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Core Equivalences for Equilibria Supported by Non-linear Prices

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Abstract

The goal of this paper is to provide some new cooperative characterizations and optimality properties of competitive equilibria supported by non-linear prices. The general framework is that of economies whose commodity space is an ordered topological vector space which need not be a vector lattice. The central notion of equilibrium is the one of *personalized equilibrium* introduced by Aliprantis, Tourky and Yannelis (2001). Following Herves-Beloso and Moreno-Garcia (2008), the veto power of the grand coalition is exploited in the original economy and in a suitable family of economies associated to the original one. The use of *Aubin coalitions* allows us to connect results with the *arbitrage free condition* due to non-linear supporting prices. The use of *rational allocations* allows us to dispense with Lyapunov convexity theorem. Applications are provided in connection with strategic market games and economies with asymmetric information.

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1 Introduction

This paper studies core and optimality properties of equilibria in models of economies with quite general commodity spaces represented by ordered topological vector spaces which need not be vector lattices. Our analysis has consequences in the case of finite as well as infinite dimensional commodity spaces. The competitive equilibrium notion refers to the one introduced by [9] in their theory of value under non-linear prices. Non-linear pricing systems are of interest for economic analysis in the presence of discrimination among agents, imperfect competition, progressive income tax tariffs, for general adverse selection problems or land markets (see, for example, the discussion in [13]). A broad range of other applications has been provided by [9], where the alternate theory of value arises naturally from a series of previous contributions in infinite dimensional equilibrium analysis.

It is due to Aliprantis and Brown [2] the new approach to the study of infinite dimensional equilibrium theory that saw its emergence at the beginning of the 1980's. In their paper they proposed Riesz dual pairs as a natural setting to represent commodity-price dualities. They addressed the main questions of existence and optimality of equilibria emphasizing the rich lattice theoretic structure that is shared by the prevalent models in economics. The relevance of the lattice structure in infinite dimensional equilibrium analysis was confirmed in the famous existence result given by Mas-Colell in [25]. However, this relevance was in some sense surprising as it was apparently in contrast with Debreu's remarks according to which the main questions of existence and optimality of equilibria could be answered, in the finite dimensional theory, using cones that are not necessarily lattice cones (see [14]). These remarks, that concern the "coordinate free theory", have to be understood in infinite dimensional spaces in terms of a "vector lattice free analysis". Hence the contrast above implicitly motivates a series of relevant subsequent papers in which lattice requirements have been weakened (see [26], [30], [31], [17] among the others). The common idea is to assume a topology on the vector lattice commodity space that is not necessarily locally solid but only locally convex. The dual space, interpreted as price space, is still required to be a vector sublattice of the order dual. As pointed out by [9], one shortcoming of this large literature is that the existence of equilibrium allocations and welfare theorems are proved at the cost of assumptions on the agents consumption sets that preclude models where location matters and differential information economies in which the consumption sets may be very small. Moreover, the vector lattice structure on the commodity space precludes a rich class of models that arise naturally when there are constraints on disposal technologies (examples include waste discharge restrictions and pollution disposal, see [27]).

In the contribution given by [9], main reference for the present paper, the limitations of the lattice approach are addressed definitively: not only the topology of the vector lattice commodity space need not be locally solid, but also the lattice structure itself can be given up. This approach affords decentralization by means of generalized pricing systems. More precisely,

- the commodity space is simply an ordered vector space which needs not be a vector lattice;
- the existence and optimality of equilibria are proved supporting the relevant allocations by means of non-linear prices induced by personalized pricing systems.

The vector lattice free analysis gives back the standard Walrasian model of general equilibrium whenever, for instance, the commodity space is a vector lattice and consumption sets coincide with the positive cone and has applications in finite as well as infinite dimensional commodity spaces. In [9], [6] non-linear decentralization is motivated as arising, respectively, from a discriminatory price auction and from models of portfolio trading. In [7], non-linear prices are reconsidered as elements of a lattice cone called the *super order dual*[1], hence the theory with ordered vector spaces may be considered as basically not so different from the one with vector lattice commodity-price duality. Finally, in [3], [4] necessary and sufficient conditions are given on the order structure of the commodity space for supporting prices to be linear.

The aim of this paper is to establish some new cooperative characterizations of *personalized equilibria*, that is competitive equilibria supported by non-linear prices. Such characterizations arise naturally as motivated by the notion itself and are helpful for a deeper understanding of the non-linear approach to the competitive theory. In particular, we provide characterizations formulated in terms of the veto power of just one coalition, namely the grand coalition. This is done:

- in a family of economies associated to the original one, in which the initial endowment of agents is modified in a precise direction;
- in the original economy, provided that agents are allowed to participate in the grand coalition using only a share of their initial endowments.

This kind of characterizations is inspired by a regularity condition, characteristic of personalized equilibria when compared with classical Walrasian equilibria, called *arbitrage free condition*, imposed in relation to the arbitrage opportunity that is due non-linearity. The arbitrage free condition is interpreted in the paper as a coalition-based arbitrage free condition using coalitions in which agents may participate with an arbitrary share of their initial resources (Aubin coalitions). According to it, in a personalized equilibrium, agents in a coalition have no incentive to sell their total initial endowment to get a revenue that is greater than the revenue from each member's assigned bundle.

First we prove that personalized equilibria coincide, under standard assumptions, with Aubin core allocations, i.e. they are exactly those allocations that are stable with respect to Aubin coalitions improvements. Then, we concentrate on the veto power of the grand coalition adopting the veto mechanism introduced by [20]. The blocking power is exercised by the grand coalition in a family of new economies associated to the original one, once the relevant allocation x has been fixed. In each of them the total initial endowment is a path from the initial endowment e of the original economy to x. We prove that under properness and irreducibility assumptions, personalized equilibria are exactly *robustly efficient* allocations, that means allocations that cannot be blocked by the grand coalition in any of these economies.

A central ingredient of our analysis is represented by *rational allocations*. Rational allocations have been introduced by [9] as a "convexification" of the notions of individual rationality and Pareto optimality. As a further contribution of the paper, we point out that, interestingly, this convexification made directly on a set of allocations, makes us able to dispense with the use of the Lyapunov convexity theorem (clearly not valid in our infinite dimensional set up)[2].

[1] It is a larger, with respect to the dual cone, lattice ordered cone containing non-linear prices.

[2] The Lyapunov convexity theorem, through an application of the Vind's theorem on the measure of blocking coalitions (see [33]), is a central tool in [20] for finite dimensional economies.

In a second step, we assume that the veto power of the grand coalition is exercised in the original economy. Of course we cannot expect that personalized equilibria coincide with Pareto optimal allocations. Hence we assume that the grand coalition is defined with reference to a larger set of individual shares of participation: we say that a Aubin coalition has *full support* if each trader takes part in the coalition with a non-zero share of his initial endowment (a share that is not necessarily equal to one). Of course the Aubin core is a subset of the core defined considering only coalitions with full support and, consequently, also personalized equilibria are stable with respect to this class of coalitions. Moreover, assuming properness and irreducibility, we show that personalized equilibria are exactly those allocations that cannot be blocked by coalitions with full support. Here, again, we use rational allocations as a central element of our analysis, in order to dispense with the use of Lyapunov convexity theorem.

In conclusion, the veto power of the grand coalition is enough to obtain personalized equilibria. Applications of our results are discussed in the last part of the paper. In particular, following [21], we provide a strategic interpretation of personalized equilibria, as Nash equilibria of a game with only two players (subsection 5.1). Moreover, in subsection 5.2 we apply results to economies with asymmetric information furnishing a suitable interpretation of the weak fine core.

The paper is organized as follows. Section 2 includes some preliminary and general facts about ordered topological vector spaces. The model and the main equilibrium notions are introduced throughout Section 3. This Section includes the notions of blocking in connection with the grand coalition. The characterizations of personalized equilibria in terms of the veto exercised by the whole set of agents are proved in Section 4 (Theorem 4.1, Theorems 4.13 and 4.14). Section 5 is devoted to applications and the final section 6 prospects the case of non convex preferences.

2 General facts

L is assumed to be an ordered topological vector space [3] under a Hausdorff locally convex topology τ. Moreover, it will be assumed that order bounded intervals of L are (topologically) bounded sets and that the positive cone is generating i. e. $L = L_+ - L_+$.[4]

We use standard notation for the topological, order and algebraic duals of L. They are, respectively, L', $L^\sim$ and L^* and each a subset of the subsequent. Also, the values $f(c)$ of real valued functions f over L are denoted by $f \cdot c$.

For any index $t \in T$, let $X_t \subseteq L_+$ be a nonempty closed convex cone. Then $0 \in X_t$. Moreover, for a finite T (in this case we use T also to denote the cardinality of T) we have that $\Pi_{t \in T} X_t$ is a closed convex cone of the suitable power of L.

Given a point $c \in L_+$, the symbols $x, y, z, ...$ are reserved for maps on T with $x_t \in X_t$ (i.e. points of $\Pi_{t \in T} X_t$) and such that $\sum_{t \in T} x_t \leq c$. Such maps form a set denoted by $\mathcal{A}_c$. Naturally the set $\mathcal{A}_c$ is convex and therefore also closed in the weak topology (in the power L^T). Observe that

$$\mathcal{A}_c + \mathcal{A}_d \subseteq \mathcal{A}_{c+d} \ \textit{ and } \mathcal{A}_{\varepsilon c} \subseteq \varepsilon \mathcal{A}_c \textit{ for all } c, d \in L_+, \ \varepsilon \geq 0.$$

[3] Remind that the positive orthant L_+ of an ordered vector space is a *proper cone* i.e. a convex cone with vertex 0 and such that $L_+ \cap (-L_+) = \{0\}$. If a topological vector space is also an ordered vector space, then we say that it is an *ordered topological vector space* if its positive cone is closed.

[4] Of course the generating property is automatic in a vector lattice as well as the boundedness of order intervals in case L is a topological vector lattice.

Set C to be the convex cone generated by all sets X_t. One has that $C = \sum_{t\in T} X_t$. Naturally, points of C are non-negative. Denote by M the linear subspace of L generated by C, i.e. $M = C - C$.

A family $p = (p_t)_{t\in T}$ of elements of L^* (i.e. of linear functionals over L) gives rise to a *value function* $\psi_p : C \to [0, \infty]$ defined by means of

$$\psi_p \cdot c := \psi_p(c) := \sup_{y\in\mathcal{A}_c} \sum_{t\in T} p_t \cdot y_t.$$

Note that $\psi_p(0) = 0$ and that, if a family p is made of order bounded functionals, then the associated value function is real valued.

Proposition 2.1. [9, Lemma 3.3] *If the family p is made of order bounded functionals, then the associated value function ψ_p has the following properties.*

1. *monotonicity, namely on C one has $c \le d \Rightarrow \psi_p \cdot c \le \psi_p \cdot d$*
2. *superadditivity, namely on C one has $\psi_p(c + d) \ge \psi_p(c) + \psi_p(d)$*
3. *positive homogeneity on C, i.e. $\psi_p(\varepsilon c) = \varepsilon\psi_p(c)$ for $\varepsilon \ge 0$*
4. *$\psi_p \cdot c = q \cdot c$ if the family p is a constant, i. e. $q = p_t$ for all $t \in T$*
5. *$p_t \cdot c \le \psi_p \cdot c$ for all $c \in X_t$*

A property that it is worthy of being highlighted is the following. We say that *decomposability property* is satisfied if we have that

$$\mathcal{A}_c + \mathcal{A}_d = \mathcal{A}_{c+d}$$

whenever c and d are in the cone C (see property **(A4)** later).

Remark 2.2. When all sets X_t coincide with the positive cone L_+, then the above decomposition property is equivalent to the Riesz decomposition property of the space L (see [9, Lemma 3.4]).

It is also of interest to observe then, having all $X_t = L_+$ and the Riesz decomposition property for L, that $L^\sim$ is a Riesz space and the supremum of $\{p_t, t \in T; 0\}$, i.e. $(\vee_{t\in T} p_t)^+$, exists in $L^\sim$. Call $q \in L^\sim$ such a supremum. By the Riesz-Kantorovich formula it can be seen that even in this case $\psi_p \cdot c = q \cdot c$ for $c \in L_+$.

The following is a remarkable fact.

Theorem 2.3. [9, Theorem 3.5] *Let F be a linear subspace of the order dual $L^\sim$. Then the following are equivalent*

1. *the function ψ_p is additive on C whenever p belongs to F^T*
2. *$\mathcal{A}_{c+d} \subseteq cl_{\sigma(L^T, F^T)}(\mathcal{A}_c + \mathcal{A}_d)$ whenever c and d are in the cone C.*

Remark 2.4. Whenever the map ψ_p is additive on C, then it can be uniquely extended to the linear space M generated by C as a linear functional. Therefore, if the decomposability property holds, then automatically the weak closure condition of Theorem 2.3 also holds (for any F) and, for any family p that belongs to F^T, ψ_p coincides with a linear map over M.

