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WORKING PAPER NO. 287

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May 2011

This version July 2012

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Hysteresis Bands and Transaction Costs

Francisco Delgado[*], Bernard Dumas[**] and Giovanni W. Puopolo[***]

Abstract
In the presence of transactions costs, no matter how small, arbitrage activity does not necessarily render equal all riskless rates of return. When two such rates follow stochastic processes, it is not optimal immediately to arbitrage out any discrepancy that arises between them. The reason is that immediate arbitrage would induce a definite expenditure of transactions costs whereas, without arbitrage intervention, there exists some, perhaps sufficient, probability that these two interest rates will come back together without any costs having been incurred. Hence, one can surmise that at equilibrium the financial market will permit the coexistence of two riskless rates that are not equal to each other. For analogous reasons, randomly fluctuating expected rates of return on risky assets will be allowed to differ even after correction for risk, leading to important violations of the Capital Asset Pricing Model. The combination of randomness in expected rates of return and proportional transactions costs is a serious blow to existing frictionless pricing models.

JEL Classification: C61, D11, D91, G11.

Acknowledgements: We thank for their comments Blaise Allaz, David Bates, George Constantinides, Jean-Pierre Danthine, Francois Degeorge, Julian Franks, Mark Garman, Philippe Henrotte, David Hsieh, Pete Kyle, Hayne Leland, David Parsley, Jean-Charles Rochet, Hans Stoll, Lars Svensson, and especially Bertrand Jacquillat and Jean-Luc Vila whose questions led to the correction of an error. Bernard Dumas acknowledges financial support from the Swiss Finance Institute and from NCCR FINRISK of the Swiss National Science Foundation.

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1 Introduction

Investors, who have to pay transactions costs, optimally rebalance their portfolio at points in times that are random and are not easily observable. Instead, the financial econometrician measures rates of return on financial assets over regular, fixed intervals in time. Investors compare the rates of return on assets over the forthcoming holding periods while the econometrician testing the validity of an asset pricing model, arbitrarily attempts to compare them over successive weeks, months or years.

We would like to know whether it is possible meaningfully to compare the rates of return on two otherwise similar assets when the rates are measured at regular intervals, while investors trade at random times. The question cannot be addressed without a model of the way in which investors choose to rebalance or not their portfolios. We first consider the case of two riskless assets in a portfolio. Then we extend the analysis to risky, long-lived assets such as equities.

If two interest rates on deposits were to remain unequal forever, it would pay to arbitrage out their difference immediately, even if transactions costs had to be incurred in doing so. In the absence of discounting, and in the absence of any costs for rolling over the deposits, the interest differential earned by the arbitrage would eventually outweigh any finite transactions costs incurred at the outset of the arbitrage operation.

If, however, the spread between the two rates fluctuates randomly, it may no longer pay to start an arbitrage. The interest differential may not last long enough to cover profitably the transactions costs. This basic idea was put forth originally in Baldwin (1990) who argues that very small transactions costs help in accounting for the failure of foreign exchange market efficiency tests and shows that the problem mathematically resembled Dixit's (1989) problem of stochastic entry and exit.

The purpose of the present paper is to re-formulate this idea of no-arbitrage spread between the rates of return on two different types of assets and exploit it in the context of an optimal portfolio choice problem with transactions costs. We first examine the portfolio choice of an investor with given relative risk aversion who has access to two riskless investments with instantaneous returns (infinitesimal maturity). One of these brings a rate of interest that is constant over time while the other yields a rate that varies according to a stochastic process. The process incorporates a reversion force, which in the long run pulls the second rate towards the first one. We approach this problem of portfolio choice in the manner of Dumas and Luciano

(1991), postponing final consumption to a point infinitely into the future, and computing the stationary optimal policy. For a given portfolio imbalance, the investors allow some gap between the two rates to survive; this gap is called "the hysteresis band". We are interested in the size of this gap. We intend to show that the gap is much larger than the transactions costs.

Because deposits are not forcibly refunded and can be rolled over costlessly, the period over which a given investor holds the deposit - the "holding period" - is a decision variable.[1] As smaller and smaller transactions costs are considered, the allowable gap (or spread) measured over the holding period is gradually compressed but the anticipated optimal holding period shrinks because smaller transactions make it less costly to switch from one asset to the other. Depending on the rates at which these two variables approach zero, the allowable annualized quoted spread may become small slowly or quickly. We show that it becomes small at a cubic-root rate.

Later on, we consider an arbitrage between a riskless asset with a constant rate and a risky asset with a stochastic mean-reverting conditionally expected

[1] The analysis is not limited to bank deposits. In fact, it applies to all assets. Shares of stock that pay no dividend are automatically "rolled over" until the investor explicitly sells them. Section 3 will be devoted to the analysis of rates of returns on equities. The analysis could, but will not, be generalized to shares that pay a dividend. Bonds would require a separate study because they are 100% refunded at the maturity date. That is one "transaction" that is forced on the bondholder.

rate of return. We find that, as transaction costs approach zero, the size of the hysteresis bands converges to zero at a slower pace and we conclude that the CAPM must be badly violated because of the existence of transactions costs. Moreover, our model is able to generate an expected time from purchase to sell one order of magnitude smaller than the holding period shown in Constantinides (1986). The difference in the two results is traceable to the difference in the assumed behavior of the conditionally expected return on the risky asset. Constantinides considers an expected return that is constant; we consider a stochastic, mean reverting one.

Mean reversion in expected stock returns has been first studied empirically by Fama and French (1988), Poterba and Summers (1988), and Bekaert and Hodrick (1992) among others.[2] We contribute to the asset-allocation literature solving a portfolio-choice problem with transaction costs and mean-reverting expected returns. In this sense, we extend the works of Davis and Norman (1990), Dumas and Luciano (1991), Akian et al. (1996), Eastham and Hastings (1988), Liu (2004) who determine the optimal portfolio policy in case of proportional transaction costs and constant investment set, and

[2] In particular, Fama and French have shown that long-holding period returns display mean reversion. The behavior of long-period returns is the combined result of short-period mean behavior and volatility behavior. In our model, short-period volatility is assumed constant.

Kim and Omberg (1996), Campbell and Viceira (1999), and Wachter (2002) who instead consider mean reverting expected returns but no transaction costs.

More recently, Jang et al. (2007) propose a regime-switching model of portfolio choice with transaction costs and show that jumps in regime, by entailing time-varying investment opportunity set, generate first-order effects on liquidity premia. Moreover, Lynch and Tan (2011) investigate the magnitude of liquidity premia using a finite-horizon discrete-time portfolio choice problem with return predictability, wealth shocks, and state-dependent transaction costs. They find that adding these real-world complications to the canonical problem can cause liquidity premia that are no longer an order of magnitude smaller than the transactions cost rate, but are instead the same order of magnitude. Finally, Bacchetta and van Wincoop (2010) contribute to this literature by examining the impact of infrequent portfolio decisions on the forward discount puzzle. They show that asset management costs discourage investors from active trade, accounting for large deviations from the uncovered interest parity.

The paper is organized as follows. In Section 1 we solve the basic portfolio problem considered by Baldwin (1990) in which investors are constrained to investing their entire wealth in one riskless asset or the other; we measure the resulting gap in interest rates. In Section 2, we allow continuous adjustment of the portfolio while still considering only two riskless assets. In Section 3, we optimize a portfolio made up of one riskless asset with a constant rate and one risky asset with a mean reverting expected return; we evaluate the deviation from the costless CAPM. Finally, Section 4 concludes.

2 The case of two riskless assets and all-or-nothing portfolio holdings

2.1 Problem Formulation

Consider two assets. One of them has a constant riskless rate of return, which, without loss of generality in our context, we can set equal to zero. The other brings, over a small, fixed period of time, a rate of return α, which is also riskless but follows a mean-reverting stochastic process:

$$d\alpha = -\lambda \alpha dt + \sigma dz. \tag{1}$$

At any given time t, the dollar value of an investor's holding of the first asset is denoted x and the dollar value of his holding of the second asset is denoted y. Proportional transactions costs at the rate $1-s$ are incurred when exchanging one asset into the other; these costs are proportional to the dollar value of the trade.

