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A Limit Theorem for Equilibria under Ambiguous Beliefs Correspondences

Giuseppe De Marco[*] and Maria Romaniello[**]

Abstract

Previous literature shows that, in many different models, limits of equilibria of perturbed games are equilibria of the unperturbed game when the sequence of perturbed games converges to the unperturbed one in an appropriate sense. The question whether such limit property extends to the equilibrium notions in ambiguous games is not yet clear as it seems; in fact, previous literature shows that the extension fails in simple examples.
The contribution in this paper is to show that the limit property holds for equilibria under ambiguous beliefs correspondences (presented by the authors in a previous paper). Key for our result is the sequential convergence assumption imposed on the sequence of beliefs correspondences. Counterexamples show why this assumption cannot be removed.

Keywords: Ambiguous games, beliefs correspondences, limit equilibria.

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Table of contents

1 Introduction

An important issue in noncooperative game theory is how the set of Nash equilibria changes when some parameter of the game changes continuously. Even in the simplest models, it can be shown that if one considers a sequence of perturbed games converging to an unperturbed game in an appropriate sense, then there might exist an equilibrium x^* of the unperturbed game such that there are no sequences of equilibria of the perturbed games converging to x^*. Conversely, many results have been obtained in the literature showing that limits of equilibria of perturbed games are, instead, equilibria of the unperturbed game even in general models (see for instance Fudenberg and Tirole (1993)). This *limit property*[2] has interesting implications; for instance, it provides a useful theoretical tool for looking at comparative statics effects of parameter changes on Nash predictions; moreover, the classical theory on Nash equilibrium refinements builds upon this kind of result (see for instance van Damme (1989) for an extensive survey). In fact, refinements theory is based on the idea that players make infinitesimal errors when playing their equilibrium strategies and, therefore, *refined* equilibria are selected as limits of sequences of equilibria of perturbed games in which perturbations represent the possibility of mistakes. The *quantal response equilibrium* (McKelvey and Palfrey (1995)) is a generalization of Nash equilibrium which gives another way to model games with noisy players. The limit property in this case says that quantal response equilibria converge to Nash equilibria as the error goes to zero (providing a unique selection of Nash equilibria in generic games). A similar approach has been considered in Friedmann and Mezzetti (2005) in which *random beliefs equilibria* have been investigated. In a random belief equilibrium player's beliefs about others' strategy choices are randomly drawn from a belief distribution that is dispersed around a central strategy profile called the *focus*. It is shown that, as players' beliefs converge to certainty, limits of random beliefs equilibria are Nash equilibria and therefore provide a (not necessarily unique) selection of Nash equilibria.

The criticism to the strength of the consistency condition that all players' beliefs are correct in equilibrium is also the main motivation for many equilibrium notions in which this consistency condition has been weakeneed by taking into account ambiguous beliefs (see for instance Dow and Werlang (1994), Lo (1996), Klibanoff (1996), Eichberger and Kelsey (2000) and Marinacci (2000)). The question whether the limit property extends to equilibria in ambiguous games has not been clarified yet in the literature. On the one hand, the nature of the definition of equilibrium makes it reasonable to expect that the extension to ambiguous games holds. On the other hand, previous literature shows that the extension fails in simple examples. In fact, Kajii and Ui (2005) consider an interim Bayesian equilibrium notion in a class of incomplete information games with ambiguous beliefs over the state space and find out, in a counterexample, that limits of equilibria of perturbed games are not necessarily equilibria in the unperturbed game, even if the parameters which describe ambiguity change continuously.

This paper studies the limit property for a notion of equilibrium in ambiguous games called *equilibrium under ambiguous beliefs correspondences* (De Marco and Romaniello (2011,b)[3]). In our approach, each player's beliefs are given by a correspondence which provides a set of subjective additive beliefs (probability distributions) over outcomes for every strategy profile. Such *beliefs correspondences* are exogenous and represent the ability of each player to put restrictions

[2]The set-valued mapping version of this property is the *closed graph property* of the Nash Correspondence.

