that $\eta(t) = u_t(g(t),z) > u_t(f(t),y)$ for almost all $t \in S$. Then (g,z) dominates (f,y) on S, hence a contradiction. So the proposition is proved. □

Since the analogous of proposition 4.1 does not hold for σ-stable sets, in line with Harsanyi (1974), we suggest the following modification of the notion of σ-stability.

Definition 4.2 *For ξ and η in V, we say that η indirectly σ-dominates ξ and write $\eta \succ\!\succ \xi$, if there exists a sequence of feasible payoff functions and coalitions with nonnull measure $\{\{\xi^\nu\}^m_{\nu=0}, \{S^\nu\}^m_{\nu=1}\}$ such that $\xi^0 = \xi$, $\xi^m = \eta$ and for $j = 1,\dots,m$ and for all $t \in S^j$ the following three conditions hold:*

$$\xi^{j,S_j} \in V_{\mathcal{C}}(S^j), \qquad \xi^{j-1}_i(t) < \xi^j_i(t), \qquad \xi^{j-1}_i(t) < \xi^m_i(t). \tag{6}$$

A set of payoffs $H^P \subseteq V_{\mathcal{C}}$ is a payoff σ-sophisticated stable set if

$$\xi \in V_{\mathcal{C}} \setminus H^P \iff \text{there exists } \eta \in H^P \text{ such that } \eta \succ\!\succ \xi.$$

Lemma 4.3 *i) Assume that $u(f,y)$ indirectly $\widehat{\sigma}$-dominates $u(g,z)$ and consider the associated payoffs in $\mathcal{E}$, $(u_1(x_1,y),\dots,u_n(x_n,y))$ and $(u_1(g_1,z),\dots,u_n(g_n,z))$ with $x_i = \frac{1}{\mu(I_i)}\int_{I_i} f d\mu$ and $g_i = \frac{1}{\mu(I_i)}\int_{I_i} g d\mu$. If in the economy $\mathcal{E}$ assumption (3) is satisfied, then $(u_1(x_1,y),\dots,u_n(x_n,y)) \succ\!\succ (u_1(g_1,z),\dots,u_n(g_n,z))$.*

ii) Assume that, in the economy $\mathcal{E}$, the payoff $(u_1(x_1,y),\dots,u_n(x_n,y))$ σ-dominates $(u_1(g_1,y),\dots,u_n(g_n,z))$, and consider the associated payoffs in $\mathcal{E}_{\mathcal{C}}$, $u(f,y)$ and $u(g,z)$ with $f(t) = x_i$ and $g(t) = g_i$ for all $t \in I_i$ and $i = 1,\dots,n$. Then $u(f,y)$ $\widehat{\sigma}$-dominates $u(g,z)$.

PROOF: *i)* Let us consider the sequence of feasible payoffs and coalitions with nonnull measure $\{\{\xi^\nu\}^m_{\nu=0}, \{S^\nu\}^m_{\nu=1}\}$ with $\xi^\nu = u(f^\nu,y)$ satisfying conditions (6), and define

the payoff $(u_1(x^\nu_1,y^\nu),\dots,u_n(x^\nu_n,y^\nu))$ with $x^\nu_i = \frac{1}{\mu(I_i)}\int_{I_i} f^\nu d$, for any $\nu = 0,\dots,m$;

the coalition $\overline{S^\nu} = \cup_{\{I_i \cap S^\nu \neq \emptyset\}} I_i$, for any $\nu = 1,\dots,m$.