We seek an optimal portfolio policy in which the objective is to maximize the utility of terminal consumption at some later date T. The utility of terminal consumption is logarithmic so that the objective is stated as:

$$L\left(x, y, t; T\right) \equiv \max\ \mathbb{E}_t \left[\ln(c_T)\right], \tag{2}$$

where $c_T = x_T$.

In an attempt to discover a stationary optimal policy, we take T to infinity. Furthermore, we assume that the function L asymptotically exhibits linear growth, at some rate, β, to be determined:

$$L\left(x, y, \alpha, t; T\right) - \beta(T-t) \quad \rightarrow \quad J\left(x, y, \alpha\right). \tag{3}$$
$$\text{as } T \quad \rightarrow \quad \infty$$

In this section we restrict the investor to holding all his wealth in the form of one asset or the other. Hence, the portfolio, apart from its size, can only be in one of two states. The only decision to be made at any given time is whether to switch or not the entire portfolio from one asset to the other. The investor will make that switch when α and the fixed rate are sufficiently far apart from each other. We seek the optimal choice of the trigger values $\underline{\alpha}$ and $\overline{\alpha}$ on each side of the constant value, 0, of the fixed rate of interest.

Exploiting the obvious homogeneity of the problem, define:

$$J(x, y, \alpha) = \ln(x+y) + I(\theta, \alpha), \qquad \text{where } \theta \equiv \frac{y}{x+y}. \tag{4}$$

In light of the restrictions imposed on the portfolio, θ is a binary variable which takes the value 0 or the value 1. For the remainder of this section we denote: $I_0(\alpha) \equiv I(0, \alpha)$ and $I_1(\alpha) \equiv I(1, \alpha)$. I_1 is the discounted utility function for a unit wealth that obtains when the investor is invested in the variable-rate asset; I_0 is the discounted utility for a unit wealth that obtains when he is invested in the fixed interest-rate asset.

2.2 Probabilistic approach: backward induction

The relationship between these two functions I_1 and I_0 is given by Equations (5) and (6) below. In Equation (5) a backward probabilistic reasoning gives the current value, $I_1(\alpha)$ of I_1. It is equal to:

- the value, $I_0(\underline{\alpha})$, of utility when the next switch out of the variable-rate asset occurs,

- plus the logarithm of the per-unit loss in wealth produced by the transactions costs,

- plus the expected extra log-earnings, $\mathbb{E}\left[\int_0^\tau \alpha_t dt \,|\alpha\right]$, produced by the variable-rate asset during the time until the switch,

- minus the effect of discounting over the expected time till the switch:

$$I_1(\alpha) = I_0(\underline{\alpha}) + \ln(s) + \mathbb{E}\left[\int_0^\tau \alpha_t dt \,|\alpha\right] - \beta\mathbb{E}\left[\tau \,|\alpha\right]; \quad \alpha > \underline{\alpha}. \tag{5}$$

Here, τ is the first-passage time of α to $\underline{\alpha}$. A similar backward reasoning, in (6), gives the current value, $I_0(\alpha)$, of the utility function I_0 when not invested:

$$I_0(\alpha) = I_1(\overline{\alpha}) + \ln(s) - \beta\mathbb{E}\left[\tau \,|\alpha\right]; \quad \alpha < \overline{\alpha}. \tag{6}$$

In (6), τ is the first-passage time of α to $\overline{\alpha}$.

2.3 Equivalent analytical approach

Parenthetically, Equations (5) and (6) can equivalently be obtained by imposing the condition that the value of the function L, defined in (2), executes a martingale process and subsequently introducing the changes of unknown function (3) and (4).

Hence, L has zero growth; J grows linearly at the unknown rate β; I_1 grows at the rate $\beta - \alpha$ because the log of earnings grows at the rate α when the portfolio is entirely made up of the variable-rate asset; similarly I_0 grows at the rate β.

These restrictions are written successively as follows:

$$L_t - \lambda\alpha L_\alpha + \frac{1}{2}\sigma^2 L_{\alpha\alpha} = 0, \tag{7}$$

$$-\lambda\alpha J_\alpha + \frac{1}{2}\sigma^2 J_{\alpha\alpha} = \beta, \tag{8}$$

$$\begin{cases} -\lambda\alpha I_1' + \frac{1}{2}\sigma^2 I_1'' = \beta - \alpha, \\ -\lambda\alpha I_0' + \frac{1}{2}\sigma^2 I_0' = \beta. \end{cases} \tag{9}$$

Equations (9), plus Value-Matching boundary conditions, are equivalent to (5) and (6) by virtue of the Feynman-Kac formula, but they are more easily generalizable to the cases of Sections 2 and 3 below than the probabilistic approach would be.

2.4 Solution

Returning to the backward, probabilistic approach, we first calculate the expected-earnings integral, $\mathbb{E}\left[\int_0^\tau \alpha_t dt \,|\alpha\right]$, which appears in Equation (5). An analogous calculation is performed in Karlin and Taylor (1981).[3] The answer in our case is:

$$\mathbb{E}\left[\int_0^\tau \alpha_t dt \,|\alpha\right] = \frac{\alpha - \underline{\alpha}}{\lambda}; \quad \alpha > \underline{\alpha}. \tag{10}$$

For the purpose of interpretation, recall that the value of this integral is the expected cumulative earnings on the variable-rate asset until the next switch to the fixed-rate asset, which will occur at time τ, the first time that α reaches $\underline{\alpha}$ from above.[4]

These expected earnings are always non negative, which may be surprising. In order to understand this result, it is important to keep in mind that

[3] On pages 196-197.

[4] We expect that $\underline{\alpha} < 0$.

the event $\alpha = \underline{\alpha}$ stops the sample paths over which the integral is calculated. Hence, earnings that are below $\underline{\alpha}$ are censored out, whereas excursions of large positive earnings are included in the sum. It may also be surprising to the reader that these expected earnings increase as $\underline{\alpha}$ is set to a lower, presumably negative value. The answer to this puzzle is again that setting $\underline{\alpha}$ lower takes the earnings into a somewhat lower negative zone but also allows some additional, possibly long excursions into positive values that would otherwise be censored out.[5]

The calculation of the expected first-passage time of an Ornstein-Uhlenbeck process, $\mathbb{E}[\tau\,|\alpha]$, is performed in Ricciardi and Sato (1988). In contrast to a standard Brownian motion, an Ornstein-Uhlenbeck process always has a finite expected hitting time. Ricciardi and Sato define a function ϕ_1 as follows:

$$\phi_1(\alpha) = \frac{1}{2\lambda}\sum_{n=1}^{\infty}\left[\frac{2\sqrt{\lambda}}{\sigma}\alpha\right]^n \frac{\Gamma(n/2)}{n!}, \tag{11}$$

where Γ is the gamma function. Depending on the situation, $\phi_1(\alpha)$ or $-\phi_1(-\alpha)$ serve to compute expected hitting time.

[5] The reader might also wonder why the investor would ever want to switch to a zero-rate of return asset when the value of his earnings on the variable-rate asset till the next switch is currently expected to be negative. He will do this (see optimization below) when α is negative enough because that will enhance his expected earnings. Earnings of the near future are negative; by switching he avoids those. Later, the switch back to the variable-rate asset will occur only when α is positive and large enough again ($\alpha = \overline{\alpha}$).

In Equation (5), the expected earnings and the expected hitting time are inserted as follows:

$$I_1(\alpha) = I_0(\underline{\alpha}) + \ln(s) + \frac{\alpha - \underline{\alpha}}{\lambda} - \beta\left[-\phi_1(-\alpha) + \phi_1(-\underline{\alpha})\right], \qquad (12)$$

while, in Equation (6), the correct expression is:

$$I_0(\alpha) = I_1(\overline{\alpha}) + \ln(s) - \beta\left[\phi_1(\overline{\alpha}) - \phi_1(\alpha)\right]. \qquad (13)$$

The functions I_0 and I_1 given by (12) and (13) are solutions of the differential equations (9).