Finally, we recall that a vector $e \in L_+$ is an *order unit* if for each $c \in L$ there exists some $\lambda > 0$ such that $c \leq \lambda e$. If e is an order unit for L_+, then so are αe and $e + c$, for all $\alpha > 0$ and $c \in L_+$. An element $e \in L_+$ is an order unit if and only if it is an internal point of L_+, that is for each $c \in L$ there exists $\lambda_o \in (0, 1)$ such that $\lambda e + (1 - \lambda)c \in L_+$, for all $\lambda \in (\lambda_o, 1)$ (see [8, Lemma 1.7]). Moreover, an order unit e of L_+ is a strictly positive vector, i.e. for every non-zero linear functional $q \in L'_+$ we have $q \cdot e > 0$. This follows from [8, Lemma 2.54] and [8, Lemma 2.65].

3 The Model

We shall deal with exchange economies. The commodity space L is assumed to be an ordered topological vector space under a topology τ that is locally convex. Moreover, it will be assumed that order bounded intervals are (topologically) bounded sets and that the positive cone is generating i. e. $L = L_+ - L_+$[5].

Let us denote by T a finite set of consumers. For any consumer $t \in T$, the initial endowment is $e_t > 0$. It belongs to the consumption set $X_t \subseteq L_+$ which is a closed convex cone. The total initial endowment is e and it belongs to C, the cone generated by all consumption sets.

Preferences of the individual t are given by means of a correspondence P_t of the consumption set in itself. The set $P_t(x)$ represents the collection of elements in X_t that are preferred to the bundle $x \in X_t$ (the "*preferred to*" set). Here are possible assumptions on preferences.

1. *irreflexivity*, namely any $P_t(x)$ does not contain x;
2. *convexity* of values;
3. *strict monotonicity*, i.e. $x + y \in P_t(x)$ whenever $x, y \in X_t$ and $y \neq 0$
4. any set $P_t(x)$ is open in X_t
5. any "*worse than*" set $P_t^{-1}(x) := \{z \in X_t : x \in P_t(z)\}$ is open in X_t with respect to the weak topology.
6. $x \in P_t(y)$ implies that $y \notin P_t(x)$ for each $t \in T$ and x, $y \in X_t$.

Naturally, when strict monotonicity holds, any set $P_t(x)$ is non-empty as it contains $x + \varepsilon e_t$ for positive numbers ε. Consequently, also $x \in \overline{P_t(x)}$ always holds true.

The set of above properties of the preference correspondence is denoted by **(A1)**.

Concerning hypothesis denoted in bold characters like **(A1)**, **(A2)** etc, throughout the sequel, an explicit appeal to them will be made whenever they are used in obtaining results. Assumption (**A1**, 6.) will be explicitly used only in subsection 5.1, assumption (**A1**, 5.) only in connection with the existence of equilibria.

An *assignment* is, by definition, a map x on T with $x_t \in X_t$ (i. e. a point in $\Pi_{t \in T} X_t$). An assignment x is said to be an *allocation* if $\sum_{t \in T} x_t \leq e$ (i.e. if feasibility is satisfied). Allocations form a set denoted by $\mathcal{A}$ or by $\mathcal{A}_e$ to emphasize the role of vector e in the feasibility. The following compactness condition is introduced to guarantee the existence of relevant allocations:

[5] If L is a vector lattice, then it would be enough to assume that the positive cone of L is τ-closed and that L' is a vector sublattice of $L^\sim$.

(A2): the set $\mathcal{A}$ is assumed to be weakly compact.

Of course **(A2)** is satisfied under the hypothesis of weak compactness of the interval $[0, e]$, since $\mathcal{A} \subseteq [0, e]^T$.

Let us introduce a *properness condition* on preferences as in [9].

(A3): There is an allocation v, with $v_t > 0$ for any $t \in T$, and a family of correspondences[6] $(\hat{P}_t)_{t \in T}$, each defined on X_t and with values in L, such that

$$t \in T,\ x \in X_t \Rightarrow \hat{P}_t(x) \cap X_t = P_t(x) \text{ and } x + v_t \in int(\hat{P}_t(x)).$$

As it is usual, the properness-like assumption on preferences compensates for the possible lack of interior points in the positive cone of the commodity space, since it ensures that infinite dimensional separation theorems can be applied. Precisely, it gives a condition sufficient for the preferred set to be supported by a continuous linear functional. The condition contained in **(A3)** and adopted in [9], is due to [31]. If the commodity space L is a vector lattice and preferences are complete preorderings on $X_t = L_+$, then it can be proved that Mas-Colell's ([25]) uniform properness condition is strictly stronger than **(A3)** (see also the extreme desirability condition in [32]). Related conditions are given in [10] and [30].

3.1 Properties of feasible allocations

We now introduce some well-known properties that a (feasible) allocation may satisfy. The following notions are price free, in the sense that they should be understood as intrinsic in the commodity space. The term coalition will be reserved to non empty subsets of the set T.

Definition 3.1. *An allocation x is*

1. *individually rational, when $e_t \notin P_t(x_t)$ for any agent t (write $x \in \mathcal{IR}$).*
2. *weakly Pareto optimal (write $x \in w\mathcal{PO}$), if $y_t \in P_t(x_t)$ for all t, entails that $(y_t)_{t \in T}$ is not feasible. Equivalently: $\mathcal{A}_e \cap \Pi_{t \in T} P_t(x_t) = \emptyset$.*
3. *in the core (write $x \in \mathcal{C}$) , if it cannot be blocked by any coalition of agents, namely there does not exist a coalition S and a map y on S with $y_t \in P_t(x_t)$, for all t, and such that $\sum_{t \in S} y_t \leq \sum_{t \in S} e_t$.*
4. *an Edgeworth equilibrium (write $x \in \mathcal{C}^E$), whenever it is in the core of any replica economy.*
5. *in the Aubin core [7] (write $x \in \mathcal{C}^{\mathrm{A}}$) , if it cannot be f-blocked by any coalition of agents, namely a coalition S does not exist such that a map $(\alpha_t, y_t)_{t \in S}$ can be found with*
$$(\alpha_t, y_t) \in]0, 1] \times P_t(x_t) \text{ and } \sum_{t \in S} \alpha_t y_t \leq \sum_{t \in S} \alpha_t e_t.$$
6. *a strong Edgeworth equilibrium (write $x \in s\,\mathcal{C}^E$) if*
$$(co \bigcup_{t \in T} [P_t(x_t) - e_t]) \cap (-L_+) = \emptyset.$$

[6] convex-valued is required if we are under the assumption (**A1,** 2.) of convexity of preferences

[7] Same as *f-core* or *fuzzy core*.

7. *in the strong Aubin core (write $x \in \mathrm{s}\mathcal{C}^{\mathrm{A}}$) , if it cannot be strongly f-blocked, namely it is impossible to find a set (necessarily finite here) of coalitions $S_1, \dots, S_R$ such that maps $(\alpha_t^r, y_t^r)_{t \in S_r}$, $r = 1, \dots, R$, can be found with*

$$(\alpha_t^r, y_t^r) \in]0,1] \times P_t(x_t) \text{ and } \sum_{r=1}^{R} \sum_{t \in S_r} \alpha_t^r y_t^r \le \sum_{r=1}^{R} \sum_{t \in S_r} \alpha_t^r e_t.$$

The optimality properties provided by Definitions 3.1 will be reinterpreted in terms of suitable coalitions in subsection 3.3. For the moment, we register that the following obvious inclusions hold true:

$$\mathcal{C}^E \subseteq \mathcal{C} \subseteq \mathrm{w}\mathcal{PO} \cap \mathcal{IR}.$$

Moreover,

Proposition 3.2. [9, Lemma 7.2] *We have $s\mathcal{C}^E \subseteq \mathcal{C}^E$. The reverse inclusion $\mathcal{C}^E \subseteq \mathrm{s}\mathcal{C}^E$ also holds under the assumption that preferences are convex-valued, i.e. (**A1**, 2.), and continuous, i.e. (**A1**, 4.).*

Proposition 3.3. *We have $s\mathcal{C}^A \subseteq \mathcal{C}^A$. The reverse inclusion $\mathcal{C}^{\mathrm{A}} \subseteq \mathrm{s}\mathcal{C}^{\mathrm{A}}$ also holds under the assumption that preferences are convex-valued, i.e. (**A1**, 2.).*

PROOF: The first inclusion simply follows by definition. Assume now that $x \notin \mathrm{s}\mathcal{C}^A$. Then there exist finitely many coalitions $S_1, \dots, S_R$ such that maps $(\alpha_t^r, y_t^r)_{t \in S_r}$, $r = 1, \dots, R$, can be found with

$$(\alpha_t^r, y_t^r) \in]0,1] \times P_t(x_t) \text{ and } \sum_{r=1}^{R} \sum_{t \in S_r} \alpha_t^r y_t^r \le \sum_{r=1}^{R} \sum_{t \in S_r} \alpha_t^r e_t.$$

Then, defining for each $r = 1, \dots, R$ and for $t \notin S_r$, α_t^r equal to zero and y_t^r arbitrary, we have also

$$\sum_{t \in T} \sum_{r=1}^{R} \alpha_t^r y_t^r \le \sum_{t \in T} \sum_{r=1}^{R} \alpha_t^r e_t.$$

Let us define for each $t \in T$, $\alpha_t = \sum_{r=1}^{R} \alpha_t^r$, $S = \{t \in T : \alpha_t \neq 0\}$ and for each $t \in S$, $y_t = \dfrac{1}{\alpha_t} \sum_{r=1}^{R} \alpha_t^r y_t^r$. Then, by convexity, $y_t \in P_t(x_t)$ and a contradiction follows from

$$\sum_{t \in S} \alpha_t y_t = \sum_{t \in T} \sum_{r=1}^{R} \alpha_t^r y_t^r \le \sum_{t \in T} \sum_{r=1}^{R} \alpha_t^r e_t = \sum_{t \in S} \alpha_t e_t.$$

□

Proposition 3.4. *We have $s\mathcal{C}^{\mathrm{A}} \subseteq s\mathcal{C}^{\mathrm{E}}$.*

PROOF:

Let $x \in \mathrm{s}\mathcal{C}^{\mathrm{A}}$ and assume that $x \notin \mathrm{s}\mathcal{C}^{\mathrm{E}}$ so that we find a point in $-L_+$ which is a convex combination of points in $\bigcup_{t \in T} [P_t(x_t) - e_t]$. Set $T_o \subseteq T$ by $t \in T_o$ if and only if $R_t > 0$ points of $P_t(x_t) - e_t$ are involved in the above mentioned convex combination. Then we can write for $t \in T_o$, $r = 1, \dots R_t$,

$$(\alpha_t^r, y_t^r) \in]0,1] \times P_t(x_t) \text{ and } \sum_{t\in T_o}\sum_{r=1}^{R_t} \alpha_t^r y_t^r \leq \sum_{t\in T_o}\sum_{r=1}^{R_t} \alpha_t^r e_t.$$

Let $R = max_{t\in T_o} R_t$ and define for $r = 1 \dots R$

$$S_r = \{t \in T_o : \alpha_t^r \neq 0\}.$$

Then we have that each S_r is non empty and writing previous inequality as

$$\sum_{t\in S_r}\sum_{r=1}^{R} \alpha_t^r y_t^r \leq \sum_{t\in S_r}\sum_{r=1}^{R} \alpha_t^r e_t$$

we contradict $x \in \mathrm{s}\,\mathcal{C}^{\mathrm{A}}$.

□

Proposition 3.5. *We have $s\mathcal{C}^{\mathrm{E}} \subseteq \mathcal{C}^{\mathrm{A}}$. The reverse inclusion $\mathcal{C}^{\mathrm{A}} \subseteq s\mathcal{C}^{\mathrm{E}}$ also holds under the assumption that preferences are convex-valued, i.e. (**A1**, 2.).*

PROOF: Let us start with $x \notin \mathcal{C}^{\mathrm{A}}$. Then a coalition S exists such that a map $(\alpha_t, y_t)_{t\in S}$ can be found with

$$(\alpha_t, y_t) \in]0,1] \times P_t(x_t) \text{ and } \sum_{t\in S} \alpha_t y_t \leq \sum_{t\in S} \alpha_t e_t.$$

Up to an obvious normalization, it is clear that the set $(\mathrm{co}\bigcup_{t\in T}[P_t(x_t) - e_t]) \cap (-L_+)$ is nonempty. Viceversa, take $x \notin \mathrm{s}\,\mathcal{C}^{\mathrm{E}}$, then a point $z \in (co\bigcup_{t\in T}[P_t(x_t) - e_t]) \cap (-L_+)$. We can write, by convexity, $0 \geq z = \sum_{t\in T} \lambda_t(z_t - e_t)$ with $z_t \in P_t(x_t)$ for all t and $\sum_{t\in T} \lambda_t = 1$. Naturally we rewrite $\sum_{t\in T} \lambda_t z_t \leq \sum_{t\in T} \lambda_t e_t$ to understand that $x \notin \mathcal{C}^{\mathrm{A}}$. □

3.2 Competitive equilibria and generalized prices

We start introducing standard notions of valuation and competitive equilibria. Since at this stage prices are required to be linear, existence and optimality of equilibria are in general not guaranteed for infinite as well as finite dimensional commodity spaces, given the possibility that the ordering of the commodity space is not a lattice order[8]. Here are the Walrasian notions.

