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# *On Fairness of Equilibria in Economies with Differential Information*

**Achille Basile, Maria Gabriella Graziano and Marialaura Pesce**

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University of Naples Federico II — University of Salerno — Bocconi University, Milan

CSEF - Centre for Studies in Economics and Finance
DEPARTMENT OF ECONOMICS – UNIVERSITY OF NAPLES
80126 NAPLES - ITALY
Tel. and fax +39 081 675372 – e-mail: csef@unisa.it

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**Abstract**

The paper proposes a notion of fairness which overcomes the conflict arising between efficiency and the absence of envy in economies with uncertainty and asymmetrically informed agents. We do it in general economies which include, as particular cases, the main differential information economies studied in the literature. The analysis is further extended by allowing the presence of large traders, which may cause the lack of perfect competition.

**JEL Classification**: C71, D51, D82

**Keywords**: Mixed markets, fairness, envy, efficiency, asymmetric information

* Università di Napoli Federico II and CSEF. E-mails: basile@unina.it, mgrazian@unina.it, marialaura.pesce@unina.it

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# 1 Introduction

The aim of this paper is to investigate *efficiency* and *envy-freeness* (here same as *equitability*) for allocations of goods in a market economy with asymmetrically informed agents.

Efficiency and equity are, of course, classical issues of economic analysis. With the papers by Foley [10], Varian [31] and the unpublished contribution by Schmeidler-Yaari [30] a systematic study of *fairness* of allocations resulting as market equilibria was developed.

Fair allocations are not uniquely defined in the literature. We shall follow here the definition according to which an allocation is fair if it is both efficient and no agent would prefer to exchange his own bundle for anyone else's. In other words, an allocation is said to be fair is it is Pareto optimal and envy free.

It is known that in a pure exchange economy a fair allocation always exists. This follows from the fact that any competitive allocation resulting from an equal sharing of the total initial endowment is fair. The situation is radically different when production is allowed as well as in economies where agents are asymmetrically informed. In such cases, envy freeness may be incompatible with efficiency and therefore the set of fair allocations may be empty (see [6], [12], [25] and [31] among others).

Dealing with differential information economies we face a complex scenario depending on: what agents know when they write contracts, what they know ex post (namely, when consumption takes place). Our goal is to provide, in a model which includes, as particular cases, different situations covered by the literature, a notion of equitable allocation which solves the conflict arising between efficiency and the absence of envy.

To achieve our objective the proposed model is rather general both as for information as for the space of agents.

Concerning information, we assign, to each trader $t$, two subalgebras $\mathcal{F}_t$ and $\mathcal{G}_t$ of $(\Omega, \mathcal{F})$, the measurable space used for depicting uncertainty about states of nature, in order to represent: by $\mathcal{F}_t$, private information at time of contracting and, by the larger algebra $\mathcal{G}_t$, information revealed to $t$ at the date of contract delivery. The interpretation is standard. Let $\omega$ be the prevailing state of nature. Agent $t$ only knows that, among those belonging to the partition generating his information algebra, the (unique) event containing $\omega$ is realized. Given that, since indistinguishable states determine the same consumption, the relevant $\mathcal{G}_t$-measurability constraint on allocations follows. Analogously, it is only with reference to information $\mathcal{F}_t$ that the agent $t$ evaluates, in signing contracts, his future consumption.

Simply by specifying the fields $\mathcal{F}_t$ and $\mathcal{G}_t$, we end up into one of the well known cases of economies with uncertainty: economies with just uncertainty, represented by a probability space $(\Omega, \mathcal{F}, \pi)$, where $\Omega$ is the set of states of nature (see [3]); ex ante asymmetric information economies (see [9], [19], [28], [33] among others); interim economies in which the occurred state of nature is commonly known ex post (see for example [5], [6] and [32]); interim asymmetric information economies (see [1], [2], [13] and [33] among others).

Concerning the space of agents, we shall adopt a general measure space. Indeed we allow the presence of large traders, which causes the lack of perfect competition. Precisely we consider two kinds of agents: *large* traders which are represented by atoms of the measure space, and an ocean of *small* traders represented by the atomless part. A general mixed model represents the natural framework to deal with coalitional fairness (see [8], [11], [15], and [31] ), and it enables us to study simultaneously the case of finite economies, non-atomic economies as well as economies that may have atoms.

Now, given our model of market economy, we observe that if an individual compares himself to another one which enters in the market with an advantage (higher initial endowment or more precise informational signal), he is fated to be envious. Thus, each agent should compare his bundle only with the bundle of agents entering in the market without any kind of advantage respect to him, that is with his same endowment and same signal. We also guess that this goes in the direction of Varian's intuition. In the notion of envy introduced by Foley and Varian for a complete information exchange economy, agents with equal income under a competitive equilibrium allocation are envy free, so that the set of fair allocations is non-empty. That is, allowing comparisons to be made only between agents with the same initial opportunities a fair redistribution is possible.

The path for achieving the proof of the existence of fair allocations is as in Varian [31]: in generalized interim models, in which agents may be even ex post asymmetrically informed (i.e. $\mathcal{F}_t$ and $\mathcal{G}_t$ are both arbitary), we define a notion of *competitive market equilibrium*; once non emptiness of the set of competitive market equilibria is proved, from it we deduce the existence of (individual) fair allocations as well as coalitionally type fair allocations.

Competitive market equilibria coincide with well established solution concepts whenever the algebras $\mathcal{F}_t$ and $\mathcal{G}_t$ are specified. In particular, in economies with just uncertainty (i.e., $\mathcal{F}_t = \{\emptyset, \Omega\}$ and $\mathcal{G}_t = 2^{\Omega}$ for all $t \in T$), we obtain the classical notion defined by Arrow and Debreu ( [3]); whenever ex post agents are asymmetrically informed (i.e. $\mathcal{F}_t = \{\emptyset, \Omega\}$ for all $t \in T$ and $\mathcal{G}_t$ is arbitrary), we end up into the notion of Walrasian expectations equilibrium introduced by Radner in [28]; in the interim model, when ex post the state of nature is commonly known (i.e. $\mathcal{G}_t = 2^{\Omega}$ for all $t \in T$ and arbitrary $\mathcal{F}_t$), our notion reduces to be a constrained market equilibrium defined by Wilson (see [32])).

In generalized interim models we propose (we emphasize that here information is not only asymmetrically distributed among agents but also may changes ex-post), our notion of equilibrium is new, therefore we furnish a proof of its existence. Thus, a further contribution of this paper is the introduction of a new equilibrium concept which is efficient. In order to show the existence of a competitive market equilibrium, we construct a suitable correspondence between the original differential information economy and an auxiliary Arrow-Debreu exchange economy with uncertainty. Agents of the auxiliary economy are defined adopting the same idea by Harsanyi [18] to define Bayesian games. A type-agent is a couple $(t, \mathcal{E})$, where $t$ is an agent and $\mathcal{E}$ is an atom of his information partition. The future state of the fictitious economy is uncertain, but each type-agent has no private information. Moreover, since contracts are contingent on the future state of the economy, standard Arrow-Debreu equilibrium notions can be applied.

The paper is organized as follows. Section 2 specifies the general framework of differential information mixed markets. In Section 3 we discuss the problem of incompatibility between efficiency and absence of envy and propose a solution, that is a new notion of fair allocation whose existence is proved in Section 4. In Section 5 we introduce the notion of c-type fairness and prove that the set of c-fair allocations is non empty. Proofs are collected in the Appendix.

## 2 The Model

The model of economy we adopt presents a twofold generality: concerning the space of agents and concerning informational asymmetries among agents.

The space of agents we consider is a complete, finite measure space $(T, \mathcal{T}, \mu)$, where: $T$ is the set of agents and $\mathcal{T}$ is the $\sigma$-field of all eligible coalitions, whose economic weight on the market is given by the measure $\mu$. An arbitrary finite measure space of agents makes us deal simultaneously with the case of discrete economies, non-atomic economies as well as economies that may have atoms. Indeed, discrete economies are covered by a finite set $T$ with a counting measure $\mu$. Atomless economies are analyzed by assuming that $(T, \mathcal{T}, \mu)$ is the Lebesgue measure space with $T = [0, 1]$. Finally, mixed markets are those for which $T$ is composed by two sets: $T_0$ and $T_1$, where $T_0$ is the atomless sector and $T_1$ the set of atoms. We will refer to $T_0$ as the set of "small" traders and to $T_1$ as the set of "large" traders.[1]

Uncertainty about nature is, as usual, depicted by means of a probability space $(\Omega, \mathcal{F}, \pi)$, where $\Omega$ is the finite set of possible states of nature. Without loss of generality, we assume that $\mathcal{F}$ is the power set of $\Omega$ (i.e., $\mathcal{F} = 2^{\Omega}$) and $\pi$ is a strictly positive common prior which describes the relative probability of the states.

[1]This terminology is, in particular, motivated when $T$ is a separable metric space. Indeed, in this case, $T_0$ is the set of traders $t \in T$ for which $\mu(t) = 0$, while $T_1$ is the set of traders such that $\mu(t) > 0$ (see [20]).

Information has the standard representation by means of partitions of $\Omega$. With an abuse of notation we use the same symbol for a partition and for the algebra of subsets of $\Omega$ it generates. The interpretation is as usual: if the prevailing state is $\omega$, an agent endowed with information $\mathcal{G}$ (a partition of $\Omega$) observes the unique element $G(\omega)$ of $\mathcal{G}$ to which $\omega$ belongs or, in other words, the agent is informed that the prevailing state is in the event $G(\omega)$. Agents may be not equally informed concerning the true state of nature both when they write contracts and when consumption takes place, therefore to each trader $t$ we assign partitions $\mathcal{F}_t$ and $\mathcal{G}_t$ of $\Omega$ and hence the related algebras generated by them. The algebra $\mathcal{F}_t$ represents the private information at the time of contracting. Beside the information $\mathcal{F}_t$, a new information, due to the larger field $\mathcal{G}_t$, i.e., $\mathcal{F}_t \subseteq \mathcal{G}_t$, is revealed to $t$ at the date of contract delivery. Notice that for each $\omega \in \Omega$, $F_t(\omega) = \cup_{\omega' \in F_t(\omega)} G_t(\omega')$, where $G_t(\omega')$ denotes the event in the partition $\mathcal{G}_t$ containing the state $\omega'$ (since partition $\mathcal{G}_t$ is finer than $\mathcal{F}_t$).