The values of $I_1(\overline{\alpha})$ and $I_0(\underline{\alpha})$ are easily eliminated between equations (12) and (13) to get a single equation:

$$0 = 2\ln(s) + \frac{\overline{\alpha} - \underline{\alpha}}{\lambda} - \beta\left[\phi_1(\overline{\alpha}) - \phi_1(-\overline{\alpha}) - \phi_1(\underline{\alpha}) + \phi_1(-\underline{\alpha})\right] \qquad (14)$$

This equation lends itself to a satisfactory interpretation. The sum of the first two terms of the right-hand side, $2\ln(s) + \frac{\overline{\alpha}-\underline{\alpha}}{\lambda}$, equals the expected net log-earnings (per unit of wealth) from a round-trip between the two assets:

$2\ln(s)$ is the per unit log-transactions costs and $\frac{\overline{\alpha}-\underline{\alpha}}{\lambda}$ is the expected log-earnings during the part of the round trip where the variable-rate asset is held. β is, of course, the expected rate of growth of wealth (or the expected increment of log utility per unit of time). β is multiplied by the term between square brackets, which is simply equal to the expected duration of a round trip.

Hence, Equation (14) serves to calculate the expected rate of growth of wealth produced by a given $(\underline{\alpha}, \overline{\alpha})$ switching policy; it is equal to the expected net earnings during a round trip divided by the expected time that the round trip takes:

$$\beta = \frac{2\ln(s) + \frac{\overline{\alpha}-\underline{\alpha}}{\lambda}}{\phi_1\left(-\underline{\alpha}\right) - \phi_1\left(-\overline{\alpha}\right) + \phi_1\left(\overline{\alpha}\right) - \phi_1\left(\underline{\alpha}\right)}. \tag{15}$$

2.5 Optimization

We need to write that the choice of $\underline{\alpha}$ and $\overline{\alpha}$ is optimal. Two Smooth-pasting conditions will accomplish that task. They are:

$$I_1'\left(\underline{\alpha}\right) = I_0'\left(\underline{\alpha}\right),$$

and

$$I_1'(\overline{\alpha}) = I_0'(\overline{\alpha}),$$

otherwise written (based on (12) and (13)) as:

$$\frac{1}{\lambda} - \beta\phi_1'(-\underline{\alpha}) = \beta\phi_1'(\underline{\alpha}) \tag{16}$$

and

$$\frac{1}{\lambda} - \beta\phi_1'(-\overline{\alpha}) = \beta\phi_1'(\overline{\alpha}). \tag{17}$$

It is easy to check that Equations (16) and (17) are the straightforward first-order conditions of the maximization of the rate of growth, β, calculated as in (15), with respect to the choice of $\underline{\alpha}$ and $\overline{\alpha}$.

Because we have been able to express the functions I_1 and I_0 explicitly in (12) and (13), the difficult variational problem that we were facing has been reduced to the solution of a system of three algebraic equations (14, 16 and 17) in three real numbers. Furthermore, in that system the unknown number β appears linearly so that it can be easily eliminated leaving two equations in two unknowns. A further simplification is reached since we can easily show symmetry: $\underline{\alpha} = -\overline{\alpha}$. Hence, we are left with just one equation in one unknown number. That number must be found numerically.

2.6 The hysteresis band

We have solved the system (14, 16 and 17) repeatedly for various values of s, from the value 1 downward, corresponding to increasing rates of transactions costs. Figure 1 shows the values of $\underline{\alpha}$ and $\overline{\alpha}$ against the value of s (outer curve).[6] The interesting result is that, as $s \to 1$, the slope of these curves approaches infinity. As the rate of transactions costs goes to zero, the spread that the investor lets survive between the two riskless rates goes to zero at a slower pace.

FIGURE 1 GOES HERE

In the absence of transactions costs, arbitrage would force α to be pegged at the value 0. Transactions costs allow wide deviations from the arbitrage result. We can quantify the rate at which the range of deviations approaches zero:

[6] For the time being, we focus on the qualitative features of the solution. We discuss the choice of parameter values in the next sections.

Statement 1: As ln(s) approaches zero, the range of fluctuations of α, over which no transaction takes place, approaches zero like $\mathbf{ln(s)}^{1/3}$.

Proof: Call $z = \overline{\alpha} = -\underline{\alpha}$ the common unknown value of the interest rate bounds. Eliminate β between (14) and (15) or (16-17), to get:

$$-\ln(s) = \frac{z}{\lambda} - \frac{1}{\lambda}\frac{\phi_1(z) - \phi_1(-z)}{\phi_1'(z) + \phi_1'(-z)}. \tag{18}$$

The expansion of $\phi_1(z)$ was provided in (11). The expansion of $\phi_1'(z)$ is:

$$\phi_1'(z) = \frac{1}{2\lambda}\sum_{n=1}^{\infty}\left[\frac{2\sqrt{\lambda}}{\sigma}\right]^n z^{n-1}\frac{\Gamma(n/2)}{(n-1)!}. \tag{19}$$

From these we can get the expansion of the right-hand side of (18). The result is:[7]

$$-\ln(s) = \frac{1}{6\lambda}\left[\frac{2\sqrt{\lambda}}{\sigma}\right]^2 z^3, \tag{20}$$

or:

$$z = \left(-\frac{3}{2}\sigma^2\ln(s)\right)^{1/3}. \tag{21}$$

[7] $\Gamma(1/2)/\Gamma(3/2) = 2$.

Q.E.D.

Cubic rates of convergence for similar limit problems have been found in different contexts by Dixit (1991), Fleming et al. (1990) and Svensson (1991).

Equation (21) shows that, for small transaction costs, only two parameters play a role in the determination of the hysteresis band, viz. σ and s. Mean reversion parameter, λ, is not present. For finite transactions costs, the band remains very insensitive to the value of λ. Figure 1 displays the approximate values of $\underline{\alpha}$ and $\overline{\alpha}$ as given by (21) (dotted line); they are virtually identical to the exact values over the range of transactions costs shown.

2.7 The expected rate of growth and the expected frequency of transactions

Equation (11) makes it plain that the expected time between two transactions is of the same order of magnitude, i.e. 1/3, as the barrier position itself. From the identity (14) and the leading term in the expansion (11) of the function ϕ_1, one can deduce that the limit of the expected rate of growth as transaction costs are taken to zero is equal to the following number β^* :

$$\beta^* = \left[\frac{2\sqrt{\lambda}}{\sigma}\Gamma(1/2)\right]^{-1}. \tag{22}$$

Substituting (21) into the first terms of (15), the expected duration of a round trip is approximately equal to:

$$\frac{\left(-6\ln(s)\sigma^2\right)^{1/3}}{\lambda\beta^*} + \frac{1}{\lambda}\left[\frac{2\sqrt{\lambda}}{\sigma}\right]^3 \frac{\Gamma(3/2)}{6}\left(-3\ln(s)\sigma^2\right). \tag{23}$$

Finally, the value of the expected growth rate in a neighborhood of $s = 1$ is given by:

$$\beta = \beta^* - 2\beta^*\left[\frac{1}{\lambda}\left(3\sigma^2\right)^{1/3}\right]^{-1}\left(-\ln(s)\right)^{2/3}. \tag{24}$$

As in the case of the boundary positions, these approximate expressions are extremely accurate over a range of transactions costs from zero to several percentage points. The assumption of "small" transactions costs allows the derivation of accurate analytical expressions.

