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Giuseppe De Marco* and Jacqueline Morgan**

Abstract
This paper studies new refinement concepts for correlated equilibria based on altruistic behavior of the players and which generalize some refinement concepts introduced by the authors in previous papers for Nash equilibria. Effectiveness of the concepts, relations with the corresponding notions for Nash equilibria and with other correlated equilibrium refinements are investigated. The analysis of the topological properties of the set of solutions concludes the paper.

Keywords: correlated equilibrium, altruistic behavior, refinement.

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1 Introduction

The correlated equilibrium concept (Aumann (1974, 1987)) appears as the appropriate solution concept in games where pre-play communication is allowed between the players. In such games, the presence of a mediator allows players to use correlated strategies: the mediator (privately) recommends actions to each player according to the realization of an (agreed upon) correlation device. Hence a correlated equilibrium is a self-enforcing correlated strategy profile in the sense that no individual has an incentive to deviate from the recommendation received.

It is well known that the set of correlated equilibria is convex and contains the set of Nash equilibria (considered as distributions on the set of strategy profiles); therefore on one hand the correlated equilibrium concept allows to reach agreements which are (sometimes) more *compelling* than Nash equilibrium agreements[1], but, on the other hand, it involves a problem of multiplicity of the set of solutions (and the corresponding *drawbacks* of the solutions) which is even more evident with respect to the Nash equilibrium concept. In fact, since in case of multiplicity a Nash equilibrium may be not robust with respect to perturbations on the strategies or on the payoffs or it might be unstable with respect to mutually beneficial deviations of coalitions of players, a correlated equilibrium may suffer from the same drawbacks even more frequently. However, only few papers in the literature have focused on the problem of correlated equilibrium selection; in particular, Myerson (1986) and Dhillon and Mertens (1996) focus on the problem of stability with respect to trembles proposing some generalizations of the perfect equilibrium concept while Milgrom and Roberts (1996), Moreno and Wooders (1996), Ray (1996) and Bloch and Dutta (2009) look at refinements of the correlated equilibrium concept based on coalitional stability. In this work, we analyze refinements of the correlated equilibria based on altruistic behavior which generalize (in some sense) the concepts introduced and investigated in De Marco and Morgan (2008;a,b).

Recent empirical and theoretical literature has shown that there exists a substantial evidence suggesting that fairness motives affect the behavior of many people. Empirical results (see for instance Fehr and Schimdt (1999) and references therein) and theoretical papers (see Rabin (1993) or Falk and Fischbacher (2003) and references therein) show how in some strategic situations altruistic behavior may emerge; in particular, most theoretical papers describe reciprocal altruism and equilibrium behavior by considering psychological game theory (Geanakoplos, Pearce and Stacchetti (1989)) which usually gives different predictions with respect to the standard notions of equilibrium in games[2].

The idea to use altruistic behavior for equilibrium selection has been firstly proposed in Rusinowska (2002) by introducing (in a class of bargaining problems) the concept of *friendly behavior*: a player is supposed to move away from the equilibrium even only to guarantee a better payoff to the others. *Friendliness equilibria* (De Marco and Morgan

[1] Correlated equilibria usually enlarges the set of outcomes providing higher payoffs with respect to some or all Nash equilibrium payoff vectors.

[2] See also De Marco and Morgan (2008;a,b) for an extensive list of references on reciprocal altruism in games.

(2008,a)) provide an application of the friendly behavior property to general strategic form games. While, the concept of *slightly altruistic equilibrium* (De Marco and Morgan (2008,b)) is based on a stability property with respect to trembles which captures an idea of reciprocal altruism: each player cares only about himself but his choice corresponds to the limit of choices he would have done in equilibrium if he had cared about the others, provided the others had done the same. In general, a slightly altruistic equilibrium is not necessarily a friendliness equilibrium and viceversa. However, sufficient conditions on the payoffs of the game guarantee that every slightly altruistic equilibrium is a friendliness equilibrium. Moreover, it is possible to enforce the robustness property of friendliness equilibria (*strong friendliness equilibria*) in order to obtain strategy profiles which are also slightly altruistic equilibria.

In this work, first, we extend the notion of slightly altruistic equilibria to correlated strategies by introducing the so-called *slightly altruistic correlated equilibrium* concept. It turns out from the examples that, on one hand, this concept can provide an effective selection device for correlated equilibria and, on the other hand, it allows to reach agreements which are more compelling than those determined by slightly altruistic (Nash) equilibria. We show that existence of slightly altruistic correlated equilibria follows easily from the existence of slightly altruistic Nash equilibria; moreover, connections with *essentiality* of equilibria are investigated[3].

In the second part of the paper we study the role of friendly behavior in the problem of correlated equilibrium selection and define the concept of *strong friendliness correlated equilibrium*. This concept, on one hand, provides an effective refinement for slightly altruistic correlated equilibria. On the other hand, it is a natural extension of the strong friendliness (Nash) equilibrium concept. We show also that there exist games with no strong friendliness correlated equilibria and that it is possible to obtain a refinement for both the slightly altruistic correlated equilibrium and the strong friendliness correlated equilibrium concepts by introducing *altruistic equilibria*. This latter concept is obtained by enforcing the stability property in the definition of slightly altruistic correlated equilibria.

Finally, since the set of correlated equilibria is closed and convex and such properties turn to be useful in the applications, we conclude the paper by analyzing such properties for the set of solutions corresponding to the refinement concepts introduced in this paper.

2 Slightly Altruistic Correlated Equilibria

2.1 Definition and existence

Let $\Gamma = \{I; X_1, \dots, X_N; f_1, \dots, f_N\}$ be a N-player game where $I = \{1, \dots, N\}$ is the set of players, X_i is the finite set of strategies of player i, and $f_i : X \to \mathbb{R}$ is the payoff function of player i, with $X = \prod_{i \in I} X_i$. Denote with Ω the set of all probability distributions on

[3]Essentiality is the strongest kind of stability (with respect to perturbation) property for a solution concept in a game (see Wu and Jiang (1962) or van Damme (1987)) .

X.