Since the space $\Omega$ is finite, there exists a finite collection $\{\mathcal{F}_i\}_{i\in I}$ of algebras on $\Omega$ such that

$$\{\mathcal{F}_t \,:\, t \in T\} = \{\mathcal{F}_i \,:\, i \in I\}.$$

We assume that the set $\Theta_i$ defined by $\Theta_i = \{t \in T \,:\, \mathcal{F}_t = \mathcal{F}_i\}$ belongs to $\mathcal{T}$ (i.e., $\Theta_i$ is measurable) and that the family $(\Theta_i)_{i\in I}$ forms a partition of $T$ satisfying $\mu(\Theta_i) > 0$ for each $i \in I$[2]. Therefore there is a finite set $I$ of information types $\Theta_i$ and every agent $t \in \Theta_i$ is of information type $i$ in the sense that $\mathcal{F}_t = \mathcal{F}_i$.

We assume that there are $\ell$ private goods, so that $\mathbb{R}^\ell_+$ (the positive cone of the Euclidean space $\mathbb{R}^\ell$) is the commodity space. Furthermore, each agent $t \in T$ is characterized not only by the two algebras $\mathcal{F}_t$ and $\mathcal{G}_t$, but also by:

- a state-dependent utility function representing his preferences:

$$\begin{aligned} u_t :\ \Omega \times \mathbb{R}^\ell_+ &\ \rightarrow\ \mathbb{R} \\ (\omega, x) &\ \rightarrow\ u_t(\omega, x). \end{aligned}$$

  The utility function $u_t(\omega, \cdot)$ of each agent $t$ is strictly increasing, continuous and concave in each state $\omega$ of the economy, moreover for all $\omega$ in $\Omega$ the mapping $(t, x) \mapsto u_t(\omega, x)$ is $\mathcal{T} \otimes \mathcal{B}$-measurable, where $\mathcal{B}$ is the $\sigma$-field of Borel subsets of $\mathbb{R}^\ell_+$.

[2]This assumption implies that the following correspondence has measurable graph: $\Phi : T \to 2^{\mathcal{F}}$ defined by $\Phi(t) = \mathcal{F}_t$. It means that the graph of $\Phi$, namely $\{(t, \mathcal{E}) : \mathcal{E} \in \mathcal{F}_t\}$ belongs to the product $\sigma$-algebras $\mathcal{T} \otimes \mathcal{B}(2^{\mathcal{F}})$, where $\mathcal{B}$ denotes the Borel $\sigma$-algebra.

- An initial endowment of physical resources represented by the function

$$e_t : \ \Omega \ \rightarrow \ \mathbb{R}_+^\ell.$$

Since the consumption of an agent that is unable to distinguish between two states must be assumed to be the same, consumption profiles, including, naturally, the initial endowment that we have just introduced, are assumed to be $\mathcal{G}_t$-measurable. In other words a *consumption profile* is a vector valued function $a(t,\omega) =: a_t(\omega)$ which is $\mathcal{G}_t$-measurable for all $t \in T$[3].

For all $t \in T$, denote by $M_t$ the set

$$M_t = \left\{ a_t : \Omega \rightarrow \mathbb{R}_+^\ell \text{ such that } a_t(\cdot) \text{ is } \mathcal{G}_t - \text{measurable} \right\}.$$

Notice that wherever $\mathcal{G}_t = 2^\Omega$, measurability constraints play no role; hence $M_t$ coincides with the set of all functions $a_t : \Omega \rightarrow \mathbb{R}_+^\ell$.

**Definition 2.1.** *A consumption profile $a : T \times \Omega \rightarrow \mathbb{R}_+^\ell$ is said to be an* **allocation** *if for each $\omega \in \Omega$, $a(\cdot,\omega)$ is $\mu$-integrable and for almost all $t \in T$, $a_t \in M_t$.*
*The allocation $a$ is said to be* **feasible** *if*

$$\int_T a_t(\omega)\, d\mu \leq \int_T e_t(\omega)\, d\mu \quad \textit{for all } \omega \in \Omega.$$

Specification now of preferences clarifies that the model we are using can be ascribed to the class of the interim models, though the double algebra allows the unified treatment also of the so-called ex-ante models. Reason why we shall speak here of generalized interim model. In **"interim models"**, agents write contracts after they have received a signal as to what is the event containing the realized state of nature (interim). Precisely, if $\omega$ is the state that is going to occur, each agent $t$ receives the signal $F_t(\omega)$, so that even if $t$ still does not know exactly which state is going to occur, he can at least exclude all those states not in the event $F_t(\omega)$. Therefore, agents evaluate their bundles by taking into account such additional information or, we say, they use the **interim expected utility** defined as follows:

$$V_t(a_t)(\omega) = \sum_{\omega' \in F_t(\omega)} u_t(\omega', a_t(\omega')) \frac{\pi(\omega')}{\pi\left(F_t(\omega)\right)}. \tag{1}$$

Hence an agent $t$ in a state $\omega$ prefers an allocation $a$ to another $a'$ if and only if $V_t(a_t)(\omega) > V_t(a'_t)(\omega)$. Notice that for each agent $t$ and each allocation $a$, $V_t(a_t)(\cdot)$ is $\mathcal{F}_t$-measurable. Moreover, whenever $\mathcal{F}_t = \{\emptyset, \Omega\}$ the utility $V_t$ is the ex ante expected utility $h_t$ defined as follows over $a_t(\cdot)$

[3]This means that the function $a_t(\cdot)$ is constant over each event in the partition $\mathcal{G}_t$.

$$h_t(a_t) = \sum_{\omega\in\Omega} u_t(\omega, a_t(\omega))\pi(\omega), \tag{2}$$

Summing up, an exchange economy $E$ is modeled in this paper by the following collection:

$$E = \big\{(\Omega, \mathcal{F}, \pi);\ (T, \mathcal{T}, \mu);\ I\!R^{\ell}_{+};\ (\mathcal{F}_t, \mathcal{G}_t, u_t, e_t)_{t\in T}\big\}.$$

As usual, we can interpret the above economy as a two period ($\tau = 1, 2$) model, where consumption takes place at $\tau = 2$. At the date of contracting $\tau = 1$ there is uncertainty over the state of nature, and agents make a contract on net trades which may be contingent on the realized state of nature at $\tau = 2$. However, each agent observes private information denoted by $\mathcal{F}_t$, with respect to the prevailing state. That is, if $\omega$ is the true state at $\tau = 2$, agent $t$ knows that the realized state belongs to the event $F_t(\omega)$. At $\tau = 2$ , agents execute the trades according to the contract previously agreed and consumption takes place. At the date of contract delivery a new information, denoted by $\mathcal{G}_t$, is revealed to agent $t$. Since $\mathcal{F}_t \subseteq \mathcal{G}_t$, no one forgets his previous information. The fact that agents are asymmetrically informed at the date $\tau = 2$ makes measurability constraints on allocations relevant.

A **price** is a non-zero function $p : \Omega \to I\!R^{\ell}_{+}$. Given a price $p$, we define the **budget set** of agent $t$ in state $\omega$ as follows:

$$B_t(p, \omega) = \left\{ y_t \in M_t\ :\ \sum_{\omega'\in F_t(\omega)} p(\omega')\cdot y_t(\omega') \le \sum_{\omega'\in F_t(\omega)} p(\omega')\cdot e_t(\omega') \right\}. \tag{3}$$

We now define the competitive equilibrium notion in our general framework.

**Definition 2.2.** *A feasible allocation $a$ is said to be a* **competitive market equilibrium allocation** *if there exists a price $p : \Omega \to I\!R^{\ell}_{+}$ such that*

(1) $\quad a_t(\cdot)\ is\ \mathcal{G}_t - measurable\ for\ almost\ all\ t \in T$

(2) $\quad for\ all\ \omega \in \Omega\ and\ t \in T,\ a_t \in argmax_{y_t\in B_t(p,\omega)} V_t(y_t)(\omega).$

*The pair $(a, p)$ is said to be a* **competitive market equilibrium**. *We denote by $CME(E)$ the set of competitive market equilibrium allocations of the economy $E$.*

**Remark 2.3.** We observe that simply by specifying information algebras we cover different models of economies.
**Case of ex-ante model**: for all $t \in T$, $\mathcal{F}_t = \{\emptyset; \Omega\}$.
In this case agents are uninformed prior to contracting and use the ex-ante

expected utility (2) over a budget set that does not depend on $\omega$ and it is the well known ex-ante budget set (see [9] for example). It is the ex-ante model that can be further specified by taking:

1. $\mathcal{G}_t = 2^{\Omega}$, as the classical Arrow-Debreu one with uncertainty and symmetry among agents. The equilibrium notion of Definition 2.2 reduces to that of [3];

2. an arbitrary $\mathcal{G}_t$, as the ex ante asymmetric information economy (see [9], [28], [33] among others). With respect to the previous model the difference consists on what happens at time $\tau = 2$. In both models contracts are written ex-ante (i.e., before the state of nature is realized). Then, when consumption takes place if there is just uncertainty, at time $\tau = 2$ agents will exactly know which state of nature occurs while if agents are asymmetrically informed, if $\omega$ is the state of nature occurred at $\tau = 2$, each individual $t$ just observes $G_t(\omega)$. In other words, he cannot distinguish between states belonging to the same event of his private partition, but he only knows that states not in the event $G_t(\omega)$ have not occurred. With agents ex post asymmetrically informed, the equilibrium notion of Definition 2.2 reduces to the notion of Walrasian expectations equilibrium introduced by Radner [28] (see also [9] for atomless economies and [14], [26] for mixed markets with asymmetric information).

**Case of interim models**: any agent $t$ has his own information $\mathcal{F}_t$ and after receiving a signal as to what is the event in $\mathcal{F}_t$ containing the realized state o f nature, signs contracts. The budget set reduces to the known interim budget set (see for example [5],[6], [8] and [32]) and this is *strictu sensu* an interim situation. Still two subcases are possible.

3. When for all $t \in T$, $\mathcal{G}_t = 2^{\Omega}$, then we analyze interim exchange economy in which the true state of nature is commonly know ex post (see [5], [6] and [32] among others). Here our equilibrium notion reduces to be a constrained market equilibrium defined by Wilson (see [32]).

4. If for all $t \in T$, we have an arbitrary algebra $\mathcal{G}_t$ larger than the initial $\mathcal{F}_t$. Cases where no further information is ex post revealed to agents have been considered in [1], [2], [13] and [33], among others, under the name of interim asymmetric information economies. We emphasize that in general we do not require that $\mathcal{F}_t$ and $\mathcal{G}_t$ coincide. In interim models in which ex post agents may still asymmetrically receive further information, to the best of our knowledge[4], Definition 2.2 is new,

[4] The unique equilibrium concept studied in this context is the **rational expectations equilibrium (REE)** (see [1], [23] and [29] among others) which is different from Definition 2.2 since according to the REE notion, agents take into account also the information generated by the equilibrium prices.

therefore we need also to prove existence of equilibria. What will be our goal in Section 4.

