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Approximating Security Values of MinSup Problems with Quasi-variational Inequality Constraints

M. Beatrice Lignola* and Jacqueline Morgan**

Abstract

We consider a two-stage model where a leader, according to its risk-averse proneness, solves a MinSup problem with constraints corresponding to the reaction sets of a follower and defined by the solutions of a quasi-variational inequality (i.e. a variational inequality having constraint sets depending on its own solutions) which appear in concrete economic models such as social and economic networks, financial derivative models, transportation network congestion and traffic equilibrium. In general the infimal value of a MinSup (or the maximal value of a MaxInf) problem with quasi-variational inequality constraints is not stable under perturbations in the sense that the sequence of optimal values for the perturbed problems may not converge to the optimal value of the original problem even in presence of nice data. Thus, we introduce different types of approximate values for this problem, we investigate their asymptotical behavior under perturbations and we emphasized the results concerning MinSup problems with variational inequality constraints as well results holding under stronger assumptions that can be more easily employed in applications.

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Table of contents

1 Introduction

Variational problems with contraints defined by solutions to other variational problems modelize numerous economic and engineering problems [35], [14], [8], [12]... Usually, in these models two levels can be distinguished: a lower level in which a variational problem is solved by a follower in reaction to every decision imposed by a leader which solves an optimization problem at an upper level. In this paper the leader is assumed to be a minimizer. When the set of solutions to the lower level (called *best responses* of the follower) is not a singleton, to choose a best decision for the leader can become problematic if he cannot predict the follower choice simply on the basis of his own rational behavior. According to the risk proneness of the leader, two possible extremes situations are:
(i) *the optimistic case*, when the leader assumes that the follower, in reaction to any decision of the leader, chooses, amongst its best responses, one which *minimizes* the objective of the leader;
(ii) *the pessimistic case*, when the leader assumes that the follower can choose, amongst its best responses, one which *maximizes* the objective of the leader.
So, according to its risk proneness, if $\mathcal{Q}(x)$ is the set of best responses to the leader decision of the follower and f is the objective function of the leader, then the leader minimizes the function $\inf_{u\in\mathcal{Q}(x)} f(x,u)$ in the optimistic case and the function $\sup_{u\in\mathcal{Q}(x)} f(x,u)$ in the pessimistic one. Therefore, the leader modelizes the upper level as a MinInf (risk-prone) problem in the first case and as a MinSup (risk-averse) problem in the second one.
When the lower level corresponds to an optimization problem, these two formulations configure, respectively, a *strong (optimistic)* and a *weak (pessimistic) Stackelberg* problem (also called bilevel optimization problem) (see, for example, [7], [10], [28]....)
When the lower level is defined more generally by the solutions set of a variational or a quasi-variational problem, the optimistic case has been extensively investigated from various point of view [40], [35], [21], [23], [10], [14].... More recently, in [26], the asymptotic behavior of the infimal values of **optimistic** bilevel programs with **variational** inequalities constraints under perturbations has been investigated.
On the contrary, in this paper, we consider a **pessimistic** two-stage model with **quasi-variational** inequality constraints. More precisely, we assume that (X,τ) is a Hausdorff topological space, $H\subseteq X$ is a nonempty closed set, $K\subseteq\mathbb{R}^h$ is a nonempty convex and closed set, A is a function from $H\times K$ to $\mathbb{R}^h$ and S is a set-valued map from $H\times K$ to K with nonempty values. Then, for every $x\in H$, we consider the following parametric quasi-variational inequality

$$(QVI)(x) \quad \text{find} \;\; u\in S(x,u) \;\; \text{such that} \;\; \langle A(x,u), u-w\rangle \;\le\; 0 \;\; \forall\, w\in S(x,u).$$

The solution map $\mathcal{Q}: x\in H\to\mathcal{Q}(x)$ associates to every $x\in H$ the set $\mathcal{Q}(x)$ of solutions to $(QVI)(x)$.
It is worth noting that $\mathcal{Q}(x)$ may be not a singleton even under very restrictive conditions on the function A ([4]).
Then, given the objective function of the leader $f: H\times K\to\mathbb{R}\cup\{-\infty\}$, the *MinSup (pessimistic bilevel) problem with quasi-variational inequality constraints*, (MS) for short, consists in finding $\hat{x}\in H$ such that

$$\sup_{u\in\mathcal{Q}(\hat{x})} f(\hat{x},u) \;=\; \min_{x\in H}\sup_{u\in\mathcal{Q}(x)} f(x,u)$$

and the corresponding infimal value, called the *security value* is

$$\omega \;=\; \inf_{x\in H}\sup_{u\in\mathcal{Q}(x)} f(x,u).$$

Differently from what concerns optimistic two-stage models, there are quite a few papers devoted to MinSup problems with quasi-variational problem constraints ([37], [38]). This is probably due to the intrinsic theoretical difficulties of the problems presented in both levels [20], [14], [11]... Indeed, referring to classical weak Stackelberg problems, it is known that they may fail to have a solution also in presence of regular data [7], even if some restricted classes of functions ensuring existence results have been determined in [34], [36] and [32]. Moreover, at the lower level, a quasi-variational inequality has to be solved, that is a problem in which a fixed-point problem is combined with a variational inequality over a set depending on the solution itself, and that amounts, therefore, to an implicit variational problem [4], [22]... Finally, note that their "bilevel" nature does not allow to get general convergence results for solutions and security values under perturbations of the data (see, for example, [32]). An attempt to overcome these difficulties consisted in defining appropriate regularized problems admitting solutions whose security values approach the initial security value under reasonable conditions. Investigations of regularizations for weak Stackelberg problems, parametric or not, have been presented in a sequential setting, [31] and [32], as well in a topological one [20], together with approximation methods (like Tykhonov, least-norm regularization, Molodtsov and interior penalty methods) [30], [29] and [28].

Aim of this paper is now to investigate the asymptotical behavior of the security values of the *pessimistic* model with *quasi-variational* inequality constraints under perturbations on the data, having in mind to obtain conditions of minimal character which guarantee the convergence of the security values of suitable *regularized* perturbed problems to the security value of the unperturbed problem. So, some useful tools of Variational and Set-valued Analysis ([3],[25],[39]...), necessary to reach results of minimal character involving possibly discontinuous data, will be recalled in Section 2.