Definition 3.6. *An allocation x is a Walrasian*

1. *valuation equilibrium, when $q \cdot P_t(x_t) \geq q \cdot x_t$ for a certain $q \in L'$ such that $q \cdot e \neq 0$;*
2. *quasi equilibrium, when $q \cdot P_t(x_t) \geq q \cdot e_t$ for a certain $q \in L'$ such that $q \cdot e \neq 0$ and $q \cdot x_t = q \cdot e_t,\ \forall t \in T$;*
3. *equilibrium, when $q \cdot P_t(x_t) > q \cdot e_t$ for a certain $q \in L'$ such that $q \cdot e \neq 0$ and $q \cdot x_t = q \cdot e_t,\ \forall t \in T$.*

[8]In [27] an example is provided of an economy with three commodities in which there is no Walrasian equilibrium and the second welfare theorem fails for linear prices.

The alternate theory of value arises from a personalized pricing system defined as follows. We just refer to a family $p = (p_t)_{t\in T}$ of elements of L^* (i.e. of linear functionals over L assigned to each trader) as a *list of personalized price*. A list of personalized price, as we have seen, gives rise to a value function $\psi_p : C \to [0, \infty]$ that will be defined as the *generalized price* associated to the list. If a list of personalized prices is made of order bounded functionals (we say that it is order bounded), then the value function is real valued. In this case, the name generalized price for the value function is naturally motivated by property 4 in Proposition 2.1 or, when all consumption sets coincide with the positive cone L_+ and the Riesz decomposition property holds for L, by Remark 2.2. Then, following the interpretation given by [9], in a first step a discriminating Walrasian auctioneer assigns to each (price-taking) consumer t a personal linear price p_t. Given this list of prices, the value of the generalized price ψ_p over a commodity bundle c, $\psi_p \cdot c$, is the highest possible value that it is attainable by decomposing the vector $c \in C$ into a consumable family $(y_t)_{t\in T}$ of vectors and assuming that each agents t pays its own personal price p_t. According to this, it is natural to say that an allocation x of the economy, given the personalized price p, is a *maximizing allocation* if the value $\psi_p \cdot e$ is exactly achieved in x, namely

$$\psi_p \cdot e := \sup_{y\in\mathcal{A}_e} \sum_{t\in T} p_t \cdot y_t = \sum_{t\in T} p_t \cdot x_t.$$

We use the symbol $\mathcal{A}^{(\max, p)}$ to denote the set of maximizing allocations.

Remark 3.7. *Notice that under the assumption of weak compactness of the set of allocations of the economy, assumption* **(A2)***, a maximizing allocation always exists when the personalized price is made of continuous functionals (we speak then of a continuous personalized price) due to weak continuity of the function* $\sum_{t\in T} p_t y_t$ *defined on* L^T*. In other words:*

$$p \in (L')^T \Rightarrow \mathcal{A}^{(\max, p)} \neq \emptyset.$$

Also observe that

$$x \in \mathcal{A}^{(\max, p)} \Rightarrow \psi_p \cdot x_t = p_t \cdot x_t, \; \forall t \in T$$

since otherwise, by Proposition 2.1 (properties 5., 2. and 1.), one has $\sum_{t\in T} p_t \cdot x_t < \sum_{t\in T} \psi_p \cdot x_t \le \psi_p \cdot e = \sum_{t\in T} p_t \cdot x_t$*, i.e. a contradiction.*

We say that the economy satisfies the **consumption decomposability property, (A4),** if it is true that $\mathcal{A}_c + \mathcal{A}_d = \mathcal{A}_{c+d}$ whenever c and d are in the cone C.

Property **(A4)** relies on the consumption sets of the economy. Indeed if $u \in \mathcal{A}_{c+d}$, and therefore for the map $u = (u_t)_{t\in T}$, one has $\sum_{t\in T} u_t \le c + d = \sum_{t\in T} (c_t + d_t)$, there is no guarantee that any u_t is $u_t = u'_t + u''_t$ with $u'_t \in [0, c_t]$ and $u''_t \in [0, d_t]$. Moreover even if the Riesz decomposition property would ensure the right decomposition, without the solidity of consumption sets **(A4)** could fail.

The consumption decomposability property is related to the possibility of a generalized price to be a linear map. The base for such a statement is in Theorem 2.3 and Remark 2.4.

Remark 3.8. *If* **(A4)** *holds, then for any order bounded personalized price,* ψ_p *coincides with a linear map over* M*. As we have seen in Remark 2.2, when the commodity space* L *enjoys the Riesz decomposition property and all consumption sets coincide with the positive cone* L_+*, then we have* **(A4)** *and the generalized price associated to an order bounded personalized price can be identified with an element of* $L^\sim$*, the order dual of* L*.*

About continuity of possible linear extensions of generalized prices we have what follows.

Theorem 3.9. *Suppose all consumption sets are L_+ and p is an order bounded personalized price. If ψ_p is the restriction to L_+ of an element q of L', then in $L^\sim$ the supremum of $\{p_t, t \in T; 0\}$, i.e. $(\vee_{t\in T} p_t)^+$, exists and it is continuous, moreover it coincides with q.*

We define now equilibria supported by non-linear prices, that is competitive equilibria naturally arising in our setting whose introduction is due to [9]. They turn out to be of interest for infinite as well as finite dimensional commodity spaces, leading to a broad range of economic applications.

Definition 3.10. *For an allocation $x \in \mathcal{A}$, we say that it is a:*

1. *personalized valuation equilibrium if it exists an order bounded personalized price p such that*

$$\begin{array}{ll} (i) & \psi_p \cdot e > 0 \\ (ii) & t \in T,\, y \in P_t(x_t) \Rightarrow \quad \psi_p \cdot y \geq \psi_p \cdot x_t \\ (iii) & \psi_p \cdot e = \sum_{t\in T} \psi_p \cdot x_t. \end{array}$$

2. *personalized quasi-equilibrium if it exists an order bounded personalized price p such that (i), (ii) above hold and (iii) is replaced by*

$$(iii)_b \qquad \psi_p \cdot \sum_{t\in T} \alpha_t e_t \leq \sum_{t\in T} \alpha_t \psi_p \cdot x_t,\ \forall \alpha \in \mathbb{R}^T_+.$$

3. *personalized equilibrium if it exists an order bounded personalized price p such that (i), $(iii)_b$ above hold and (ii) is replaced by*

$$(ii)_b \qquad t \in T,\, y \in P_t(x_t) \Rightarrow \quad \psi_p \cdot y > \psi_p \cdot x_t.$$

Condition *(iii)* is an *arbitrage-free* condition motivated by nonlinearity of prices. What is relevant in it is the $\leq$ sign since the reverse inequality holds due to: feasibility of x, monotonicity and superadditivity of the generalized price [9]. Clearly in the above definition 3. $\Rightarrow$ 2. $\Rightarrow$ 1. Moreover, if we denote by p$\mathcal{VE}^p$ the set of allocations that are personalized valuation equilibria with respect to a list p, then it is trivially true that:

$$x \in \mathrm{p}\mathcal{VE}^p \text{ and } \psi_p \cdot x_t = p_t \cdot x_t,\ \forall t \in T \Rightarrow x \in \mathcal{A}^{(\max,p)}.$$

Let us denote now by p$\mathcal{QE}^p$ and by p$\mathcal{E}^p$ the sets of allocations that are, respectively, personalized quasi-equilibria and personalized equilibria with respect to the personalized price p. Sets of all allocations that are, respectively, personalized valuation equilibria, quasi-equilibria and equilibria with respect to some personalized price p are p$\mathcal{VE}$, p$\mathcal{QE}$ and p$\mathcal{E}$. The irreducibility condition needed in this setting to convert quasi equilibria in equilibria supported by nonlinear prices (i.e. to show that 2. $\Rightarrow$ 3.), looks as an assumption on decomposability of all initial endowments. We say that a vector $0 < c \in L_+$ is *decomposable* if an always nonzero element of $\mathcal{A}_c$ can be found. Let us introduce assumption
(A5): All initial endowments are decomposable.

[9] $\psi_p \cdot e \geq \psi_p \cdot (\sum_{t\in T} x_t) \geq \sum_{t\in T} \psi_p \cdot x_t$. So when *(iii)* holds, then necessarily the inequalities become equalities. We shall discuss the condition in details in subsection 3.3.
Note also that from $(iii)_b$ it follows that $\psi_p \cdot e_t \leq \psi_p \cdot x_t$ for any agent t.

Remark 3.11. *Notice that if the consumption sets are equal each other, then* $\frac{e_t}{T}\mathbf{1} \in \mathcal{A}_{e_t}$, *so assumption* **(A5)** *is fulfilled. A similar conclusion is achieved for different consumption sets if one can find a number* $k > 0$ *such that* $e \leq ke_t$ *for all* $t \in T$. *Indeed* $\frac{e}{k} \in \mathcal{A}_{e_t}$ *for all* $t \in T$.

Lemma 3.12. *Assume strict monotonicity and continuity of preferences (i.e. (***A1,** *3. and 4.)). Assume further the decomposability condition* **(A5)**.
If $x \in \mathrm{p}\mathcal{QE}^p \cap \mathcal{A}^{(\max,p)}$ *and, for all agent, one has* $(*)\, p_t \cdot x_t \leq p_t \cdot P_t(x_t)$, *then* $x \in \mathrm{p}\mathcal{E}^p$.

PROOF: It is only necessary to prove the above condition $(ii)_b$.

We first claim that: for any t necessarily $\psi_t \cdot x_t > 0$.

Since x is maximizing, by condition (i) we have: $0 < \psi_p \cdot e = \sum_{t\in T} p_t \cdot x_t$ and necessarily an s exists with $0 < p_s \cdot x_s = \psi_s \cdot x_s$.

Now by continuity take for any $y \in P_s(x_s)$ a number $k \in]0,1[$ such that $ky \in P_s(x_s)$ and apply assumption $(*)$:

$$0 < p_s \cdot x_s \leq p_s \cdot ky = kp_s \cdot y$$

obtaining that $p_s \cdot y$ is non zero and therefore $p_s \cdot y > kp_s \cdot y$ and also $p_s \cdot y > p_s \cdot x_s$ In other words we have obtained: $(**) p_s \cdot x_s < p_s \cdot P_s(x_s)$. Using **(A5)** to decompose e_t, we can find $z \in X_s$ with $0 < z \leq e_t$ and by $(**)$ applied to $y = x + z_s$ ($\in P_s(x_s)$ because of monotonicity) we have $p_s \cdot z > 0$. Now: $\psi_p \cdot x_t \geq \psi_p \cdot e_t \geq \psi_p \cdot z \geq p_s \cdot z$, the three inequalities coming from, respectively, condition $(iii)_b$, 1. and 5. of Proposition 2.1.

The claim is therefore proved and to obtain $(ii)_b$ again take for any $y \in P_t(x_t)$ a number $k \in]0,1[$ such that $ky \in P_t(x_t)$ and apply assumption (ii): $k\psi_p \cdot y = \psi_p \cdot ky \geq \psi_p \cdot x_t$ discovering that $\psi_p \cdot y > 0$ and therefore $\psi_p \cdot y > k\psi_p \cdot y \geq \psi_p \cdot x_t$, as desired. □

Theorem 3.13. ([9, Theorem 4.3]) *A Walrasian equilibrium* x *with respect to the positive equilibrium price* q *necessarily is a personalized equilibrium with respect to the continuous personalized price* $p_t = q\,(\forall t)$ *that gives rise to the generalized price* ψ_q *additive on* C. *Reciprocally, a personalized equilibrium* x, *with respect to an order bounded personalized price* p *such that* ψ_p *is additive and continuous on* C, *is a Walrasian equilibrium.*

PROOF: It is just by assumption that $q \in L_+,\ q \cdot e \neq 0,\ q \cdot x_t = q \cdot e_t,\ q \cdot P_t(x_t) > q \cdot e_t$ for all $t \in T$. Applying Proposition 2.1 to the family constantly equal to q, one has that $\psi_p \cdot c = q \cdot c$ on C. Then (i), $(ii)_b$ and $(iii)_b$ of Definition 3.10 obviously hold.