## 3 Fair and Fair$^*$ allocations

In this section we first illustrate the problem of the non existence of a fair distribution of resources among asymmetrically informed agents, and then, we provide a solution (the proposed concept of **fair$^*$ allocations**) to the conflict between envy freeness and efficiency in our general context which includes as a particular case all the situations described before (see Remark 2.3). To this end, we first introduce the notion of Pareto optimality and envy freeness.

**Definition 3.1.** *A feasible allocation $a'$* **Pareto dominates** *an allocation $a$ if everybody weakly prefers (given his private information) $a'$ over $a$ in each state and there exist a state $\bar{\omega}$ and a coalition $S$ (i.e., $S \in \mathcal{T}$ with $\mu(S) > 0$) whose members strictly prefer $a'$ over $a$ in $\bar{\omega}$, that is*

$$(1) \qquad V_t(a'_t)(\omega) \geq V_t(a_t)(\omega) \quad for\ almost\ all\ t \in T\ and\ for\ all\ \omega \in \Omega$$
$$(2) \qquad V_t(a'_t)(\bar{\omega}) > V_t(a_t)(\bar{\omega}) \quad for\ almost\ all\ t \in S.$$

*A feasible allocation $a$ is said to be* **efficient (or Pareto optimal)** *if it is not Pareto dominated by any other allocation. We denote by $PO(E)$ the set of efficient allocations.*

**Remark 3.2.** From Definition 3.1 one may obtain well known efficiency notions simply by specifying the fields $\mathcal{F}_t$ and $\mathcal{G}_t$ as in Remark 2.3.

Below, we extend Varian's definition of fairness (see [31]) to our general model.

Given: an allocation $a$ , an agent $t \in T$ and $\omega \in \Omega$, we define some sets useful in analyzing equitability of allocations. Here they are:

$$\begin{aligned} A_t(\omega, a) &= \{s \in T \,:\, V_t(a_t)(\omega) < V_t(a_s)(\omega)\}. \\ A(\omega, a) &= \{t \in T \,:\, \mu(A_t(\omega, a)) > 0\}\ and \\ A(a) &= \bigcup_{\omega \in \Omega} A(\omega, a). \end{aligned}$$

The meaning is clear:
$A_t(\omega, a)$ is the set of individuals that $t$ envies at $a$ in state $\omega$.
$A(\omega, a)$ is the set of envious agents at $a$ in state $\omega$
$A(a)$ is the set of envious traders under the allocation $a$.

**Definition 3.3.** *An allocation $a$ is said to be* **envy free or equitable** *if $\mu(A(a)) = 0$.*

We denote by $E_E(E)$ the set of envy free allocations of the economy $E$.

**Definition 3.4.** *A feasible allocation $a$ is said to be* **fair** *if it is Pareto optimal and envy free.*

We denote by $IF(E)$ the set of fair allocations, i.e.,

$$IF(E) = PO(E) \cap E_E(E).$$

Notice that the above definition includes as a particular case Varian's notion, simply by allowing $T$ to be finite, and hence $E$ to be a finite economy. Moreover, if for all $t \in T$, $\mathcal{G}_t = 2^{\Omega}$, then we obtain the notion of envy freeness given in [6].

### 3.1 The incompatibility between efficiency and envy freeness

In [6], it is proved that whenever agents are asymmetrically informed, envy freeness may be incompatible with efficiency and therefore the set of fair allocations may be empty (see also [8] and [27]). De Clippel considers a differential information economy assuming that the true state of nature is commonly known at the time of implementing contracts (i.e., $\mathcal{G}_t = 2^{\Omega}$ for each agent $t \in T$), a requirement that makes both measurability and incentive compatibility constraints irrelevant.

Here we report de Clippel's example in [6] and revise it for our purposes in the general context of differential information economies where the true state of nature is not necessarily commonly known at the time of implementing contracts and hence measurability condition must be required.

**Example 3.5.** See [6]. Consider an economy $E$ with two equiprobable states of nature $\Omega = \{a, b\}$ and three agents $T = \{1, 2, 3\}$. Assume that agents' characteristics at the time of contracting are given as follows

$$\begin{array}{rr}
\mathcal{F}_1 = \{\{a, b\}\} & u_1(a, x_1) = u_1(b, x_1) = x_1 \\
\mathcal{F}_2 = \{\{a, b\}\} & u_2(a, x_2) = u_2(b, x_2) = \sqrt{x_2} \\
\mathcal{F}_3 = \{\{a\}, \{b\}\} & u_3(a, x_3) = u_3(b, x_3) = x_3.
\end{array}$$

The initial endowment totally amounts to $(e(a), e(b)) = (1200, 1800)$. An allocation $x$ is envy free if and only if it solves the following system

$$\begin{cases}
x_1(a) + x_1(b) \geq \max\{x_2(a) + x_2(b);\ x_3(a) + x_3(b)\} \\
\sqrt{x_2(a)} + \sqrt{x_2(b)} \geq \max\left\{\sqrt{x_1(a)} + \sqrt{x_1(b)};\ \sqrt{x_3(a)} + \sqrt{x_3(b)}\right\} \\
x_3(a) \geq \max\{x_1(a);\ x_2(a)\} \\
x_3(b) \geq \max\{x_1(b);\ x_2(b)\},
\end{cases}$$

whose solution is given below:

$$x_1^*(a) = x_2^*(a) = x_3^*(a)$$
$$x_1^*(b) = x_2^*(b) = x_3^*(b).$$

Assuming that $\mathcal{G}_t = 2^{\Omega}$, for $t = 1, 2, 3$, as in [6], and imposing the feasibility constraints, one obtains that

$$\begin{aligned} x_1^*(a) &= 400; \quad x_1^*(b) = 600 \\ x_2^*(a) &= 400; \quad x_2^*(b) = 600 \\ x_3^*(a) &= 400; \quad x_3^*(b) = 600; \end{aligned}$$

but, $x^*$ is Pareto dominated by the following feasible allocation

$$\begin{aligned} x_1(a) &= 301; \quad x_1(b) = 701 \\ x_2(a) &= 498; \quad x_2(b) = 498 \\ x_3(a) &= 401; \quad x_3(b) = 601. \end{aligned}$$

On the other hand, in the remaining cases, for each possible choice of the information fields $\mathcal{G}_t$ such that $\mathcal{F}_t \subseteq \mathcal{G}_t$ for $t = 1, 2, 3$, by imposing also the measurability constraints, we obtain

$$x_1^*(a) = x_1^*(b) = x_2^*(b) = x_2^*(a) = x_3^*(a) = x_3^*(b),$$

and by feasibility

$$x_1^*(a) + x_2^*(a) + x_3^*(a) = x_1^*(b) + x_2^*(b) + x_3^*(b) \leq 1200.$$

It follows that the best among these feasible allocations is the following

$$\begin{aligned} x_1^*(a) &= x_1^*(b) = 400 \\ x_2^*(a) &= x_2^*(b) = 400 \\ x_2^*(a) &= x_3^*(b) = 400 \end{aligned}$$

which is Pareto dominated by the allocation: $(e_1(a), e_1(b)) = (400, 400)$ $(e_2(a), e_2(b)) = (400, 400)$ $(e_3(a), e_3(b)) = (400, 1000)$.

Therefore, independently by the final information of agents, the set of envy free allocations is empty, i.e., $IF(E) = \emptyset$.

### 3.2 Envy-free$^*$ allocations

The reason of the incompatibility between efficiency and envy freeness in differential information economies is clear: in a complete information exchange economy when agents have the same initial endowment they face the same budget set. This implies that any competitive equilibrium allocation resulting from an equal sharing on the total initial endowment is envy free. Hence from the existence of a competitive equilibrium, one can deduce the non emptiness of the set of fair allocations. This is no longer true, for example when agents are asymmetrically informed and contracts are made in an interim stage, because even if the total initial endowment is equally shared among agents, different agents may have different private information, and hence different budget set . Therefore, the related notion of competitive equilibrium (constrained market equilibrium), may not be envy free (see [27]). The same problem arises even when ex post agents are still asymmetrically informed, as illustrated in the above example. For this reason, in [6] envy freeness is evaluated only in common knowledge events[5]. Precisely, it has been proved that *there exists an interim Pareto optimal allocation such that it is impossible to find two agents $i$ and $j$ for which it is common knowledge that $i$ interim envies $j$* (see Proposition 1 in [6]). Behind this result there is a clear point: in order to have a non empty set of fair allocations, we have to weaken the notion of envy freeness by reducing the number of bundle-comparisons among agents. In other words, each agent $t$ compares his own bundle $x_t$ with the bundle $x_s$ of another guy $s$ not in each possible state of nature, but only in the events commonly known.

We agree that the number of bundle-comparisons must be reduced in order to test whether the economy may exhibit an empty set of fair allocations. However, we think, more precisely, that each agent should compare his bundle only with the bundle of agents entering in the market without any kind of advantage with respect to him, even from the informational point of view. In the notion of envy introduced by Foley and Varian for a complete information exchange economy, agents with equal income under a competitive equilibrium allocation are envy free, so that the set of fair allocations is non-empty. That is, allowing comparisons to be made only between agents with the same initial opportunities a fair market redistribution is possible. To support our intuition, in the following notion of envy-freeness, we adapt the idea to a setup that explicitly encompasses not only uncertainty, but also an allocation procedure taking place before the resolution of uncertainty.

Given an allocation $a$ of $E$, define for all $t \in T$ and $\omega \in \Omega$ the following sets:

$$\begin{aligned} C_t(\omega, a) &= \{s \in A_t(\omega, a) \ : \ G_t(\omega^{'}) = G_s(\omega') \text{ for each } \omega^{'} \in F_t(\omega) = F_s(\omega)\}. \\ C(\omega, a) &= \{t \in T \ : \ \mu(C_t(\omega, a)) > 0\} \textit{ and} \\ C(a) &= \bigcup_{\omega \in \Omega} C(\omega, a). \end{aligned}$$

**Definition 3.6.** *An allocation $a$ is said to be* **envy free**$^*$ *or* **equitable**$^*$ *if* $\mu(C(a)) = 0$.

We denote by $E_E^*(E)$ the set of envy free$^*$ allocations of the economy $E$.