Definition 2.1 (Aumann): A probability distribution μ on X (also called correlated strategy) is a correlated equilibrium if for every player i and every strategy $\bar{x}_i \in X_i$,

$$\sum_{x_{-i} \in X_{-i}} \mu\left(x_{-i}|\bar{x}_i\right) f_i(\bar{x}_i, x_{-i}) \geq \sum_{x_{-i} \in X_{-i}} \mu\left(x_{-i}|\bar{x}_i\right) f_i(x_i, x_{-i}) \quad \forall x_i \in X_i. \tag{1}$$

where $\mu\left(x_{-i}|\bar{x}_i\right)$ is the conditional probability of x_{-i} given $\bar{x}_i$, that is

$$\mu\left(x_{-i}|\bar{x}_i\right) = \frac{\mu\left(\bar{x}_i, x_{-i}\right)}{\sum_{\widetilde{x}_{-i} \in X_{-i}} \mu\left(\bar{x}_i, \widetilde{x}_{-i}\right)}$$

whenever $\sum_{\widetilde{x}_{-i} \in X_{-i}} \mu\left(\bar{x}_i, \widetilde{x}_{-i}\right) \neq 0$ and $\mu\left(x_{-i}|\bar{x}_i\right) = 0$ otherwise.
In other words, $\mu\left(x_{-i}|\bar{x}_i\right)$ is the probability that player i assigns to the strategy profile x_{-i} of his opponents once the mediator has communicated player i to play $\bar{x}_i$. Hence, μ is a correlated equilibrium if the expected payoff from playing the recommended strategy is no worse than playing any other strategy.

Now we introduce the main features of our approach. Let $g_i : X \to \mathbb{R}$ be the function defined by:

$$g_i(x) = \sum_{j \in I \setminus \{i\}} f_j(x) \quad \text{for all } x \in X. \tag{2}$$

Definition 2.2: Let ε be a positive real number and, for each player i, let $h_{i,\varepsilon} : X \to \mathbb{R}$ be the function, called *ε-altruistic payoff*, defined by:

$$h_{i,\varepsilon}(x) = f_i(x) + \varepsilon g_i(x) \qquad \text{for all } x \in X. \tag{3}$$

For every $\varepsilon > 0$, the game $\Gamma_\varepsilon = \{I; X_1, \ldots, X_N;\ h_{1,\varepsilon}, \ldots, h_{N,\varepsilon}\}$ is called the *ε-altruistic game* associated to Γ and C_ε denotes the set of its correlated equilibria.

Each $h_{i,\varepsilon}$ represents the utility function of player i supposed to take into account the sum of the payoffs of the opponents with weight ε.

Therefore:

Definition 2.3: A correlated equilibrium μ of the game Γ is said to be a *slightly altruistic correlated equilibrium* if there exist a sequence of positive real numbers $\left(\varepsilon_n\right)_n$ decreasing to 0 and a sequence of correlated strategies $\left(\mu_n\right)_n$, such that

i) μ_n is a correlated equilibrium of the ε_n-altruistic game Γ_{ε_n} associated to Γ, for every $n \in \mathbb{N}$.

ii) μ_n converges to μ as $n \to \infty$.

Denote with $\mathcal{SA}^C$ the set of all slightly altruistic correlated equilibria.

Now let us give an illustrative example showing the effectiveness of this concept.

EXAMPLE 2.4: Let us consider the following 2×2-game:

Player 1, Player 2	L	R
T	3,0	1,0
B	1,3	1,3

the set of Nash equilibria is $E = ([0,1] \times \{0\}) \cup (\{1\} \times [0,1])$.

Denote with $(\alpha, \beta, \gamma, \delta)$ a correlated strategy whose probability assignments are given by the following matrix

	L	R
T	α	β
B	γ	δ

Therefore the correlated strategies corresponding to the Nash equilibria in mixed strategies of the game are

$$N = \{(\alpha, \beta, 0, 0) \mid \alpha, \beta \geq 0,\ \alpha + \beta = 1\} \cup \{(0, \beta, 0, \delta) \mid \beta, \delta \geq 0,\ \beta + \delta = 1\}$$

while it can be checked that the set of correlated equilibria is

$$C = \{(\alpha, \beta, 0, \delta) \mid \alpha, \beta, \delta \geq 0,\ \alpha + \beta + \delta = 1\}.$$

The $\varepsilon-$altruistic game associated to Γ is given by

Player 1, Player 2	L	R
T	3,3ε	1,ε
B	1+3ε,3+ε	1+3ε,3+ε

In order to calculate the set of correlated equilibria C_ε of these $\varepsilon-$altruistic game, write the conditions (1) corresponding to the perturbed games:

$$3\alpha + \beta \geq \alpha(1 + 3\varepsilon) + \beta(1 + 3\varepsilon) \quad \Leftrightarrow \quad \alpha(2 - 3\varepsilon) \geq 3\varepsilon\beta \tag{4}$$
$$\gamma(1 + 3\varepsilon) + \delta(1 + 3\varepsilon) \geq 3\gamma + \delta \quad \Leftrightarrow \quad 3\varepsilon\delta \geq \gamma(2 - 3\varepsilon) \tag{5}$$
$$3\alpha\varepsilon + (3 + \varepsilon)\gamma \geq \varepsilon\alpha + (3 + \varepsilon)\gamma \quad \Leftrightarrow \quad 2\varepsilon\alpha \geq 0 \tag{6}$$
$$\varepsilon\beta + (3 + \varepsilon)\delta \geq 3\beta\varepsilon + (3 + \varepsilon)\delta \quad \Leftrightarrow \quad 2\varepsilon\beta \leq 0 \tag{7}$$

Therefore the $\varepsilon-$altruistic correlated equilibria are correlated strategies satisfying (4) to (7). Taking the limit as $\varepsilon \to 0$ we get

$$\mathcal{SA}^C = \{(\alpha, 0, 0, \delta) \mid \alpha, \delta \geq 0,\ \alpha + \delta = 1\}.$$

and therefore $\mathcal{SA}^C \subset C$, i.e., slightly altruistic equilibria properly refine correlated equilibria.

Correlated versus Nash

Given the finite game Γ denote with $\overline{\Gamma} = \{I; S_1, \ldots, S_N; F_1, \ldots, F_N\}$ its mixed extension where each mixed strategy $s_i \in S_i$ is a vector $s_i = (s_i(x_i))_{x_i \in X_i} \in \mathbb{R}_+^{|X_i|}$ such that $\sum_{x_i \in X_i} s_i(x_i) = 1$ and the expected payoff function $F_i : S \to \mathbb{R}$ is defined by: $F_i(s) = \sum_{x_i \in X_i} \left[\prod_{i \in I} s_i(x_i)\right] f_i(\varphi)$ for all $s \in S$.

Definition 2.5 (De Marco and Morgan (2008,b)): A Nash equilibrium s^* of the game $\overline{\Gamma}$ is said to be a *slightly altruistic equilibrium* if there exist a sequence of positive real numbers $(\varepsilon_n)_n$ decreasing to 0 and a sequence of strategy profiles $(s_n)_n \subseteq S$, such that

i) s_n is a Nash equilibrium of Γ_n for every $n \in \mathbb{N}$.

ii) s_n converges to s^* as $n \to \infty$.

Denote with $\mathcal{SA}^N$ the set of all slightly altruistic equilibria.