[3]Many motivating examples can be found in this paper together with existence results. An application to coalition formation is the subject of De Marco and Romaniello (2011,a).

on beliefs over outcomes consistently with the strategy profile. In particular, this model embodies ambiguity about beliefs over opponents' strategies and a class of incomplete information games with multiple priors. In this paper we show that limits of equilibria under perturbed beliefs correspondences are equilibria under unperturbed beliefs correspondences when the sequence of perturbed beliefs correspondences converge to the unperturbed ones in the appropriate sense. In particular, key for our result is the sequential convergence assumption (Lignola and Morgan (1992);a,b) imposed on the sequence of beliefs correspondences. Two counterexamples show why this assumption cannot be removed and provide the intuition why the equilibrium notion in Kajii and Ui (2005) fails to satisfy the limit property in their counterexample. Finally, since our equilibrium notion coincides with the classical Nash equilibrium concept when beliefs are correct, our limit result implies that sequences of equilibria under ambiguous beliefs correspondences converge to Nash equilibria if ambiguity converges to zero in the appropriate sense. Therefore, this latter remark makes it possible to construct selection mechanisms for Nash equilibria based on a stability property with respect to ambiguous perturbations of beliefs.

2 Games, beliefs correspondences and equilibria

We consider a finite set of players $I = \{1, \dots n\}$; for every player i, $\Psi_i = \{\psi_i^1, \dots, \psi_i^{k(i)}\}$ is the (finite) pure strategy set of player i, $\Psi = \prod_{i \in I} \Psi_i$ and $\Psi_{-i} = \prod_{j \neq i} \Psi_j$. Denote with X_i the set of mixed strategies of player i, that is, each strategy $x_i \in X_i$ is a vector $x_i = (x_i(\psi_i))_{\psi_i \in \Psi_i} \in \mathbb{R}_+^{k(i)}$ such that $\sum_{\psi_i \in \Psi_i} x_i(\psi_i) = 1$. Denote also with $X = \prod_{j=1}^n X_j$ and with $X_{-i} = \prod_{j \neq i} X_j$.

Differently from the classical literature on games, in this work we do not assume the existence of a one to one correspondence between strategies and outcomes of a game. Instead, we denote with $\Omega \subseteq \mathbb{R}^n$ the set of outcomes of the game, where ω_i represents the payoff of player i when outcome ω is realized. Let $\mathcal{P}$ be the set of all probability distributions on Ω, we consider the general situation in which each player is endowed with a set-valued map $\mathcal{B}_i : X \rightsquigarrow \mathcal{P}$, called *beliefs correspondence*, which gives to player i the set $\mathcal{B}_i(x)$ of subjective beliefs over outcomes, for every strategy profile $x \in X$. We consider the (extreme) situation in which players are either *pessimistic* or *optimistic*, where a player is pessimistic if, in the presence of ambiguity, emphasizes the lower payoffs while he is optimistic if he emphasizes the higher ones instead.

More precisely, if we denote with $E_i(\varrho) = \sum_{\omega \in \Omega} \varrho(\omega)\omega_i$, then a pessimistic player has the pessimistic payoff $F_i^P : X \to \mathbb{R}$ defined by

$$F_i^P(x) = \min_{\varrho \in \mathcal{B}_i(x)} E_i(\varrho) \quad \forall x \in X, \tag{1}$$

while an optimistic player has the optimistic payoff $F_i^O : X \to \mathbb{R}$ defined by

$$F_i^O(x) = \max_{\varrho \in \mathcal{B}_i(x)} E_i(\varrho) \quad \forall x \in X. \tag{2}$$

Assuming that players are partitioned in optimistic and pessimistic ones, that is, $I = O \cup P$ with $O \cap P = \emptyset$; we consider the game

$$\Gamma^{O,P} = \{I; (X_i)_{i \in I}; (\mathcal{B}_i)_{i \in I}; (F_i^O)_{i \in O}, (F_i^P)_{i \in P}\}.$$

This game is a classical strategic form game and

Definition 2.1: A Nash equilibrium of $\Gamma^{O,P}$ is called *equilibrium under beliefs correspondences* $\mathcal{B}_i$ with optimistic players O and pessimistic players P.

For the sake of completeness we report the existence result presented in De Marco and Romaniello (2011). Moreover, it will be clear in the next section that the assumptions imposed in the existence theorem are not required to prove the limit theorem.