Thus, we restrict the possibility of comparisons, assuming that an agent may compare his bundle only with others starting not only with the same

[5]An event $\mathcal{E}$ is said to be common knowledge if it can be written as a union of elements of $\mathcal{F}_i$ for each $i \in I$; i.e., $\mathcal{E} \in \bigwedge_{i \in I} \mathcal{F}_i$.

endowment, but also with the same informational opportunities. Precisely, each agent in each state is allowed to make comparisons only with individuals: 1. receiving the same signal in the state; 2. receiving the same information (independently by the state that actually occurs) at the time of implementing contracts.

Another reason, not less important than the previous one, why we propose to replace Definition 3.3 with Definition 3.6 is the following: in differential information economies in which ex post agents are still asymmetrically informed, the consumption set of each individual $t$ consists of $\mathcal{G}_t$-measurable profiles. So, how can agent $t$ compare his bundle $x_t$, which is $\mathcal{G}_t$-measurable, with the $\mathcal{G}_s$-measurable bundle $x_s$ of agent $s$? Notice that in Definition 3.6 we do not require that $t$ can envy only agents with his same information $\mathcal{F}_t$ and $\mathcal{G}_t$. We only require that at least in the states where envy is evaluated, $t$ and $s$ receive the same signals. In other words, the condition $G_t(\omega^{'}) = G_s(\omega')$ for each $\omega^{'} \in F_t(\omega)$ ensures that each agent may envy only commodity bundles that are compatible with his private information. That is, as it is natural, no additional information becomes available to an agent due to the utility comparisons he makes to evaluate an allocation on an ethical basis. Notice that in the case $\mathcal{F}_t = \mathcal{G}_t$ as well as $\mathcal{G}_t = 2^{\Omega}$, for each $t \in T$, the restriction on utility comparisons only requires that an agent may envy individuals receiving the same signal at the time of contracting.

**Remark 3.7.** Observe that our assumption on information types ensures that for every agent $t \in T$ and for each state $\omega \in \Omega$, the set of traders $s$ such that $F_t(\omega) = F_s(\omega)$ has positive measure. Moreover, for every allocation $a$ of $E$, $C(a) \subseteq A(a)$. This inclusion implies that any envy free allocation is also envy free*.

An allocation $a$ is said to be **fair*** if it is efficient and envy free*. We denote by $IF^*(E)$ the set of fair* allocations, i.e.,

$$IF^*(E) = PO(E) \cap E_E^*(E).$$

## 3.3 Some comparisons

From Remark 3.7, it follows that $IF(E) \subseteq IF^*(E)$. However, in models with only uncertainty, since for all $t \in T$ $\mathcal{F}_t = \{\emptyset, \Omega\}$ and $\mathcal{G}_t = 2^{\Omega}$, $IF(E) = IF^*(E)$, simply because all the agents have the same signals, that is none. In other situations the above inclusion may be strict, as the next example shows. It also proves that in the same economy of Example 3.5, the conflict between efficiency and envy freeness* ceases to exist.

**Example 3.8.** Consider the same differential information economy illustrated in Example 3.5 with $\mathcal{F}_t = \mathcal{G}_t$, for $t = 1, 2, 3$, and notice that the only guys who can envy each other according to Definition 3.6 are 1 and 2, since

$F_1(a) = F_1(b) = F_2(b) = F_2(a)$. It is easy to show that an allocation $x$ is envy free* if and only if it solves the following system

$$\begin{cases} x_1(a) + x_1(b) \geq x_2(a) + x_2(b) \\ \sqrt{x_2(a)} + \sqrt{x_2(b)} \geq \sqrt{x_1(a)} + \sqrt{x_1(b)} \\ x_1(a) = x_1(b) \\ x_2(a) = x_2(b), \end{cases}$$

that is $x_1(a) = x_1(b) = x_2(b) = x_2(a)$. These inequalities are satisfied, in particular, by the endowment $(e_1(a), e_1(b)) = (400, 400)$, $(e_2(a), e_2(b)) = (400, 400)$, $(e_3(a), e_3(b)) = (400, 1000)$ which is Pareto optimal . Therefore, the set of fair* allocations $IF^*(E)$ is non empty since it contains the above endowment. Hence,

$$\emptyset = IF(E) \subset IF^*(E).$$

Notice that if $\mathcal{F}_1 = \mathcal{G}_1$ and $\mathcal{F}_2 \subset \mathcal{G}_2 = \{\{a\}, \{b\}\}$, then since there are no agents with the same signals, any allocation is envy free* and hence the set of fair* allocations coincides with the set of efficient allocations which is clearly non empty. The same holds true if $\mathcal{F}_2 = \mathcal{G}_2$ and $\mathcal{F}_1 \subset \mathcal{G}_1 = \{\{a\}, \{b\}\}$. Moreover, if $\mathcal{F}_t \subset \mathcal{G}_t = \{\{a\}, \{b\}\}$ for $t = 1, 2$, then it is easy to show that the following feasible allocation

$$\begin{aligned} (x_1(a), x_1(b)) &= (500, 500) \\ (x_2(a), x_2(b)) &= (500, 500) \\ (x_3(a), x_3(b)) &= (200, 800) \end{aligned}$$

is envy free* and Pareto optimal. Thus, the set of fair* allocations is non empty anyway.

The following proposition shows that, in the particular case of interim differential information economies in which ex post the true state of nature is commonly known (i.e., where for all $t \in T$, $\mathcal{G}_t = 2^{\Omega}$), our notion of envy freeness is not comparable with the one introduced by De Clippel. Ideed, while De Clippel restricts the set of states of nature in which an agent is allowed to be envious; we reduce the number of agents that each individual may envy.

**Proposition 3.9.** *Let $E$ be a finite (interim) economy with asymmetric information. Assume that the true state of nature is publicly verifiable. The weaker notion of envy freeness introduced by De Clippel in [6] is not compatible with that of Definition 3.6.*

PROOF: Consider $\mathcal{G}_t = 2^{\Omega}$, for $t = 1, 2, 3$ in the example 3.5. We already observed in Example 3.8 that the endowment $(e_1(a), e_1(b)) = (400, 400)$, $(e_2(a), e_2(b)) = (400, 400)$, $(e_3(a), e_3(b)) = (400, 1000)$ is fair*, therefore Pareto optimal. We claim that it is not envy free in the sense of common knowledge event[6]. The only event that is common knowledge is $\mathcal{E} = \{a, b\}$, thus we want to show that there is envy in each state of $\mathcal{E}$. Since

$$\begin{aligned} V_1(e_1)(a) &= V_1(e_1)(b) = \frac{1}{2}e_1(a) + \frac{1}{2}e_1(b) = 400 \\ &< 700 = \frac{1}{2}e_3(a) + \frac{1}{2}e_3(b) = V_1(e_3)(a) = V_1(e_3)(b), \end{aligned}$$

it is common knowledge that agent 1 envies agent 3 under the initial endowment $e$.

On the other hand, consider a three agents economy $T = \{1, 2, 3\}$ with three states of nature $\Omega = \{a, b, c\}$. Assume that

$$\begin{array}{lll} \mathcal{F}_1 = \{\{a, b\}, \{c\}\} & u_1(\cdot, x_1) = x_1 & (e_1(a), e_1(b), e_1(c)) = (400, 600, 0) \\ \mathcal{F}_2 = \{\{a, b\}, \{c\}\} & u_2(\cdot, x_2) = \sqrt{x_2} & (e_2(a), e_2(b), e_3(c)) = (400, 600, 0) \\ \mathcal{F}_3 = \{\{a\}, \{b, c\}\} & u_3(\cdot, x_3) = x_3 & (e_3(a), e_3(b), e_3(c)) = (400, 600, 0) \end{array}$$

and $\pi(a) = \pi(b) = \pi(c) = \frac{1}{3}$, $\mathcal{G}_t = 2^{\Omega}$, for $t = 1, 2, 3$. Consider the allocation $x$ defined by

$$\begin{aligned} x_1 &= (300, 600, 0) \\ x_2 &= (500, 500, 0) \\ x_3 &= (400, 700, 0) \end{aligned}$$

and notice that $x$ is efficient. Indeed, otherwise there exists a feasible allocation $y$ such that

$$\begin{cases} y_1(a) + y_1(b) \geq 900 \\ y_1(c) \geq 0 \\ \sqrt{y_2(a)} + \sqrt{y_2(b)} \geq 2\sqrt{500} \\ y_2(c) \geq 0 \\ y_3(a) \geq 400 \\ y_3(b) + y_3(c) \geq 700 \end{cases}$$

with at least one strict inequality and, by feasibility, $\sum_t y_t(a) \leq 1200$, $\sum_t y_t(a) \leq 1800$, $\sum_t y_t(c) \leq 0$, which implies that $y_t(c) = 0$ for each $t$.

Then $3000 \geq \sum_i (y_t(a) + y_t(b)) \geq 2000 + y_2(a) + y_2(b)$, and consequently

$$\begin{cases} y_2(a) + y_2(b) \leq 1000 \\ \sqrt{y_2(a)} + \sqrt{y_2(b)} \geq 2\sqrt{500} \end{cases}$$

where at least one inequality is strict. It follows that

$$\sqrt{1000 - y_2(b)} + \sqrt{y_2(b)} > 2\sqrt{500}$$

[6]An agent $i$ envies agent $j$ in the commonly known event $\mathcal{E}$ if $i$ envies $j$ in each state of $\mathcal{E}$ (see Proposition 1 in [6]).

and in turn

$$(y_2(b) - 500)^2 < 0$$

which is absurd. Thus, $x \in PO(E)$. Let us verify now that $x$ is envy free in the sense of common knowledge event. The only common knowledge event is $\mathcal{E} = \{a, b, c\}$. Since agent 1 does not envy 2 and 3 in state $c$, and in the same state agent 2 does not envy agents 1 and 3, the only agent who could envy in the sense of common knowledge event is 3. From $V_3(x_3)(a) = \frac{1}{3}400 > \frac{1}{3}300 = V_3(x_1)(a)$ it follows that agent 3 does not envy agent 1 in $\mathcal{E}$. On the other hand, from

$$V_3(x_3)(b) = \frac{1}{2}x_3(b) + \frac{1}{2}x_3(c) = \frac{700}{2} > \frac{500}{2} = \frac{1}{2}x_2(b) + \frac{1}{2}x_2(c) = V_3(x_2)(b)$$

it follows that agent 3 does not envy agent 2 in the common knowledge event. Hence the allocation $x$ is fair in the sense of common knowledge event. Let us show that $x$ is not fair*. The only agents that can envy each other are 1 and 2 in the state $a$ or $b$. Since

$$V_1(x_1)(a) = \frac{1}{2}x_1(a) + \frac{1}{2}x_1(b) = \frac{900}{2} < \frac{1000}{2} = \frac{1}{2}x_2(a) + \frac{1}{2}x_2(b) = V_1(x_2)(a)$$

the allocation $x$ is not fair*. $\square$

**Remark 3.10.** In the case of perfect information, that is whenever all agents are fully informed, both notions of fair allocations coincide with the Varian's one. Indeed, if $\mathcal{F}_t = 2^{\Omega}$ for all $t \in T$, then $F_t(\omega) = F_s(\omega) = \{\omega\}$ and $V_t(\cdot)(\omega) = u_t(\omega, \cdot)$ for all $\omega \in \Omega$ and $t, s \in T$.