3 The case of two riskless assets and continuous portfolio holdings

When the two asset holdings x and y are allowed to vary continuously, the state transition equations are:

$$dx = sdl - du; \tag{25}$$

$$dy = \alpha y dt - dl + sdu; \tag{26}$$

$$d\alpha = -\lambda \alpha dt + \sigma dz. \tag{27}$$

Here u and l are two nondecreasing stochastic processes which increase only when (respectively) some amount of fixed-rate, or variable-rate asset is sold. We call $\overline{\alpha}(\theta)$ and $\underline{\alpha}(\theta)$ the upper and lower trigger values of α, which depend on the current composition, $\theta \equiv \frac{y}{x+y}$, of the portfolio.

Between transactions, $dx = 0$ and $dy = \alpha y dt$ so that the portfolio composition, θ, satisfies the following time-differential equation:

$$d\theta = \alpha\theta\left(1-\theta\right)dt. \tag{28}$$

Over the domain of no transactions, therefore, the value function, $I\left(\alpha, \theta\right)$, satisfies the following partial differential equation:[8]

$$-\beta + \alpha\theta - \lambda\alpha I_{\alpha} + \frac{1}{2}\sigma^{2} I_{\alpha\alpha} + \alpha\theta\left(1-\theta\right) I_{\theta} = 0. \tag{29}$$

We solve this partial differential equation by first discretizing it over the values of θ. We pick $\theta \in \left\{\theta_i; i = 0, 1, ..., n\right\}$. Then we need to find $n+1$ functions $I\left(\alpha, \theta_i\right)$, analogous to the two functions $I_0\left(\alpha\right)$ and $I_1\left(\alpha\right)$ in the previous section. At any time t, and for any portfolio composition θ_i, the agent drops her holdings to θ_{i-1} whenever α reaches $\underline{\alpha}_{i-1} \equiv \underline{\alpha}\left(\theta_i\right)$, whereas she increases the portfolio proportion to θ_{i+1} when α reaches $\overline{\alpha}_i \equiv \overline{\alpha}\left(\theta_i\right)$.[9]

Given the existence of proportional transactions costs, the utility impact of switching may be computed as follows. First, on the way down from θ_i to

[8] This P.D.E. is analogous to the pair of Equations (9) above.

[9] This also means that two agents characterized by the same log-utility function and the same investment opportunity set, but endowed with a different initial-portfolio composition, will necessary have the same portfolio policy.

θ_{i-1} :

$$\theta_i = \frac{y}{x+y}; \qquad \theta_{i-1} = \frac{y - \Delta y}{x + s\Delta y + y - \Delta y}, \tag{30}$$

which implies:

$$\Delta y = (x+y)\,\frac{\theta_i - \theta_{i-1}}{\theta_{i-1}(s-1)+1}. \tag{31}$$

Matching the values of the indirect utility before and after the change in portfolio composition, we have:

$$\ln(x+y) + I\left(\underline{\alpha}, \theta_i\right) = \ln\left(x + s\Delta y + y - \Delta y\right) + I\left(\underline{\alpha}, \theta_{i-1}\right). \tag{32}$$

From (31):

$$\frac{x + s\Delta y + y - \Delta y}{x+y} = 1 + (s-1)\frac{\theta_i - \theta_{i-1}}{\theta_{i-1}(s-1)+1}. \tag{33}$$

Call $\underline{\pi}_{i-1}$ the right-hand side of (33). Since (32) may be rewritten as:[10]

$$I\left(\underline{\alpha}_{i-1}, \theta_i\right) = \ln\left(\underline{\pi}_{i-1}\right) + I\left(\underline{\alpha}_{i-1}, \theta_{i-1}\right), \tag{34}$$

[10] Assume $\theta_{i-1} < \frac{1}{1-s}$, so that $\underline{\pi}_{i-1} > 1$.

we conclude that the transaction-cost related utility loss on the way down is $\ln\left(\underline{\pi}_{i-1}\right)$. Equation (34) is a Value-Matching condition.

The transition on the way up from θ_i to θ_{i+1} is handled in a similar way. Let:[11]

$$\overline{\pi}_i = 1 + (s-1)\frac{\theta_i - \theta_{i+1}}{\theta_{i+1}(s-1) - s}, \quad i = 0, ..., n-1. \tag{35}$$

The transaction-cost related utility loss on the way up is $\ln\left(\overline{\pi}_i\right)$, resulting in a second set of Value-Matching conditions.

Finally, we need to write that the choice of $\underline{\alpha}_{i-1}$ and $\overline{\alpha}_i$ is optimal for each i. Smooth-pasting necessary conditions accomplish that task:

$$I_\alpha\left(\underline{\alpha}_{i-1}, \theta_i\right) = I_\alpha\left(\underline{\alpha}_{i-1}, \theta_{i-1}\right), \quad i = 1, .., n; \tag{36}$$

$$I_\alpha\left(\overline{\alpha}_i, \theta_i\right) = I_\alpha\left(\overline{\alpha}_i, \theta_{i+1}\right), \quad i = 0, .., n-1. \tag{37}$$

At $\theta = 0$ and $\theta = 1$, the P.D.E. (29) is locally an ordinary differential equation. Hence, the functions $I(\alpha, 0)$ and $I(\alpha, 1)$, already obtained in Section 1, namely:

[11] Assume $\theta_i > \frac{-s}{1-s}$, so that $\overline{\pi}_i > 1$.

$$I(\alpha, 1) = I\left(\underline{\alpha}(1), 1\right) + \frac{\alpha - \underline{\alpha}(1)}{\lambda} - \beta\left[-\phi_1\left(-\alpha\right) + \phi_1\left(-\underline{\alpha}(1)\right)\right], \tag{38}$$

$$I\left(\alpha, 0\right) = I\left(\overline{\alpha}(0), 0\right) - \beta\left[\phi_1\left(\overline{\alpha}(0)\right) - \phi_1\left(\alpha\right)\right], \tag{39}$$

serve as boundary conditions at these values of θ.

The solution to the system (29, 34 - 39), i.e. the functions $I(\alpha, \theta)$, $\overline{\alpha}(\theta)$ and $\underline{\alpha}(\theta)$, is obtained numerically using a finite-difference method and considering two different scenarios for the portfolio composition: one (Scenario 1) in which we assume no short-selling, i.e. $\theta_0 = 0$ and $\theta_n = 1$, and the other (Scenario 2) in which we allow θ to take values outside the $[0, 1]$ range, i.e. $\theta_0 < 0$ and $\theta_n > 1$.

We proceed in three steps. First, for every θ_i, and for trial functions $\underline{\alpha}(\theta)$ and $\overline{\alpha}(\theta)$, we discretize the values of α within the range $[\underline{\alpha}(\theta_i), \overline{\alpha}(\theta_i)]$, therefore obtaining a grid of points in the variables θ and α. Then, for each arbitrary position of the barriers, we compute $I(\alpha, \theta)$ by solving the system of simultaneous linear equations given by (29, 34 - 35, and 38 - 39). More precisely, in Scenario 1 we use Condition (29) at all points strictly inside the grid, the Value-Matching conditions corresponding to the upper and the lower barriers, and Equations (38) and (39) at $\theta = 0$ and $\theta = 1$. In Scenario 2, for $\theta \in [0, 1]$ we use the same conditions as for Scenario 1, while, outside this range, only Equation (29) and Value-Matching conditions are employed. Furthermore, at $\theta = \theta_0$ and at $\theta = \theta_n$, we compute the partial derivative I_θ using exclusively the information inside the grid, thus implicitly assuming as a side condition that the second partial derivative is zero.[12] In the last step, for both scenarios we determine the functions $\overline{\alpha}(\theta)$ and $\underline{\alpha}(\theta)$ using an iterative procedure, updating the position of the barriers on the basis of the violations of the smooth-pasting conditions (36 - 37).

The resulting portfolio-adjustment boundary is shown in Figure 2.

FIGURE 2 GOES HERE

Figure 2 shows the optimal position of the barrier corresponding to both scenarios. It is worthwhile to mention that, when short-selling is allowed, and for every values of θ_0 and θ_n, the no-arbitrage spread between the interest rates is simply the continuation of the portfolio-adjustment boundary obtained when θ is restricted to be in the $[0, 1]$ range. This result is not a

[12] In Scenario 2, we have $\theta_0 < 0$ and $\theta_n > 1$.

surprise and is consistent with the myopic behavior of agents exhibiting a logarithmic utility function. Myopic investors do not look ahead to time at which they will be constrained to not sell short.