It immediately follows that

Proposition 2.6: *Let s be a slightly altruistic (Nash) equilibrium then the probability distribution on X induced by s is a slightly altruistic correlated equilibrium.*

Since slightly altruistic equilibria in mixed strategies do always exist in finite games we immediately conclude that

Theorem 2.7: *Every finite game has a slightly altruistic correlated equilibrium.*

Example 2.8: Consider the game in Example 2.4. It can be checked that the set of slightly altruistic equilibria is (in terms of correlated strategies)

$$\mathcal{SA}^N = \{(1,0,0,0), (0,0,0,1)\}.$$

It can be obviously recognized that $(1,0,0,0)$ is favorable for player 1 but not for player 2 and, conversely, $(0,0,0,1)$ is favorable for player 2 but not for player 1. However, the presence of a mediator allows to reach an agreement which is fair for both the players and which is stable with respect to altruism, by choosing, for instance the slightly altruistic correlated equilibrium $(1/2, 0, 0, 1/2)$.

2.2 Essentiality

In this subsection, we analyze the relations between slightly altruistic correlated equilibria and essentiality. Roughly speaking and analogously with essential Nash equilibria, a correlated equilibrium μ of a game Γ is essential if every game *nearby* Γ has a correlated equilibrium *nearby* μ. Hence essentiality is a very strong stability concept. We show that every essential correlated equilibrium is a slightly altruistic correlated equilibrium but the set of essential correlated equilibria might be empty, implying that we cannot retrict the attention only to essentiality.

Let $|X| = K$ denote the cardinality of the set of all pure strategy profiles, then every payoff function $f_i : X \to \mathbb{R}$ has finite range, in particular $y_i = (f_i(x))_{x \in X}$ is a

K-dimensional vector for every player i. Then, it is possible to identify each game Γ with and only one point $y = (y_1, \ldots, y_n) \in \mathbb{R}^{nK}$. Therefore, denoting with $\mathcal{G}(X_1, \ldots, X_n)$ the set of n-player finite games with pure strategy sets $(X_1, \ldots, X_n)$, there is a one to one correspondence between $\mathbb{R}^{nK}$ and $\mathcal{G}(X_1, \ldots, X_n)$. Then, one can define a distance, denoted by $d(\Gamma', \Gamma'')$, between the games Γ' and Γ'' using the classical Euclidean distance between the corresponding vectors in $\mathbb{R}^{nK}$. Hence

Definition 2.9: A correlated equilibrium μ of Γ is said to be *essential* if for every $\eta > 0$ there exists $\delta > 0$ such that for every game Γ' with $d(\Gamma, \Gamma') < \delta$ there exists a correlated equilibrium μ' with $d(\mu, \mu') < \eta$.

Given a set valued map $F : Z \rightrightarrows Y$ recall that (Aubin and Frankowska(???))

$$\text{Lim inf}_{z \to z'} F(z) = \left\{ y \in Y \mid \lim_{z \to z'} d(y, F(z)) = 0 \right\}$$

and

$$\text{Lim sup}_{z \to z'} F(z) = \left\{ y \in Y \mid \liminf_{z \to z'} d(y, F(z)) = 0 \right\}.$$

Then

Remark 2.10: Denote with Ω the set of all probability distributions on X. Let $\mathcal{C} : \mathcal{G}(X_1, \ldots, X_n) \to \Omega$ be the set-valued map associating to every game $\Gamma \in \mathcal{G}(X_1, \ldots, X_n)$ the set $\mathcal{C}(\Gamma)$ of all correlated equilibria of Γ; then, by definition, μ is an essential correlated equilibrium of Γ if and only if $\mu \in \text{Lim inf}_{\Gamma' \to \Gamma} \mathcal{C}(\Gamma')$

Proposition 2.11: *Every essential correlated equilibrium is a slightly altruistic correlated equilibrium.*

Proof. Since μ is an essential correlated equilibrium for Γ, for every $\nu \in \mathbb{N}$ there exists $\delta_\nu > 0$ such that any game Γ' satisfying $d(\Gamma, \Gamma') < \delta_\nu$ has an equilibrium μ' such that $d(\mu, \mu') < 1/\nu$. Let ε_ν be a positive real number such that the corresponding ε_ν-altruistic game Γ_{ε_ν} satisfies $d(\Gamma, \Gamma_{\varepsilon_\nu}) < \delta_\nu$. Hence, for every $\nu \in \mathbb{N}$ there exists a correlated equilibrium μ_ν of Γ_{ε_ν}, which satisfies $d(\mu, \mu_\nu) < 1/\nu$. Consider a converging subsequence $(\mu_{\nu_k})_{k \in \mathbb{N}}$ of the sequence $(\mu_\nu)_{\nu \in \mathbb{N}}$. Then, $\lim_{k \to \infty} \mu_{\nu_k} = \mu$ and μ is a slightly altruistic correlated equilibrium of Γ. $\square$

In the next example we show that essential correlated equilibria may not exist.

Example 2.12: Consider the game in Example 2.4. The set of slightly altruistic correlated equilibria is $\mathcal{SA}^C = \{(\alpha, 0, 0, \delta) \mid \alpha, \delta \geq 0,\ \alpha + \delta = 1\}$. Therefore, in light of Proposition 2.11, essential correlated equilibria must belong to this set. Consider the following perturbation $\Gamma(\varrho, \psi)$ of the game Γ

Player 1, Player 2	L	R
T	3,0	ϱ, ψ
B	1,3	1,3

with $\varrho > 1$ and $\psi > 0$. The correlated equilibrium conditions (1) read

$$\begin{array}{rlrl} \text{i)} & 3\alpha + \varrho\beta \geq \alpha + \beta, & \text{ii)} & \gamma + \delta \geq 3\gamma + \varrho\delta, \\ \text{iii)} & 3\gamma \geq \psi\alpha + 3\gamma, & \text{iv)} & \psi\beta + 3\delta \geq 3\delta \end{array}$$

Condition ii) implies that $-2\gamma \geq (\varrho - 1)\delta \implies \delta = 0$. Condition iii) implies $\alpha = 0$ and conditions i) and iv) impose no restrictions. This means that the set of correlated equilibria $C_{\varrho,\psi}$ of $\Gamma(\varrho, \psi)$ is given by

$$C_{\varrho,\psi} = \{(0, \beta, \gamma, 0) \mid \beta, \gamma \geq 0,\ \beta + \gamma = 1\}.$$

Since

$$\begin{gathered} \text{Lim inf}_{(\varrho,\psi)\to(1,0)} C_{\varrho,\psi} = \text{Lim sup}_{(\varrho,\psi)\to(1,0)} C_{\varrho,\psi} = \\ \text{Lim}_{(\varrho,\psi)\to(1,0)} C_{\varrho,\psi} = \{(0, \beta, \gamma, 0) \mid \beta, \gamma \geq 0,\ \beta + \gamma = 1\}. \end{gathered}$$

and

$$\mathcal{SA}^C \cap \text{Lim}_{(\varrho,\psi)\to(1,0)} C_{\varrho,\psi} = \emptyset$$

then Γ does not have essential correlated equilibria.