More precisely, let $(A_n)_n$ be a sequence of functions from $H \times K$ to $\mathbb{R}^h$, $(S_n)_n$ be a sequence of set-valued maps from $H \times K$ to K and $(f_n)_n$ be a sequence of functions defined on $H \times K$ and valued in $\mathbb{R} \cup \{-\infty\}$. For each positive integer n, we denote by $\mathcal{Q}_n$ the map that associates to $x \in H$ the solutions set of the problem

$$(QVI)_n(x) \quad \text{find } u \in K \ : \ u \in S_n(x,u) \text{ and } \langle A_n(x,u), u-w \rangle \ \leq \ 0 \ \ \forall\, w \in S_n(x,u).$$

For each $n \in \mathbb{N}$, we denote by ω_n the security value for the corresponding problem $(MS)_n$

$$\omega_n \ = \ \inf_{x \in H} \sup_{u \in \mathcal{Q}_n(x)} \ f_n(x,u),$$

We show in Section 3 that the sequence of the exact security values ω_n may not converge to the exact security value ω even under "nice" assumptions on the data. So, we define approximate security values for MinSup problems with quasi-variational inequality constraints (without perturbations) by the security values of suitable regularized MinSup problems with quasi-variational inequality constraints.

More precisely, we assume that $\varepsilon = (\varepsilon_1, \varepsilon_2)$, $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ and we consider the following *approximate solutions* map

$$\mathcal{Q}_\varepsilon : x \in H \to \mathcal{Q}_\varepsilon(x) = \{u \in K : d(u, S(x,u)) \leq \varepsilon_2 \text{ and } \langle A(x,u), u-w \rangle \ \leq \ \varepsilon_1 \, \forall\, w \in S(x,u)\}\,. \tag{1}$$

Then, we formulate the following *regularized* MinSup problem

$$\text{find } \ \hat{x} \ \in H \quad \text{such that } \sup_{u \in \mathcal{Q}_\varepsilon(\hat{x})} f(\hat{x}, u) = \min_{x \in H} \sup_{u \in \mathcal{Q}_\varepsilon(x)} f(x,u),$$

whose corresponding *approximate security value* is

$$\omega_\varepsilon \ = \ \inf_{x \in H} \sup_{u \in \mathcal{Q}_\varepsilon(x)} f(x,u),$$

and we investigate the convergence of ω_ε to the exact security value ω as ε is converging to 0. Finally, in Section 4, we assume the presence of perturbations as defined above and we investigate the asymptotic behavior of the approximate security values $\omega_\varepsilon^n \ = \ \inf_{x \in H} \sup_{u \in \mathcal{Q}_\varepsilon^n(x)} f_n(x,u)$. The case where H and S are described by inequalities is analyzed and, in both sections 3 and 4, results concerning MinSup problems with variational inequality constraints are enlightened. We emphasize that such results can open a way for motivate the use of numerical approximations as discretizations and penalizations since they allow to define a general scheme for approaching the security value ω by the sequence ω_ε^n of security values of regularized perturbed problems.

2 Preliminaries

The following notions ([3], [21]) will be used in the paper. Let $(K_n)_n$ be a sequence of nonempty subsets of $\mathbb{R}^h$.

The Painlevé-Kuratowski upper and lower limits of the sequence $(K_n)_n$ are defined respectively by
• $z \in \limsup_n K_n$ if there exists a sequence $(z_k)_k$ converging to z such that $z_k \in K_{n_k}$ for a subsequence $(K_{n_k})_k$ of $(K_n)_n$ and for each $k \in \mathbb{N}$
• $z \in \liminf_n K_n$ if there exists a sequence $(z_n)_n$ converging to z such that $z_n \in K_n$ for n sufficiently large.
We recall that both these sets are closed and may be empty.

A function $h : H \to \mathbb{R} \cup \{-\infty\}$ is *coercive* on H if for every $t \in \mathbb{R}$ there exists a sequentially compact set $C_t \subseteq X$ such that

$$Lev_t\, h \ = \{x \in H \ : h(x) \le t\} \ \subseteq \ C_t.$$

A function $g : H \times K \to \mathbb{R} \cup \{-\infty\}$ is *coercive with respect to* u on K *uniformly with respect to* $x \in H$ (*coercive in* u *on* K for short) if for every $t \in \mathbb{R}$ there exists a compact set $Y_t \subseteq \mathbb{R}^h$ such that

$$(Lev_t\, g)\,(x) \ = \{u \in K \ : g(x,u) \le t\} \ \subseteq \ Y_t \text{ for every } x \in H.$$

A set-valued map F from X to K is *sequentially lower semicontinuous* over X, *lower semicontinuous* for short, if for every $x \in X$ and every sequence $(x_n)_n$ converging to x in X

$$F(x) \ \subseteq \ \liminf_n \ F(x_n).$$

A set-valued map F from X to K is *sequentially closed* over X, *closed* for short, if for every $x \in X$ and every sequence $(x_n)_n$ converging to x in X

$$\limsup_n \ F(x_n) \ \subseteq \ F(x).$$

A set-valued map F from X to K is *sequentially subcontinuous* over X, *subcontinuous* for short, if for every $x \in X$ and every sequence $(x_n)_n$ converging to x in X, every sequence $(u_n)_n$ such that

$u_n \in F(x_n)$, for every $n \in \mathbb{N}$, has a convergent subsequence;

A sequence $(F_n)_n$ of set-valued maps from X to K *lower converges* to F in X if for every x and every sequence $(x_n)_n$ converging to x in X

$$F(x) \subseteq \liminf_n F_n(x_n).$$

A sequence $(F_n)_n$ of set-valued maps from X to K *upper converges* to F in X if for every $x \in X$ and every sequence $(x_n)_n$ converging to x in X

$$\limsup_n F_n(x_n) \subseteq F(x).$$

A sequence $(T_n)_n$ of functions from K to $\mathbb{R}^h$:

$-$ is G^-*$-$converging* to T in K if for every $u \in K$ there exists a sequence $(u'_n)_n$ converging to u in K such that $\lim_n T_n(u'_n) = T(u)$, that is

$$graph\, T \subseteq \liminf_n \; graph\, T_n.$$

$-$ is *equi-coercive* on K if there exist a point $v_o \in K$ and, for every $t \in \mathbb{R}$, a compact set $Z_t \subseteq \mathbb{R}^h$ such that

$$\{u \in K \; : \langle T_n(u), u - v_o\rangle \leq t\} \subseteq Z_t \text{ for all } n \in \mathbb{N}.$$

A sequence of functions $(g_n)_n$, $g_n : H \times K \to \mathbb{R} \cup \{-\infty\}$:

$-$ *sequentially continuously converges* to a function g in $H \times K$, *c$-$converges* to g for short, if for every $(x, u) \in H \times K$ and every sequence $(x_n, u_n)_n$ converging to (x, u), in $H \times K$, one has $\lim_n g_n(x_n, u_n) = g(x, u)$;

$-$ is *equi-coercive* on $H \times K$ if for every $t \in \mathbb{R}$ there exists a sequentially compact set $W_t \subseteq X \times \mathbb{R}^h$ such that

$$Lev_t\, g_n = \{(x, u) \in H \times K \; : g_n(x, u) \leq t\} \subseteq W_t \text{ for all } n \in \mathbb{N}.$$

For examples that illustrate and compare the above concepts see [21].
The next lemma is a basic result for the next sections and can be proved as in [26].

Lemma 2.1 *Let $(F_n)_n$ be a sequence of set-valued maps from $H \times K$ to K.*
If $(F_n)_n$ lower converges to F in $H \times K$, then, for every $x \in H$, every $u \in K$, every sequence $(x_n, u_n)_n$ converging towards (x, u), in $H \times K$, one has

$$\limsup_n d(u_n, F_n(x_n, u_n)) \leq d(u, F(x, u)).$$

If $(F_n)_n$ upper converges to F in $H \times K$ and the following holds:
$\diamond$ given a convergent sequence $(x_n, u_n)_n$, $(x_n, u_n) \in H \times K$, every sequence $(w_n)_n$, such that $w_n \in F_n(x_n, u_n)$ for all $n \in \mathbb{N}$, has a convergent subsequence,
then, for every $x \in H$, every $u \in K$, every sequence $(x_n, u_n)_n$ converging towards (x, u), in $H \times K$, one has

$$d(u, F(x, u)) \leq \liminf_n d(u_n, F_n(x_n, u_n)).$$

3 Approximating security values

Assuming that A_n, S_n, f_n are perturbations of A, S and f respectively, for each positive integer n, as in the introduction, we denote by $\mathcal{Q}_n$ the map that associates to $x \in H$ the solutions set of the problem

$$(QVI)_n(x) \quad \text{find } u \in K \ : \ u \in S_n(x,u) \text{ and } \langle A_n(x,u), u-w\rangle \ \leq \ 0 \ \ \forall\, w \in S_n(x,u).$$

Throughout the paper, the sets of solutions to the lower level problems are assumed to be nonempty. Conditions ensuring the existence of solutions to quasi-variational inequalities, or to variational inequalities, in finite dimensional spaces can be found, for example, in [15] and in [13].
For each $n \in \mathbb{N}$, we denote by ω_n the security value for the corresponding perturbed problem $(MS)_n$

$$\omega_n \ = \ \inf_{x\in H} \sup_{u\in \mathcal{Q}_n(x)} f_n(x,u),$$

First, we show that the sequence of the perturbed exact security values ω_n may not converge to the exact value ω even under "nice" assumptions on the data.

Example 3.1 Assume that $a \in \mathbb{R}$, $X = H = [0,a]$, $h = 1$, $K = [0,+\infty[$, $S_n(x,u) = S(x,u) = K$, $A_n(x,u) = 1/n$ and $f_n(x,u) = u + x + 1/n$. The sequences $(A_n)_n$ and $(f_n)_n$ uniformly converge, and therefore also continuously converge, to the functions A and f defined, respectively, by: $A(x,u) = 0$ and $f(x,u) = u + x$. One easily checks that $\mathcal{Q}_n(x) = \{0\}$, $\mathcal{Q}(x) = [0,+\infty[$, so that $\omega_n = \inf\limits_{x\in[0,a]} (x + 1/n) = 1/n$, $\omega = +\infty$ and the sequence $(\omega_n)_n$ does not converge to ω.

Therefore, we assume that $\varepsilon = (\varepsilon_1, \varepsilon_2)$, $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ and in line with previous papers (see, for example, [29], [19], [37], [16], [25], [26]...) we consider the following *approximate solutions* map

$$\mathcal{Q}_\varepsilon : x \in H \to \mathcal{Q}_\varepsilon(x) = \{u \in K : d(u, S(x,u)) \leq \varepsilon_2 \text{ and } \langle A(x,u), u-w\rangle \ \leq \ \varepsilon_1 \ \forall\, w \in S(x,u)\} \quad (2)$$

Then, we formulate the following *regularized* MinSup problem

$$\text{find } \ \hat{x} \ \in H \quad \text{such that } \sup_{u\in \mathcal{Q}_\varepsilon(\hat{x})} f(\hat{x},\, u) = \min_{x\in H} \sup_{u\in \mathcal{Q}_\varepsilon(x)} f(x,u)$$

whose corresponding *approximate value* is

$$\omega_\varepsilon \ = \ \inf_{x\in H} \sup_{u\in \mathcal{Q}_\varepsilon(x)} f(x,u).$$

We show that ω_ε can be used to determine the security value ω under suitable conditions.