Assume now that x is a personalized equilibrium under p. We have seen already in footnote that $\psi_p \cdot e_t \leq \psi_p \cdot x_t$ for any agent t. If ψ_p is additive on C then we have by feasibility that

$$\sum_{t\in T} \psi_p \cdot x_t = \psi_p \cdot \sum_{t\in T} x_t \leq \psi_p \cdot e = \sum_{t\in T} \psi_p \cdot e_t$$

and therefore $\psi_p \cdot e_t = \psi_p \cdot x_t$ for al t. Since ψ_p is uniquely extendible as a linear and continuous functional over M, by the Hahn-Banach theorem let $q \in L'$ be an extension of it. Then x is a Walrasian equilibrium under q. □

Remark 3.14. *In view of Theorem 3.13, we are in the Arrow-Debreu-McKenzie model whenever the generalized price is additive and continuous. This is for example the case when the personalized price is continuous and the consumption decomposability property holds true or the consumption sets coincide with the positive cone of a vector lattice* L *in which* $L^{'}$ *is a vector sublattice of* $L^{\sim}$.

We can see now conditions under which the set p$\mathcal{VE}$ of allocations that are personalized valuation equilibria can be compared with the set w$\mathcal{PO}$ of weakly Pareto optimal allocations. For the proof of the next two theorems we refer to [9, Theorem 5.1].

Theorem 3.15. *Assume preferences are convex-valued. Assume further that* **(A3)** *holds (the properness condition) and* $x \in \mathrm{w}\mathcal{PO}$*.*

1. *A continuous personalized price* p *can be found such that*

$$p_t \cdot x_t \leq p_t \cdot P_t(x_t),\ \forall t \in T.$$

2. *Assume further that preferences are strictly monotone, then the same* p *supports* x *as a personalized valuation equilibrium (i.e.* $\mathrm{w}\mathcal{PO} \subseteq \mathrm{p}\mathcal{VE}$*) and, moreover,* $x \in \mathcal{A}^{(\max, p)}$ *or, what is the same here,* $\psi_p \cdot x_t = p_t \cdot x_t$*,* $\forall t \in T$*.*

Of course, one can also summarize the thesis (analogous to that of the second welfare theorem) of the above Theorem as: $\mathrm{w}\mathcal{PO} \subseteq \bigcup_{p \in (L')^T} \mathrm{p}\mathcal{VE}^p$. Under continuity of preferences the inclusion of Theorem 3.15 can be reversed.

Theorem 3.16. *Assume that any set* $P_t(x_t)$ *is open in* X_t*, then*

$$\mathrm{p}\mathcal{VE} \subseteq \mathrm{w}\mathcal{PO}.$$

We conclude this subsection by proving the following inclusion.

Proposition 3.17. *We have* $p\mathcal{E} \subseteq s\mathcal{C}^{\mathrm{A}}$*.*

PROOF:

Let $x \in \mathrm{p}\mathcal{E}$ be supported by a generalized price ψ_p and assume that $x \notin \mathrm{s}\mathcal{C}^{\mathrm{A}}$ so that we find coalitions $S_1, \ldots, S_R$ and maps $(\alpha_t^r, y_t^r)_{t \in S_r}$, $r = 1, \ldots, R$, with $(\alpha_t^r, y_t^r) \in]0,1] \times P_t(x_t)$ and

$$(+) \quad \sum_{r=1}^{R} \sum_{t \in S_r} \alpha_t^r y_t^r \leq \sum_{r=1}^{R} \sum_{t \in S_r} \alpha_t^r e_t.$$

By condition $(ii)_b$ of Definition 3.10 we have for $r = 1, \ldots, R$ and $t \in S_r$

$$\psi_p \cdot y_t^r > \psi_p \cdot x_t$$

and then

$$(++) \quad \sum_{r=1}^{R} \sum_{t \in S_r} \alpha_t^r \psi_p \cdot y_t^r > \sum_{r=1}^{R} \sum_{t \in S_r} \alpha_t^r \psi_p \cdot x_t.$$

Observe now that by superadditivity, positive homogeneity and monotonicity of the functional ψ_p, due to (+), one has

$$\sum_{r=1}^{R} \sum_{t \in S_r} \alpha_t^r \psi_p \cdot y_t^r \leq \psi_p \cdot \left(\sum_{r=1}^{R} \sum_{t \in S_r} \alpha_t^r y_t^r \right) \leq \psi_p \cdot \left(\sum_{r=1}^{R} \sum_{t \in S_r} \alpha_t^r e_t \right).$$

If the latter quantity is written as

$\psi_p \cdot \left(\sum_{r=1}^R \sum_{t\in S_r} \alpha_t^r e_t\right) = \psi_p \cdot \left(\sum_{r=1}^R \sum_{t\in T} \beta_t^r e_t\right) = \psi_p \cdot \left(\sum_{t\in T} \left(\sum_{r=1}^R \beta_t^r\right) e_t\right)$, where β_t^r is null on $T \setminus S_r$ and coincide with α_t^r on S_r, condition $(iii)_b$ of Definition 3.10 can be used in order to get

$$\psi_p \cdot \left(\sum_{t\in T} \left(\sum_{r=1}^R \beta_t^r\right) e_t\right) \leq \sum_{t\in T} \left(\sum_{r=1}^R \beta_t^r\right) \psi_p \cdot x_t = \sum_{r=1}^R \sum_{t\in S_r} \alpha_t^r \psi_p \cdot x_t.$$

that contradicts (++).

□

3.3 Notions of blocking

Reconsider the additional condition in the notion of personalized equilibrium denoted by $(iii)_b$ and called arbitrage free condition:

$$\psi_p \cdot \sum_{t\in T} \alpha_t e_t \leq \sum_{t\in T} \alpha_t \psi_p \cdot x_t, \ \forall \alpha \in I\!R_+^T$$

and observe that, due to the positive homogeneity of ψ_p, it can be rewritten in the form

$$\psi_p \cdot \sum_{t\in T} \alpha_t e_t \leq \sum_{t\in T} \alpha_t \psi_p \cdot x_t, \ \forall \alpha \in [0,1]^T.$$

This remark allows us to interpret $(iii)_b$ as a coalition based arbitrage free condition. Indeed, define the set

$$\Sigma = \{\alpha : T \to [0,1] : \{t \in T : \alpha_t > 0\} \neq \emptyset\}$$

and call: any element α in the set Σ an *Aubin* (or *fuzzy*) *coalition*, the set $S_\alpha = \{t \in T : \alpha_t > 0\}$ the *support* of α.

The function $(\alpha) \in [0,1]^T$ is interpreted as a coalition in which agent t takes part employing only a share α_t of is initial resources e_t. Of course ordinary coalitions form a subset of Σ, since they can be identified with their characteristic functions.

Looking now at condition $(iii)_b$ above, it is clear that $\sum_{t\in T} \alpha_t e_t$ is the initial endowment of the Aubin coalition and the arbitrage free condition can be interpreted by saying that: under a personalized equilibrium allocation, Aubin coalitions have no nominal incentive to sell their total initial endowment to get a revenue that is greater than the revenue from each member's assigned bundle. Hence the personalized equilibrium notion implicitly contains a requirement of stability with respect to deviations of Aubin coalitions, in the sense that agents in a Aubin coalition have no incentive to deviate from their optimal choice in the generalized budget set[10]. Is the allocation also stable with respect to deviations when agents in the coalition are free to redistribute their initial endowments among theirself?

To answer to this question, we shall give or reinterpret in the following the blocking mechanisms and the corresponding cooperative notions of equilibria in terms of Aubin coalitions.

[10]the interpretation of condition $(iii)_b$ proposed in [9] relies on coalitions of a continuum economy canonically associated to the finite one.

Definition 3.18. *The coalition $\alpha \in \Sigma$ of support S_α blocks an allocation x, if there exists, over S_α, an assignment y s.t.*

i) $\sum_{t \in S_\alpha} \alpha_t y_t \leq \sum_{t \in S_\alpha} \alpha_t e_t$*;*

ii) $y_t \in P_t(x_t)$*, for all* $t \in S_\alpha$*.*

Once this is fixed, it is clear that the notion of Aubin core introduced in Definition 3.1 can be reconsidered from the perspective of Aubin coalitions: $x \in \mathcal{C}^{\mathrm{A}}$ is the same as saying that x cannot be blocked by an Aubin coalition. Since each agent t may participate to a coalition α employing only a share α_t of his initial resources, the intuition behind is that a feasible allocation belongs to the Aubin core if it is not possible for agents to join such a coalition and to redistribute their initial endowment among themselves according to these shares letting each member obtain a strictly preferred bundle. The feasibility over the coalition has to account of these shares as it is expressed by the weighted sum in condition ii)[11].

Similarly, if we take a finite set $A = \{\alpha^1, \ldots, \alpha^R\}$ of Aubin coalitions, we say that x is blocked by A if we can find assignments $y^1 \ldots, y^R$ enjoying the following properties:

$$t \in S_{\alpha^r} \Rightarrow y_t^r \in P_t(x_t) \qquad \text{and} \qquad \sum_{r=1}^{R} \sum_{t \in T} \alpha_t^r y_t^r \leq \sum_{r=1}^{R} \sum_{t \in T} \alpha_t^r e_t.$$

Let us call A a *generalized Aubin coalition* for supporting the intuition that an agent t may join simultaneously several coalitions $\alpha^1, \ldots, \alpha^R$ by participating in each with a portion α_t^r of his resources.

Once this is fixed, we can clearly say that $x \in \mathrm{s}\,\mathcal{C}^{\mathrm{A}}$ is the same as saying that x cannot be blocked by any generalized coalition.

The main results in the paper aim to characterize personalized equilibria as the only allocations that are stable with respect to the veto power of the grand coalition (in the usual or Aubin sense). Personalized equilibria are strong Edgeworth equilibria (see [9, Lemma 7.4], proved here as a combination of Propositions 3.4 and 3.17). Consequently, they are in the Aubin core and weakly Pareto optimal, that means stable with respect to improvements of the grand coalition, assuming that agents may employ a share or the whole initial endowment joining it. In general these inclusions do not end up in equivalences. So the veto power of the grand coalition is not enough to characterize competitive equilibria unless the veto mechanisms is modified in a suitable direction.

To show this, we shall deal with the blocking mechanism due to [20], in which the veto is exercised by the grand coalition in a family of economies associated to the original one. The next definitions are formulated following their approach.

Definition 3.19. *Let x be a given assignment. We say that it is non-dominated when the grand coalition can't block it, i.e.* $\mathcal{A}_e \cap \Pi_{t \in T} P_t(x_t) = \emptyset$ *(write* $x \in \mathcal{ND}_e$*).*

It is clear that $\mathrm{w}\mathcal{PO} = \mathcal{ND}_e \cap \mathcal{A}_e$.

We introduce now a notation. Fix maps $y \in \Pi_{t \in T} X_t$ and $\alpha \in [0,1]^T$ then define the new assignment $e(\alpha, y)$ by means of $e(\alpha, y)_t = \alpha_t e_t + (1 - \alpha_t) y_t$.

[11] the notion of Aubin core is due to [11] for finite economies and has been extended in [29] to arbitrary measure spaces of agents, in [12], [16], [18] it has been studied for economies with infinitely many goods and allowing for production.

The economy coincident with the original one except for the replacement of the initial endowment e by $e(\alpha, y)$ is just referred to as $e(\alpha, y)$ for simplicity. Obviously $e(\mathbf{1}, y) = e$ and $e(0, y)$ is the economy that has y as initial endowment. The economies $e(\alpha, y)$ describe a path connecting these two.

Call *robustly efficient* (write $x \in \mathcal{R}o\mathcal{E}f$) a feasible allocation x that is non dominated in each economy $e(\alpha, x)$, i.e. an allocation for which $\mathcal{A}_{e(\alpha,x)} \cap \Pi_{t\in T} P_t(x_t) = \emptyset$ for all $\alpha \in [0,1]^T$. In other terms, set

$$\mathcal{ND}(x) := \bigcap_{\alpha\in[0,1]^T} \mathcal{ND}_{e(\alpha,x)},$$

the definition of robust efficiency is

$$\mathcal{R}o\mathcal{E}f = \{x \in \mathcal{A}_e : x \in \mathcal{ND}(x)\}.$$

4 Further properties and characterizations of personalized equilibria

We are going to show in this section that personalized equilibria can be characterized in terms of the veto power of the grand coalition. This will be done under two different approaches. In subsection 4.1 we will see that personalized equilibria coincide with robustly efficient allocations, that is they are robust with respect to the veto power exercised in each of the auxiliary economy in which the initial endowment is modified in a precise direction. Then, in subsection 4.2, it will be proved that personalized equilibria are exactly those allocations that cannot be f-blocked by the grand coalition. In both cases, the central element of our analysis will be represented by the class of *rational allocations*.