# 4 Existence and fairness* of equilibria

In the previous section we have introduced a new notion of envy freeness in a very general setting and shown that at least in the economy illustrated in [6], a fair* allocation exists (see Example 3.8). We are now ready to prove that our new notion provides a solution to the conflict between absence of envy and Pareto optimality not only in the particular case of differential information economies described in [6], but in all possible situations. To this end, we prove that a competitive market equilibrium allocation resulting from an equal sharing of the total initial endowment is fair*, and from the existence of a competitive market equilibrium we deduce the non emptiness of the set of fair* allocations.

As already noted in Remark 2.3, the notion of competitive market equilibrium includes as particular cases the equilibrium concepts used in the frameworks summarized in Section 2. Existence of such equilibria is a well established achievement of the theory. In interim models in which ex post agents are still asymmetrically informed, to the best of our knowledge, it is still an open question the existence of a competitive market equilibrium . This is our next goal. To this end, the following irreducibility condition is needed.

We say that the differential information economy $E$ is **irreducible** if the following condition is satisfied:

(IC*) For each family of partitions of $T$, $\{T_1(\omega), T_2(\omega)\}_{\omega\in\Omega}$ with

$$\sum_{\omega\in\Omega} \mu(T_1(\omega)) \cdot \mu(T_2(\omega)) > 0$$

and for each allocation $a : T \times \Omega \to I\!R^{\ell}_{+}$, there exists an allocation $b : T \times \Omega \to I\!R^{\ell}_{+}$ such that

(1) $\displaystyle\int_{T_1(\omega)} e_t(\omega) d\mu + \int_{T_2(\omega)} a_t(\omega) d\mu \geq \int_{T_2(\omega)} b_t(\omega) d\mu$, for each $\omega \in \Omega$;

(2) $V_t(b_t)(\omega) \geq V_t(a_t)(\omega)$, for each $\omega \in \Omega$ and each $t \in T_2(\omega)$, the inequality being strict in at least one state.

**Remark 4.1.** Assume that the final information $\mathcal{G}_t$ of each trader is complete at the time of implementing the contract. If each trader is endowed with a strictly positive amount of each good in each state (i.e., $e_t(\omega) \gg 0$ for all $t \in T$ and all $\omega \in \Omega$), then the economy $E$ satisfies the irreducible condition (IC*). Indeed, whenever $T_1(\omega)$ has positive measure, $\displaystyle\int_{T_1(\omega)} e_t(\omega) d\mu \gg 0$. Then, there exists a strictly positive vector $v(\omega) \in R^{\ell}$ such that $\displaystyle\int_{T_1(\omega)} e_t(\omega) d\mu \gg v(\omega)\mu(T_2(\omega))$. Hence for each allocation $a$, the allocation $b$ defined by

$$b_t(\omega) = \begin{cases} a_t(\omega) & \text{if } \mu(T_1(\omega)) = 0 \\ a_t(\omega) + v(\omega) & \text{if } \mu(T_1(\omega)) > 0 \end{cases}$$

satisfies properties (1) and (2) of (IC*). Also, notice that the above irreducibility condition (IC*) reduces to the usual one in exchange economies without uncertainty.

Now we are ready to state the existence theorem for competitive market equilibria in our general framework.

**Theorem 4.2.** *Let $E$ be a mixed market with asymmetric information satisfying the irreducible condition (IC*). Assume that $\bigwedge_{t\in T} \mathcal{F}_t = \{\emptyset, \Omega\}$* [7] *and*

[7] Notice that since $\Omega$ is finite, there is only a finite number $I$ of information types; therefore $\bigwedge_{t\in T} \mathcal{F}_t = \bigwedge_{i\in I} \mathcal{F}_i$. Moreover, the assumption $\bigwedge_{t\in T} \mathcal{F}_t = \{\emptyset, \Omega\}$ has been already used by Kobayashi in [22], and it is satisfied in each example illustrated in this paper.

*that each agent has a strictly positive amount of each good in each state (i.e., $e_t(\omega) \gg 0$ for each $t \in T$ and each $\omega \in \Omega$). Then, a competitive market equilibrium exists, i.e., $CME(E) \neq \emptyset$.*

PROOF: See Appendix.

Thus, as a consequence of Theorem 4.2, a further contribution of this paper is to introduce a new equilibrium concept which exists and, as proved next, it is efficient, contrary to the rational expectations equilibrium which may not exist neither be efficient (see [13] and [23] among others).

In order to prove that the set of fair* allocations is non empty, we need the following proposition, which states that any competitive market equilibrium allocation resulting from an equal sharing of the total initial endowment is fair*.

**Proposition 4.3.** *Any equal income competitive market equilibrium allocation is fair*.*

Before proving it, we exhibit the efficiency of equilibria

**Proposition 4.4.** *Any competitive market equilibrium is efficient.*

PROOF: Let $a$ be a competitive market equilibrium allocation. We first prove that $a$ is Pareto optimal. Assume, on the contrary, that there exists another feasible allocation $a'$ such that

$$V_t(a'_t)(\omega) \geq V_t(a_t)(\omega) \quad \text{for almost all } t \in T \text{ and for all } \omega \in \Omega,$$

and for a coalition $S \subseteq T$ and a state $\bar{\omega} \in \Omega$

$$V_t(a'_t)(\bar{\omega}) > V_t(a_t)(\bar{\omega}) \quad \text{for almost all } t \in S.$$

Since $a$ is a competitive market equilibrium allocation, then there exists an equilibrium price $p$ such that $a'_t \notin B_t(p, \bar{\omega})$ for almost all $t \in S$, that is

$$\sum_{\omega' \in F_t(\bar{\omega})} p(\omega') \cdot a'_t(\omega') > \sum_{\omega' \in F_t(\bar{\omega})} p(\omega') \cdot \frac{e_t(\omega')}{\mu(T)} \quad \text{for almost all } t \in S.$$

Moreover, by continuity and monotonicity of the expected utility, for almost all $t \in T$ and for all $\omega \in \Omega$

$$\sum_{\omega' \in F_t(\omega)} p(\omega') \cdot a'_t(\omega') \geq \sum_{\omega' \in F_t(\omega)} p(\omega') \cdot \frac{e_t(\omega')}{\mu(T)}.$$

Hence, for almost all $t \in T$,

$$\sum_{\omega\in\Omega} p(\omega)\cdot a'_t(\omega) \geq \sum_{\omega\in\Omega} p(\omega)\cdot \frac{e_t(\omega)}{\mu(T)}$$

the inequality being strict for agents in the coalition $S$. This implies that

$$\sum_{\omega\in\Omega} p(\omega)\cdot \int_T \left[a'_t(\omega) - \frac{e_t(\omega)}{\mu(T)}\right] d\mu > 0,$$

which contradicts the feasibility requirement of $a'$. Thus, $a$ is efficient. □

Now we go to the proof of Proposition 4.3

PROOF: Indeed, assume on the contrary that $\mu(C(a)) > 0$. This means that there exists a state $\omega$ such that $\mu(C(a,\omega)) > 0$. For all $t \in C(\omega, a)$, $\mu(C_t(\omega, a)) > 0$. Thus, for all $s \in C_t(\omega, a)$,

$G_t(\omega^{'}) = G_s(\omega^{'})$ for each $\omega^{'} \in F_t(\omega) = F_s(\omega)$ and $V_t(a_t)(\omega) < V_t(a_s)(\omega)$.

For all $s \in C_t(\omega, a)$, consider the allocation $b_s$ defined as follows:

$$b_s(\omega') = \begin{cases} a_s(\omega') & \text{if } \omega' \in F_t(\omega) = F_s(\omega) \\ a_t(\omega') & \text{otherwise.} \end{cases}$$

Notice that from $G_t(\omega^{'}) = G_s(\omega^{'})$ for each $\omega^{'} \in F_t(\omega) = F_s(\omega)$, it follows that $b_s \in M_t$ and $V_t(a_t)(\omega) < V_t(b_s)(\omega)$. Since $a$ is a competitive market equilibrium allocation, $b_s \notin B_t(p,\omega)$. Since $b_s \in M_t$, this means that for almost all $s \in C_t(\omega, a)$

$$\sum_{\omega'\in F_t(\omega)} p(\omega')\cdot b_s(\omega') > \sum_{\omega'\in F_t(\omega)} p(\omega')\cdot \frac{e(\omega')}{\mu(T)}.$$

Hence, by definition of $b_s$ in $F_t(\omega)$, it follows that

$$\sum_{\omega'\in F_t(\omega)} p(\omega')\cdot a_s(\omega') > \sum_{\omega'\in F_t(\omega)} p(\omega')\cdot \frac{e(\omega')}{\mu(T)}.$$

Moreover, $F_t(\omega) = F_s(\omega)$ for $s \in C_t(\omega, a)$ implies that

$$\sum_{\omega'\in F_s(\omega)} p(\omega')\cdot a_s(\omega') > \sum_{\omega'\in F_s(\omega)} p(\omega')\cdot \frac{e(\omega')}{\mu(T)},$$

that is $a_s \notin B_s(p,\omega)$, which is a contradiction. Hence, $a$ is fair*. □

Thus, from Proposition 4.3 and Theorem 4.2, it follows the non emptiness of the set of fair* allocations, as the following corollary states.

**Corollary 4.5.** *Let the economy $E$ satisfy the irreducibility condition (IC*). Assume that $\bigwedge_{t\in T}\mathcal{F}_t = \{\emptyset, \Omega\}$ and that each agent has a strictly positive amount of each good in each state (i.e., $e_t(\omega) \gg 0$ for each $t \in T$ and each $\omega \in \Omega$). Then, a fair* allocation exists, i.e., $IF^*(E) \neq \emptyset$.*

## 5 Coalitional notion of fairness

We have observed that any competitive market equilibrium allocation resulting from an equal sharing of the total initial endowment is fair*. The following example shows that the converse may not be true.