Proposition 3.1 *Assume that the following hold:*
L_1) *the set-valued map S is subcontinuous, lower semicontinuous and closed on $H \times K$;*
L_2) *the function A is continuous on $H \times K$;*
U_1) *the function $-f$ is coercive in u on K;*
U_2) *for every $x \in H$ there exists a sequence $(x_n)_n$ converging to x in H such that for every $u \in K$ and every sequence $(u_n)_n$ converging to u in K one has*

$$\limsup_n \ f(x_n, u_n) \ \leq \ f(x,u).$$

Then,

$$\omega = \lim_{\varepsilon\to 0} \omega_\varepsilon.$$

Proof
Since $\lim\limits_{\varepsilon\to 0}\, \omega_\varepsilon \;=\inf\limits_{\varepsilon>0}\, \omega_\varepsilon$ and $\omega \le \omega_\varepsilon$, it is sufficient to prove that

$$\inf_{\varepsilon>0}\, \omega_\varepsilon \;\le\; \omega.$$

Assume that this inequality is not true and let a be a real number such that

$$\omega \;<\; a \;<\; \inf_{\varepsilon>0}\, \omega_\varepsilon.$$

There exists a point $\bar{x}\in H$ such that

$$\sup_{u\in\mathcal{Q}(\bar{x})} f(\bar{x},u) \;<\; a. \tag{3}$$

So, $f(\bar{x},u)<a$ for every $u\in\mathcal{Q}(\bar{x})$. Let $(\bar{x}_n)_n$ be a sequence converging to $\bar{x}$ and satisfying condition $U_2)$, and let $(\varepsilon_n)_n \;=\; (\varepsilon_{1,n},\varepsilon_{2,n})_n$ be a sequence of pairs of positive real numbers decreasing to 0 such that $a \;+\; \varepsilon_{1,n} \;<\; \omega_{\varepsilon_n} \;\le\; \sup\limits_{u\in\mathcal{Q}_{\varepsilon_n}(\bar{x}_n)} f(\bar{x}_n,u)$ for every $n\in\mathbb{N}$. There exists a sequence $(\bar{u}_n)_n$ such that

$$\bar{u}_n\in\mathcal{Q}_{\varepsilon_n}(\bar{x}_n) \quad\text{and}\quad a \;+\; \varepsilon_{1,n} \;<\; f(\bar{x}_n,\bar{u}_n) \tag{4}$$

for every $n\in\mathbb{N}$. Assumption $U_1)$ implies that a subsequence $(\bar{u}_{n'})_{n'}$ of $(\bar{u}_n)_n$ converges to a point $\bar{u}\in K$ that must belong to $S(\bar{x},\bar{u})$ since $d(\bar{x},S(\bar{x},\bar{u}))\le \liminf\limits_{n'} d(\bar{x}_{n'},S(\bar{x}_{n'},\bar{u}_{n'}))\le \lim\limits_{n'} \varepsilon_{2,n'} \;=\; 0$ by Lemma 2.1.
Now, let $w\in S(\bar{x},\bar{u})$ and let $(w_{n'})_{n'}$ be a sequence converging to $w\in K$ such that $w_{n'}\in S(\bar{x}_{n'},\bar{u}_{n'})$ for n' sufficiently large. Since the function A is continuous one has

$$\langle A(\bar{x},\bar{u}),\bar{u}-w\rangle \;\;=\;\; \lim_{n'} \langle A(\bar{x}_{n'},\bar{u}_{n'}),\bar{u}_{n'}-w_{n'}\rangle \;\le\; \lim_{n'} \varepsilon_{1,n'} \;\;=\;\; 0.$$

Therefore, $\bar{u}\in\mathcal{Q}(\bar{x})$ and $f(\bar{x},\bar{u})<a$ by (3). However, conditions $U_2)$ and (3) imply that $a\le \limsup\limits_n f(\bar{x}_n,\bar{u}_n)\le f(\bar{x},\bar{u})$ and one has a contradiction. $\square$

Remark 3.1 Assumption $U_2)$ is satisfied if the function $f(x,\cdot)$ is upper semicontinuous (usc for short) on K for every $x\in H$, but the following example shows that these two conditions are not equivalent in general.

Example 3.2 Assume that $X=H=[0,+\infty[$, $h=1$, $K=[0,+\infty[$, $f(x,u)=u^{-x}$ when $x>0$, $f(0,u)=1$ if $u\in[0,1]$ and $f(0,u)=2$ if $u>1$. The function $f(0,\cdot)$ is not usc at $u=1$ since $f(0,1)=1<\limsup\limits_n f(0,u_n)=2$ for every sequence $(u_n)_n$, $u_n>1$, converging to 1. However, condition $U_2)$ is satisfied for $x=0$ because there exists the sequence $(x_n)_n=(1/n)_n$ such that for every $u\in K$ and every sequence $(u_n)_n$ converging to u one has: $\lim\limits_n f(1/n,u_n)\le 1\le f(0,u)$. In fact, $\lim\limits_n\, f_n(1/n,u_n)=\lim\limits_n\,(u_n)^{-1/n}$ is equal to 1 when u is positive and is equal to 0 when $u=0$.

Having in mind to approach the security value ω also in the presence of perturbations of the data, it is useful to introduce the *strict approximate solutions* map ([29], [19], [37])

$$\mathfrak{S}_\varepsilon : x\in H\to\mathfrak{S}_\varepsilon(x)=\{u\in K : d(u,S(x,u))<\varepsilon_2 \text{ and } \langle A(x,u),u-w\rangle \;<\; \varepsilon_1\,\forall\, w\in S(x,u)\} \tag{5}$$

and the corresponding MinSup problem

$$\text{find } \hat{x} \in H \text{ such that } \sup_{u \in \mathfrak{S}_\varepsilon(\hat{x})} f(\hat{x}, u) = \min_{x \in H} \sup_{u \in \mathfrak{S}_\varepsilon(x)} f(x, u)$$

whose *security value* is

$$\sigma_\varepsilon = \inf_{x \in H} \sup_{u \in \mathfrak{S}_\varepsilon(x)} f(x, u).$$

Since, for every $x \in H$, $\mathcal{Q}(x) \subseteq \mathfrak{S}_\varepsilon(x) \subseteq \mathcal{Q}_\varepsilon(x)$ one has

$$\omega \leq \sigma_\varepsilon \leq \omega_\varepsilon \tag{6}$$

and these inequalities imply that assumptions of Proposition 3.1 also guarantee that

$$\lim_{\varepsilon \to 0} \sigma_\varepsilon = \omega.$$

The following corollary is a simplified version of Proposition 3.1, easier to use in the applications.