4.1 Personalized equilibria and robust efficiency

The main aim in this section is to show that personalized equilibria coincide with robustly efficient allocations. This characterization has been introduced for competitive equilibria of finite dimensional economies by [20]. There the proof relies on the use of Vind's theorem about the measure of blocking coalitions in suitable atomless economies, that in its turn applies the Lyapunov convexity theorem. To extend the result to infinite dimensional economies we proceed first proving that Aubin core allocations are robustly efficient. Then we show that robustly efficient allocations belong to the class of rational allocations. Rational allocations have been introduce by [9] as a "convexification"of the notions of individual rationality and Pareto optimality. They turn out to have a central role even in our analysis. Then we use the equivalence between personalized equilibria and rational allocations to prove our result. Notice that by means of rational allocations and properness assumption, we can dispense with the use of Lyapunov convexity theorem and Vind's result.

Theorem 4.1. *Assume preferences are irreflexive, convex-valued, strictly monotone and continuous, i.e. (***A1***,1., 2., 3. and 4.). Assume further that* **(A3)** *(the properness condition) and* **(A5)** *hold. Take x to be an allocation, i.e. $x \in \mathcal{A}_e$. Then x is a personalized equilibrium if and only if it is robustly efficient i.e.*

$$\mathrm{p}\mathcal{E} = \mathcal{R}o\mathcal{E}f.$$

The proof will be given in several steps

Theorem 4.2. *Assume convexity (**A1**, 2.), strict monotonicity (**A1**, 3.) and continuity (**A1**, 4.) of preferences. Take x to be an allocation, i.e. $x \in \mathcal{A}_e$. If x is in the Aubin core, then x is robustly efficient, i.e.*

$$\mathcal{C}^{\mathrm{A}} \subseteq \mathcal{R}o\mathcal{E}f.$$

PROOF:

Take a point $y \in \mathcal{A}_{e(\alpha,x)} \cap \Pi_{t\in T} P_t(x_t)$ and set S to be the support of α. Clearly

$$\sum_{t\in T} e_t(\alpha, x) = \sum_{t\in S} \alpha_t e_t + \sum_{t\in T} x_t - \sum_{t\in S} \alpha_t x_t \leq \sum_{t\in S} \alpha_t e_t + \sum_{t\in T} e_t - \sum_{t\in S} \alpha_t x_t$$

and therefore

$$\sum_{t\notin S} y_t + \sum_{t\in S} y_t + \sum_{t\in S} \alpha_t x_t \leq \sum_{t\in S} \alpha_t e_t + \sum_{t\in T} e_t$$

that can be rewritten as

$$(\star) \quad \sum_{t\notin S} y_t + \sum_{t\in S} (y_t + \alpha_t x_t) \leq \sum_{t\in S} (1+\alpha_t) e_t + \sum_{t\notin S} e_t.$$

Up to a normalization factor, the function β equals to 1 outside S and to $1+\alpha$ over S is a fuzzy coalition and we see now that it blocks the allocation x so that we have proved

$$x \in \mathcal{C}^{\mathrm{A}} \Rightarrow x \in \bigcap_{\alpha\in[0,1]^T} \mathcal{ND}_{e(\alpha,x)}.$$

Fix $\varepsilon \in]0,1[$ such that $1-\alpha_t < \varepsilon$ and also (by (**A1**, 4.)) $\varepsilon y_t \in P_t(x_t)$, $\forall t \in S$. Observe the following

$$y_t + \alpha_t x_t = \varepsilon y_t + \alpha_t x_t + (1-\varepsilon) y_t = \varepsilon y_t + \alpha_t z_t$$

once we set $x_t + \frac{1-\varepsilon}{\alpha_t} y_t =: z_t \in P_t(x_t)$ because of assumption (**A1**, 3.). Also note that

$$y_t + \alpha_t x_t = \varepsilon y_t + \alpha_t z_t = (1+\alpha_t)\left(\frac{1}{1+\alpha_t}\varepsilon y_t + \frac{\alpha_t}{1+\alpha_t} z_t\right)$$

and that $\left(\frac{1}{1+\alpha_t}\varepsilon y_t + \frac{\alpha_t}{1+\alpha_t} z_t\right) =: w_t \in P_t(x_t)$ by assumption (**A1**, 2.). It is now clear by $(\star)$ that the fuzzy coalition β blocks via the assignment w that outside S coincide with y. □

As consequence of Theorem 4.2, since personalized equilibria are in the Aubin core, we obtain the first inclusion.

Proposition 4.3. *Assume convexity (**A1**, 2.), strict monotonicity (**A1**, 3.) and continuity (**A1**, 4.) of preferences. Take x to be a personalized equilibrium. Then x is robustly efficient, i.e.*

$$\mathrm{p}\mathcal{E} \subseteq \mathcal{R}o\mathcal{E}f.$$

To fully characterize personalized equilibria as robustly efficient allocations, we introduce the notion of rational allocations, a class of allocations isolated by [9]. Some notation helpful to define them now follows. By $\mathcal{L}$ we mean the class of functions $x : T \to L$ such that all values sum no more than e, the total initial endowment. Clearly such set is nonempty, closed and convex and contains $\mathcal{A}$. By θ^t we denote the indicator function of the singleton $\{t\}$, therefore $\theta^s x := x + \theta^s(e_s - x_s)$ coincides, given the map x, with x everywhere on T but s, where the value is e_s, the initial endowment of agent s.

Now, given an allocation x, set

$$\mathcal{Z}(x) = \operatorname{co}(\mathcal{L} \cup \{\theta^t x : t \in T\}) \bigcap \Pi_{t\in T} X_t.$$

The above set is trivially convex and closed, naturally contains $\mathcal{A}$. A map $y \in \mathcal{Z}(x)$ means both that $y_t \in X_t$ for all t and that $y \in \operatorname{co}(\mathcal{L} \cup \{\theta^t x : t \in T\})$.

Definition 4.4. *An allocation x is said to be rational if $\mathcal{Z}(x) \cap \Pi_{t\in T} P_t(x_t) = \emptyset$.*

Naturally, the defining equality can be written as: $\operatorname{co}(\mathcal{L} \cup \{\theta^t x : t \in T\}) \cap \Pi_{t\in T} P_t(x_t) = \emptyset$. We may refer to rational allocations also as *rational equilibria* and denote the class of all of them by $\mathcal{RA}$.

Remark 4.5. *Notice that: $\mathcal{RA} \subseteq$ w$\mathcal{PO}$. Indeed $\mathcal{Z}(x)$ includes $\mathcal{A}$ and $\mathcal{A} \cap \Pi_{t\in T} P_t(x_t) = \emptyset$ is the weak optimality of an allocation x.*

We present below the main properties of rational allocations.

Proposition 4.6. ([9, Lemma 6.2]) *If we assume strict monotonicity (**A1**, 3.) and continuity (**A1**, 4.) of preferences and **(A5)**, then: $\mathcal{RA} \subseteq \mathcal{IR}$.*

Remark 4.5 joint with Theorem 3.15 says that

$$\mathcal{RA} \subseteq \mathrm{w}\,\mathcal{PO} \subseteq \bigcup_{p\in(L')^T} (\mathrm{p}\mathcal{VE}^p \cap \mathcal{A}^{(\max,\, p)})$$

of course under convexity, monotonicity of preferences and under properness **(A3)**. The further properties enjoyed by rational allocations with respect to weakly Pareto optimal, namely that disjoint from $\Pi_{t\in T} P_t(x_t)$ is not just $\mathcal{A}$ but the bigger set $\mathcal{Z}(x)$, are mirrored in what follows.

Theorem 4.7. *Assume preferences are convex-valued. Assume further that* **(A3)** *holds (the properness condition) and $x \in \mathcal{RA}$.*

1. *A continuous personalized price p can be found such that*

$$p_t \cdot x_t \leq p_t \cdot P_t(x_t),\ \forall t \in T.$$

2. *Assume further that preferences are strictly monotone, then the same p supports x as a personalized quasi equilibrium (i.e. $\mathcal{RA} \subseteq \mathrm{p}\mathcal{QE}$). Moreover, $x \in \mathcal{A}^{(\max,\, p)}$.*

PROOF: Let x be a rational allocation. As in the proof of Theorem 3.15, a separating $p = (p_t)_{t\in T}$, i.e. a family $p \in (L')^T$ such that

$$p \cdot \mathcal{Z}(x) \leq p \cdot (\Pi_{t\in T} \hat{P}_t(x_t))$$

is found. Since p separates a fortiori $\mathcal{A}$ from $(\Pi_{t\in T} \hat{P}_t(x_t))$ it remains only to prove condition $(iii)_b$ because all the rest was already proved in Theorem 3.15. For its proof we refer to [9, Lemma 6.3]. □

Corollary 4.8. *Assume preferences are convex-valued, strictly monotone and continuous, i.e. (***A1***, 2., 3. and 4.). Assume further that* **(A3)** *(the properness condition) and* **(A5)** *hold. Then:*

$$\mathcal{RA} \subseteq \bigcup_{p \in (L')^T} (\mathrm{p}\mathcal{E}^p \cap \mathcal{A}^{(\max, p)}).$$

PROOF: Let x be a rational allocation. By Theorem 4.7 $x \in (\mathrm{p}\mathcal{QE}^p \cap \mathcal{A}^{(\max, p)})$ and by an appeal to Lemma 3.12 it is possible to conclude.

□

Remark 4.9. *In [9, Emaple 7.7] an economy is presented in which all assumptions* **(Ai)** *but* **(A5)** *hold. For this economy a personalized quasi equilibrium can be exhibited (and therefore also a rational equilibrium) which is not individually rational (and therefore it is not a core allocation and a fortiori not a personalized equilibrium). It is worth noticing that the commodity space is 2-dimensional, and there are two agents with different consumption sets both non coinciding with the positive orthant.*

Theorem 4.10. *Assume preferences are irreflexive, convex-valued, strictly monotone and continuous, i.e. (***A1***, 1., 2., 3. and 4.). Then a feasible allocation x is rational if it is robustly efficient, i.e.*

$$\mathcal{R}o\mathcal{E}f \subseteq \mathcal{RA}.$$

PROOF: Assume that x is robustly efficient. If it is not rational, then take $z \in \mathcal{Z}(x) \cap \Pi_{t \in T} P_t(x_t)$. Since z is the convex combination of a point $u \in \mathcal{L}$ and of the vectors θ_x^t, $t \in T$, we can write

$$z = \alpha u + \sum_{t \in T} \beta_t \theta_x^t$$

where $\alpha + \sum_{t \in T} \beta_t = 1$ and $\alpha, \beta_t \geq 0$, for all $t \in T$. According to a standard argument for rational allocations (see for example the proof of [9, Lemma 7.3]), we derive:

$$\sum_{t \in T} z_t + \sum_{s \in J} \beta_s x_s \leq \sum_{t \in T} e_t + \sum_{s \in J} \beta_s e_s$$

once denoted by J the set of $s \in T$ for which β_s is different from zero. This follows from the inequalities

$$\sum_{t \in T} (z_t - e_t) = \alpha \sum_{t \in T} (u_t - e_t) + \sum_{s \in T} \sum_{t \in T} \beta_s ((\theta_x^s)_t - e_t) \leq \sum_{s \in T} \sum_{t \in T} \beta_s ((\theta_x^s)_t - e_t) \leq -\sum_{s \in T} \beta_s (x_s - e_s).$$

By continuity assumption on preferences, there exists $\lambda \in (0,1)$ such that $\lambda z_t \in P_t(x_t)$ for all $t \in T$. Then, by obvious calculation, observing that

$$\sum_{t \notin J} \frac{1}{\gamma + T} (1 - \lambda) z_t \geq 0,$$

we get

$$\sum_{t \in T} \frac{1}{\gamma + T} \lambda z_t + \sum_{s \in J} \frac{\beta_s}{\gamma + T} \left(x_s + \frac{1 - \lambda}{\beta_s} z_s \right) \leq \sum_{t \in T} \frac{1}{\gamma + T} e_t + \sum_{s \in J} \frac{\beta_s}{\gamma + T} e_s$$

where

$$\gamma = \sum_{t\in J} \beta_t \geq 0, \qquad \sum_{t\in T} \frac{1}{\gamma+T} + \sum_{t\in J} \frac{\beta_t}{\gamma+T} = \frac{T+\gamma}{\gamma+T} = 1 \qquad \text{and} \qquad \lambda z_t \in P_t(x_t).$$

Observe now that $x_s + \frac{1-\lambda}{\beta_s} z_s \in P_s(x_s)$ by monotonicity assumption, when $z_s \neq 0$. Otherwise, from $z_s = 0 \in P_s(x_s)$ it follows $x_s \neq 0$ and, by continuity, there would exist $\delta_s \in]0,1[$ such that $\delta_s x_s \in P_s(x_s)$. So we can replace x_s with such $\delta_s x_s$ still keeping the inequality. Formally,

$$\sum_{t\in T} \frac{1}{\gamma+T} \lambda z_t + \sum_{s\in J} \frac{\beta_s}{\gamma+T} w_s \leq \sum_{t\in T} \frac{1}{\gamma+T} e_t + \sum_{s\in J} \frac{\beta_s}{\gamma+T} e_s$$

where w_s is equal to $x_s + \frac{1-\lambda}{\beta_s} z_s$, when $z_s \neq 0$, it is equal to $\delta_s x_s$ otherwise.