Theorem 2.2: *Assume that for every player i, $\mathcal{B}_i$ is continuous*[4] *with not empty, compact and convex images for every $x \in X$. If, for every player $i \in O$, $\mathcal{B}_i(\cdot, x_{-i})$ is concave*[5] *in X_i for every $x_{-i} \in X_{-i}$ and, for every player $i \in P$, $\mathcal{B}_i(\cdot, x_{-i})$ is convex in X_i for every $x_{-i} \in X_{-i}$, then, the game $\Gamma^{O,P}$ has at least an equilibrium.*

Remark 2.3 (Related concepts): A particular case of beliefs correspondences can be obtained in the classical framework in which each player i is endowed with a payoff function $f_i : \Psi \to \mathbb{R}$ and a beliefs correspondence from strategy profiles to correlated strategies, i.e. $\mathcal{K}_i : X \rightsquigarrow \Delta$, where Δ is the set of probability distributions on Ψ. In that case, denoting with $\Omega = \{(f_1(\psi), \ldots, f_n(\psi)) \,|\, \psi \in \Psi\}$ the set of outcomes, the beliefs correspondence over outcomes $\mathcal{B}_i : X \rightsquigarrow \mathcal{P}$ is defined by

$$\mathcal{B}_i(x) = \{\varrho \in \mathcal{P} \,|\, \exists \mu \in \mathcal{K}_i(x) \text{ with } \varrho(f_i(\psi)) = \mu(\psi) \; \forall \psi \in \Psi\}.$$

Note that if $\mathcal{K}_i(x) = x$ then the game $\Gamma^{O,P}$ coincides with the mixed extension of Γ so that the set of equilibria of $\Gamma^{O,P}$ coincides with the set of mixed strategy Nash equilibria of Γ.
Finally, in De Marco and Romaniello (2011,b) it is shown that our concept is related to the *equilibrium with uncertainty aversion* notion (Klibanoff (1996)) whenever the beliefs correspondence take the following form

$$\mathcal{B}_i(x) = \{\varrho \in \mathcal{P} \,|\, \exists p \in B_i \text{ with } \varrho(f_i(\psi)) = x_i(\psi_i)p(\psi_{-i}) \; \forall \psi \in \Psi\} \quad \forall x \in X. \tag{3}$$

where B_i is a nonempty, closed and convex subset of probability distributions over Ψ_{-i} for every $i = 1, \ldots, n$.

3 The limit theorem

The problem we address in this section is the following: Given the n-tuple of beliefs correspondence $(\mathcal{B}_1, \ldots, \mathcal{B}_n)$ with corresponding game $\Gamma^{O,P}$ and a sequence of n-tuples of perturbed beliefs

[4]Given the set valued map $F : Z \rightsquigarrow Y$, then (see Aubin and Frankowska (1990))

i) F is lower semicontinuous in z' if for every $y \in F(z')$ and every sequence $(z_\nu)_{\nu\in\mathbb{N}}$ converging to z' there exists a sequence $(y_\nu)_{\nu\in\mathbb{N}}$ converging to y such that $y_\nu \in F(z_\nu)$ for every $\nu \in \mathbb{N}$.

ii) F is *upper semicontinuous* in z' if for every open set U such that $F(z') \subseteq U$ there exists $\eta > 0$ such that $F(z) \subseteq U$ for all $z \in B_Z(z', \eta) = \{\zeta \in Z \,|; ||\zeta - z'|| < \eta\}$.

iv) F is *continuous* (in the sense of Painlevé-Kuratowski) in z' if it is lower semicontinuous and upper semicontinuous in z'.

[5]Given a convex set Z and a set valued map $F : Z \rightsquigarrow Y$ then

i) F is a said to be *concave* if $tF(\overline{z}) + (1-t)F(\widehat{z}) \subseteq F(t\overline{z} + (1-t)\widehat{z}) \quad \forall\, \overline{z}, \widehat{z} \in Z, \; \forall\, t \in [0,1]$

ii) F is a said to be *convex* if $F(t\overline{z} + (1-t)\widehat{z}) \subseteq tF(\overline{z}) + (1-t)F(\widehat{z}) \quad \forall\, \overline{z}, \widehat{z} \in Z, \; \forall\, t \in [0,1]$.

correspondences $(\mathcal{B}_{1,\nu},\dots,\mathcal{B}_{n,\nu})_{\nu\in\mathbb{N}}$ with corresponding sequence of perturbed games $(\Gamma_\nu^{O,P})_{\nu\in\mathbb{N}}$; we look for conditions of convergence of the sequence of perturbed beliefs correspondences to the beliefs correspondences of $\Gamma^{O,P}$ which guarantee that converging sequences of equilibria of the perturbed games have their limits in the set of equilibria of the unperturbed game. This problem looks especially interesting in the particular case in which the unperturbed game corresponds to a classical game with no ambiguity. In fact, the question in this case is to understand whether sequences of equilibria under ambiguous beliefs correspondences converge to Nash equilibria if ambiguity converges to zero in an appropriate sense.