Corollary 3.1 *Assume that the sets H and K are compact. If conditions $L_1)$, $L_2)$ and*
$U_2')$ for every $x \in H$, the function $f(x, \cdot)$ is usc on K,
hold, then

$$\omega = \lim_{\varepsilon \to 0} \omega_\varepsilon = \lim_{\varepsilon \to 0} \sigma_\varepsilon.$$

Variational inequality constraints case

If the map S does not depend on u, i.e. variational inequality constraints are considered at the lower level, the above approximation scheme leads to consider the approximate values

$$\nu_\varepsilon = \inf_{x \in H} \sup_{u \in \mathcal{V}_\varepsilon(x)} f(x, u) \qquad \tau_\varepsilon = \inf_{x \in H} \sup_{u \in \mathcal{S}_\varepsilon(x)} f(x, u)$$

where

$$\mathcal{V}_\varepsilon : x \in H \to \mathcal{V}_\varepsilon(x) = \{u \in K : d(u, S(x)) \leq \varepsilon_2 \text{ and } \langle A(x, u), u - w \rangle \leq \varepsilon_1 \, \forall \, w \in S(x)\}$$

$$\mathcal{S}_\varepsilon : x \in H \to \mathcal{S}_\varepsilon(x) = \{u \in K : d(u, S(x)) < \varepsilon_2 \text{ and } \langle A(x, u), u - w \rangle < \varepsilon_1 \, \forall \, w \in S(x)\}$$

and one has the following result:

Proposition 3.2 *Assume that the following hold:*
$L_1)$ the set-valued map S is subcontinuous, lower semicontinuous and closed on H;
$L_2)$ the function A is continuous on $H \times K$;
$U_2)$ for every $x \in H$ there exists a sequence $(x_n)_n$ converging to x in H such that for every $u \in K$ and every sequence $(u_n)_n$ converging to u in K one has

$$\limsup_n f(x_n, u_n) \leq f(x, u).$$

Then,

$$\nu = \lim_{\varepsilon \to 0} \nu_\varepsilon = \lim_{\varepsilon \to 0} \tau_\varepsilon.$$

Proof
As in Proposition 3.1, assume that there exists $a \in \mathbb{R}$ such that $\nu < a < \inf_{\varepsilon>0} \nu_\varepsilon$. Then, there exists $\bar{x} \in H$ such that $f(\bar{x}, u) < a$ for every $u \in \mathcal{V}(\bar{x})$, and a sequence $(\bar{x}_n)_n$ converging to $\bar{x}$ satisfying condition U_2). If $(\varepsilon_n)_n = (\varepsilon_{1,n}, \varepsilon_{2,n})_n$ is a sequence of pairs of positive real numbers decreasing to 0 such that $a + \varepsilon_{1,n} < \nu_{\varepsilon_n}$ for every $n \in \mathbb{N}$, there exists a sequence $(\bar{u}_n)_n$ such that

$$\bar{u}_n \in \mathcal{V}_{\varepsilon_n}(\bar{x}_n) \quad \text{and} \quad a + \varepsilon_{1,n} < f(\bar{x}_n, \bar{u}_n) \tag{7}$$

for every $n \in \mathbb{N}$. The map S being subcontinuous, the set $S(x_n)$ is compact for every n. So, from $d(\bar{u}_n, S(\bar{x}_n)) \leq \varepsilon_{2,n}$ one infers that there exists $z_n \in S(\bar{x}_n)$ such that

$$||\bar{u}_n - z_n|| = \min_{z \in S(\bar{x}_n)} ||u_n - z|| \leq \varepsilon_{2,n}.$$

A subsequence of $(z_n)_n$ must converge to a point $\bar{u} \in S(\bar{x})$ since S is closed and subcontinuous. Therefore, a subsequence of $(\bar{u}_n)_n$ converges to the same point $\bar{u}$ that can be proved to solve the variational inequality $(VI)(\bar{x})$, so that $\bar{u} \in \mathcal{V}(\bar{x})$ and $f(\bar{x}, \bar{u}) < a$. Then, conditions U_2) and (7) lead to a contradiction since $a \leq \limsup_n f(\bar{x}_n, \bar{u}_n) \leq f(\bar{x}, \bar{u}) < a$. □

Note that, in order to approximate the security value ν, assumption U_1) can be eliminated, so a "compactness" condition (that is: S is subcontinuous) is present only on the lower level problem.

Corollary 3.2 *Assume that the set K is compact. If conditions L_1), L_2), U_2') hold, then*

$$\nu = \lim_{\varepsilon \to 0} \nu_\varepsilon = \lim_{\varepsilon \to 0} \tau_\varepsilon.$$

4 Asymptotically approximating security values

Assuming that A_n, S_n, f_n are perturbations of A, S and f respectively, we define the following approximate solutions maps

$\mathcal{Q}_\varepsilon^n : x \in H \to \mathcal{Q}_\varepsilon^n(x) = \{u \in K \ : \ d(u, S_n(x, u)) \leq \varepsilon_2 \text{ and } \langle A_n(x, u), u - w\rangle \leq \varepsilon_1 \ \forall\, w \in S_n(x, u)\}$

$\mathfrak{S}_\varepsilon^n : x \in H \to \mathfrak{S}_\varepsilon^n(x) = \{u \in K \ : \ d(u_n, S_n(x, u)) < \varepsilon_2 \text{ and } \langle A_n(x, u), u - w\rangle < \varepsilon_1 \ \forall\, w \in S_n(x, u)\}$

and we consider the regularized perturbed MinSup problems with constraints described by the maps $\mathcal{Q}_\varepsilon^n$ and $\mathfrak{S}_\varepsilon^n$, whose security values are

$$\omega_\varepsilon^n = \inf_{x \in H} \sup_{u \in \mathcal{Q}_\varepsilon^n(x)} f_n(x, u) \qquad \sigma_\varepsilon^n = \inf_{x \in H} \sup_{u \in \mathfrak{S}_\varepsilon^n(x)} f_n(x, u)$$

In this section, as in the unperturbed case considered in Section 3, we wish to approximate the security value ω with the approximate security values ω_ε^n and/or σ_ε^n and we start by showing that the data of Example 3.1 guarantee that the sequences $(\omega_\varepsilon^n)_n$ and $(\sigma_\varepsilon^n)_n$ approach asymptotically the security value ω even if the exact security values sequence $(\omega_n)_n$ does not converge to ω.