Hence we can write

$$\sum_{t\in T} \left[a_t \overbrace{(s_t)}^{\in P_t(x_t)} + b_t \overbrace{(w_t)}^{\in P_t(x_t)} \right] \leq \sum_{t\in T} (a_t + b_t) e_t$$

where even though all $a_t \neq 0$ but not all b_t (for $t \notin J$), surely all $a_t + b_t =: \rho_t \neq 0$. Therefore

$$\sum_{t\in T} \rho_t \overbrace{\left[\frac{a_t}{\rho_t} s_t + \frac{b_t}{\rho_t} w_t \right]}^{:= y_t \in P_t(x_t)} \leq \sum_{t\in T} \rho_t e_t$$

and in $\sum_{t\in T} \rho_t y_t \leq \sum_{t\in T} \rho_t e_t$ we can assume all $\rho_t \in]0,1]$. Choose now by continuity $\varepsilon > 0$ such that all $\varepsilon y_t \in P_t(x_t)$. From

$$\sum_{t\in T} \rho_t y_t + \sum_{t\in T} (1-\rho_t) x_t \leq \sum_{t\in T} \rho_t e_t + \sum_{t\in T} (1-\rho_t) x_t$$

we derive

$$\sum_{t\in T} \rho_t \varepsilon y_t + \sum_{t\in T} (1-\rho_t) x_t + \sum_{t\in T} \rho_t (1-\varepsilon) y_t \leq \sum_{t\in T} \left[\rho_t e_t + (1-\rho_t) x_t \right],$$

and also

$$\sum_{t\in T} \rho_t \varepsilon y_t + \sum_{t\in S} (1-\rho_t) \left[x_t + \frac{\rho_t (1-\varepsilon)}{(1-\rho_t)} y_t \right] + \sum_{t\notin S} (1-\varepsilon) y_t \leq \sum_{t\in T} \left[\rho_t e_t + (1-\rho_t) x_t \right]$$

where S denotes the set of traders t for which ρ_t is not equal to one.

Define now $u_t = x_t + \dfrac{\rho_t(1-\varepsilon)}{(1-\rho_t)} y_t$ for those $t \in S$ such that y_t is not zero and $u_t = \delta_t x_t \in P(x_t)$, when $t \in S$ and $y_t = 0$. Then we can write

$$\sum_{t\notin S} \varepsilon y_t + \sum_{t\in S} \left[\rho_t \varepsilon y_t + (1-\rho_t) u_t \right] \leq \sum_{t\in T} \rho_t e_t + (1-\rho_t) x_t$$

that implies, by convexity assumption, that $x \notin \mathcal{R}o\mathcal{E}f$. □

Remark 4.11. *With reference to the assumptions on preferences, we saw that: under convexity, the sets* $\mathcal{C}^{\mathrm{A}}, \mathrm{s}\,\mathcal{C}^{\mathrm{A}}, \mathrm{s}\,\mathcal{C}^{\mathrm{E}}$ *do coincide and, by adding continuity, they also coincide with* $\mathcal{C}^{\mathrm{E}}$ *(Propositions 3.2, 3.3 and 3.5).*
As we have just seen in Theorem 4.2, then, we can even write that the previous set is contained in $\mathcal{R}o\mathcal{E}f$ *if we assume further strict monotonicity.*
Theorem 4.10 gives, with no extra conditions than irreflexivity, the inclusion of $\mathcal{R}o\mathcal{E}f$ *in* $\mathcal{RA}$*.*
We derive therefore the two conclusions that follows.
1 - With continuous, convex, strictly monotone irreflexive preferences and assuming properness, by means of an appeal to Theorem 4.7 and to [9, Lemma 6.4], we get:

$$\mathcal{C}^{\mathrm{A}} = \mathrm{s}\,\mathcal{C}^{\mathrm{A}} = \mathrm{s}\,\mathcal{C}^{\mathrm{E}} = \mathcal{C}^{\mathrm{E}} \subseteq \mathcal{R}o\mathcal{E}f \subseteq \mathcal{RA} = \mathrm{p}\,\mathcal{QE}$$

2 - With continuous, convex, strictly monotone irreflexive preferences, assuming properness and **(A5)***, by means of Corollary 4.8, since personalized equilibria are in the Aubin core, we get :*

$$\mathcal{C}^{\mathrm{A}} = \mathrm{s}\,\mathcal{C}^{\mathrm{A}} = \mathrm{s}\,\mathcal{C}^{\mathrm{E}} = \mathcal{C}^{\mathrm{E}} = \mathcal{R}o\mathcal{E}f = \mathcal{RA} = \mathrm{p}\mathcal{E} = \mathrm{p}\,\mathcal{QE}.$$

Note that the proof of Theorem 4.1 is now achieved.

4.2 Personalized equilibria and coalitions with full support

In this section we concentrate on the veto power of the grand coalition. It will be exercised allowing all agents to take part in the coalition with a (non-zero) share of their endowment. This means that we shall concentrate on the set Σ_f of coalitions with full support (i.e. the support is all of T) defining a core, namely the set of allocations that cannot be f-blocked by the grand coalition, denoted by $\mathcal{C}^{\mathrm{A}}_f$.

Naturally, $\mathcal{C}^{\mathrm{A}} \subseteq \mathcal{C}^{\mathrm{A}}_f$. Our aim is to find mild conditions under which the equality holds true. Then, we will derive a characterization of personalized equilibria in terms of the blocking power of the grand coalition.

Lemma 4.12. *Let us assume (***A1***, 4) and that the initial endowments* e_t*, for all agents* t*, are o-units (or internal points). Then, given an allocation* $x \notin \mathcal{C}^{\mathrm{A}}$*, a map* $\alpha : T \to [0, 1]$ *and an assignment* y *exist with:*
$S_\alpha \neq \emptyset$*,* $\sum_{t\in T} \alpha_t(e_t - y_t)$ *is an o-unit and, for all* $t \in S_\alpha, y_t$ *is an o-unit belonging to* $P_t(x_t)$*.*

PROOF: First let us show that α and y exist such that $S := S_\alpha(=$ support of $\alpha) \neq \emptyset$, $\sum_{t\in T} \alpha_t y_t \leq \sum_{t\in T} \alpha_t e_t$, and, for all $t \in S, y_t$ is an o-unit with $y_t \in P_t(x_t)$. Note that, for this, we must only prove that the assignment y can be chosen with the property that y_t is an o-unit $\forall t \in S$, the rest being simply the definition of the Aubin core.

Consider that $\varepsilon y_t + (1 - \varepsilon)e_t$ tends to y_t as ε tends to 1, so, by continuity assumption, we can find $\varepsilon \in]0, 1[$ with $\varepsilon y_t + (1 - \varepsilon)e_t \in P_t(x_t)$ for all $t \in S$. Naturally, the inequality $\sum_{t\in T} \alpha_t y_t \leq \sum_{t\in T} \alpha_t e_t$ can be written as $\sum_{t\in S} \frac{\alpha_t}{\varepsilon} (\varepsilon y_t) \leq \sum_{t\in S} \frac{\alpha_t}{\varepsilon}(\varepsilon e_t)$ and, consequently, as $\sum_{t\in S} \frac{\alpha_t}{\varepsilon} [\varepsilon y_t + (1 - \varepsilon)e_t] \leq \sum_{t\in S} \frac{\alpha_t}{\varepsilon} e_t$. Clearly $\varepsilon y_t + (1 - \varepsilon)e_t$ is an o-unit and we are done.

Now, to obtain that $\sum_{t\in T} \alpha_t(e_t - y_t)$ is an o-unit, assume $\sum_{t\in T} \alpha_t y_t \leq \sum_{t\in T} \alpha_t e_t$, and, for all $t \in S, y_t \in P_t(x_t)$, y_t o-unit. It is then enough to take a positive ε with $\varepsilon y_t \in$

$P_t(x_t)$ for all $t \in S$ to have that $\sum_{t\in S} \alpha_t y_t - \sum_{t\in S} \alpha_t(\varepsilon y_t) = \sum_{t\in S} \alpha_t y_t(1-\varepsilon)$ is an o-unit and therefore that even

$$\sum_{t\in S} \alpha_t y_t - \sum_{t\in S} \alpha_t(\varepsilon y_t) + \sum_{t\in T} \alpha_t(e_t - y_t) = \sum_{t\in T} \alpha_t(e_t - \varepsilon y_t)$$

is an o-unit.

□

Observe that in the above lemma, under the weaker assumption that all initial endowments are strictly positive vectors, then in the conclusion we can replace o-units by strictly positive vectors.

Theorem 4.13. *Let us assume (**A1**, 3. and 4.) and that the initial endowments e_t, for all agents t, are o-units (or internal points). Then, an allocation $x \notin \mathcal{C}^{\mathrm{A}}$, can be f-blocked by the grand coalition or, in other words, can be blocked by a fuzzy coalition with full support. Consequently,*

$$\mathcal{C}^{\mathrm{A}} = \mathcal{C}^{\mathrm{A}}_f.$$

PROOF: An appeal to above Lemma 4.12 gives a blocking fuzzy coalition α with a support S. Let the vector v be an o-unit and such that $v = \sum_{t\in S} \alpha_t e_t - \sum_{t\in S} \alpha_t y_t$. Assuming that the support S is not full, for any $t \in T \setminus S$, monotonicity says that $x_t + e_t \in P_t(x_t)$ and choose, according to the definition of internal point, $\lambda > 0$ with $v - \lambda \sum_{t\notin S} x_t \in L_+$. Now modify α and y replacing, for $t \notin S$, α_t by λ and y_t by $x_t + e_t$. The modified fuzzy coalition α has full support and blocks x:

$$\sum_{t\in S} \alpha_t y_t + \sum_{t\notin S} \lambda(x_t + e_t) = -v + \sum_{t\in S} \alpha_t e_t + \sum_{t\notin S} \lambda(x_t + e_t) =$$

$$\sum_{t\in S} \alpha_t e_t - v + (\lambda \sum_{t\notin S} x_t) + \sum_{t\notin S} \lambda e_t \leq \sum_{t\in S} \alpha_t e_t + \sum_{t\notin S} \lambda e_t.$$

□

Order units of the commodity space L are precisely the internal points of L_+. When L is also completely metrizable, then order units are precisely the interior points of L_+ ([9, Theorem 2.8]), and then for the validity of Theorem 4.13 we need to assume that L_+ has a non-empty interior. That non fuzzy core allocations can be f-blocked by the grand coalition even when the initial endowments are not necessarily internal points, is an achievement of next theorem. In this case, properness assumption is required and the role of the grand coalition is clarified also in connection with personalized equilibria.

Theorem 4.14. *Assume preferences are irreflexive, convex-valued, strictly monotone and continuous, i.e. (**A1**, 1., 2., 3. and 4.). Assume further that* **(A3)** *(the properness condition) and* **(A5)** *hold. Then, an allocation that cannot be f-blocked by the grand coalition or, in other words, cannot be blocked by a fuzzy coalition with full support, is a personalized equilibrium. Consequently,*

$$\mathcal{C}^{\mathrm{A}}_f = \mathrm{p}\mathcal{E}.$$

PROOF: Assume that x cannot be f-blocked by the grand coalition and that it is not rational. Following line by line the same proof of Theorem 4.10 we find that

$$\sum_{t\in T}\rho_t y_t \le \sum_{t\in T}\rho_t e_t$$

where we can assume all $\rho_t \in]0,1]$ and $y_t \in P_t(x_t)$, that means a contradiction. Since x is therefore rational, the conclusion now follows from Corollary 4.8. □

5 Applications

In this section we provide some applications of our results on the veto power of the grand coalition. Subsection 5.1 provides a game theoretic approach to equilibria supported by non-linear prices, showing a characterization of personalized equilibria as Nash equilibria of a game with only two players. Subsection 5.2 applies our main results to differential information economies, offering an interpretation of the weak fine core.