THEOREM 3.1: *Given the n-tuple of beliefs correspondence $(\mathcal{B}_1,\dots,\mathcal{B}_n)$ and the corresponding game $\Gamma^{O,P}$. Assume that,*

i) For every player i $(\mathcal{B}_{i,\nu})_{\nu\in\mathbb{N}}$ is a sequence of correspondences, with $\mathcal{B}_{i,\nu}: X \rightsquigarrow \mathcal{P}$ for every $\nu\in\mathbb{N}$, which is sequentially convergent to $\mathcal{B}_i$, that is, for every $x\in X$ and every sequence $(x_\nu)_{\nu\in\mathbb{N}}$ converging to x,

$$\operatorname*{Lim\,sup}_{\nu\to\infty} \mathcal{B}_{i,\nu}(x_\nu) \subseteq \mathcal{B}_i(x) \subseteq \operatorname*{Lim\,inf}_{\nu\to\infty} \mathcal{B}_{i,\nu}(x_\nu) \tag{4}$$

where

$$\operatorname*{Lim\,inf}_{\nu\to\infty} \mathcal{B}_{i,\nu}(x_\nu) = \{\varrho\in\mathcal{P} \mid \forall\varepsilon>0\ \exists\overline{\nu}\ s.t.\ for \nu\geq\overline{\nu}\ S(\varrho,\varepsilon)\cap\mathcal{B}_{i,\nu}(x_\nu)\neq\emptyset\},$$
$$\operatorname*{Lim\,sup}_{\nu\to\infty} \mathcal{B}_{i,\nu}(x_\nu) = \{\varrho\in\mathcal{P} \mid \forall\varepsilon>0\ \forall\overline{\nu}\in\mathbb{N}\ \exists\nu\geq\overline{\nu}\ s.t.\ S(\varrho,\varepsilon)\cap\mathcal{B}_{i,\nu}(x_\nu)\neq\emptyset\}.$$

and $S(\varrho,\varepsilon)$ is the ball in $\mathbb{R}^{|\Omega|}$ with center ϱ and radius ε.

ii) The sequence $(x_\nu^*)_{\nu\in\mathbb{N}}\subset X$ converges to $x^*\in X$ and, for every $\nu\in\mathbb{N}$, x_ν^* is an equilibrium of the game $\Gamma_\nu^{O,P}$ corresponding to the n-tuple of beliefs correspondences $(\mathcal{B}_{1,\nu},\dots,\mathcal{B}_{n,\nu})$.

Then, x^* is an equilibrium of the game $\Gamma^{O,P}$.

Proof. For every $\nu\in\mathbb{N}$, let $F_{i,\nu}^O$ and $F_{i,\nu}^P$ be the payoffs for the optimistic and pessimistic player i in the game $\Gamma_\nu^{O,P}$ corresponding to the n-tuple of beliefs correspondences $(\mathcal{B}_{1,\nu},\dots,\mathcal{B}_{n,\nu})$. First, we prove that for a fixed player i the sequences $(F_{i,\nu}^O)_{\nu\in\mathbb{N}}$ and $(F_{i,\nu}^P)_{\nu\in\mathbb{N}}$ continuously converge respectively to F_i^O and F_i^P, that is, for every $x\in X$ and every sequence $(x_\nu)_{\nu\in\mathbb{N}}$ converging to x it holds that $\lim_{\nu\to\infty} F_{i,\nu}^O(x_\nu) = F_i^O(x)$ and $\lim_{\nu\to\infty} F_{i,\nu}^P(x_\nu) = F_i^P(x)$. In fact, for every ν there exist $\overline{\varrho}_\nu$ and $\widehat{\varrho}_\nu$ in $\mathcal{B}_{i,\nu}(x_\nu)$ such that $E_i(\overline{\varrho}_\nu) = F_{i,\nu}^O(x_\nu)$ and $E_i(\widehat{\varrho}_\nu) = F_{i,\nu}^P(x_\nu)$. By assumption, the sequence $(\mathcal{B}_{i,\nu})_{\nu\in\mathbb{N}}$ is sequential upper convergent; that is, for every $x\in X$ and every sequence $(x_\nu)_{\nu\in\mathbb{N}}$ converging to x,

$$\operatorname*{Lim\,sup}_{\nu\to\infty} \mathcal{B}_{i,\nu}(x_\nu) \subseteq \mathcal{B}_i(x).$$