**Example 5.1.** Consider the same economy described in Example 3.8 with $\mathcal{F}_t = \{\{a,b\}\}$ for $t = 1,2$ and $\mathcal{F}_3 = \mathcal{G}_t = \{\{a\},\{b\}\}$ for all $t = 1,2,3$. We have already observed that the allocation

$$\begin{aligned}(x_1(a), x_1(b)) &= (500, 500)\\ (x_2(a), x_2(b)) &= (500, 500)\\ (x_3(a), x_3(b)) &= (200, 800)\end{aligned}$$

is fair*. Clearly, it is not a competitive market equilibrium allocation, simply because for any price $p$,

$$500(p(a) + p(b)) > 400(p(a) + p(b)) = p(a)e_t(a) + p(b)e_t(b),$$

that is $x_t \notin B_t(p,\omega)$ for any $p$, any $\omega \in \Omega$ and any $t = 1,2$.

Therefore, the competitive market equilibrium allocation are not the only fair* allocations. Moreover, a competitive market equilibrium with no equal income is not necessarily envy free*, as the following example illustrates.

**Example 5.2.** Consider the following differential information economy with two equiprobable states of nature $\Omega = \{a,b\}$, one good and two agents $T = \{1,2\}$, whose characteristics are as follows:

$$\begin{aligned}\mathcal{F}_1 = \{\{a,b\}\} \quad \mathcal{G}_1 &= \{\{a\},\{b\}\} \quad (e_1(a), e_1(b)) = (400, 400)\\ \mathcal{F}_2 = \{\{a,b\}\} \quad \mathcal{G}_2 &= \{\{a\},\{b\}\} \quad (e_2(a), e_2(b)) = (600, 600).\end{aligned}$$

Moreover, $u_t = \sqrt{x_t}$ for any $t = 1,2$.
Notice that agents have different initial endowment, which is the unique competitive market equilibrium allocation. However, it is not fair*, since agent 1 envies 2 in states $a$ and $b$, i.e.,

$$V_1(e_1)(a) = V_1(e_1)(b) = \sqrt{400} < \sqrt{600} = V_1(e_2)(a) = V_1(e_2)(b).$$

In this section, we highlight the following questions: "which fairness concept characterized the competitive market equilibrium allocations?"; "which fairness concept should we consider when the total initial endowment is not equally shared among agents?".

The key is to allow groups of agents to make utility comparisons, that is we need coalitional fairness notions. In the coalitional envy that we are going to introduce, an allocation $a$ is not equitable if it treats coalitions in a discriminatory way: there exists in each state a coalition whose agents prefer to receive the net trade assigned by $a$ to some other disjoint coalition instead of keeping their own net trade. The reason is that they would be able to redistribute this net trade in such a way that each of them, given his private information, is better off. This kind of notion of envy is closer to a blocking notion. Since there are no individual comparisons between agents, we do not need to impose equal sharing of the total initial endowment neither restrictions on informational opportunities. Moreover, the absence of individual envy, makes impossible for agents of a potentially envious coalition to receive additional information from the comparisons.

Let $a$ be an allocation that one can see as a potential outcome and let $a'$ be an alternative allocation. Denote by $D(a, a', \omega)$ the set of deviators in state $\omega$, that is the set of agents that, given their private information, would prefer to receive $a'$ instead of keeping $a$, i.e.,

$$D(a,\, a',\, \omega) = \{t \in T \;:\; V_t(a'_t)(\omega) > V_t(a_t)(\omega)\}.$$

Notice that by measurability assumption of the mapping $(t, x) \mapsto u_t(\omega, x)$, the set of deviators is always measurable.

**Definition 5.3.** *An allocation $a$ is said to be blocked in the c-type fair sense by $a'$ if for almost all $t \in T$, $a'_t(\cdot)$ is $\mathcal{G}_t$-measurable and if for all $\omega \in \Omega$ there exists a coalition $S(a,\, a',\, \omega) \subseteq T$ such that*

(1) $\quad \mu(D(a,\, a',\, \omega) \cap S(a,\, a',\, \omega)) = 0$

(2) $\quad \displaystyle\int_{D(a,\, a',\, \omega)} (a'_t(\omega) - e_t(\omega))\; d\mu \leq \int_{S(a,\, a',\, \omega)} (a_t(\omega) - e_t(\omega))\; d\mu,$

*with $\mu(D(a,\, a',\, \omega)) > 0$ in at least one state of nature.*

Notice that we do not require that $S(a,\, a',\, \omega)$ has positive measure.

The interpretation goes as follows: $a$ is blocked in the c-type fair sense by $a'$ if in each state $\omega$ the set of deviators can redistribute among its members the net trade of another disjoint coalition $S(a, a', \omega)$ by using only their own private information. In this case, we say that for each state $\omega$, the deviators $t \in D(a, a', \omega)$ envy the net trade of coalition $S(a, a', \omega)$.

**Definition 5.4.** *An allocation $a$ is c-type fair if it is feasible and it is not blocked in the c-type fair sense. We denote by $\mathcal{C}^{fair}_{type}(E)$ the set of c-type fair allocations for the economy $E$.*

An allocation is qualified *c-type fair* if there is zero probability of a set of deviators envying the net trade of any other coalition. Under such distribution of resources, there does not exist an alternative allocation and a state at which potentially deviators are treated in a discriminatory way by the market. According to the c-type fair criterion, agents in a potentially envious coalition make comparisons given their private information and not after the observation of the state.

Notice that the c-fairness notions introduced in [5], [11] and [15] can be viewed as particular case of Definition 5.4, by specifying properly the algebras $\mathcal{F}_t$ and $\mathcal{G}_t$ as in Remark 2.3. In economies in which the Core-Walras equivalence theorem has been proved, one can obtained a characterization of competitive equilibria in terms of coalitional fair allocations. In other contexts, like in interim models in which ex post agents are still asymmetrically informed (i.e., $\mathcal{F}_t \subseteq \mathcal{G}_t$ for all $t \in T$), such an equivalence is still an open question. We will work on it in a future paper.

It is easy to show that any c-type fair allocation is efficient in the sense that we specify below.

**Definition 5.5.** *An allocation $a'$* **Pareto* dominates** *an allocation $a$ if $a'$ is feasible and if everybody strictly prefers (given his private information) $a'$ over $a$ in each state, that is*

$$(1) \qquad V_t(a'_t)(\omega) > V_t(a_t)(\omega) \quad for\ almost\ all\ t \in T\ and\ for\ all\ \omega \in \Omega$$

$$(2) \qquad \int_T a'(\omega)\, d\mu \leq \int_T e_t(\omega)\, d\mu \quad for\ all\ \omega \in \Omega.$$

*An allocation $a$ is said to be* **efficient* (or Pareto* optimal)** *if it is not Pareto* dominated by any other allocation. We denote by $PO^*(E)$ the set of interim Pareto* optimal allocations.*

**Remark 5.6.** We can easily notice that $PO(E) \subseteq PO^*(E)$. Moreover, Example 3.5 can be also used to prove that $PO(E) \subsetneq PO^*(E)$. Indeed, $x^* \notin PO(E)$, but it cannot be Pareto* dominated by any other feasible allocation.

To show that a c-type fair allocation is Pareto* optimal, we just need, for all $\omega$, that the set of deviators is taken equal to the whole set of agents $T$ and the disjoint coalition equal to the empty set.

Moreover, competitive market equilibrium allocations are c-type fair.

**Proposition 5.7.** *Any competitive market equilibrium allocation is c-type fair*[8]*.*

PROOF: See Appendix.

[8]Notice that contrary to Proposition 4.3, we do not need to assume that agents have the same initial endowment in each state of nature.

**Proposition 5.8.** *Let the economy $E$ satisfy the irreducibility condition ($IC^*$). Assume that $\bigwedge_{t\in T}\mathcal{F}_t = \{\emptyset, \Omega\}$ and that each agent has a strictly positive amount of each good in each state (i.e., $e_t(\omega) \gg 0$ for all $t \in T$ and all $\omega \in \Omega$). Then, a c-type fair allocation exists, i.e., $\mathcal{C}^{fair}_{type}(E) \neq \emptyset$.*

# 6 Appendix

## 6.1 Fictitious economy

In order to prove the main results, following Wilson's idea [32] (see also [5] and [8]), we consider a type-agent representation $E^*$ of the original differential information economy $E$ described before. It is a fictitious market for state-contingent claims that can be formalized as follows:

$$E^* = \left\{(T^*, \mathcal{T}^*, \mu^*);\ I\!R_+^{\ell\times|\Omega|};\ \left(X_{(t,\mathcal{E})}, V_{(t,\mathcal{E})}, e_{(t,\mathcal{E})}\right)_{(t,\mathcal{E})\in T^*}\right\}$$

where:

1. $T^*$ is the set of the type agents. More precisely, $T^*$ coincides with the graph of the correspondence $\Phi : T \to 2^{\mathcal{F}}$ defined by $\Phi(t) = \mathcal{F}_t$, i.e., $T^*$ is the set of couple $(t, \mathcal{E})$, where $t$ is an agent and $\mathcal{E}$ is an atom of his information partition.
   $\mathcal{T}^*$ is the family of coalitions: a coalition $S^*$ is a measurable subset of $T^*$, i.e., $S^* \in \mathcal{T} \otimes \mathcal{B}(2^{\mathcal{F}})$, where $\mathcal{B}(2^{\mathcal{F}})$ denotes the Borel $\sigma$-algebra on the discrete topological space $2^{\mathcal{F}}$ and $\otimes$ denotes the product $\sigma$-algebra. Finally, the measure $\mu^*$ on $\mathcal{T}^*$ is defined as the product measure of $\mu$ and of the counting measure.

2. $X_{(t,\mathcal{E})}$ is the consumption set of the type agent $(t, \mathcal{E})$ defined as follows:

$$X_{(t,\mathcal{E})} := \bigcup_{\beta\in M_t} \{\alpha : \Omega \to I\!R_+^{\ell} :\ \alpha(\omega) = \beta(\omega)\chi_{\mathcal{E}}(\omega) \text{ for all } \omega \in \Omega\}.$$

3. The couple $(V_{(t,\mathcal{E})}, e_{(t,\mathcal{E})})$ characterizes the type-agent $(t, \mathcal{E})$: given a type-agent $(t, \mathcal{E})$, his preference for a commodity $\alpha \in X_{(t,\mathcal{E})}$ is defined by

$$V_{(t,\mathcal{E})}(\alpha) = \sum_{\omega\in\mathcal{E}} u_t(\omega, \alpha(\omega))\pi(\omega),$$

   while $e_{(t,\mathcal{E})}$ represents his initial endowment of physical resources defined as follows

$$e_{(t,\mathcal{E})}(\omega) = \begin{cases} e_t(\omega) & \text{if } \omega \in \mathcal{E} \\ 0 & \text{otherwise.} \end{cases}$$

Notice that since $e_t(\cdot)$ is $\mathcal{G}_t$-measurable for all $t \in T$, it follows that for all $(t, \mathcal{E}) \in T^*$, $e_{(t,\mathcal{E})} \in X_{(t,\mathcal{E})}$.