5.1 Personalized equilibria and two-player games

Our aim in this section is to show that personalized equilibria of the economy coincide with Nash equilibria of a two-player game G. This result represents a first attempt to provide a strategic interpretation of competitive equilibia supported by non-linear prices. The characterization will follow from Theorem 4.14 and requires the introduction of a suitable game G. In presenting the game, we adapt to our general framework the main ideas in [21]. Contrary to usual strategic interpretation of competitive equilibria, the approach proposed by [21] does not consider money and prices. The game associated to the market economy is played by two players regardless of the number of agents. It will be referred as a society game because, according to the interpretation of [21], the society plays the game in two different roles. The results of this section extend the main theorems in [21] to the case of infinite dimensional commodity spaces, allowing for non-linear prices supporting competitive allocations. Notice that when the personalized price become linear, for example if the commodity space is a vector lattice and the consumption sets coincide with the positive cone, the characterization result in [21] is extended to classical competitive equilibria.

There are two players: the strategy set for player 1 coincides with the set of allocations that are non zero in each coordinate, i.e.

$$S_1 = \{x \in \mathcal{A} \,:\, x_t \neq 0 \,\text{for all}\; t \in T\}$$

while the strategy set of player 2 is defined as

$$S_2 = \left\{(\alpha, y) \in \Sigma_f \times \Pi_{t\in T} X_t \,:\, \sum_{t\in T}\alpha_t y_t \le \sum_{t\in T}\alpha_t y_t\right\},$$

that is player 2 chooses a strategy in the set of allocations that are feasible in the Aubin sense, considering a strictly positive participation of each member of the society. Observe that S_1 and S_2 are non empty since they contain, respectively, the initial endowment allocation e and the pair $(\mathbf{1}, e)$, where $\mathbf{1}$ denotes the characteristic function of the whole set T of traders. The set of strategy profiles is $S = S_1 \times S_2$, hence a strategy profile is $s \equiv (x, \alpha, y) \in S$, where $x \in S_1$ is the strategy of player 1, $(\alpha, y) \in S_2$ is the strategy of player 2. Let us denote by $\Pi_1 : S \to I\!R$ and $\Pi_2 : S \to I\!R$ the payoff functions for players 1 and 2, respectively.

We say that the game (G, Π_1, Π_2) is a *society game* if it satisfies the following properties: for $s \equiv (x, \alpha, y) \in S$

- G.1 if $x = y$, then $\Pi_1(s) = \Pi_2(s) = 0$;
- G.2 $x_t \in P_t(y_t)$, for all $t \in T$ iff $\Pi_1(s) > 0$;
- G.3 $y_t \in P_t(x_t)$, for all $t \in T$ iff $\Pi_2(s) > 0$;
- G.4 if $s^{'} \equiv (z, \alpha, y) \in S$ and $z_t \in P_t(x_t)$ for all $t \in T$, then $\Pi_1(s^{'}) > \Pi_1(s)$.

From the definition of a society game, some immediate consequences follow.

Proposition 5.1. *Assume that the economy satisfies (***A1***, 1., 3., 6.) and let* (G, Π_1, Π_2) *be the associated society game. Then the following properties hold true:*

1. *for each* $s \in S$*, if* $\Pi_1(s) > 0$ *then* $\Pi_2(s) < 0$ *and conversely;*
2. *if* $s^* \equiv (x^*, \alpha^*, y^*)$ *is a Nash equilibrium for* G*, then* $x^* \in \mathrm{w}\mathcal{PO}$*;*
3. *if* $s \equiv (x, \alpha, x) \in S$ *and* $x \in \mathrm{w}\mathcal{PO}$*, then agent 1 cannot improve his payoff;*
4. *if* $x \in S_1$*, then* $x \notin \mathcal{C}_f^{\mathrm{A}}$ *iff* $\Pi_2(x, \alpha, y) > 0$*, for a strategy* $(\alpha, y) \in S_2$*.*

PROOF: The first item is a consequence of assumption (**A1**, 6.) and of the properties G.2 and G.3 of the society game.

To prove the second statement, observe that if in a Nash equilibrium $s^* \equiv (x^*, \alpha^*, y^*)$ the allocation $x^* \notin \mathrm{w}\mathcal{PO}$, then there would exists $z \in \mathcal{A}$ such that $z_t \in P_t(x_t^*)$ for all $t \in T$. Assumptions (**A1**, 1.) and (**A1**, 6.) ensure that, if for a $t \in T$ $z_t = 0$, then $x_t^* \neq 0$ and $x_t^* \notin P_t(0)$ contrary to (**A1**, 3.). Hence $z \in S_1$. Now from G.4, we have $\Pi_1(z, \alpha^*, y^*) > \Pi_1(s^*)$ and a contradiction.

Let $s \equiv (x, \alpha, x) \in S$ and $x \in \mathrm{w}\mathcal{PO}$, by G.1 the payoff of player 1 in s is equal to zero. Assume then that $\Pi(z, \alpha, x) > 0$ for an allocation $z \in S_1$. Then by $G.2$ we have a contradiction.

The last statement simply follows by definition of S_2 and property G.3. □

Hence a society game can be interpreted as a game in which the society plays in two different roles. As player 1, it chooses feasible allocations and tries to make Pareto improvements. As player 2, the society comes up with an alternative allocation trying to dominate the allocation proposed by 1. Theorem 5.1 tells us that player 1 has incentive to deviate whenever the selected strategy is not efficient. Player 2 has incentive to deviate whenever the strategy of player 1 can be dominated by the grand coalition. What we will see next is that in a Nash equilibrium the first player chooses a strategy that is efficient, while the second player forces the strategy of 1 to be competitive.

First we state the existence Theorem of a Nash equilibrium for a society game.

Theorem 5.2. *Assume that the economy satisfies (***A1***, 1., 2., 4., 5., 6.), (***A2***), (***A3***) and (***A5***). Let* (G, Π_1, Π_2) *be the associated society game. Then the set of Nash equilibria in pure strategies is non-empty.*

PROOF: By [9, Theorem 8.3] and Remark 4.11, there exists a personalized equilibrium allocation x^*. Clearly $s^* \equiv (x^*, 1, x^*)$ belongs to S[12]. Since x^* is in w$\mathcal{PO}$, from Proposition 5.1, it follows that player 1 cannot improve his payoff. On the other hand, by Theorem 4.14 , $x^* \in \mathcal{C}_f^{\mathrm{A}}$ and then from 4. of Proposition 5.1 the conclusion follows. □

Our last result characterizes personalized equilibria as Nash equilibria of a society game.

Theorem 5.3. *Assume that the economy satisfies (**A1**,1., 2., 3., 4. and 6.). Assume further that* **(A3)** *and* **(A5)** *hold. Let (G, Π_1, Π_2) be the associated society game.*

1. *If $s^* \equiv (x^*, \alpha^*, y^*) \in S$ is a Nash equilibrium with $\Pi_1(s^*) = \Pi_2(s^*) = 0$, then $x^* \in \mathrm{p}\mathcal{E}$;*
2. *if $x^* \in \mathrm{p}\mathcal{E}$, then any $s^* \equiv (x^*, \alpha^*, y^*) \in S$ with $\Pi_1(s^*) = \Pi_2(s^*) = 0$ is a Nash equilibrium.*

In particular, $x^* \in \mathrm{p}\mathcal{E}$ iff $(x^*, \mathbf{1}, x^*)$ is a Nash equilibrium for (G, Π_1, Π_2).

PROOF: To prove the first implication, assume that $x^* \notin \mathrm{p}\mathcal{E}$. Then by Theorem 4.14, $x^* \notin \mathcal{C}_f^{\mathrm{A}}$ and, by 4. of Proposition 5.1, $\Pi_2(x^*, \alpha, y) > 0 = \Pi_2(x^*, \alpha^*, y^*)$, for some $(\alpha, y) \in S_2$, that is a contradiction.

Conversely, assume that $x^* \in \mathrm{p}\mathcal{E}$, and that for $s^* \equiv (x^*, \alpha^*, y^*) \in S$ with $\Pi_1(s^*) = \Pi_2(s^*) = 0$, s^* is not a Nash equilibrium. Then it may be the case that for some $x \in S_1$, $\Pi_1(x, \alpha^*, y^*) > 0 = \Pi_1(s^*)$. But then by G.2, $x^* \notin$ w$\mathcal{PO}$ and a contradiction. Or it may be the case that $\Pi_2(x^*, \alpha, y) > 0 = \Pi_2(s^*)$ for some $(\alpha, y) \in S_2$. Then by G.3 $x^* \notin \mathcal{C}_f^{\mathrm{A}}$ and a contradiction follows from Theorem 4.14. □

Example 5.4. *An example of a society game associated to the economy will be now provided. Since the strategy sets of both players have been already fixed, we have only to define the payoff functions.*
Assume that for each trader $t \in T$ $X_t = L_+$ and that for $x \in L_+$

$$P_t(x) = \{y \in L_+ : u_t(y_t) > u_t(x_t)\}$$

for some continuous concave and strictly monotone utility function u_t. Define the payoffs of players 1 and 2 as follows

$$\Pi_1(x, \alpha, y) = min_{t \in T} \ [u_t(x_t) - u_t(y_t)]$$

$$\Pi_2(x, \alpha, y) = min_{t \in T} \, \alpha_t \, [u_t(y_t) - u_t(x_t)]$$

for each $(x, \alpha, y) \in S$. Then it is easy to show that the payoff functions define a society game (G, Π_1, Π_2) for which Proposition 5.1 holds true. Under properness assumption **(A3)**, *the existence of a Nash equilibrium for the game G is guaranteed by Theorem 5.2 (notice that assumption* **(A5)** *in this case is obviously true). Finally, following verbatim the proof of [21, Proposition 4.1], one can show that in each Nash equilibrium s^* of the game G it is true that $\Pi_1(s^*) = \Pi_2(s^*) = 0$. Hence Theorem 5.3 guarantees a one to one correspondence between Nash equilibria of the game G and personalized equilibria.*

[12]Notice that $x_t^* = 0$ for a $t \in T$ would imply, by monotonicity, that x^* can be blocked by t and this contradicts the fact that $x*$ belongs to $\mathcal{C}^{\mathrm{A}}$

5.2 Economies with differential information

We introduce for a Differential Information Economy (DIE, for short) the notion of *Aubin weak fine core* in line with the notion of weak fine core introduced by [24]. Our aim is to show that the Aubin weak fine core coincides with the *Aubin private core* (see [19] and [22]) of a suitable economy with symmetric information and therefore with the personalized equilibria of that economy. With respect to the analogous result proved by [15], we consider a more general ordered topological vector space of contingent commodities. Moreover, we emphasize here the use of the previous full support results in order to overcome the unavailability of Lyapunov's Theorem.

For simplicity we consider a DIE in which the exogenous uncertainty is represented by a probability space $(\Omega, \mathcal{F}, I\!P)$, the finite set Ω collecting all possible states of nature and $I\!P$ representing a strictly positive common prior. The private information of a trader $t \in T$ is represented by a sub-algebra $\mathcal{F}_t$ of $\mathcal{F}$ and it is assumed that $\mathcal{F} = \vee_{t\in T}\mathcal{F}_t$. We assume that the space of physical commodities is an ordered topological vector space B as described in Section 3 and that B_+ has a non empty interior. We shall denote by L the product space $L = B^\Omega$. Utility that a trader $t \in T$ derives from physical consumption is given by the real function $u_t(\omega, v)$, with $\omega \in \Omega$ and $v \in B_+$. Standard assumptions on u_t are that $u_t(\omega, \cdot)$ are continuous, concave and (strictly) increasing. The same properties are then inherited by the (ex-ante) expected utility of the random consumption utility $u_t(\omega, x(\omega))$. Such expected utility is denoted by $U_t(x)$, for each random bundle $x \in L_+$.

For each trader $t \in T$, we shall denote by X_t the set of random bundles that are measurable with respect to the private information of trader t, i.e.

$$X_t = \{x \in L_+ : x \text{ is measurable wrt } \mathcal{F}_t\},$$

a closed convex cone in L_+. We complete the description of our DIE by defining the initial endowment as a profile $(e_t)_{t\in T}$ such that $e_t \in X_t$.

Notice that by introducing the correspondence $x \in X_t \mapsto P_t(x) = \{y \in L_+ : U_t(y) > U_t(x)\} \cap X_t$, DIE is a special case of an economy as given in Section 3. Consequently, we see that Aubin core as given in Definition 3.1 is identical to the Aubin private core as defined in [19] and [22]. "Private" since in the blocking mechanism agents in a blocking coalition don not share their private information.

It is natural to say that a list $(p_t)_{t\in T} \in (L')^T$ of continuous personalized prices is *individually measurable* if each p_t is $\mathcal{F}_t$-measurable. The next notion is due to [9].