Parameter values were obtained from the empirical literature on mean reversion in interest rates. Our principal sources is Chan *et al.* (1992). The crucial parameters are the degree of mean reversion, λ and the volatility, σ, in expected returns. For the case of two riskless assets, we have chosen the values: $\lambda = 1.125\%/\text{year}$ and $\sigma = 2.6\%/\text{year}$. The value of λ implies that it takes eighty years on average for the interest rate to revert to its long-run value. This is a very small value of the reversion parameter, which is equal to half of the value λ estimated by Chan *et al.*.

We consider the situation where transaction costs on bank deposits are levied at the rate of 0.1%, $s = 0.999$. Figure 2 highlights that the combined effect of such small transactions costs and fluctuating expected returns is enough to produce a wide hysteresis effect in the rebalancing of the portfolio. Specifically, a gap of 1.005%/year in interest rates must exist before a decision is made to switch from one asset to the other.

For both scenarios, numerical experiments indicate that the barrier has the following property:

Statement 2: The optimal barrier is a flat straight line whose middle point is located at the optimal switching point of the binary policy.

The location of the boundary implies that the "cubic" property (Statement 1) applies equally to the width of the hysteresis band in this case and confirms the symmetric behavior (around the line $\alpha = 0$) for the thresholds functions $\overline{\alpha}(0)$ and $\underline{\alpha}(1)$ found in Section 1.

FIGURE 3 GOES HERE

To further illustrate this last point we plot the optimal position of the barriers as function of the transaction cost parameter s. For the sake of simplicity, we only show the case in which θ is restricted to be between 0 and 1 (Scenario 1). Figure 3 confirms Statement 2 and shows that, for the same percentage reduction in the transaction costs $1 - s$, the size of the hysteresis bands shrinks at a decreasing pace. For instance, this reduction is 0.0056 when the transaction costs pass from 2% to 1%, while it is only 0.0044 in case of further halving, i.e. from 1% to 0.5%.

4 The case of one riskless and one risky, mean-reverting asset

4.1 Problem formulation and solution

When not only the expected rate of return on one of the two assets follows a stochastic process but its rate of return is also risky, the state transition equations are:

$$dx = sdl - du; \tag{40}$$

$$dy = \mu ydt + \sigma_1 ydz_1 - dl + sdu; \tag{41}$$

$$d\mu = \lambda\left(\gamma - \mu\right)dt + \sigma dz. \tag{42}$$

Here, μ is the conditionally expected rate of return on the risky asset, σ_1 is the conditional standard deviation of the rate of return on that asset. The expected rate of return, μ, is assumed to be mean reverting. We call the long-run mean γ the center of reversion. The white-noise shock, dz_1, affecting the rate of return on the asset is assumed independent of the white noise shock, dz, affecting the expected rate of return.

Consistently with the notation used in Section 2, we introduce a change of state variable:[13]

$$\alpha_t = \mu_t - \gamma, \tag{43}$$

and observe that the investor's frictionless demand for the risky asset at any given time would be given by:

$$\theta_t = \frac{\mu_t}{\sigma_1^2} = \frac{\alpha_t + \gamma}{\sigma_1^2},$$

or:

$$\theta_t - \frac{\gamma}{\sigma_1^2} = \frac{\alpha_t}{\sigma_1^2}, \tag{44}$$

which means that the frictionless demand schedule is symmetric around the point $\left(\alpha = 0,\ \theta = \frac{\gamma}{\sigma_1^2}\right)$. The portfolio demand with transactions costs will inherit the same symmetry property.

[13]The case of two riskless assets studied in Section 2 is obviously a special case of the model described here. In fact, under the assumption that $\gamma = 0$ and $\sigma_1 = 0$, Equations (40-42) are equivalent to (25-27).

Using the new state variable, between transactions, the stochastic differential equation governing the evolution of the portfolio composition, θ, is:

$$d\theta = \theta\left(1-\theta\right)\left(\alpha+\gamma-\theta\sigma_1^2\right)dt + \sigma_1\theta\left(1-\theta\right)dz_1. \tag{45}$$

Over the domain of no transactions, the value function, $I\left(\alpha,\theta\right)$, satisfies the following partial differential equation:

$$\begin{aligned} 0 &= -\beta+\left(\alpha+\gamma\right)\theta-\frac{1}{2}\theta^2\sigma_1^2-\lambda\alpha I_\alpha+\frac{1}{2}\sigma^2 I_{\alpha\alpha} \\ &\quad +\theta\left(1-\theta\right)\left(\alpha+\gamma-\theta\sigma_1^2\right)I_\theta+\frac{1}{2}\sigma_1^2\theta^2\left(1-\theta\right)^2 I_{\theta\theta}. \end{aligned} \tag{46}$$

The Value-matching and Smooth-pasting boundary conditions remain as in (34-37). The conditions at $\theta = 0$ and $\theta = 1$ become now:

$$I\left(\alpha,1\right) = I\left(\underline{\alpha}(1),1\right)+\frac{\alpha-\underline{\alpha}(1)}{\lambda}+\left(-\beta+\gamma-\sigma_1^2/2\right)\left[-\phi_1\left(-\alpha\right)+\phi_1\left(-\underline{\alpha}(1)\right)\right], \tag{47}$$

$$I\left(\alpha,0\right) = I\left(\overline{\alpha}(0),0\right)-\beta\left[\phi_1\left(\overline{\alpha}(0)\right)-\phi_1\left(\alpha\right)\right]. \tag{48}$$

The optimal policy that solves this system is obtained numerically by the method outlined in the previous section. The same scenarios seen in Section 2 are analyzed here.

FIGURE 4 GOES HERE

FIGURE 5 GOES HERE

In Figure 4 we plot the position of the barrier assuming that short-selling is not allowed, while the hysteresis bands resulting from extending the portfolio holdings outside the $0-1$ range are plotted in Figure 5. As in Section 2, when short-selling is allowed, the no-arbitrage spread between the interest rates is simply the continuation of the portfolio-adjustment boundary obtained when θ is restricted to be in the $[0,1]$ range, thus confirming the myopic behavior of the agent. Obviously, in order to get the corresponding barriers as function of the expected rate of return on the risky asset, μ, it suffices to add the long-run mean γ to the portfolio boundaries $\underline{\alpha}(\theta)$ and $\overline{\alpha}(\theta)$ computed above, that is

$$\underline{\mu}(\theta) = \underline{\alpha}(\theta) + \gamma,$$

$$\overline{\mu}(\theta) = \overline{\alpha}(\theta) + \gamma.$$

The case of one risky and one riskless asset is calibrated in a manner similar to Section 2. Our principal source is Jegadeesh (1991). The evidence concerning mean reversion in stock returns is not as conclusive as that concerning interest rates; nonetheless, we used the same value of the mean reversion parameter, which was in any case a low one. The parameter values chosen are: $\lambda = 1.125\%$/year, $\sigma_1 = 15\%$/year, $\sigma = 2.6\%$/year $\gamma = 15\%$/year. Here we use transactions costs of 0.5%, i.e. $s = 0.995$, consistent with the evidence provided by Chordia et al. (2001).[14] With these values, at $\theta = 0.5$, some wealth is transferred from the risky asset to the riskless as soon as the expected return on the risky asset falls below -0.77%/year. It must be 2.92% before the investor wishes to transfer some wealth from riskless to risky asset.

The following statement summarizes the property of the portfolio-adjustment boundary.[15]

Statement 3: The optimal barrier is a straight line parallel to the

[14] Chordia et al. (2001) report effective bid-ask spreads between 50 and 100 basis points for NYSE stocks.

[15] This property also holds when transactions costs are levied at the rate 0.1%, implying $s = 0.999$, which is the same used to generate the hysteresis band in the case of two riskless assets. Results are not shown and are available upon request.

frictionless demand and positioned outside the hysteresis band of the riskless case constructed around the frictionless demand itself.