Clearly, assuming that $\mathcal{F}_t = \mathcal{G}_t$ for each agent $t \in T$, then $X_{(t,\mathcal{E})}$ is made by all functions that are constant on $\mathcal{E}$ and zero on $\Omega \setminus \mathcal{E}$. In the case the true state of nature is publicly announced, for each type-agent $(t, \mathcal{E})$ the consumption set is simply made by functions on $\Omega$ that are null outside $\mathcal{E}$. Notice also that if in the differential information economy $E$ there is equal income, that is agents have the same initial endowment, the equal sharing of initial resources among the type-agents in the associated economy $E^*$ may not hold.

An *allocation* in the fictitious economy is a function $\alpha : \Omega \times T^* \to \mathbb{R}^\ell_+$ such that for each $(t, \mathcal{E}) \in T^*$, $\alpha_{(t,\mathcal{E})} : \Omega \to \mathbb{R}^\ell_+$ belongs to $X_{(t,\mathcal{E})}$, where $\alpha_{(t,\mathcal{E})}$ represents the bundle that the type-agent $(t, \mathcal{E})$ receives under the allocation $\alpha$.
An allocation $\alpha$ is feasible for the coalition $S^*$ if

$$\int_{S^*} \alpha_{(t,\mathcal{E})}(\omega)\, d\mu^* \leq \int_{S^*} e_{(t,\mathcal{E})}(\omega)\, d\mu^* \qquad \text{for all } \omega \in \Omega,$$

and it is feasible if it is feasible for the whole coalition of agents $T^*$.

**Definition 6.1.** *An allocation $\alpha$ is c-fair blocked by an assignment $\alpha'$ if there exist two disjoint coalitions $S_1^*$ and $S_2^*$ (i.e., $\mu^*(S_1^* \cap S_2^*) = 0$), such that*

$$(1) \qquad \mu^*(S_1^*) > 0$$
$$(2) \qquad \alpha'_{(t,\mathcal{E})} \in X_{(t,\mathcal{E})} \quad \textit{for almost all } (t, \mathcal{E}) \in S_1^*$$
$$(3) \qquad V_{(t,\mathcal{E})}(\alpha'_{(t,\mathcal{E})}) > V_{(t,\mathcal{E})}(\alpha_{(t,\mathcal{E})}) \quad \textit{for almost all } (t, \mathcal{E}) \in S_1^*$$
$$(4) \qquad \int_{S_1^*} \left[\alpha'_{(t,\mathcal{E})}(\omega) - e_{(t,\mathcal{E})}(\omega)\right] d\mu^* \leq \int_{S_2^*} \left[\alpha_{(t,\mathcal{E})}(\omega) - e_{(t,\mathcal{E})}(\omega)\right] d\mu^* \quad \textit{for all } \omega \in \Omega.$$

*An allocation $\alpha$ is* **c-fair** *for $E^*$ if it is feasible and it is not c-fair blocked.*

**Definition 6.2.** *An allocation $\alpha$ is a Walrasian (or Arrow-Debreu) allocation of the type-agent economy $E^*$ if there exists a price $p : \Omega \to \mathbb{R}^\ell_+$ such that for almost all $(t, \mathcal{E}) \in T^*$,*

(1) $\alpha_{(t,\mathcal{E})} \in X_{(t,\mathcal{E})}$

(2) $\alpha_{(t,\mathcal{E})} \in \arg\max_{\alpha' \in B_{(t,\mathcal{E})}(p)} V_{(t,\mathcal{E})}(\alpha')$, *where*

$$B_{(t,\mathcal{E})}(p) = \left\{ \alpha' \in X_{(t,\mathcal{E})} \mid \sum_{\omega \in \Omega} p(\omega) \cdot \alpha'(\omega) \leq \sum_{\omega \in \Omega} p(\omega) \cdot e_{(t,\mathcal{E})}(\omega) \right\}.$$

*The pair $(\alpha, p)$ is said to be a Walrasian (or Arrow-Debreu) equilibrium.*

*The allocation $\alpha$ is said to be a quasi-Walrasian allocation of the type agent economy when condition (2) is replaced by the following*

(2) $\alpha_{(t,\mathcal{E})} \in \arg\max_{\alpha' \in B_{(t,\mathcal{E})}(p)} V_{(t,\mathcal{E})}(\alpha')$, *for almost all* $(t, \mathcal{E})$ *such that* $inf\, p \cdot X_{(t,\mathcal{E})} < inf\, p \cdot e_{(t,\mathcal{E})}$.

**Remark 6.3.** With standard arguments one can easily show that any Walrasian equilibrium allocation is c-fair in $E^*$.

We are going to show that any competitive market equilibrium of the economy $E$ is a Walrasian equilibrium of the associated type economy $E^*$. To this end, we construct a natural isomorphism between $E$ and its type-agent representation $E^*$.

Given an allocation $a$ of $E$, its type-agent representation is the allocation $\alpha$ such that for each $(t, \mathcal{E})$ in $T^*$

$$\alpha_{(t,\mathcal{E})}(\omega) = a_t(\omega)\chi_{\mathcal{E}}(\omega).$$

Notice that since $a_t(\cdot)$ is $\mathcal{G}_t$-measurable, then $\alpha_{(t,\mathcal{E})} \in X_{(t,\mathcal{E})}$.

Given an allocation $\alpha$ of $E^*$, its associated allocation $a$ in the original economy $E$ is such that for each $t$ in $T$ and each $\omega$ in $\Omega$

$$a_t(\omega) = \alpha_{(t,F_t(\omega))}(\omega).$$

Since $\alpha$ is an allocation of $E^*$, by definition $\alpha_{(t,\mathcal{E})} \in X_{(t,\mathcal{E})}$ for all $(t, \mathcal{E}) \in T^*$. Thus, $\alpha_{(t,F_t(\omega))}(\omega) = \alpha_{(t,F_t(\omega))}(\omega')$ for all $\omega' \in G_t(\omega)$. This implies that $a_t(\cdot)$ is $\mathcal{G}_t$-measurable.

Therefore, $a_t(\cdot)$ is $\mathcal{G}_t$-measurable if and only if $\alpha_{(t,\mathcal{E})} \in X_{(t,\mathcal{E})}$ for all $(t, \mathcal{E}) \in T^*$.

**Remark 6.4.** By adopting similar arguments used in [5] and [8], it is easy to show that there exists a one to one correspondence between a differential information economy $E$ and the associated economy $E^*$ in terms of competitive equilibrium and c-fair allocations. Precisely, if $(a, p)$ is a competitive market equilibrium for $E$, then the associated allocation $\alpha$ is such that the pair $(\alpha, p)$ is an Arrow-Debreu equilibrium for $E^*$. Conversely, if $(\alpha, p)$ is an Arrow-Debreu equilibrium for $E^*$, then the associated allocation $a$ is such that the pair $(a, p)$ is a competitive market equilibrium for $E$. Analogously, to any c-type fair allocation $a$ in $E$ corresponds a c-fair allocation $\alpha$ in $E^*$ and vice versa.

From the above remark we can deduce Proposition 5.7 as follows.

## 6.2 Proof of Proposition 5.7

Let $a$ be a competitive market equilibrium allocation. We need to show that it is c-type fair. To this end, consider the associated allocation $\alpha$ in the type-agent economy $E^*$, which is a Walrasian equilibrium allocation (see Remark 6.4). From Remark 6.3 it follows that $\alpha$ is c-fair for $E^*$, and hence by coming back to the original differential information economy $E$ we get that $a$ is c-type fair (see Remark 6.4). $\square$

## 6.3 Proof of Theorem 4.2

Our next goal is to prove the existence of a competitive market equilibrium. We will proceed by following the steps below:

1. Starting from the mixed differential information economy $E$, we construct the associated mixed type-agent economy $E^*$ as before. For each atom $t \in T_1$, the type agent $A^* = (t, \mathcal{E})$ for any $\mathcal{E}$ is an atom of $E^*$. Thus, the set of type agents $T^*$ can be decomposed into the disjoint union of the nonatomic sector $T_0^*$ and the atomic part $T_1^*$, where $T_1^*$ is the disjoint union of at most countable many atoms $A_i^*$. Precisely,

$$\begin{aligned} T_0^* &= \{(t, \mathcal{E}) \in T_0 \times \mathcal{F} \,:\, \mathcal{E} \in \mathcal{F}_t\} \quad \text{and} \\ T_1^* &= \{(A, \mathcal{E}) \in T_1 \times \mathcal{F} \,:\, \mathcal{E} \in \mathcal{F}_A\} \quad \text{that is } T_1^* = T^* \setminus T_0^*. \end{aligned}$$

We have already observed that there is a one to one correspondence between $E$ and $E^*$ in terms of competitive equilibrium (Remark 6.4; see also [5] and [8]), that is if $(a, p)$ is a competitive market equilibrium for $E$, then the associated allocation $\alpha$ is such that the pair $(\alpha, p)$ is an Arrow-Debreu equilibrium for $E^*$. Conversely, if $(\alpha, p)$ is an Arrow-Debreu equilibrium for $E^*$, then the associated allocation $a$ is such that the pair $(a, p)$ is a competitive market equilibrium for $E$.

2. It has been proved in [16] that $E^*$ can be identified with the atomless economy $\tilde{E}^*$ in which the set of agents $\tilde{T}^*$ is the union of $T_0^*$ and the intervals $\tilde{A}_i^*$ corresponding to atoms $A_i^*$, such that $\mu^*(A_i^*) = \tilde{\mu}^*(\tilde{A}_i^*)$. Agents of $\tilde{A}_i^*$ have the same characteristics (i.e. the same utility function, consumption set and initial endowment) of atom $A_i^*$ (see also [7] for economies with infinitely many commodities and [26] for differential information economies). It is easy to show that given a quasi equilibrium $(\tilde{\alpha}, p)$ of the atomless economy $\tilde{E}^*$, the pair $(\alpha, p)$ where $\alpha_t = \tilde{\alpha}_t$ for all $t \in T_0^*$ and $\alpha(A_i^*) = \int_{\tilde{A}_i^*} \tilde{\alpha}_t d\tilde{\mu}^*$ is a quasi equilibrium of $E^*$.