Remark 2.13: We emphasize that the previous example shows that $\mathcal{C}(\Gamma) \nsubseteq \text{Lim inf}_{\Gamma'\to\Gamma} \mathcal{C}(\Gamma')$; implying that, in general, $\mathcal{C}(\cdot)$ is not a lower semicontinuous set-valued map.

3 Correlated Equilibria and Friendliness

In this section we show that slightly altruistic equilibria can be further refined by considering a different kind of altruistic behavior, namely the so called *friendly behavior* (see Rusinowska (2002)). The application to correlated equilibria of the friendliness behavior approach allows to obtain a sharper selection device based on altruistic behavior even if this approach does not permit to define a refinement concept for correlated equilibria satisfying a general existence theorem.

In fact, the next example shows that the concept of slightly altruistic equilibrium is not always able to select a proper subset of the set of correlated equilibria. While, the next subsection shows that strong friendliness correlated equilibria provide an effective selection device in this game.

Example 3.1: Consider the following game:

Player 1, Player 2	L	R
T	3,0	1,0
B	1,2	1,3

the correlated strategies corresponding to the Nash equilibria of the game are

$$N = \{(\alpha, \beta, 0, 0) \mid \alpha, \beta \geq 0,\ \alpha + \beta = 1\} \cup \{(0, \beta, 0, \delta) \mid \beta, \delta \geq 0,\ \beta + \delta = 1\}$$

while it can be checked that the set of correlated equilibria is

$$C = \{(\alpha, \beta, 0, \delta) \mid \alpha, \beta, \delta \geq 0,\ \alpha + \beta + \delta = 1\}.$$

The $\varepsilon-$altruistic game associated to Γ is given by

Player 1, Player 2	L	R
T	3,3ε	1,ε
B	1+2ε,2+ε	1+3ε,3+ε

In order to calculate the set of correlated equilibria C_ε of these $\varepsilon-$altruistic games, write the conditions (1) corresponding to the perturbed games:

$$3\alpha + \beta \geq \alpha(1+2\varepsilon) + \beta(1+3\varepsilon) \quad \Leftrightarrow \quad 2\alpha(1-\varepsilon) \geq 3\varepsilon\beta \tag{8}$$

$$\gamma(1+2\varepsilon) + \delta(1+3\varepsilon) \geq 3\gamma + \delta \quad \Leftrightarrow \quad 3\varepsilon\delta \geq 2\gamma(1-\varepsilon) \tag{9}$$

$$3\alpha\varepsilon + (2+\varepsilon)\gamma \geq \varepsilon\alpha + (3+\varepsilon)\gamma \quad \Leftrightarrow \quad 2\varepsilon\alpha \geq \gamma \tag{10}$$

$$\varepsilon\beta + (3+\varepsilon)\delta \geq 3\beta\varepsilon + (2+\varepsilon)\delta \quad \Leftrightarrow \quad 2\varepsilon\beta \leq \delta \tag{11}$$

Therefore the set of $\varepsilon-$altruistic correlated equilibria are correlated strategies satisfying (8-11). Taking the limit as $\varepsilon \to 0$ we observe that the unique inequality which imposes conditions is (10) which finally implies that

$$\mathcal{SA}^C = C = \{(\alpha, \beta, \delta, 0) \mid \alpha, \delta \geq 0,\ \alpha + \beta + \delta = 1\}. \tag{12}$$

and therefore the concept of slightly altruistic correlated equilibrium provides no selection in this example.

3.1 Strong Friendliness Correlated Equilibria

An individual can be considered to be well inclined towards other individuals if whenever he is indifferent between more alternatives he chooses the most favorable to the others. Previous literature shows that this kind of behavior (called *friendly behavior*), can affect the outcome of a game (see Rusinowska (2002)). In particular, in the framework of Nash equilibrium selection, the *strong friendliness (Nash) equilibrium* concept (De Marco and Morgan (2008,a)) says that if there are multiple best replies to opponents' strategy profile of s_{-i}, then player i selects (one of) the strategies in this set maximizing the sum of opponents' payoffs[4].

Definition 3.2: A Nash equilibrium in mixed strategies s^* of the game Γ is a *strong friendliness (Nash) equilibrium* if for every player i, the following property is satisfied:

$$(\mathcal{SFB}): \qquad G_i(s_i^*, s_{-i}^*) \geq G_i(s_i, s_{-i}^*) \qquad \text{for all } s_i \in BR_i(s_{-i}^*)$$

where $BR_i(s_{-i}^*)$ is the set of the best replies in Γ of player i to his opponents' strategy profile s_{-i}^* and $G_i(s) = \sum_{x \in X} \left[\prod_{j \in I} s_j(x_j)\right] g_j(x)$ for every mixed strategy profile s.

[4]In De Marco and Morgan (2008,a) the *friendliness (Nash) equilibrium* concept has also been investigated. This concept roughly says that if there are multiple Nash equilibria in which opponents play a given profile of s_{-i}, then player i selects (one of) the equilibria in this set maximizing the sum of opponents' payoffs. The interesting results concerning this concept in the framework of Nash equilibria do not hold for the possible extensions of this concept to the case of correlated equilibria.

We now extend this concept to correlated equilibria.

DEFINITION 3.3: A correlated equilibrium μ of the game Γ is said to be a *strong friendliness correlated equilibrium* if

$$\begin{array}{rcl}\sum_{x_{-i}\in X_{-i}} \mu\left(x_{-i}|\bar{x}_i\right) f_i(\bar{x}_i, x_{-i}) & = & \sum_{x_{-i}\in X_{-i}} \mu\left(x_{-i}|\bar{x}_i\right) f_i(\widetilde{x}_i, x_{-i}) \\ & \Downarrow & \\ \sum_{x_{-i}\in X_{-i}} \mu\left(x_{-i}|\bar{x}_i\right) g_i(\bar{x}_i, x_{-i}) & \geq & \sum_{x_{-i}\in X_{-i}} \mu\left(x_{-i}|\bar{x}_i\right) g_i(\widetilde{x}_i, x_{-i})\end{array} \tag{13}$$

Denote with $\sigma\mathcal{F}^C$ the set of strong friendliness correlated equilibria.