Example 4.1 Assume that $a \in \mathbb{R}$, $X = H = [0, a]$, $h = 1$, $K = [0, +\infty[$, $S_n(x, u) = S(x, u) = K$, $A_n(x, u) = 1/n$, $f_n(x, u) = u + x + 1/n$, $A(x, u) = 0$ and $f(x, u) = u + x$. One easily checks that $\mathcal{Q}_\varepsilon^n(x) = [0, n\varepsilon]$, $\mathfrak{S}_\varepsilon^n(x) = [0, n\varepsilon[$, so that $\omega_\varepsilon^n = \sigma_\varepsilon^n = \inf_{x \in [0,a]} (x + 1/n + n\varepsilon) = 1/n + n\varepsilon$. Then, we have:

$$\lim_{\varepsilon \to 0} \lim_n \omega_\varepsilon^n = \lim_{\varepsilon \to 0} \lim_n \sigma_\varepsilon^n = \omega = +\infty.$$

Remark 4.1 In Example 4.1 one also has:

$$\lim_{n} \lim_{\varepsilon \to 0} \omega_{\varepsilon}^{n} = 0$$

so, it is clear that to define a satisfactory approximation scheme one has to study the behavior of $(\omega_{\varepsilon}^{n})_n$ and $(\sigma_{\varepsilon}^{n})_n$ first for n going to $+\infty$ and second for ε going to 0 and not the contrary.
Moreover, in line with classical methods in Variational Analysis ([1], [9], [39]), we approximate ω by the sequences $(\omega_{\varepsilon}^{n})_n$ and $(\sigma_{\varepsilon}^{n})_n$ separately from above and from below because this allows to individuate assumptions of minimal character on the upper level data (see [18],[17]).

We start by approximating ω from above.

Proposition 4.1 *Assume that the following hold:*
L_3) *the sequence $(S_n)_n$ upper and lower converges to S on $H \times K$;*
L_4) *for every $(x, u) \in H \times K$ and every sequence $(x_n, u_n)_n$ converging to (x, u) in $H \times K$, any sequence $(w_n)_n$, such that $w_n \in S_n(x_n, u_n)$, has a convergent subsequence;*
L_5) *for every $x \in H$ and every sequence $(x_n)_n$ converging to x in H, the sequence $(A_n(x_n, \cdot))_n$ $G^{-}-$converges to $A(x, \cdot)$ in K;*
U_3) *the sequence $(f_n)_n$ is equicoercive on $H \times K$;*
U_4) *for every $(x, u) \in H \times K$ and every sequence $(x_n, u_n)_n$ converging to (x, u) in $H \times K$ one has*

$$f(x, u) \ \leq \ \liminf_{n} \ f_n(x_n, u_n).$$

Then, we have

$$\sigma_{\varepsilon} \ \leq \ \liminf_{n} \sigma_{\varepsilon}^{n} \qquad \forall \ \varepsilon > 0 \tag{8}$$

and consequently

$$\omega \ \leq \ \liminf_{\varepsilon \to 0} \ \liminf_{n} \sigma_{\varepsilon}^{n}.$$

Proof
Assume that (8) fails to be true. There exist $\varepsilon > 0$ and a real number a such that

$$\liminf_{n} \sigma_{\varepsilon}^{n} \ < \ a \ < \ \sigma_{\varepsilon}.$$

Then, there exist an increasing sequence of positive integers $(n_k)_k$ and a sequence $(x_k)_k$, $x_k \in H$, such that

$$\sup_{u \in \mathfrak{S}_{\varepsilon}^{n_k}(x_k)} \ f_{n_k}(x_k, u) \ < \ a < \ \sigma_{\varepsilon} \quad \forall \ k \in \mathbb{N}. \tag{9}$$

By assumption U_3) we can assume that a subsequence of $(x_k)_k$, still denoted by $(x_k)_k$, converges to a point $\bar{x} \in H$.
Consider $\bar{u} \in \mathfrak{S}_{\varepsilon}(\bar{x})$ and a sequence $(u_k)_k$, whose existence is guaranteed by L_5), converging to $\bar{u}$ in K and such that

$$\lim_{k} \ A_{n_k}(x_k, u_k) \ = A(\bar{x}, \bar{u}). \tag{10}$$

Since $d(\bar{u}, S(\bar{x}, \bar{u})) < \varepsilon_2$, Lemma 2.1 ensures that there exists $k_o \in \mathbb{N}$ such that $d(u_k, S_k(x_k, u_k)) < \varepsilon_2$ for $k \geq k_o$ and we claim that $\langle A_{n_k}(x_k, u_k), u_k - w \rangle \ < \varepsilon_1$ for every $w \in S_k(x_k, u_k)$ and for k sufficiently large. Indeed, if it is not true, there exists an infinite set of positive integers $\mathbb{N}'$ and sequence $(w_{k'})_{k'}$ such that $w_{k'} \in S_{k'}(x_{k'}, u_{k'})$ and $\langle A_{n_{k'}}(x_{k'}, u_{k'}), u_{k'} - w_{k'} \rangle \ \geq \varepsilon_1$ for every $k' \in \mathbb{N}'$. By assumptions L_3) and L_4), a subsequence of $(w_{k'})_{k'}$ must converge towards a point $w \in S(\bar{x}, \bar{u})$ and, by (10), $\langle A(\bar{x}, \bar{u}), \bar{u} - w \rangle \geq \varepsilon_1$ which is in contradiction with $\bar{u} \in \mathfrak{S}_{\varepsilon}(\bar{x})$. Therefore, $u_k \in \mathfrak{S}_{\varepsilon}^{n_k}(x_k)$ for

k sufficiently large, so, conditions U_4) and (9) imply that $f(\bar{x},\bar{u}) \le \liminf_k f_{n_k}(x_k,u_k) \ \le \ a < \sigma_\varepsilon$. As $\bar{u}$ is an arbitrary point in $\mathfrak{S}_\varepsilon(\bar{x})$, we also have

$$\sup_{u\in\mathfrak{S}_\varepsilon(\bar{x})} f(\bar{x},u) \ \le \ a < \ \sigma_\varepsilon$$

and we get a contradiction. $\square$

The next result gives an approximation of ω from below.