Definition 5.5. *An allocation x of DIE is said to be an individually measurable personalized equilibrium if there exists an individually measurable list of continuous prices $(p_t)_{t\in T} \in (L'_+)^T$ supporting x as a personalized equilibrium (see Definition 3.10).*

According to [9], in an individually measurable personalized equilibrium the auctioneer assigns to agents personalized prices in such a way that none of them may infer additional information from his individual price. The generalized price is the same for every trader and, since agents maximize their ex-ante utility functions subject to informational constraints, agents that are better informed will be in general better off.
Following [9] and results of subsection 4.2 we derive.

Proposition 5.6. *Assume that the total initial endowment $e = \sum_{t\in T} e_t$ is strictly positive and that a number $k > 0$ exists such that $e \leq ke_t$ for all $t \in T$. Then, the Aubin private core of DIE coincides with the set of individually measurable personalized equilibria.*

PROOF: Since $e \leq ke_t$ each e_t is decomposable (see Remark 3.11). Moreover each e_t belongs to the interior of L_+ and then the economy is proper with respect to the initial endowment allocation e_t.

We know that $p\mathcal{E} = \mathcal{C}^{\mathrm{A}}$ (see Remark 4.11, 2.) and, clearly, individually measurable personalized equilibria are in the Aubin core. We want to show the converse. So take $x \in p\mathcal{E} = \mathcal{C}^{\mathrm{A}}$ and the list of prices $p_t \in L'$ that support x as a personalized equilibrium. By definition, the conditional expectation $E[p_t|\mathcal{F}_t] =: q_t$[13] is $\mathcal{F}_t$-measurable and $p_t \cdot x_t = q_t \cdot x_t$. More generally $p_t \cdot X_t = q_t \cdot X_t$. What above is enough to prove that ψ_p is the same as ψ_q namely that x is an individually measurable personalized equilibrium with the generalized price ψ_q. □

A notion of core in which agents in a blocking coalition share private information will be introduced below. It will be in the spirit of the so called *weak fine core* and therefore called *Aubin weak fine core*. For the rest of the present section we denote by $(\mathcal{F}_i)_{i \in I}$ the finitely many distinct information partitions of Ω and we shall identify a partition with the information algebra it generates. The set T of agents can be naturally partitioned into finitely many subsets $T_i = \{t \in T : \mathcal{F}_t = \mathcal{F}_i\}$. Set, for any group S of agents, $I(S)$ to denote the subset of I made of those indices i for which $T_i \cap S \neq \emptyset$.

Definition 5.7. *An $x \in L_+^T$ is called an Aubin weak fine core allocation, we write then $x \in \mathcal{C}^{\mathrm{Awf}}$, if any x_t is measurable with respect to $\mathcal{F}$ and*

1. $\sum_{t \in T} x_t(\omega) \leq \sum_{t \in T} e_t(\omega)$, *for all state ω*
2. *a fuzzy coalition γ and $(y_t) \in L_+$ for any $t \in S_\gamma$ such that*
   - *(a) for any $t \in S_\gamma, y_t$ is $(\vee_{i \in I(S_\gamma)} \mathcal{F}_i)$-measurable*
   - *(b) $\sum_{t \in S_\gamma} \gamma_t y_t(\omega) \leq \sum_{t \in S_\gamma} \gamma_t e_t(\omega)$, for all state ω*
   - *(c) $U_t(y_t) > U_t(x_t)$, for all $t \in S_\gamma$.*

   do no exist.

Notice, in the above definition, the request of measurability of y_t with respect to the joint information of members of the coalition. If $x \in \mathcal{C}^{\mathrm{A}}$, the Aubin (private) core, then x_t is $\mathcal{F}_t$-measurable and therefore also $\mathcal{F}$-measurable as required in the definition above. However, a direct comparison between $\mathcal{C}^{\mathrm{A}}$ and $\mathcal{C}^{\mathrm{Awf}}$ is not possible since from the existence of a blocking y in the weak fine core sense, it does not follow that y blocks x in the private sense.

While private core is the appropriate core notion when agents have no access to any communication systems, the weak fine core is the appropriate notion of core when agents do have access to a communication system allowing them to fully share their information in a blocking coalition. We want to show now that the Aubin weak fine core of the DIE coincides with the Aubin (private) core of a suitable economy with symmetric information. The latter coincides with DIE except for the replacement of any $\mathcal{F}_t$ with $\mathcal{F}$. As a consequence, the weak fine core of DIE coincides with the set of personalized equilibria in the new economy. This result has been proved for large differential information economies with finitely many commodities by [15] with a proof that relies on the application of Vind's theorem (and therefore Lyapunov's).

[13] $q_t(\omega)$ is defined by $E[p_t|\mathcal{F}_t](\omega) = \frac{1}{\sharp A_t(\omega)} \sum_{\omega' \in A_t(\omega)} p(\omega')$, where $A_t(\omega)$ is the unique element of $\mathcal{F}_t$ containing ω

Theorem 5.8. *Assume that e_t is an interior point of L_+, for each $t \in T$. Then the Aubin weak fine core of* DIE *is the same as the Aubin private core of the economy in all equals to the* DIE *except for the assumption that all agents have the same information $\mathcal{F}_t := \mathcal{F}$.*

PROOF: By writing $\mathcal{F} = \{E_1, ..., E_k\}$, random bundles in the set $X = \{x \in B_+^\Omega : x \text{ is } \mathcal{F}-$ measurable$\}$ are identifiable with element of B_+^k. The ex-ante utilities U_t with utilities defined over B_+^k as

$$U_t(x_1, \ldots, x_k) = \sum_{s=1}^{k} \left(\sum_{\omega \in E_s} u_t(\omega, x_s) I\!P(\omega) \right).$$

Denote by $\star - \mathcal{C}^{\mathrm{A}}$ the Aubin core of the symmetric economy in all equals to DIE except for the assumption that all agents have the same information $\mathcal{F}$.

The inclusion $\star - \mathcal{C}^{\mathrm{A}} \subseteq \mathcal{C}^{\mathrm{Awf}}$ is obvious, so let us show the converse. For that purpose, consider the complete information economy where, for each trader $t \in T$, the commodity space is B_+^k, the utility is the U_t given above and the initial endowments e_t are reconsidered through representatives in B_+^k.

Let $x \in \mathcal{C}^{\mathrm{Awf}}$ and suppose $x \notin \star - \mathcal{C}^{\mathrm{A}}$. The latter circumstance entails that there are γ and for any $t \in S_\gamma$ a $y_t \in X$ such that $\sum_{t \in S_\gamma} \gamma_t y_t(\omega) \leq \sum_{t \in S_\gamma} \gamma_t e_t(\omega)$, for all ω, and $U_t(y_t) > U_t(x_t)$, for all $t \in S_\gamma$.

Modulo the identifications described above, we have, in B_+^k: $\sum_{t \in S_\gamma} \gamma_t y_t \leq \sum_{t \in S_\gamma} \gamma_t e_t$ and $U_t(y_t) > U_t(x_t)$, for all $t \in S_\gamma$.

As a consequence of Theorem 4.13, the non fuzzy core allocation x can be blocked by a fuzzy coalition with full support, i. e. we can find for any trader $t \in T$ a pair $(\alpha_t, z_t) \in]0, 1] \times L_+^k$ with $\sum_{t \in T} \alpha_t z_t \leq \sum_{t \in T} \alpha_t e_t$ and $U_t(z_t) > U_t(x_t)$, for all $t \in T$.

Going back to DIE, to see that our assumption $x \in \mathcal{C}^{\mathrm{Awf}}$ is violated by the fuzzy coalition α and assignment $(z_t)_{t \in T}$ is enough to observe that trivially $I(S_\alpha) = I$. □

Our final result guarantees that the weak fine core coincide with personalized equilibria. Hence, while the Aubin private core coincides with the set of personalized equilibria, and then it rewards the information advantage of a trader, in a weak fine core allocation, the information advantage is worthless.

Theorem 5.9. *Assume that the total initial endowment $e = \sum_{t \in T} e_t$ is interior to the positive cone and that a number $k > 0$ exists such that $e \leq k e_t$ for all $t \in T$. Then, the Aubin weak fine core of* DIE *coincides with individually measurable personalized equilibria of the economy in all equals to* DIE *except for the assumption that all agents have the same information $\mathcal{F}_t := \mathcal{F}$.*

PROOF: Let $x \in \mathcal{C}^{\mathrm{Awf}}$ and suppose x is not an individually measurable personalized equilibrium. By Theorem 4.14 and Proposition 5.6, $x \notin \star - \mathcal{C}^{\mathrm{A}}$. Then as in the proof of Theorem 5.8 we get a contradiction. □

6 Final remarks

Our final remarks deal with the possible extension of core equivalences presented in this paper to the case of non-convex economies. Again the central element is represented by rational allocations, even if slightly modified. This class will be related to strong Aubin core allocations. The use of the strong Aubin core to characterize competitive allocations in the presence of non-convex preferences is due to [23] and to [18] in Banach lattice commodity spaces.

A possible approach to cover the case in which assumption (**A1**, 2.) is not fulfilled follows.
Remind, from subsections 3.1 and 3.2 , that the inclusions $\mathrm{p}\mathcal{E} \subseteq \mathrm{s}\mathcal{C}^{\mathrm{A}} \subseteq \mathrm{s}\mathcal{C}^{\mathrm{E}}$ are valid in general. Now, the following holds true.

Proposition 6.1. *Under the assumptions (**A1**, 3. and 4.), any allocation $x \in \mathrm{s}\mathcal{C}^{\mathrm{E}}$ satisfies the condition*

$$(\diamond) \quad \mathcal{Z}(x) \cap \Pi_{t\in T} \operatorname{co} P_t(x_t) \;=\; \emptyset.$$

PROOF: Assume not. The same proof of [9, Lemma 7.3] in which $P_t(x_t)$ is replaced by $\operatorname{co}P_t(x_t)$, guarantees that co $(\bigcup_{t\in T}[(\operatorname{co}P_t(x_t)) - e_t]) \cap (-L_+) \neq \emptyset$. This clearly implies that $(\operatorname{co}\bigcup_{t\in T}[(P_t(x_t)) - e_t]) \cap (-L_+) \neq \emptyset$ and a contradiction. □

Naturally, an allocation x for which the condition $(\diamond)$ is true is a rational allocation and the above condition $(\diamond)$ can be considered as defining rational allocations of an economy equals to the original one except for the replacement of the sets $P_t(x_t)$ by $\operatorname{co}P_t(x_t)$.

To characterize personalized equilibria as strong Aubin core allocations, we consider the following stronger version of properness assumption:

(A3)$'$: There is an allocation v, with $v_t > 0$ for any $t \in T$, and a family of convex correspondences $(\hat{Q}_t)_{t\in T}$, each defined on X_t and with values in L, such that

$$t \in T,\; x \in X_t \;\Rightarrow \hat{Q}_t(x) \cap X_t \;=\; \operatorname{co} P_t(x) \;\text{ and }\; x + v_t \in \mathit{int}(\hat{Q}_t(x)).$$

Theorem 6.2. *Assume that the economy $\mathcal{E}$ (**A1**, 3. and 4.). Assume further that* **(A3)$'$** *and* **(A5)** *hold. Then $\mathrm{p}\mathcal{E} = \mathrm{s}\mathcal{C}^{\mathrm{A}} = \mathrm{s}\mathcal{C}^{\mathrm{E}}$.*

PROOF: It is enough to show that an allocation x for which $(\diamond)$ is true belongs to $\mathrm{p}\mathcal{E}$. Let x be such an allocation. Due to the above observation we just apply Corollary 4.8 replacing the sets $P_t(x_t)$ by $\operatorname{co}P_t(x_t)$. To conclude, just note that personalized quasi-equilibria and equilibria do not change when we modify the original economy replacing preferences by their convex hull. Indeed,

$$\psi_p \cdot P_t(x_t) \geq \psi_p \cdot x_t \Leftrightarrow \psi_p \cdot \operatorname{co} P_t(x_t) \geq \psi_p \cdot x_t$$

and

$$\psi_p \cdot P_t(x_t) > \psi_p \cdot x_t \Leftrightarrow \psi_p \cdot \operatorname{co} P_t(x_t) > \psi_p \cdot x_t$$

as it can be seen by using properties of ψ_p. □

A natural question arises whether a suitable properness condition required directly on the preferences $P_t(x)$ implies a condition like **(A3)$'$**. For example, if this is the case in the properness assumption formulated in [18] for non-convex preferences (the commodity space is a Banach lattice and prices are linear) or in the uniform monotonicity assumption for non-convex preferences required in [13] (the commodity space is an ordered topological vector space and prices are superadditive). Finally, for a study of non-linear prices in non-convex economies with the tools of variational analysis, we refer to [28].

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