PROPOSITION 3.4: *Let s^* be a strong friendliness Nash equilibrium and μ^* the probability distribution on X induced by s^*. Then, μ^* is a strong friendliness correlated equilibrium.*

Proof. Fix a player i. Since s^* is a Nash equilibrium, it follows that the probability distribution μ^* on X induced by s^* (i.e. $\mu^*(x) = \prod_{i\in I} s_i^*(x_i)$ for all $x \in X$) is a correlated equilibrium. Then, for every player i and every strategy $\bar{x}_i \in X_i$,

$$\sum_{x_{-i}\in X_{-i}} \mu^*\left(x_{-i}|\bar{x}_i\right) f_i(\bar{x}_i, x_{-i}) \geq \sum_{x_{-i}\in X_{-i}} \mu^*\left(x_{-i}|\bar{x}_i\right) f_i(x_i, x_{-i}) \quad \forall x_i \in X_i.$$

Suppose that for a player j and strategies $\bar{x}_j, \widetilde{x}_j \in X_j$ it results that

$$\sum_{x_{-j}\in X_{-j}} \mu^*\left(x_{-j}|\bar{x}_j\right) f_j(\bar{x}_j, x_{-j}) = \sum_{x_{-j}\in X_{-j}} \mu^*\left(x_{-j}|\bar{x}_j\right) f_j(\widetilde{x}_j, x_{-j}). \tag{14}$$

Denote with $supp(s_j^*)$ the support of strategy s_j^*, i.e.

$$supp(s_j^*) = \{x_j \in X_j \mid s_j^*(x_j) > 0\}.$$

If $\bar{x}_j \notin supp(s_j^*)$ then $\mu^*\left(x_{-j}|\bar{x}_j\right) = 0$ for all $x_{-j} \in X_{-j}$ which obviously implies that

$$\sum_{x_{-j}\in X_{-j}} \mu^*\left(x_{-j}|\bar{x}_j\right) g_j(\bar{x}_j, x_{-j}) = 0 = \sum_{x_{-j}\in X_{-j}} \mu\left(x_{-j}|\bar{x}_j\right) g_j(\widetilde{x}_j, x_{-j}).$$

Suppose now that $\bar{x}_j \in supp(s_j^*)$, then it follows that $\bar{x}_j \in BR_j(s_{-j}^*)$. Since $\mu^*\left(x_{-j}|\bar{x}_j\right) = s_{-j}^*(x_{-j}) = \prod_{i\neq j} s_i^*(x_i)$ then

$$\sum_{x_{-j}\in X_{-j}} \mu^*\left(x_{-j}|\bar{x}_j\right) f_j(\bar{x}_j, x_{-j}) = F_j(\bar{x}_j, s_{-j}^*) \text{ and } \sum_{x_{-j}\in X_{-j}} \mu^*\left(x_{-j}|\bar{x}_j\right) f_j(\widetilde{x}_j, x_{-j}) = F_j(\widetilde{x}_j, s_{-j}^*)$$

where $F_j(\bar{x}_j, s_{-j}^*)$ (resp. $F_j(\widetilde{x}_j, s_{-j}^*)$) gives the expected payoff to player from playing the pure strategy $\bar{x}_j$ (resp. $\widetilde{x}_j$) when his opponents are playing s_{-j}^*. Then, (14) implies that $\widetilde{x}_j \in BR_j(s_{-j}^*)$. Since s^* is a strong friendliness Nash equilibrium then

$$\sum_{x_{-j}\in X_{-j}} s_{-j}^*(x_{-j}) g_j(\bar{x}_j, x_{-j}) = G_i(\bar{x}_j, s_{-j}^*) \geq G_i(\widetilde{x}_j, s_{-j}^*) = \sum_{x_{-j}\in X_{-j}} s_{-j}^*(x_{-j}) g_j(\widetilde{x}_j, x_{-j})$$

Substituting $s^*_{-j}(x_{-j})$ with $\mu^*(x_{-j}|\bar{x}_j)$ in the previous inequality we get

$$\sum_{x_{-j}\in X_{-j}} \mu^*(x_{-j}|\bar{x}_j)\, g_j(\bar{x}_j, x_{-j}) \geq \sum_{x_{-j}\in X_{-j}} \mu^*(x_{-j}|\bar{x}_j)\,(x_{-j}) g_j(\widetilde{x}_j, x_{-j}).$$

Hence the assertion follows. □

Example 3.5: Consider again the game in Example 3.1 and recall that in this case the set of correlated equilibria and slightly altruistic correlated equilibria coincide, i.e. $\mathcal{SA}^C = C$ (see (12)). Now, apply conditions (13) to calculate strong friendliness correlated equilibria:

$$3\alpha + \beta = \alpha + \beta \;\Rightarrow\; 0 \geq 2\alpha + 3\beta \tag{15}$$
$$\gamma + \delta = 3\gamma + \delta \;\Rightarrow\; 2\gamma + 3\delta \geq 0 \tag{16}$$
$$2\gamma = 3\gamma \;\Rightarrow\; 3\alpha + \gamma \geq \alpha + \gamma \tag{17}$$
$$3\delta = 2\delta \;\Rightarrow\; \beta + \delta \geq 3\beta + \delta \tag{18}$$

If $(\alpha, \beta, \gamma, \delta)$ is a correlated equilibrium, that is $\gamma = 0$, then conditions (16) and (17) impose no restrictions while (15) means that $\alpha = 0 \implies \beta = 0$ and (19) means that $\delta = 0 \implies \beta = 0$ which implies that the set of strong friendliness equilibria $\sigma\mathcal{F}^C$ properly refines $\mathcal{SA}^C$ since

$$\sigma\mathcal{F}^C = \{(\alpha, 0, 0, \delta) \mid \alpha, \delta \geq 0,\ \alpha + \delta = 1\} \cup \{(\alpha, \beta, 0, \delta) \mid \alpha, \beta, \delta > 0 \text{ and } \alpha + \beta + \delta = 1\}$$

Remark 3.6: The set $\sigma\mathcal{F}^C$ might be empty[5]. In fact, consider the following game

Player 1, Player 2	L	R
T	1,-1	0,0
B	0,0	1,-2

This game has a unique correlated equilibrium which is not a strong friendliness correlated equilibrium. In fact the correlated equilibrium conditions (1) read

$$i)\ \alpha \geq \beta, \qquad ii)\ \delta \geq \gamma$$
$$iii)\ -\alpha \geq -2\gamma \quad iv)\ -2\delta \geq -\beta$$

[5]The conditions in the definition of strong friendliness (Nash) equilibria are somehow similar to those in the definition of *evolutionary stable strategies (ESS)* (see Mynard Smith and Price (1973)), even if there are no general connections between the two concepts. However, since it is well known that ESS may not exist, the analogous lack of existence of strong friendliness Nash equilibria and of strong friendliness correlated equilibria seems to be natural. The definition of ESS is the following: Consider a symmetric two player game $\Gamma = \{\{1,2\}; X_1, X_2; f_1, f_2\}$, where $X_1 = X_2 = A$ and $f_1(z,y) = f_2(y,z) = f(z,y)$. Denote with S the set of probability distribution on X and with $F(p,q) = \sum_{x\in A^2} p(x_1)q(x_2)f(x_1,x_2)$. Then a strategy $p \in S$ is an evolutionary stable strategy if: 1) $F(p,p) \geq F(q,p)$ for all $q \in S$; 2)$F(p,p) = F(q,p)$ and $p \neq q \implies F(p,q) > F(q,q)$. While the definition of symmetric strong friendliness Nash equilibrium in symmetric 2-player game collapses to 1) $F(p,p) \geq F(q,p)$ for all $q \in S$; 2)$F(p,p) = F(q,p) \implies F(p,p) \geq F(p,q)$.