Then, for every converging subsequence $(\overline{\varrho}_k)_{k\in\mathbb{N}}\subset(\overline{\varrho}_\nu)_{\nu\in\mathbb{N}}$ with $\overline{\varrho}_k\to\overline{\varrho}$ and for every converging subsequence $(\widehat{\varrho}_k)_{k\in\mathbb{N}}\subset(\widehat{\varrho}_\nu)_{\nu\in\mathbb{N}}$ with $\widehat{\varrho}_k\to\widehat{\varrho}$, the limits $\overline{\varrho}$ and $\widehat{\varrho}$ belong to $\mathcal{B}_i(x)$. This implies that $E_i(\overline{\varrho})\geq F_i^P(x)$ and $E_i(\widehat{\varrho})\leq F_i^O(x)$. Hence

$$\limsup_{\nu\to\infty} F_{i,\nu}^O(x_\nu) \leq F_i^O(x) \quad \text{and} \quad F_i^P(x) \leq \liminf_{\nu\to\infty} F_{i,\nu}^P(x_\nu).$$

Conversely, let $\overline{\varrho}$ and $\widehat{\varrho}$ be elements of $\mathcal{B}_i(x)$ such that $E_i(\overline{\varrho}) = F_i^P(x)$ and $E_i(\widehat{\varrho}) = F_i^O(x)$. By assumption, the sequence $(\mathcal{B}_{i,\nu})_{\nu\in\mathbb{N}}$ is sequential lower convergent; that is, for every $x \in X$ and every sequence $(x_\nu)_{\nu\in\mathbb{N}}$ converging to x, it follows that

$$\mathcal{B}_i(x) \subseteq \operatorname*{Lim\,inf}_{\nu\to\infty} \mathcal{B}_{i,\nu}(x_\nu).$$

Then, there exist sequences $(\overline{\varrho}_\nu)_{\nu\in\mathbb{N}}$ with $\overline{\varrho}_\nu \to \overline{\varrho}$ and $(\widehat{\varrho}_\nu)_{\nu\in\mathbb{N}}$ with $\widehat{\varrho}_\nu \to \widehat{\varrho}$, such that $\overline{\varrho}_\nu$ and $\widehat{\varrho}_\nu$ in $\mathcal{B}_{i,\nu}(x_\nu)$ for all $\nu \in \mathbb{N}$. Since $E_i(\overline{\varrho}_\nu) \geq F_{i,\nu}^P(x_\nu)$ and $E_i(\widehat{\varrho}_\nu) \leq F_{i,\nu}^O(x_\nu)$ for all $\nu \in \mathbb{N}$. Hence

$$F_i^P(x) = \limsup_{\nu\to\infty} E_i(\overline{\varrho}_\nu) \geq \limsup_{\nu\to\infty} F_{i,\nu}^P(x_\nu)$$

and

$$F_i^O(x) = \liminf_{\nu\to\infty} E_i(\widehat{\varrho}_\nu) \leq \liminf_{\nu\to\infty} F_{i,\nu}^O(x_\nu).$$

Therefore

$$F_i^P(x) \leq \liminf_{\nu\to\infty} F_{i,\nu}^P(x_\nu) \leq \limsup_{\nu\to\infty} F_{i,\nu}^P(x_\nu) \leq F_i^P(x) \implies F_i^P(x) = \lim_{\nu\to\infty} F_{i,\nu}^P(x_\nu)$$

and

$$F_i^O(x) \leq \liminf_{\nu\to\infty} F_{i,\nu}^O(x_\nu) \leq \limsup_{\nu\to\infty} F_{i,\nu}^O(x_\nu) \leq F_i^O(x) \implies F_i^O(x) = \lim_{\nu\to\infty} F_{i,\nu}^O(x_\nu)$$

which finally implies that the sequences $(F_{i,\nu}^O)_{\nu\in\mathbb{N}}$ and $(F_{i,\nu}^P)_{\nu\in\mathbb{N}}$ continuously converge respectively to F_i^O and F_i^P.