In light of Liapunov arguments and previous results, we can immediately prove that, using the sequence of feasible payoffs and coalitions

$$\left\{\{(u_1(x_1^\nu, y^\nu), \ldots, u_n(x_n^\nu, y^\nu))\}_{\nu=0}^m, \{\overline{S^\nu}\}_{\nu=1}^m\right\},$$

the payoff $(u_1(x_1, y), \ldots, u_n(x_n, y))$ indirectly σ-dominates $(u_1(g_1, z), \ldots, u_n(g_n, z))$.
ii) Let us consider the sequence $\{\{(u_1(x_1^\nu, y^\nu), \ldots, u_n(x_1^\nu, y^\nu))\}_{\nu=0}^m, \{S^\nu\}_{\nu=1}^m\}$ satisfying conditions (4), and define

the payoff $u(f^\nu, y^\nu)$ with $f^\nu(t) = x_i^\nu$, for any $\nu = 0, \ldots, m$ and $t \in I_i$;

the coalition $\widehat{S}^\nu = \cup_{i \in S^\nu} I_i$, for any $\nu = 1, \ldots, m$.

We can immediately prove that, thanks to sequence $\left\{\{u(f^\nu, y^\nu)\}_{\nu=0}^m, \left\{\widehat{S}^\nu\right\}_{\nu=1}^m\right\}$, the payoff $u(f, y)$ $\widehat{\sigma}$-dominates $u(g, z)$.

□

From the previous lemma, it follows that

Proposition 4.4 *Assume (3). If $H_{\mathcal{C}}^P$ is a payoffs $\widehat{\sigma}$-sophisticated stable set in $\mathcal{E}_{\mathcal{C}}$, then the corresponding set*

$$H^P = \left\{(u_1(x_1, y), \ldots, u_n(x_n, y)) \mid x_i = \frac{1}{\mu(I_i)} \int_{I_i} f d\mu \text{ and } u(f, y) \in H_{\mathcal{C}}^P\right\}$$

is a payoffs σ- sophisticated stable set in $\mathcal{E}$. Reciprocally, if H^P is a payoffs σ-sophisticated stable set in $\mathcal{E}$, then the corresponding set

$$H_{\mathcal{C}}^P = \left\{u(f, y) \mid f(t) = x_i \ \forall t \in I_i \text{ and } (u_1(x_1, y), \ldots, u_n(x_n, y)) \in H^P\right\}$$

is a payoffs $\widehat{\sigma}$- sophisticated stable set in $\mathcal{E}$.

Extending the definition to the allocations space, we have

Definition 4.5 *For (f, y) and (g, z) in $A_{\mathcal{C}}$, we say that (g, z) indirectly σ-dominates (f, y) and write $(g, z) \succ\succ (f, y)$, if there exists a sequence of feasible allocations and coalitions with nonnull measure $\{\{(f^\nu, y^\nu)\}_{\nu=0}^m, \{S^\nu\}_{\nu=1}^m\}$ such that $(f^0, y^0) = (f, y)$, $(f^m, y^m) = (g, z)$ and for $j = 1, \ldots, m$ and for all $t \in S^j$ the following three conditions hold:*

$$\begin{aligned} &(f^{j,S_j}, y^j) \in A_{\mathcal{C}}(S^j), \\ &u_t(f^{j-1}(t), y^{j-1}) < u_t(f^j(t), y^j), \\ &u_t(f^{j-1}(t), y^{j-1}) < u_t(f^m(t), y^m) \end{aligned} \tag{7}$$

A set of allocations $H^A \subseteq A_{\mathcal{C}}$ is an allocation σ-sophisticated stable set if

$$(g, z) \in A_{\mathcal{C}} \setminus H^A \iff \text{there exists } (g, z) \in H^P \text{ such that } (f, y) \succ\succ (g, z).$$

Lemma 4.6 *i) Assume (f, y) indirectly $\widehat{\sigma}$-dominates (g, z) and consider the associated allocations in $\mathcal{E}$, $(x_1, \ldots, x_n, y)$ and $(g_1, \ldots, g_n, z)$ with $x_i = \frac{1}{\mu(I_i)} \int_{I_i} f d\mu$ and $g_i = \frac{1}{\mu(I_i)} \int_{I_i} g d\mu$. If in the economy $\mathcal{E}$ assumption (3) is satisfied, then $(x_1, \ldots, x_n, y)$ σ-dominates $(g_1, \ldots, g_n, z)$.*

ii) Assume that, in the economy $\mathcal{E}$, $(x_1, \ldots, x_n, y)$ σ-dominates $(g_1, \ldots, g_n, z)$, and consider the associated allocations in $\mathcal{E}_{\mathcal{C}}$, (f, y) and (g, z) with $f(t) = x_i$ and $g(t) = g_i$ for all $t \in I_i$. Then (f, y) $\widehat{\sigma}$-dominates (g, z).