which imply that

$$\alpha = \beta = 2\gamma = 2\delta.$$

These latter inequalities together with $\alpha + \beta + \gamma + \delta = 1$ imply that $\alpha = \beta = 1/3$ and $\gamma = \delta = 1/6$. Then $(1/3, 1/3, 1/6, 1/6)$ is the unique correlated equilibrium. Now, apply conditions (13), we get

$$\begin{aligned} \alpha = \beta &\Rightarrow -\alpha \geq -2\beta \\ \delta = \gamma &\Rightarrow -2\delta \geq -\gamma \\ -\alpha = -2\gamma &\Rightarrow \alpha \geq \gamma \\ -2\delta = -\beta &\Rightarrow \delta \geq \beta \end{aligned}$$

The latter of those condition implies that $\delta \geq \beta$ which is a contradiction since $1/6 = \delta < \beta = 1/3$. Hence $(1/3, 1/3, 1/6, 1/6)$ is not a strong friendliness correlated equilibrium and $\sigma\mathcal{F}^C = \emptyset$.

Relation between $\mathcal{SA}^C$ and $\sigma\mathcal{F}^C$

Example 3.1 already shows that slightly altruistic correlated equilibria and strong friendliness correlated equilibria may differ. Now we analyze the connections between the two concepts: the next result shows that $\sigma\mathcal{F}^C \subseteq \mathcal{SA}^C$ while, in the next subsection, enforcing the stability condition in the definition of slightly altruistic correlated equilibrium (by introducing the *altruistic equilibrium* concept) allows to find elements in the intersection $\sigma\mathcal{F}^C \cap \mathcal{SA}^C$.

PROPOSITION 3.7: *If μ is a strong friendliness correlated equilibrium then it is a slightly altruistic correlated equilibrium.*

Proof. Let μ be a strong friendliness correlated equilibrium. Fix a strategy $\overline{x}_i$; if

$$\sum_{x_{-i} \in X_{-i}} \mu\left(x_{-i}|\bar{x}_i\right) f_i(\bar{x}_i, x_{-i}) - \sum_{x_{-i} \in X_{-i}} \mu\left(x_{-i}|\bar{x}_i\right) f_i(\widetilde{x}_i, x_{-i}) = \delta(\bar{x}_i, \widetilde{x}_i) > 0$$

then there exists $\varepsilon(\bar{x}_i, \widetilde{x}_i) > 0$ such that, for every $0 < \varepsilon < \varepsilon(\bar{x}_i, \widetilde{x}_i)$, it results that

$$\delta(\bar{x}_i, \widetilde{x}_i) \geq \varepsilon \left[\sum_{x_{-i} \in X_{-i}} \mu\left(x_{-i}|\bar{x}_i\right) g_i(\widetilde{x}_i, x_{-i}) - \sum_{x_{-i} \in X_{-i}} \mu\left(x_{-i}|\bar{x}_i\right) g_i(\bar{x}_i, x_{-i}) \right].$$

Denote with $\Delta(\bar{x}_i) = \{\widetilde{x}_i \in X_i \mid \delta(\bar{x}_i, \widetilde{x}_i) > 0\}$ and let $\varepsilon(\bar{x}_i) = \min_{\widetilde{x}_i \in \Delta(\bar{x}_i)} \varepsilon(\bar{x}_i, \widetilde{x}_i)$. Let $\widetilde{x}_i \in X_i \setminus \Delta(\bar{x}_i)$, (i.e. $\delta(\bar{x}_i, \widetilde{x}_i) = 0$); since μ is a strong friendliness correlated equilibrium and $\varepsilon(\bar{x}_i) > 0$ then, for every $0 < \varepsilon < \varepsilon(\bar{x}_i)$ it results that

$$0 \geq \varepsilon \left[\sum_{x_{-i} \in X_{-i}} \mu\left(x_{-i}|\bar{x}_i\right) g_i(\widetilde{x}_i, x_{-i}) - \sum_{x_{-i} \in X_{-i}} \mu\left(x_{-i}|\bar{x}_i\right) g_i(\bar{x}_i, x_{-i}) \right].$$

Summarizing, for every $0 < \varepsilon < \varepsilon(\bar{x}_i)$, we get that,

$$\sum_{x_{-i}\in X_{-i}} \mu\left(x_{-i}|\bar{x}_i\right) f_i(\bar{x}_i, x_{-i}) - \sum_{x_{-i}\in X_{-i}} \mu\left(x_{-i}|\bar{x}_i\right) f_i(\widetilde{x}_i, x_{-i}) \geq$$

$$\varepsilon\left[\sum_{x_{-i}\in X_{-i}} \mu\left(x_{-i}|\bar{x}_i\right) g_i(\widetilde{x}_i, x_{-i}) - \sum_{x_{-i}\in X_{-i}} \mu\left(x_{-i}|\bar{x}_i\right) g_i(\bar{x}_i, x_{-i})\right] \quad \forall \widetilde{x}_i \in X_i$$

Hence μ is a ε-altruistic correlated equilibrium for every $\varepsilon \leq \min_{i\in N, x_i\in X_i} \varepsilon(x_i)$ and we get the assertion □

3.2 Altruistic Correlated Equilibria

DEFINITION 3.8: A correlated equilibrium μ is said to be an *altruistic correlated equilibrium* if there exists $\delta > 0$ such that $\mu \in \bigcap_{0\leq\varepsilon\leq\delta} C_\varepsilon$. Denote with $\mathcal{A}^C$ the set of altruistic correlated equilibria.