Now, let $(x_\nu^*)_{\nu\in\mathbb{N}} \subset X$ be a sequence converging to $x^* \in X$ such that, for every $\nu \in \mathbb{N}$, x_ν^* is an equilibrium of the game $\Gamma_\nu^{O,P}$ corresponding to the n-tuple of beliefs correspondences $(\mathcal{B}_{1,\nu}, \ldots, \mathcal{B}_{n,\nu})$. Suppose i is an optimistic player then for every ν it follows that

$$F_{i,\nu}^O(x_{i,\nu}^*, x_{-i,\nu}^*) \geq F_{i,\nu}^O(x_i', x_{-i,\nu}^*) \quad \forall x_i' \in X_i$$

taking the limit as $\nu \to \infty$ we get

$$F_i^O(x_i^*, x_{-i}^*) = \lim_{\nu\to\infty} F_{i,\nu}^O(x_{i,\nu}^*, x_{-i,\nu}^*) \geq \lim_{\nu\to\infty} F_{i,\nu}^O(x_i', x_{-i,\nu}^*) = F_i^O(x_i', x_{-i}^*) \quad \forall x_i' \in X_i$$

which implies that x_i^* is a best reply to x_{-i}^*. Since analogous arguments hold for pessimistic players we get the assertion. □

Remark 3.2: As it will be clarified in the counterexamples in the next section, the sequential convergence assumption (4) is crucial to obtain the limit theorem. This property (namely the *upper and lower sequential convergence* properties) have been previously defined and used to obtain epicontinuity of marginal functions in Lignola and Morgan (1992,a) and convergence for minsup problems in Lignola and Morgan (1992,b). Finally, Theorem 5.44 in Rockafellar and Wets (1998) shows that the convergence assumption (4) (therein called *continuous convergence* of set-valued maps) differs from the convergence (in the sense of Painlevé-Kuratowski) of the graphs of the correspondences (*graphical convergence*).

Remark 3.3: If the beliefs correspondences over outcomes are built upon beliefs correspondences over the set of pure strategy profiles as illustrated in remark 2.3, then our limit result implies that sequences of equilibria under ambiguous beliefs correspondences converge to Nash equilibria of the game without ambiguity, whenever the sequences of perturbed beliefs correspondences sequentially converge to the identity mappings. Therefore, it could be possible to construct selection mechanisms for Nash equilibria based on a stability property with respect to ambiguous perturbations on beliefs.

4 Two counterexamples

In this section we give two examples showing that the limit property of the equilibria fails by removing from the hypothesis of the previous theorem only the sequential upper convergence assumption or only the sequential lower convergence assumption. The first example is just a reformulation of the original counterexample given in Kajii and Ui (2005) while, the second example is slight variation of the first one.

4.1 Example 1

This example is substantially the one presented in Kajii and Ui (2005). Consider the following 2-player incomplete information game with ambiguous priors.

	a_2	b_2
a_1	1,-2	0,0
b_1	0,-2	1,0

Game 1

	a_2	b_2
a_1	1,1	0,0
b_1	0,1	1,0

Game 2

Fix $\varepsilon \in [0, 1]$, assume that the prior probability that player 1 assigns to Game 1 is a function of ε given by

$$\mathbb{P}_1(\varepsilon) = \varepsilon \quad \forall \varepsilon \in [0, 1],$$

while, the prior probability that player 1 assigns to Game 1 is ambiguous[6] and given by a correspondence defined by

$$\begin{cases} \mathbb{P}_2(\varepsilon) \in \left[\frac{\varepsilon}{2-\varepsilon}, 1\right] & \text{if } \varepsilon \in]0, 1] \\ \mathbb{P}_2(\varepsilon) = 0 & \text{if } \varepsilon = 0 \end{cases}$$

Therefore, for $\varepsilon > 0$ the game is

	a_2	b_2
a_1	1,1-3$\mathbb{P}_2(\varepsilon)$	0,0
b_1	0,1-3$\mathbb{P}_2(\varepsilon)$	1,0

where $\mathbb{P}_2(\varepsilon) \in \left[\frac{\varepsilon}{2-\varepsilon}, 1\right]$.

We denote the mixed strategies as follows: $x_1 = prob(a_1)$, $1 - x_1 = prob(b_1)$, $x_2 = prob(a_2)$ and $1 - x_2 = prob(b_2)$. Therefore, with an abuse of notation, (x_1, x_2) identifies a strategy profile for every $x_1 \in [0, 1]$ and $x_2 \in [0, 1]$.