Proposition 4.2 *Assume that the following hold:*
L_3) *the sequence $(S_n)_n$ upper and lower converges to S on $H\times K$;*
L_4) *for every $(x,u)\in H\times K$ and every sequence $(x_n,u_n)_n$ converging to (x,u) in $H\times K$, any sequence $(w_n)_n$, such that $w_n\in S_n(x_n,u_n)$, has a convergent subsequence;*
L_6) *the sequence $(A_n)_n$ $c-$converges to A on $H\times K$;*
L_7) *the sequence $(A_n)_n$ is equicoercive on $H\times K$;*
U_5) *for every $x\in H$ there exists a sequence $(x_n)_n$ converging to x such that for every $u\in K$ and every sequence $(u_n)_n$ converging to u in K one has*

$$\limsup_n f_n(x_n,u_n)\le f(x,u).$$

Then,

$$\limsup_n \omega_\varepsilon^n \le \omega_\varepsilon \qquad \forall\ \varepsilon>0 \tag{11}$$

and consequently

$$\limsup_{\varepsilon\to 0}\ \limsup_n \omega_\varepsilon^n \ \le\ \omega.$$

Proof
Assume that (11) fails to be true. Let a be a real number such that $\omega_\varepsilon < a < \limsup_n \omega_\varepsilon^n$, let $\bar{x}\in H$ be such that $\sup_{u\in\mathcal{Q}_\varepsilon(\bar{x})} f(\bar{x},u)<a$ and let $(x_n)_n$ be a sequence converging to $\bar{x}$ in H satisfying assumption U_5).
Being $a<\limsup_n \omega_\varepsilon^n \le \limsup_n \sup_{u\in\mathcal{Q}_\varepsilon^n(x_n)} f_n(x_n,u)$, there exist an increasing sequence $(n_k)_k$ of positive integers and a sequence $(u_{n_k})_k$ such that $u_{n_k}\in\mathcal{Q}_\varepsilon^{n_k}(x_{n_k})$ and $f_{n_k}(x_{n_k},u_{n_k})>a$ for every k.
Then, $d\,(u_{n_k},S_{n_k}(x_{n_k},u_{n_k}))\le\varepsilon_2$ and $\langle A_{n_k}(x_{n_k},u_{n_k}),u_{n_k}-w\rangle\le\varepsilon_1$ for every $w\in S_{n_k}(x_{n_k},u_{n_k})$ and every k. The sequence $(A_n)_n$ is equicoercive on $H\times K$, so, a subsequence of $(u_{n_k})_k$, still denoted by $(u_{n_k})_k$, must converge to a point $u_o\in K$. Assumptions L_3) and L_4) guarantee that Lemma 2.1 applies and one has $d(u_o,S(\bar{x},u_o))\le\liminf_n d\,(u_{n_k},S_{n_k}(x_{n_k},u_{n_k}))\le\varepsilon_2$.
Given a point $w\in S(\bar{x},u_o)$, by the lower convergence of $(S_n)_n$ to S, there exists a sequence $(w_k)_k$ converging to w such that $w_k\in S_{n_k}(x_{n_k},u_{n_k})$ for k sufficiently large. Since

$$\langle A_{n_k}(x_{n_k},u_{n_k}),u_{n_k}-w_k\rangle\le\varepsilon_1$$

and $(A_n)_n$ $c-$converges to A, one has $\langle A(\bar{x},u_o),u_o-w\rangle\le\varepsilon_1$, which implies that $u_o\in\mathcal{Q}_\varepsilon(\bar{x})$. Therefore, by U_5) we infer that $a\le\limsup_k f_{n_k}(x_{n_k},u_{n_k})\le f(\bar{x},u_o)<a$ which gives a contradiction. $\square$

Remark 4.2 Condition U_5), that amounts to a sort of convergence of the sequence $(f_n)_n$ towards f, has been introduced by Attouch and Wets in [2] to get the upper limit of the sets of MinSup points for the functions $(f_n)_n$ contained in the set of the Minsup points for the function f. It has been further employed by Loridan and Morgan ([32], [33]) in order to get convergence of solutions to weak Stackelberg problems in a sequential setting and by the authors for stability of constrained MinSup points [17] and of approximate solutions to weak Stackelberg problems [20]. As observed in [17], this convergence cannot be set in the framework of epiconvergence ([9], [1]) differently from the convergence considered in condition U_4) of Proposition 4.1. Also note that U_5) amounts to U_2) when $f_n = f$ for every $n \in \mathbb{N}$.

Remark 4.3 If we strengthen the assumptions on the set-valued maps S_n we can weaken the assumptions on the functions A_n. Namely, the following result, alternative to Proposition 4.2, holds.

Proposition 4.3 *Assume that the following hold:*
L_0) *the set-valued map S_n is convex-valued on $H \times K$ for every n;*
L_3) *the sequence $(S_n)_n$ upper and lower converges to S on $H \times K$;*
L_4) *for every $x \in H$ and every sequence $(x_n, u_n)_n$ converging to (x, u) in $H \times K$, any sequence $(w_n)_n$, such that $w_n \in S_n(x_n, u_n)$, has a convergent subsequence;*
L_6) *the sequence $(A_n)_n$ c-converges to A on $H \times K$;*
U_5) *for every $x \in H$ there exists a sequence $(x_n)_n$ converging to x such that for every $u \in K$ and every sequence $(u_n)_n$ converging to u in K one has*

$$\limsup_n f_n(x_n, u_n) \leq f(x, u).$$

Then,

$$\limsup_n \omega_\varepsilon^n \leq \omega_\varepsilon \qquad \forall\ \varepsilon > 0 \tag{12}$$