PROPOSITION 3.9: *If μ is an altruistic correlated equilibrium then it is a strong friendliness correlated equilibrium and a slightly altruistic correlated equilibrium.*

Proof. From the assumption it follows that μ is a correlated equilibrium then for every player i and every strategy $\bar{x}_i \in X_i$,

$$\sum_{x_{-i}\in X_{-i}} \mu\left(x_{-i}|\bar{x}_i\right) f_i(\bar{x}_i, x_{-i}) \geq \sum_{x_{-i}\in X_{-i}} \mu\left(x_{-i}|\bar{x}_i\right) f_i(x_i, x_{-i}) \quad \forall x_i \in X_i.$$

Suppose that for a player j and strategies $\bar{x}_j, \widetilde{x}_j \in X_j$ it results that

$$\sum_{x_{-j}\in X_{-j}} \mu\left(x_{-j}|\bar{x}_j\right) f_j(\bar{x}_j, x_{-j}) = \sum_{x_{-j}\in X_{-j}} \mu\left(x_{-j}|\bar{x}_j\right) f_j(\widetilde{x}_j, x_{-j}). \tag{19}$$

From the assumption there exists $0 < \delta$ such that, for $0 < \varepsilon \leq \delta$, μ is a correlated equilibrium of Γ_ε. Then

$$\sum_{x_{-j}\in X_{-j}} \mu\left(x_{-j}|\bar{x}_j\right) h_{j,\varepsilon}(\bar{x}_j, x_{-j}) \geq \sum_{x_{-j}\in X_{-j}} \mu\left(x_{-j}|\bar{x}_j\right) h_{j,\varepsilon}(\widetilde{x}_j, x_{-j}).$$

Since $h_{j,\varepsilon} = f_j + \varepsilon g_j$, in light of (19) it follows that

$$\sum_{x_{-j}\in X_{-j}} \mu\left(x_{-j}|\bar{x}_j\right) g_j(\bar{x}_j, x_{-j}) \geq \sum_{x_{-j}\in X_{-j}} \mu\left(x_{-j}|\bar{x}_j\right) g_j(\widetilde{x}_j, x_{-j}).$$

Then μ is a a strong friendliness correlated equilibrium. Since μ is also a slightly altruistic equilibrium, we get the assertion. □

EXAMPLE 3.10: Consider the game in Example 3.1. The set of altruistic correlated equilibria coincides with the set of strong friendliness correlated equilibria. In fact for every $\alpha, \delta \geq 0$ with $\alpha + \delta = 1$, $(\alpha, 0, 0, \delta)$ satisfies conditions (8),(9),(10 and (11). Rewrite conditions (8) and (11) as follows

$$\beta \leq \frac{2\alpha(1-\varepsilon)}{3\varepsilon}, \quad \beta \leq \frac{\delta}{2\varepsilon} \tag{20}$$

Since the terms $\frac{2(1-\varepsilon)}{3\varepsilon}$ and $\frac{1}{2\varepsilon}$ diverge to $+\infty$ as $\varepsilon \searrow 0$ then for every $\alpha, \beta, \delta > 0$ with $\alpha + \beta + \delta = 1$ it is possible to find $\overline{\varepsilon}(\alpha, \beta, \delta)$ such that for every $\varepsilon \leq \overline{\varepsilon}(\alpha, \beta, \delta)$ conditions in (20) are satisfied. That is $(\alpha, \beta, 0, \delta) \in C_\varepsilon$ for every $\varepsilon \leq \overline{\varepsilon}(\alpha, \beta, \delta)$. Hence every strong friendliness equilibrium of this game is an altruistic equilibrium. Then, the previous proposition implies that the two concept coincide in this example, that is $\sigma\mathcal{F}^C = \mathcal{A}^C$.

4 Properties of the Set of Solutions

The concept of correlated equilibrium seems to be so appealing also because the corresponding set of solutions satisfies two important properties which turn to be very useful in the applications: convexity and closedness. In this section we investigate these properties for the set of solutions given by the equilibrium concepts defined above. Unfortunately none of them satisfies both the properties.

4.1 Slightly Altruistic Correlated Equilibria

Even if in all the investigated examples the set of slightly altruistic correlated equilibria $\mathcal{SA}^C$ is convex, we are not able to establish a general convexity result for this set of solutions. However, we show that convexity appears by enforcing the stability property in the definition of slightly altruistic equilibria. In fact:

PROPOSITION 4.1: *If the set of slightly altruistic correlated equilibria $\mathcal{SA}^C$ satisfies the following Strong Slightly Altruistic Stability Property:*

[$\sigma\mathcal{SA}$ **Property**]: *For every $\mu \in \mathcal{SA}^C$ and for every sequence of positive real numbers $(\varepsilon_n)_n$ decreasing to 0 there exists a sequence of correlated strategies $(\mu_n)_n$, such that*

i) μ_n is a correlated equilibrium of the ε_n-altruistic game Γ_{ε_n} associated to Γ, for every $n \in \mathbb{N}$.

ii) μ_n converges to μ as $n \to \infty$.

Then, $\mathcal{SA}^C$ is convex.

Proof. Let ω and γ two correlated equilibria of Γ and $(\varepsilon_n)_n$ a sequence of positive real numbers decreasing to 0. Then there exist sequences of correlated strategies $(\omega_n)_n$ and $(\gamma_n)_n$, such that ω_n and γ_n are correlated equilibria of the ε_n-altruistic game Γ_{ε_n} for every

$n \in \mathbb{N}$, and ω_n converges to ω and γ_n converges to γ as $n \to \infty$. Let $t \in]0,1[$, now we show that $\mu = t\omega + (1-t)\gamma$ is a slightly altruistic correlated equilibrium of Γ. In fact, since the sets of correlated equilibria are convex, $\mu_n = t\omega_n + (1-t)\gamma_n$ is a correlated equilibrium of Γ_{ε_n} for every n. Moreover μ_n converges to μ as $n \to \infty$ and then $\mu \in \mathcal{SA}^C$. □

REMARK 4.2: The fact that the $\sigma\mathcal{SA}$ Property is required for the convexity of the set $\mathcal{SA}^C$ comes from well known properties of upper and lower limits of set valued maps (see Aubin and Frankowska (1990) or Rockafellar and Wets (1997)).

Given a set valued map $F : Z \rightrightarrows Y$ recall that Recall that given a set K, $co(K)$ denotes the convex hull of K. Then we have

$$co\left(\mathrm{Lim\,inf}_{z\to z'}F(z)\right) \subseteq \mathrm{Lim\,inf}_{z\to z'}co(F(z))$$

but in general

$$co\left(\mathrm{Lim\,sup}_{z\to z'}F(z)\right) \nsubseteq \mathrm{Lim\,sup}_{z\to z'}co(F(z)).$$

By definition, $\mathcal{SA}^C = \mathrm{Lim\,sup}_{\varepsilon\to 0}C_\varepsilon = \mathrm{Lim\,sup}_{\varepsilon\to 0}co(C_\varepsilon)$, being C_ε convex for every ε. If $\mathcal{SA}^C$ satisfies the $\sigma\mathcal{SA}$ Property then $\mathcal{SA}^C = \mathrm{Lim\,inf}_{\varepsilon\to 0}C_\varepsilon = \mathrm{Lim\,inf}_{\varepsilon\to 0}co(C_\varepsilon)$ and therefore $\mathcal{SA}^C = co\left(\mathrm{Lim\,inf}_{\varepsilon\to 0}C_\varepsilon\right)$ which implies that $\mathcal{SA}^C$ is convex.