From Statement 3, it follows that the cubic property (Statement 1) is valid again.

FIGURE 6 GOES HERE

As before, to provide evidence of the cubic property we plot the optimal position of the barrier as function of the transaction cost parameter s. Figure 6 confirms Statement 3 showing that, as $\ln(s)$ approaches zero, the size of the hysteresis bands converges to zero at a slower pace.

4.2 Holding period

In this section we examine the impact of transaction costs and return predictability on the frequency of trade. Let $b_{\underline{\alpha}}(\theta)$ denote the optimal lower barrier, i.e. the sequence of lower thresholds $\underline{\alpha}$ as function of the portfolio composition θ, in the case of one riskless and one risky, mean-reverting asset. In this case, we can define the next sale time to be

$$\tau_s = \inf\left\{t \geq 0 : (\alpha_t, \theta_t) = b_{\underline{\alpha}}(\theta_t)\right\}, \tag{49}$$

and the expected time to the next sale starting from the initial position (α, θ) to be

$$T(a, \theta) = E\left[\tau_s \left| (a_0, \theta_0) = (a, \theta)\right.\right]. \tag{50}$$

Then, the expected holding period solves the following partial differential equation

$$-\lambda\alpha T_\alpha + \frac{1}{2}\sigma^2 T_{\alpha\alpha} + \theta(1-\theta)(a+\gamma-\theta\sigma_1^2)T_\theta + \frac{1}{2}\sigma_1^2\theta^2(1-\theta)^2 T_{\theta\theta} = -1, \tag{51}$$

with boundaries

$$T\left(\overline{a}(\theta_i), \theta_i\right) = T\left(\overline{a}(\theta_i), \theta_{i+1}\right), \qquad i = 0, 1, .., n-1, \tag{52}$$

$$T\left(\underline{a}(\theta_i), \theta_i\right) = 0, \qquad i = 1, .., n. \tag{53}$$

We solved the system (51-53) and computed the expected holding time from buy to sell, that is from the upper to the lower barrier, in both scenarios. Table 1 below shows the results (expressed in years) corresponding to

Scenario 1 as function of the initial portfolio composition θ.[16]

TABLE 1 GOES HERE

When short selling is not allowed, the expected holding periods are particularly prolonged because the likelihood that θ_t gets stuck at $\theta = 1$ for a long time is very high. In fact, from Equation (42) α_t exhibits a positive expected variation since it is much smaller than its long-run mean. This implies that the agent will keep increasing her proportion of the risky asset until she does not have holdings of the riskless security any more. Then, in correspondence of $\theta = 1$, a further buy is not possible, whereas selling takes place only when α_t hits $\underline{\alpha}(1)$. However, giving the positive drift of α_t, the probability that α_t will take long excursions into values higher than $\overline{\alpha}(1)$ before hitting $\underline{\alpha}(1)$ is very high.

On the contrary, the expected time from buy to sell (expressed in years) corresponding to Scenario 2 (shown in Figure 7) highlights the fact that the investor buys and sells the risky asset much more frequently.[17] This is not

[16] Just for clarification: in Scenario 1, whenever a_t hits $\underline{a}(0)$, no trading takes place and a_t could well assume values smaller than $\underline{a}(0)$ (without any change in θ). Similarly, when a_t hits $\overline{a}(1)$, nothing happens to the portfolio composition and a_t could well take values higher than $\overline{a}(1)$. In other words, in this scenario there is no lower trading barrier when $\theta = 0$ and no upper barrier for α when $\theta = 1$.

[17] When short selling is allowed, when a_t hits $\overline{a}(1)$ a trading activity takes place and θ

surprising because in Scenario 1, at $\theta = 0$ and $\theta = 1$, only one side trading takes place, specifically only the trading that keeps θ inside [0,1]. On the contrary, in Scenario 2 the agent can increase her holding of the risky asset much above $\theta = 1$. Therefore, starting from the initial position $\overline{\alpha}(1)$, it is possible that selling takes place in correspondence of any value of $b_{\underline{\alpha}}(\theta_t)$, thus lowering the expected holding time.

FIGURE 7 GOES HERE

We also computed the sensitivity of the expected holding period to the paramenters λ and σ controlling the dynamics of the expected return on the risky asset. Table 2 shows the results corresponding to Scenario 2, assuming $\theta = 0.5$ as initial position.

TABLE 2 GOES HERE

As expected, the holding period increases as the process becomes less persistent (high λ) and decreases as the variance σ of the rate of return increases. Our model generates an expected time from purchase to sell one

increases. On the contrary, when a_t reaches $\underline{a}(0)$ the agent invests more in the riskless security and θ decreases.

order of magnitude smaller than the holding period shown in Constantinides (1986). More interestingly, our results show that return predictability and transaction costs are consistent with the empirical evidence that trading frequencies around the world have increased significantly in the last decades.[18]

4.3 Deviations from the C.A.P.M

We now discuss the equilibrium of an economy with two production technologies available in infinitely elastic supply (constant returns to scale). One is riskless and brings a zero return; the other is risky and brings a mean-reverting expected return. The economy is populated with identical logarithmic investors, each one of them choosing a portfolio of investments, i.e. θ, in the manner that we have just described. In such an economy, any value of the variable θ, provided that it is between 0 and 1,[19] is consistent with the composition of the aggregate, "market" portfolio.[20] This variable changes over time as the expected return, μ, on the risky technology fluctuates. The

[18] For example, the turnover ratio for US stocks reported by The World Bank increased from about 0.5% (late 80s') to about 200% (2010).

[19] When $\theta = 0$ or 1, a corner occurs: only one asset is available to all investors and no portfolio decision has to be made by any of them.

[20] This is precisely due to the fact that the two technologies are available in infinitely elastic supply.

classic Capital Asset Pricing Model would say that:[21]

$$\mu_t = \sigma_1^2 \theta_t, \tag{54}$$

which is simply the inverse of the frictionless demand (44), and which is shown as the straight line on Figure 4.

In an economy with transactions costs, the expected return, $\mu_t = \alpha_t + \gamma$, is allowed to fluctuate within a wide interval given vertically by the hysteresis band of Figure 4, without any adjustment in the aggregate portfolio. Any fluctuation within that band is interpretable as a deviation from the CAPM. This shows that deviations from the CAPM can be large, even with small transactions costs, provided expected returns fluctuate randomly.

5 Conclusion

We propose a portfolio choice problem with returns predictability and transactions costs to investigate the size of the no-arbitrage gap between the two rates to survive and the expected holding period from purchase to sell. We

[21] Because there is only one risky asset, the portfolio composition, θ, and the risk measurement, beta, traditionally used in writing the CAPM, would be equal to each other in our case example.

find that hysteresis bands tend to remain large even when the costs that created them become small. Moreover, we are able to generate an expected time from buy to sell consistent with the empirical evidence. These ideas apply to pricing models so that classic CAPMs are subject to wide hysteresis-band violations when conditionally expected returns follow a stochastic, mean reverting process. Our results also imply that arbitrage models must be drastically revised to take into account the combined effect of stochastic expected returns and transactions costs.

The qualitative point made by this paper regarding violations of frictionless pricing models does not depend on our assumption that investors have unit risk aversion. Regardless of his degree of risk aversion, a risk averse investor would always choose hysteretic rebalancing decisions. The unit risk aversion assumption has only simplified the calculations and has even allowed us, in the simplest case, to obtain solutions.