Consider the case of a pessimistic player 2. In this case the payoff of player 2 for a given mixed strategy profile (x_1, x_2) is

$$F^P_{2,\varepsilon}(x_1, x_2) = \min_{\mathbb{P}_2(\varepsilon) \in \left[\frac{\varepsilon}{2-\varepsilon}, 1\right]} x_2(1 - 3\mathbb{P}_2(\varepsilon)) = -2x_2$$

Hence, best reply of player 2 is always $x_2 = 0$ regardless of player 1 strategy. This implies that the set of equilibria with a pessimistic player 2 is given by $N^P(\varepsilon) = \{(b_1, b_2)\} = \{(0, 0)\}$. For $\varepsilon = 0$, the game coincides with Game 2 and therefore $N^P(0) = \{(a_1, a_2)\} = \{(1, 1)\}$. Hence, for

[6] Kajii and Ui (2005) consider an interim Bayesian equilibrium notion in a class of incomplete information games with ambiguous beliefs over the state space. What here we call priors, in their counterexample are indeed posteriors coming from the observation of a signal and from the choice of the full Bayesian updating rule for multiple priors.

every sequence $\varepsilon_\nu \to 0$, the corresponding sequences of equilibria do not converge to the unique equilibrium in $N^P(0)$.

The failure of the limit property in this example depends on a lack of sequential upper convergence of the sequence of beliefs correspondences of player 2. In fact, being the three possible outcomes -2, 1 and 0 and denoted respectively with p_1, p_2, p_3 their probabilities, the beliefs correspondence for a given $\varepsilon > 0$ is

$$\mathcal{B}_{2,\varepsilon}(x_1, x_2) = \left\{ (p_1, p_2, p_3) \,|\, p_1 \in \left[\frac{x_2\varepsilon}{2-\varepsilon}, x_2\right],\, p_2 = x_2 - p_1,\, p_3 = 1 - x_2 \right\}$$

while

$$\mathcal{B}_{2,0}(x_1, x_2) = \{(p_1, p_2, p_3) \,|\, p_1 = 0,\, p_2 = x_2,\, p_3 = 1 - x_2\}$$

Hence, for every sequence $\varepsilon_\nu \searrow 0$ and every sequence $(x_{1,\nu}, x_{2,\nu}) \to (x_1, x_2)$ it follows that

$$\operatorname*{Lim\,sup}_{\nu\to\infty} \mathcal{B}_{2,\varepsilon_\nu}(x_{1,\nu}, x_{2,\nu}) = \{(p_1, p_2, p_3) \,|\, p_1 \in [0, x_2]\,,\, p_2 = x_2 - p_1,\, p_3 = 1 - x_2\}$$

which obviously implies that

$$\mathcal{B}_{2,0}(x_1, x_2) \subset \operatorname*{Lim\,sup}_{\nu\to\infty} \mathcal{B}_{2,\varepsilon_\nu}(x_{1,\nu}, x_{2,\nu})$$

and the sequence of $(\mathcal{B}_{2,\varepsilon_\nu})_\nu$ is not sequential upper convergent to $\mathcal{B}_{2,0}$. However, since

$$\operatorname*{Lim\,sup}_{\nu\to\infty} \mathcal{B}_{2,\varepsilon_\nu}(x_{1,\nu}, x_{2,\nu}) = \operatorname*{Lim\,inf}_{\nu\to\infty} \mathcal{B}_{2,\varepsilon_\nu}(x_{1,\nu}, x_{2,\nu})$$

for every sequence $(x_{1,\nu}, x_{2,\nu}) \to (x_1, x_2)$ it follows that the sequence of $(\mathcal{B}_{2,\varepsilon_\nu})_\nu$ is sequential lower convergent to $\mathcal{B}_{2,0}$

4.2 Example 2

Consider the game in the previous example in which the prior probability that player 1 assigns to Game 1 is still given by

$$\mathbb{P}_1(\varepsilon) = \varepsilon \quad \forall \varepsilon \in [0, 1]$$

while the prior probability that player 2 assigns to Game 1 is still ambiguous but it is given by the new correspondence defined by

$$\begin{cases} \mathbb{P}_2(\varepsilon) = \varepsilon & \text{if } \varepsilon \in]0, 1] \\ \mathbb{P}_2(\varepsilon) \in [0, 1] & \text{if } \varepsilon = 0 \end{cases}$$

therefore,

	a_2	b_2
a_1	1,1-3ε	0,0
b_1	0,1-3ε	1,0

$\varepsilon > 0$

	a_2	b_2
a_1	1,1-3$\mathbb{P}_2(\varepsilon)$	0,0
b_1	0,1-3$\mathbb{P}_2(\varepsilon)$	1,0