PROPOSITION 4.3: *The set $\mathcal{SA}^C$ of slightly altruistic correlated equilibria is closed.*

Proof. Let $(\mu_\nu)_{\nu\in\mathbb{N}}$ be a sequence converging to μ with $\mu_\nu \in \mathcal{SA}^C$ for every $\nu \in \mathbb{N}$. By definition of slightly altruistic equilibria it follows that for every $\nu \in \mathbb{N}$ there exist a sequence of positive real numbers $(\varepsilon_n^\nu)_{n\in\mathbb{N}}$ converging to 0 and a sequence of strategy profiles $(\eta_n^\nu)_{n\in\mathbb{N}}$ converging to μ_ν such that $\eta_n^\nu \in C_{\varepsilon_n^\nu}$ for every $n \in \mathbb{N}$. For every $\nu \in \mathbb{N}$ there exist n_ν such that $|\mu_\nu - \eta_n^\nu| < \frac{1}{\nu}$ and $\varepsilon_n^\nu < \frac{1}{\nu}$ for every $n \geq n_\nu$. With an abuse of notation denote with $\eta_\nu = \eta_{n_\nu}^\nu$ and with $\varepsilon_\nu = \varepsilon_{n_\nu}^\nu$ for every $\nu \in \mathbb{N}$. By compactness we can extract converging subsequences, with an abuse of notation again denoted with $(\varepsilon_\nu)_{\nu\in\mathbb{N}}$ and $(\eta_\nu)_{\nu\in\mathbb{N}}$, such that $\eta_\nu \in C_{\varepsilon_\nu}$ for every $\nu \in \mathbb{N}$. Clearly $\varepsilon_\nu \to 0$. Moreover, $\eta_\nu \to \mu$; in fact, for every $\varepsilon > 0$ there exists $\overline{\nu} \in \mathbb{N}$ such that $|\mu_\nu - \mu| < \frac{\varepsilon}{2}$ and $\frac{1}{\nu} < \frac{\varepsilon}{2}$ for every $\nu \geq \overline{\nu}$. Hence for every ε there exists $\overline{\nu}$ such that, for every $\nu \geq \overline{\nu}$,

$$|\eta_\nu - \mu| \leq |\eta_\nu - \mu_\nu| + |\mu_\nu - \mu| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and $\mu \in \mathcal{SA}^C$. □

4.2 Strong Friendliness Correlated Equilibria

PROPOSITION 4.4: *The set $\sigma\mathcal{F}^C$ of strong friendliness correlated equilibria is convex.*

Proof. Let ω and γ be equilibria in $\sigma\mathcal{F}^C$, then $\omega, \gamma \in C$, that is for every player i, every strategy $\bar{x}_i \in X_i$,

$$\sum_{x_{-i}\in X_{-i}} \omega\left(x_{-i}|\bar{x}_i\right) f_i(\bar{x}_i, x_{-i}) \geq \sum_{x_{-i}\in X_{-i}} \omega\left(x_{-i}|\bar{x}_i\right) f_i(x_i, x_{-i}) \quad \forall x_i \in X_i, \tag{21}$$

$$\sum_{x_{-i}\in X_{-i}} \gamma\left(x_{-i}|\bar{x}_i\right) f_i(\bar{x}_i, x_{-i}) \geq \sum_{x_{-i}\in X_{-i}} \gamma\left(x_{-i}|\bar{x}_i\right) f_i(x_i, x_{-i}) \quad \forall x_i \in X_i. \tag{22}$$

Therefore if $\mu = t\omega + (1-t)\gamma$ it follows that $\mu \in C$, that is for every player i and every strategy $\bar{x}_i \in X_i$

$$\sum_{x_{-i}\in X_{-i}} \mu\left(x_{-i}|\bar{x}_i\right) f_i(\bar{x}_i, x_{-i}) \geq \sum_{x_{-i}\in X_{-i}} \mu\left(x_{-i}|\bar{x}_i\right) f_i(x_i, x_{-i}) \quad \forall x_i \in X_i. \tag{23}$$

Moreover, if for a pair of strategies $\bar{x}_i, \widetilde{x}_i \in X_i$, at least one of the inequalities in (21) or (22) is strict then the inequality in (23) is also strict. If $\bar{x}_i, \widetilde{x}_i \in X_i$ are such that the inequalities in (21) and (22) are satisfied simultaneously as equalities then from (13) it follows that

$$\sum_{x_{-i}\in X_{-i}} \omega\left(x_{-i}|\bar{x}_i\right) g_i(\bar{x}_i, x_{-i}) \geq \sum_{x_{-i}\in X_{-i}} \omega\left(x_{-i}|\bar{x}_i\right) g_i(\widetilde{x}_i, x_{-i}),$$
$$\sum_{x_{-i}\in X_{-i}} \gamma\left(x_{-i}|\bar{x}_i\right) g_i(\bar{x}_i, x_{-i}) \geq \sum_{x_{-i}\in X_{-i}} \gamma\left(x_{-i}|\bar{x}_i\right) g_i(\widetilde{x}_i, x_{-i}).$$

and hence it follows that

$$\sum_{x_{-i}\in X_{-i}} \mu\left(x_{-i}|\bar{x}_i\right) g_i(\bar{x}_i, x_{-i}) \geq \sum_{x_{-i}\in X_{-i}} \mu\left(x_{-i}|\bar{x}_i\right) g_i(\widetilde{x}_i, x_{-i}).$$

which finally implies that μ is a strong friendliness correlated equilibrium. Hence $\sigma\mathcal{F}^C$ is convex. □

REMARK 4.5: The set of strong friendliness equilibria is not necessarily closed since the set $\sigma\mathcal{F}^C$ in Example 3.1 is not closed.

4.3 Altruistic Correlated Equilibria

PROPOSITION 4.6: *The set $\mathcal{A}^C$ of altruistic correlated equilibria is convex.*

Proof. Let $\omega, \mu \in \mathcal{A}^C$, then there exist $\delta_\omega, \delta_\mu > 0$ such that $\omega \in \bigcap_{0\leq\varepsilon\leq\delta_\omega} C_\varepsilon$ and $\mu \in \bigcap_{0\leq\varepsilon\leq\delta_\mu} C_\varepsilon$. If $\delta = \min\{\delta_\omega, \delta_\mu\}$, then $\omega, \mu \in \bigcap_{0\leq\varepsilon\leq\delta} C_\varepsilon$. For every ε, the set C_ε of correlated equilibria of the $\varepsilon-$altruistic games is convex so $t\omega + (1-t)\mu \in \bigcap_{0\leq\varepsilon\leq\delta} C_\varepsilon$ for every $t \in]0,1[$ which implies that $t\omega + (1-t)\mu \in \mathcal{A}^C$ for every $t \in]0,1[$. Hence $\mathcal{A}^C$ is convex. □

REMARK 4.7: The set of altruistic equilibria is not necessarily closed since the set $\mathcal{A}^C$ in Example (3.1) is not closed.

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