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Table 1: The expected holding period under Scenario 1. Case of one riskless and one risky, mean reverting asset

θ	0	0.11	0.22	0.33	0.44	0.55	0.66	0.77	0.88	1
$\overline{a}(\theta)$	-0.132	-0.129	-0.127	-0.124	-0.122	-0.119	-0.117	-0.114	-0.112	-0.109
$\underline{a}(\theta)$	-0.168	-0.166	-0.163	-0.161	-0.158	-0.156	-0.153	-0.151	-0.148	-0.146
Time	27.70	29.36	31.17	33.11	35.19	37.41	39.78	42.31	44.92	47.69

Table 1 shows the expected time from buy to sell (expressed in years) corresponding to Scenario 1 as function of the initial portfolio composition θ. The parameters are: $s = 0.995$, $\sigma_1 = 0.15$, $\lambda = 0.0125$, $\sigma = 0.026$ and $\gamma = 0.15$.

Table 2: The expected holding period for several values of the parameters λ and σ. Case of one riskless and one risky, mean reverting asset

	$\sigma = 0.02$	$\sigma = 0.026$	$\sigma = 0.05$	$\sigma = 0.1$	$\sigma = 0.15$	$\sigma = 0.2$
$\lambda = 0.005$	2.114	1.771	1.073	0.653	0.492	0.403
$\lambda = 0.0125$	2.171	1.773	1.083	0.656	0.493	0.404
$\lambda = 0.025$	2.281	1.838	1.102	0.661	0.495	0.405
$\lambda = 0.05$	2.526	1.984	1.141	0.671	0.500	0.408
$\lambda = 0.1$	3.045	2.307	1.228	0.692	0.510	0.413
$\lambda = 0.15$	3.484	2.638	1.322	0.715	0.520	0.419

Table 2 shows the expected time from buy to sell (measured in years) corresponding to Scenario 2, for several values of the parameters λ and σ controlling the dynamics of the expected return on the risky asset. The initial position is assumed to be $\theta = 0.5$. The other parameters are $s = 0.995$, $\sigma_1 = 0.15$ and $\gamma = 0.15$.

Figure 1: The effect of transaction costs

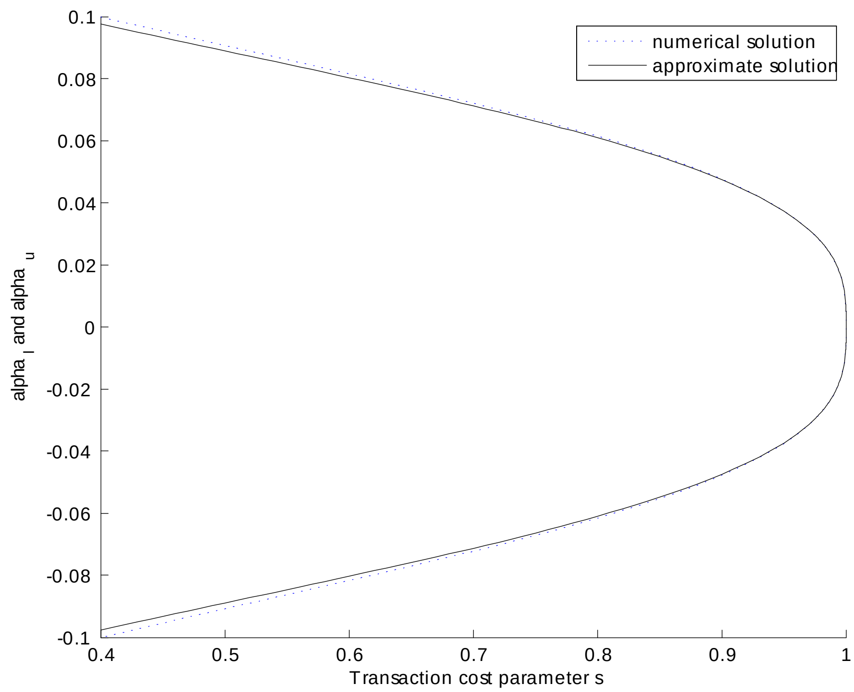

Figure 1 shows the effect of the transaction cost $1 - s$ on the size of the hysteresis bands. The mean reversion λ is set to 1.125%/year, while the conditional volatility is 2.6%/year.

Figure 2: The optimal position of the barrier. Case of two riskless assets and continuous portfolio holdings

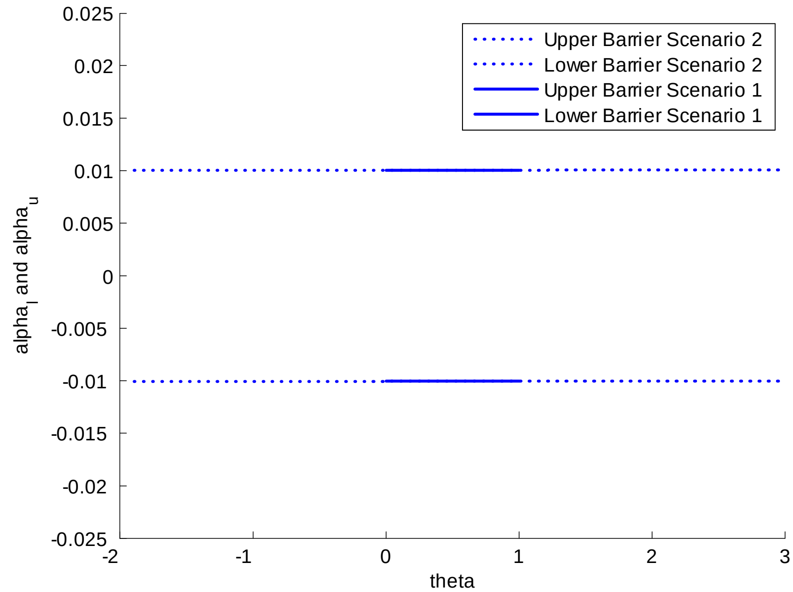

Figure 2 shows the optimal position of the thresholds functions $\overline{\alpha}$ and $\underline{\alpha}$ corresponding to Scenario 1 and 2. The mean reversion λ is set to 1.125%/year, while the conditional volatility is 2.6%/year. The transaction costs are levied at the rate of 0.1%, implying $s = 0.999$.

Figure 3: The effect of transaction costs on the portfolio-adjustment boundary. Case of two riskless assets and continuous portfolio holdings

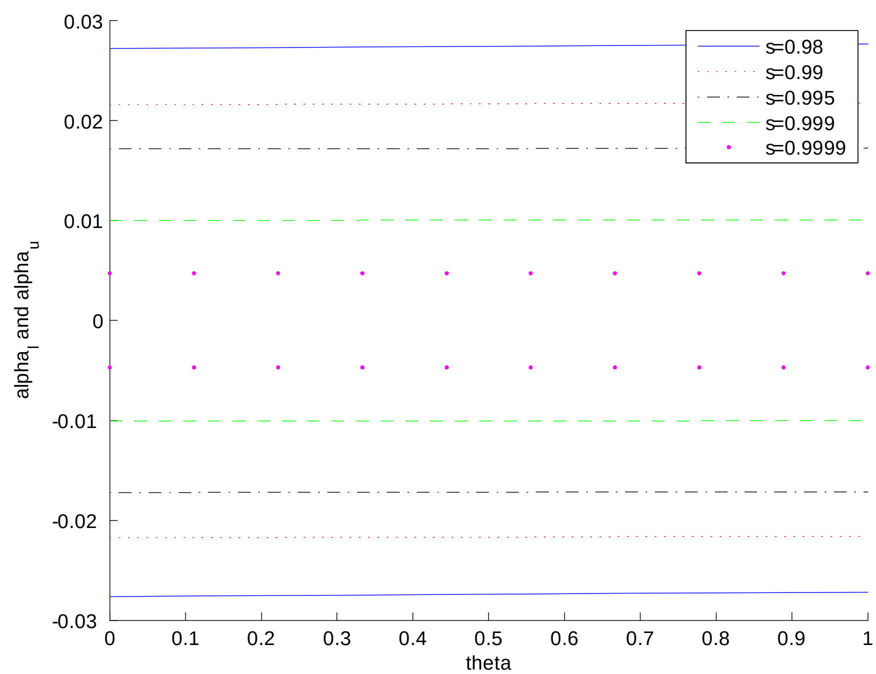

Figure 3 shows the optimal position of the barrier for different values of the transaction cost parameter s. To simplify the analysis we restrict θ between zero and one. The mean reversion λ is set to 1.125%/year, while the conditional volatility is 2.6%/year.