PROOF:

i) Let us consider the sequence of feasible allocations and coalitions with nonnull measure $\{\{(f^\nu, y^\nu)\}_{\nu=0}^m, \{S^\nu\}_{\nu=1}^m\}$ satisfying conditions (7), and define

the allocation (x^ν, y^ν) with $x_i^\nu = \frac{1}{\mu(I_i)} \int_{I_i} f^\nu d\mu$, for any $\nu = 0, \ldots, m$;

the coalition $\overline{S^\nu} = \cup_{\{I_i \cap S^\nu \neq \emptyset\}} I_i$, for any $\nu = 1, \ldots, m$.

In light of Liapunov arguments and previous results (as in the proof of lemma 2.5), we can immediately prove that, using the sequence of feasible allocations and coalitions $\left\{\{(x^\nu, y^\nu)\}_{\nu=0}^m, \{\overline{S^\nu}\}_{\nu=1}^m\right\}$, the allocation $(x_1, \ldots, x_n, z)$ indirectly σ-dominates $(g_1, \ldots, g_n, z)$.

ii) Let us consider the sequence $\{\{(x^\nu, y^\nu)\}_{\nu=0}^m, \{S^\nu\}_{\nu=1}^m\}$ satisfying conditions (5), and define

the allocation (f^ν, y^ν) with $f^\nu(t) = x_i^\nu$, for any $\nu = 0, \ldots, m$;

the coalition $\widehat{S}^\nu = \cup_{i \in S^\nu} I_i$, for any $\nu = 1, \ldots, m$.

We can immediately prove that, thanks to sequence $\left\{\{(f^\nu, y^\nu)\}_{\nu=0}^m, \left\{\widehat{S}^\nu\right\}_{\nu=1}^m\right\}$, the allocation (f, y) $\widehat{\sigma}$-dominates (g, z).

□

From the previous lemma, it follows that

Proposition 4.7 *Assume (3). If $H_{\mathcal{C}}^A$ is an allocations $\widehat{\sigma}$-sophisticated stable set in $\mathcal{E}_{\mathcal{C}}$, then the corresponding set $H^A = \left\{(x_1, \ldots, x_n, y) \mid x_i = \frac{1}{\mu(I_i)} \int_{I_i} f d\mu \text{ and } (f, y) \in H_{\mathcal{C}}^A\right\}$ is an allocations σ-sophisticated stable set in $\mathcal{E}$. Reciprocally, if H^A is an allocation σ-sophisticated stable set in $\mathcal{E}$, then the corresponding set*

$$H_{\mathcal{C}}^A = \left\{(f, y) \mid f(t) = x_i \ \forall t \in I_i \text{ and } (x_1, \ldots, y) \in H^A\right\}$$

is an allocations $\widehat{\sigma}$- sophisticated stable set in $\mathcal{E}$.

Theorem 4.8 *If $H_{\mathcal{C}}^A$ is an allocations $\widehat{\sigma}$-sophisticated stable set then $u(H_{\mathcal{C}}^A)$ is a payoffs $\widehat{\sigma}$-sophisticated stable set and conversely, if $H_{\mathcal{C}}^P$ is a payoff $\widehat{\sigma}$- sophisticated stable set then $u^{-1}(H_{\mathcal{C}}^P)$ is an allocations $\widehat{\sigma}$-sophisticated stable set.*

PROOF: The theorem follows from propositions 4.4 and 4.4. □

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