3. Consider the atomless economy $\tilde{E}^*$ and notice each consumption set $X_{(t,\mathcal{E})}$ and each utility function $V_{(t,\mathcal{E})}$ satisfy standard requirements ensuring

the existence of a quasi equilibrium $(\tilde{\alpha}, p)$ of $\tilde{E}^*$ (see [20]), that is for almost all $(t, \mathcal{E}) \in \tilde{T}^*$,

(1) $\tilde{\alpha}_{(t,\mathcal{E})} \in B_{(t,\mathcal{E})}(p)$ where

$$B_{(t,\mathcal{E})}(p) = \left\{ \alpha' \in X_{(t,\mathcal{E})} \mid \sum_{\omega \in \Omega} p(\omega) \cdot \alpha'(\omega) \leq \sum_{\omega \in \Omega} p(\omega) \cdot e_{(t,\mathcal{E})}(\omega) \right\}.$$

(2) $\tilde{\alpha}_{(t,\mathcal{E})} \in \arg\max_{\alpha' \in B_{(t,\mathcal{E})}(p)} V_{(t,\mathcal{E})}(\alpha')$, whenever $p \cdot e_{(t,\mathcal{E})} \neq 0$.

This pair $(\tilde{\alpha}, p)$ corresponds to a quasi equilibrium $(\alpha, p)$ of the mixed economy $E^*$.

4. Prove that $(\alpha, p)$ is actually an Arrow-Debreu equilibrium of $E^*$; use Remark 6.4 and observe that $(\alpha, p)$ corresponds to a competitive market equilibrium $(a, p)$ of the "original" mixed differential information economy $E$. Thus, a competitive market equilibrium for $E$ exists.

We now show the details of our proof. Let $(\tilde{\alpha}, p)$ be a quasi equilibrium of the atomless economy $\tilde{E}^*$, whose existence is proved in [20]; and let $(\alpha, p)$ be the associated quasi equilibrium of the economy $E^*$. We now want to prove that $(\alpha, p)$ is an Arrow-Debreu equilibrium for $E^*$. To this end, let us denote by $C_1^*$ the set[9]

$$C_1^* = \{(t, \mathcal{E}) \in T^* : p \cdot e_{(t,\mathcal{E})} = 0\}$$

and assume that $\mu^*(C_1^*) \in (0, \mu^*(T^*))$. Denote by $T_1(\omega)$ the set

$$T_1(\omega) = \{t \in T : (t, F_t(\omega)) \in C_1^*\}$$

and by $T_2(\omega)$ the set $T_2(\omega) = T \setminus T_1(\omega)$.

We claim that in at least one state $\omega \in \Omega$ it is true that

$$\mu(T_1(\omega)) \cdot \mu(T_2(\omega)) > 0,$$

that is both sets have positive measure. Since the pair $\{T_1(\omega), T_2(\omega)\}$ is a partition of $T$, it is enough to show that in at least one state $\omega \in \Omega$ the set $T_1(\omega)$ has measure in the interval $(0, \mu(T))$.

Assume by contradiction that in each state $\omega$ it is true that $\mu(T_1(\omega)) \in \{0, \mu(T)\}$. Define the two sets

$$A = \{\omega \in \Omega : \mu(T_1(\omega)) = 0\}$$

[9] Notice that the assumption $e_t \gg 0$ does not ensure that $e_{(t,\mathcal{E})} \gg 0$ and hence the set of type-agents for which $p \cdot e_{(t,\mathcal{E})} = 0$ may have positive measure.

and

$$B = \Omega \setminus A.$$

Then $A$ is not empty, otherwise from $\mu(T_1(\omega)) = \mu(T)$ for each $\omega$ it would follow $\mu^*(C_1^*) = \mu^*(T^*)$ a case that is excluded by strict positivity of initial endowments in each state. On the other hand, the case in which the set $B$ is empty is excluded by the assumption $\mu^*(C_1^*) > 0$. Let $\omega \in B$. Then $\mu(T_1(\omega)) = \mu(T)$ means that for almost all agents $t \in T$

$$p \cdot e_{(t,F_t(\omega))} = \sum_{\omega' \in F_t(\omega)} p(\omega') \cdot e_t(\omega') = 0.$$

Now observe that from $\omega \in B$ it follows that $F_i(\omega) \subseteq B$, for each information type $i \in I$. If not, there would exist $i \in I$ and $\bar{\omega} \in F_i(\omega)$ such that $\bar{\omega} \notin B$, that is $\mu(T_2(\bar{\omega})) = \mu(T)$, and hence

$$\sum_{\omega' \in F_i(\bar{\omega})} p(\omega') \cdot e_t(\omega') > 0 \quad \text{for each agent } t \in T.$$

Since for each agent $t$ with information type $i$, $F_i(\bar{\omega}) = F_i(\omega) = F_t(\omega)$, we have a contradiction.

Also observe that from $\omega \in A$ it follows that $F_i(\omega) \subseteq A$, for each information type $i \in I$. If not, there would exist $i \in I$ and $\bar{\omega} \in F_i(\omega)$ such that $\bar{\omega} \in B$ that is, by the previous argument, $F_i(\omega) \subseteq B$ and this is impossible.

Then $\{A, B\}$ is a measurable partition of $\Omega$ and $\mathcal{F}_i \subseteq \{A, B\}$, for each $i \in I$ contradicting the assumption that $\bigwedge_i \mathcal{F}_i = \{\emptyset, \Omega\}$. This proves our claim, that is there exists at least one state $\omega \in \Omega$ in which $\mu(T_1(\omega)) \cdot \mu(T_2(\omega)) > 0$.

Let us denote by $a$ the allocation of $E$ corresponding to the quasi equilibrium allocation $\alpha$ of $E^*$ and apply the condition $(IC^*)$ to the partitions $\{T_1(\omega), T_2(\omega)\}_{\omega \in \Omega}$ and to the allocation $a$. Let us denote by $b$ the allocation defined by $(IC^*)$ in $E$ and let $\beta$ be the corresponding allocation in $E^*$. Then for each $\omega \in \Omega$ and for each $t \in T_2(\omega)$ we have

$$V_{(t,\mathcal{E})}(\beta_{(t,\mathcal{E})}) \geq V_{(t,\mathcal{E})}(\alpha_{(t,\mathcal{E})}),$$

and in at least one state $\omega_0 \in \Omega$ the inequalities is strict for each $t \in T_2(\omega_0)$ and $(t, F_t(\omega_0)) \in C_2^*$. Hence there exists a coalition $C_0^* \subseteq C_2^*$ for which $p \cdot \beta_{(t,\mathcal{E})} > p \cdot e_{(t,\mathcal{E})}$, while for the remaining type-agents $(t, \mathcal{E}) \in C_2^* \setminus C_0^*$, by continuity and monotonicity $p \cdot \beta_{(t,\mathcal{E})} \geq p \cdot e_{(t,\mathcal{E})}$.

Since for each state $\omega \in \Omega$, $\displaystyle\int_{C_1^*} e_{(t,\mathcal{E})}(\omega)\, d\mu^* = \int_{T_1(\omega)} e_t(\omega)\, d\mu$, and

$$\int_{T_1(\omega)} e_t(\omega)\, d\mu + \int_{T_2(\omega)} a_t(\omega)\, d\mu \geq \int_{T_2(\omega)} b_t(\omega)\, d\mu$$

then

$$\int_{C_1^*} e_{(t,\mathcal{E})\,d\mu^*} + \int_{C_2^*} \alpha_{(t,\mathcal{E})\,d\mu^*} \geq \int_{C_2^*} \beta_{(t,\mathcal{E})\,d\mu^*}.$$

It follows that

$$\begin{aligned} p \cdot \int_{C_2^*} \alpha_{(t,\mathcal{E})}\,d\mu^* &= p \cdot \int_{C_1^*} e_{(t,\mathcal{E})}\,d\mu^* + p \cdot \int_{C_2^*} \alpha_{(t,\mathcal{E})}\,d\mu^* \\ &\geq p \cdot \int_{C_2^*} \beta_{(t,\mathcal{E})}\,d\mu^* > p \cdot \int_{C_2^*} e_{(t,\mathcal{E})}\,d\mu^* \\ &\geq p \cdot \int_{C_2^*} \alpha_{(t,\mathcal{E})}\,d\mu^*, \end{aligned}$$

which is a contradiction. The previous argument shows that the case $\mu^*(C_1^*) \in (0, \mu^*(T^*))$ is excluded. Since $\mu^*(C_1^*) = \mu^*(T^*)$ is excluded by the assumption of strictly positive initial endowments, then $\mu^*(C_1^*) = 0$ and hence $(\alpha, p)$ is an Arrow-Debreu equilibrium. Coming back to the original mixed market $E$, from Remark 6.4, one can deduce that the corresponding allocation $a$ is a competitive market equilibrium of $E$. Hence, the set $CME(E)$ is non empty. □

Whenever in the economy $E$ there is a group of fully informed agents at the time of implementing contracts, the irreducible condition $(IC^*)$, needed in Theorem 4.2, can be weakened as the next proposition indicates.

**Proposition 6.5.** *Let us denote by $S$ the set of the ex post fully informed traders. Assume $\bigwedge_{t \in T} \mathcal{F}_t = \{\emptyset, \Omega\}$, that $\mu(S \cap \Theta_i) > 0$ for each $i \in I$ and that each trader is endowed with a strictly positive amount of each good in each state. Then, the economy $E$ has a competitive market equilibrium.*

PROOF: We only need to prove that the economy $E$ satisfies the irreducible condition $(IC^*)$ and apply Theorem 4.2. Thus, let $\{T_1(\omega), T_2(\omega)\}_{\omega \in \Omega}$ be a partition as in the $(IC^*)$. Let us denote for each $\omega \in \Omega$, by $S_2(\omega)$ the set $S_2(\omega) = T_2(\omega) \cap S$. Then there exists a state $\omega$ in which $\mu(T_1(\omega)) > 0$ and $\mu(S_2(\omega)) > 0$. Again, whenever $T_1(\omega)$ has positive measure, $\int_{T_1(\omega)} e_t(\omega) d\mu \gg 0$. Consequently, there exists a strictly positive vector $v(\omega) \in R^\ell$ such that $\int_{T_1(\omega)} e_t(\omega) d\mu \gg v(\omega)\mu(S_2(\omega))$. Hence for each allocation $a$, the allocation $b$ defined on $T_2(\omega)$ by

$$b_t(\omega) = \begin{cases} a_t(\omega) & \text{if } \mu(T_1(\omega)) = 0 \text{ or } \mu(T_1(\omega)) > 0 \text{ and } t \notin S_2(\omega) \\ a_t(\omega) + v(\omega) & \text{otherwise} \end{cases}$$

satisfies properties required by $(IC^*)$. □

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