$\varepsilon = 0$ and $\mathbb{P}_2(\varepsilon) \in [0, 1]$

Since for $\varepsilon = 0$ the payoff of a pessimistic player 2 for a given mixed strategy profile (x_1, x_2) is

$$F^P_{2,0}(x_1, x_2) = \min_{\mathbb{P}_2(\varepsilon)\in[0,1]} x_2(1 - 3\mathbb{P}_2(0)) = -2x_2.$$

Hence, best reply of player 2 is always $x_2 = 0$ regardless of player 1 strategy. This implies that the set of equilibria with a pessimistic player 2 is given by $N^P(0) = \{(b_1, b_2)\} = \{(0, 0)\}$.

Denote with

$$N_1 = \{(1, x_2 \mid x_2 \in [1/2, 1])\}, \quad N_2 = \{(x_1, 1/2) \mid x_1 \in [0, 1])\}, \quad N_3 = \{(0, x_2) \mid x_2 \in [0, 1/2])\}.$$

Since for $\varepsilon \in]0, 1]$ the set of equilibria $N^P(\varepsilon$ obviously coincides with the set of classical Nash equilibria of the corresponding games, then the equilibrium correspondence $\varepsilon \rightsquigarrow N^P(\varepsilon)$ is defined by:

$$N^P(\varepsilon) = \begin{cases} \{(0,0)\} & \text{if } \varepsilon \in]1/3, 1] \\ N_1 \cup N_2 \cup N_3 & \text{if } \varepsilon = 1/3 \\ \{(1,1)\} & \text{if } \varepsilon \in]0, 1/3, 1[\\ \{(0,0)\} & \text{if } \varepsilon = 0 \end{cases}$$

Hence, for every sequence $\varepsilon_\nu \to 0$, the corresponding sequences of equilibria do not converge to the unique equilibrium in $N^P(0)$.

The failure of the limit property in this case depends instead on a lack of sequential lower convergence of the sequence of beliefs correspondences of player 2. In fact, being the three possible outcomes -2, 1 and 0 and denoted respectively with p_1, p_2, p_3 their probabilities, the beliefs correspondence for a given $\varepsilon = 0$ is

$$\mathcal{B}_{2,0}(x_1, x_2) = \{(p_1, p_2, p_3) \mid p_1 \in [0, x_2]\,,\, p_2 = x_2 - p_1,\, p_3 = 1 - x_2\}$$

while

$$\mathcal{B}_{2,\varepsilon}(x_1, x_2) = \{(p_1, p_2, p_3) \mid p_1 = \varepsilon x_2,\, p_2 = x_2 - p_1,\, p_3 = 1 - x_2\} \quad \forall \varepsilon \in]0, 1]$$

Hence, for every sequence $\varepsilon_\nu \searrow 0$ and every sequence $(x_{1,\nu}, x_{2,\nu}) \to (x_1, x_2)$ it follows that

$$\operatorname*{Lim\,inf}_{\nu\to\infty} \mathcal{B}_{2,\varepsilon_\nu}(x_{1,\nu}, x_{2,\nu}) = \{(p_1, p_2, p_3) \mid p_1 = 0,\, p_2 = x_2,\, p_3 = 1 - x_2\}$$

which obviously implies that

$$\operatorname*{Lim\,inf}_{\nu\to\infty} \mathcal{B}_{2,\varepsilon_\nu}(x_{1,\nu}, x_{2,\nu}) \subset \mathcal{B}_{2,0}(x_1, x_2)$$

and the sequence of$(\mathcal{B}_{2,\varepsilon_\nu})_\nu$ is not sequential lower convergent to $\mathcal{B}_{2,0}$. However, since

$$\operatorname*{Lim\,sup}_{\nu\to\infty} \mathcal{B}_{2,\varepsilon_\nu}(x_{1,\nu}, x_{2,\nu}) = \operatorname*{Lim\,inf}_{\nu\to\infty} \mathcal{B}_{2,\varepsilon_\nu}(x_{1,\nu}, x_{2,\nu})$$

for every sequence $(x_{1,\nu}, x_{2,\nu}) \to (x_1, x_2)$ it follows that the sequence of $(\mathcal{B}_{2,\varepsilon_\nu})_\nu$ is sequential upper convergent to $\mathcal{B}_{2,0}$.

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