Figure 4: The optimal position of the barrier under no short-selling. Case of one riskless and one risky, mean-reverting asset

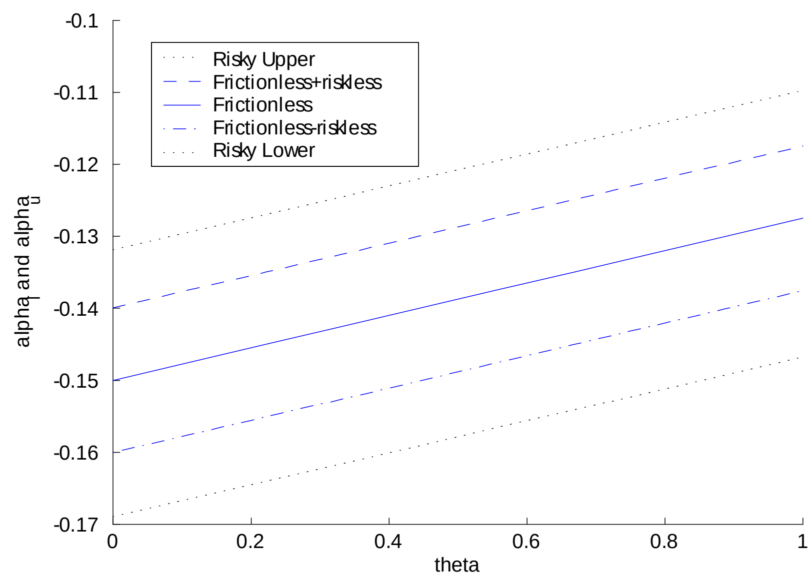

Figure 4 shows the optimal position of the thresholds functions $\overline{\alpha}$ and $\underline{\alpha}$ when short-selling is not allowed. The mean reversion λ is set to 1.125%/year, while the conditional volatility is 2.6%/year. The transaction costs are levied at the rate of 0.5%, implying $s = 0.995$. Finally, both the conditional volatility of the risky asset σ_1 and the risk premium γ are set equal to 15%/year.

Figure 5: The optimal position of the barrier when short-selling is allowed. Case of one riskless and one risky, mean-reverting asset

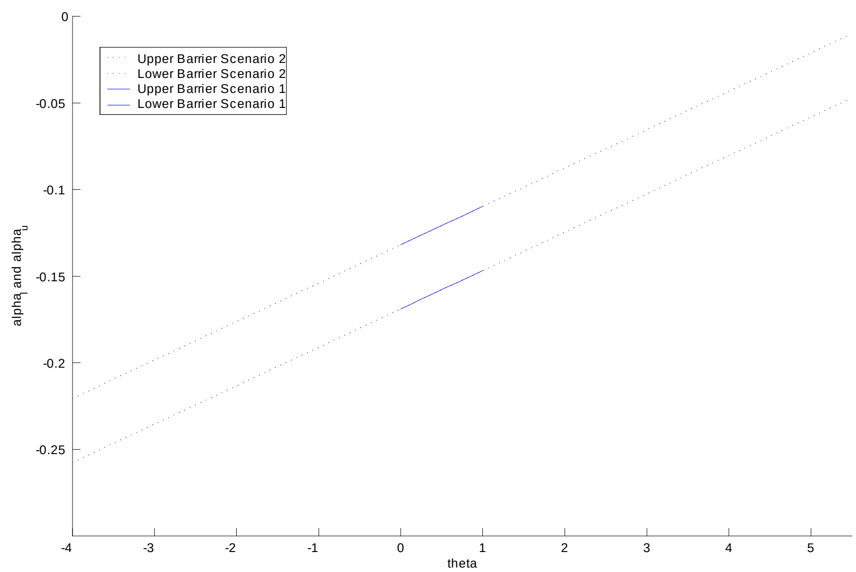

Figure 5 shows the optimal position of the thresholds functions $\overline{\alpha}$ and $\underline{\alpha}$ when short-selling is allowed. The mean reversion λ is set to 1.125%/year, while the conditional volatility is 2.6%/year. The transaction costs are levied at the rate of 0.5%, implying $s = 0.995$. Finally, both the conditional volatility of the risky asset σ_1 and the risk premium γ are set equal to 15%/year.

Figure 6: The effect of transaction costs on the portfolio-adjustment boundary. Case of one riskless and one risky, mean-reverting asset

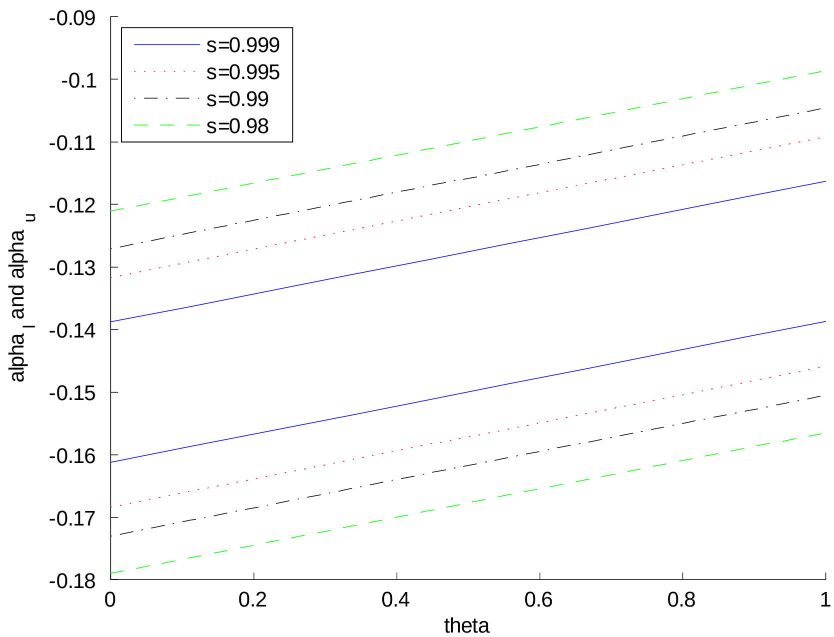

Figure 6 shows the optimal position of the barrier for different values of the transaction cost parameter s. No short-selling is allowed. The mean reversion λ is set to 1.125%/year, while the conditional volatility is 2.6%/year. Finally, both the conditional volatility of the risky asset σ_1 and the risk premium γ are set equal to 15%/year.

Figure 7: The expected holding period under Scenario 2. Case of one riskless and one risky, mean reverting asset

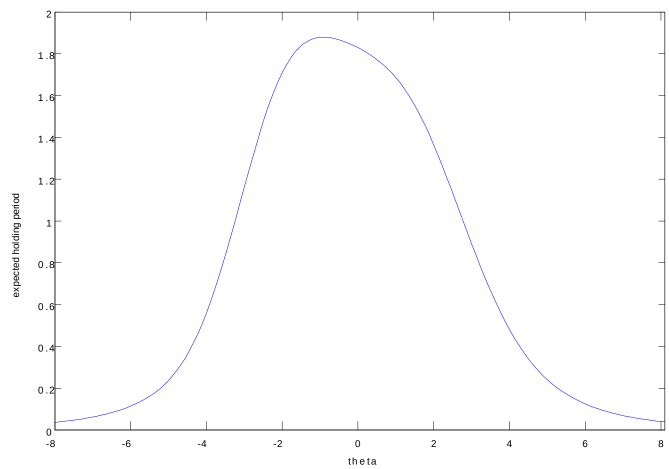

Figure 7 shows the expected time from buy to sell (expressed in years) corresponding to Scenario 2 as function of the initial portfolio composition θ. The parameters are: $s = 0.995$, $\sigma_1 = 0.15$, $\lambda = 0.0125$, $\sigma = 0.026$ and $\gamma = 0.15$.