and consequently

$$\limsup_{\varepsilon \to 0} \limsup_n \omega_\varepsilon^n \leq \omega.$$

Proof
Assume that (12) fails to be true. Let a be a real number such that $\omega_\varepsilon < a < \limsup_n \omega_\varepsilon^n$, let $\bar{x} \in H$ be such that $\sup_{u \in \mathcal{Q}_\varepsilon(\bar{x})} f(\bar{x}, u) < a$ and let $(x_n)_n$ be a sequence converging to $\bar{x}$ in H satisfying assumption U_5). Being $a < \limsup_n \omega_\varepsilon^n \leq \limsup_n \sup_{u \in \mathcal{Q}_\varepsilon^n(x_n)} f_n(x_n, u)$, there exist an increasing sequence $(n_k)_k$ of positive integers and a sequence $(u_{n_k})_k$ such that $u_{n_k} \in \mathcal{Q}_\varepsilon^{n_k}(x_{n_k})$ and $f_{n_k}(x_{n_k}, u_{n_k}) > a$ for every k. Then, $d(u_{n_k}, S_{n_k}(x_{n_k}, u_{n_k})) \leq \varepsilon_2$ and $\langle A_{n_k}(x_{n_k}, u_{n_k}), u_{n_k} - w \rangle \leq \varepsilon_1$ for every $w \in S_{n_k}(x_{n_k}, u_{n_k})$ and every k. There exists a sequence $(z_k)_k$ such that $z_k \in S_{n_k}(x_{n_k}, u_{n_k})$ and $||z_k - u_{n_k}|| \leq \varepsilon_2$ for every k, so that, by condition L_4), a subsequence of $(u_{n_k})_k$, still denoted by $(u_{n_k})_k$, must converge to a point $u_o \in K$ and, by Lemma 2.1, $d(u_o, S(\bar{x}, u_o)) \leq \varepsilon_2$. Given a point $w \in S(\bar{x}, u_o)$, by the lower convergence of $(S_n)_n$ to S, there exists a sequence $(w_k)_k$ converging to w such that $w_k \in S_{n_k}(x_{n_k}, u_{n_k})$ for k sufficiently large. Since

$$\langle A_{n_k}(x_{n_k}, u_{n_k}), u_{n_k} - w_k \rangle \leq \varepsilon_1$$

and $(A_n)_n$ $c-$converges to A, $\langle A(\bar{x}, u_o), u_o - w\rangle \leq \varepsilon_1$ and we have $u_o \in \mathcal{Q}_\varepsilon(\bar{x})$. Therefore, by U_5) we infer that $a \leq \limsup_k f_{n_k}(x_{n_k}, u_{n_k}) \leq f(\bar{x}, u_o) < a$ which gives a contradiction. $\square$

Now, from propositions 3.1, 4.1 and 4.2 (resp. propositions 3.1, 4.1 and 4.3) we infer that the exact value ω can be globally approximated by both sequences $(\omega_\varepsilon^n)_n$ and $(\sigma_\varepsilon^n)_n$.

Proposition 4.4 *Assume that assumptions $L_1) - L_4)$, $L_6) - L_7)$, $U_1) - U_5)$ (resp. $L_0) - L_4)$, $L_6)$, $U_1) - U_5)$) hold.*
Then

$$\omega \;=\; \lim_{\varepsilon\to 0} \lim_n \omega_\varepsilon^n = \lim_{\varepsilon\to 0} \lim_n \sigma_\varepsilon^n. \tag{13}$$

Proof
Inequalities in (6) imply that

$$\sigma_\varepsilon^n \leq \omega_\varepsilon^n$$

for every $n \in \mathbb{N}$. Therefore, by propositions 4.1 and 4.2 (resp. 4.1 and 4.3) one gets

$$\sigma_\varepsilon \leq \liminf_n \sigma_\varepsilon^n \leq \limsup_n \sigma_\varepsilon^n \leq \limsup_n \omega_\varepsilon^n \leq \omega_\varepsilon$$

as well as

$$\sigma_\varepsilon \leq \liminf_n \sigma_\varepsilon^n \leq \liminf_n \omega_\varepsilon^n \leq \limsup_n \omega_\varepsilon^n \leq \omega_\varepsilon.$$

So, applying Proposition 3.1 one infers that (13) is true. $\square$

Corollary 4.1 *Assume that the sets H and K are compact. If conditions $L_1) - L_3)$, $L_6)$, $U_2')$, and the following hold*
$U_6)$ the sequence $(f_n)_n$ c-converges to f,
then

$$\omega \;=\; \lim_{\varepsilon\to 0} \lim_n \omega_\varepsilon^n = \lim_{\varepsilon\to 0} \lim_n \sigma_\varepsilon^n.$$

The following example shows a set of data that satisfy all assumptions of Proposition 4.4 and that does not satisfy all assumptions of Corollary 4.1.

Example 4.2 Let $X = H = [0,1]$, $h = 1$, $K = [0,1]$, $S_n(x,u) = S(x,u) = [0,u]$, $A_n(x,u) = 1/n$, $f_n(x,u) = (x - 1/n)^2 - (u - x)^2 - 1$ for $x \in [0, 2/n]$ and $f_n(x,u) = -(u-x)^2$ for $x \in]2/n, 1]$, $f(0,u) = -(u^2+1)$ and $f(x,u) = -(u-x)^2$ for $x \in]0,1]$. One easily checks that all assumptions of Proposition 4.4 are satisfied. However the sequence $(f_n)_n$ does not continuously converge to f since, for every u, the sequence $(2/\sqrt{n}, u)_n$ converges to $(0,u)$ but the sequence $(f_n(2/\sqrt{n}, u))_n = (-(u - 2/\sqrt{n})^2)_n$ does not converge to $f(0,u) = -u^2 - 1$.

Finally, we assume that the constraint set H and the constraint maps S and S_n are described by inequalities

$$H \;=\; \{x \in X \;:\; h_i(x) \leq 0,\; i = 1,..,m\}$$

$$S(x,u) \;=\; \{v \in K \;:\; s_j(x,u,v) \leq 0,\; j = 1,..,p\} = \bigcap_{j=1}^{p} \{v \in K \;:\; s_j(x,u,v) \leq 0\}$$

$$S_n(x,u) \;=\; \{v \in K \;:\; s_{j,n}(x,u,v) \le 0,\; j=1,..,p\} = \bigcap_{j=1}^{p} \{v \in K \;:\; s_{j,n}(x,u,v) \le 0\}\,,$$

where h_i, s_j and $s_{j,n}$ are real-valued functions defined, respectively, in X and in $H \times K \times K$ and we are interested in determining sufficient conditions on the data for assumptions $L_1)$, $L_3)$ and $L_4)$. It is obvious that the set H is closed whenever the functions h_i are lower semicontinuous and that H is compact if the functions h_i are coercive. However, getting continuity properties for the map S, as well convergence results for the sequence $(S_n)_n$, needs more specific arguments. This is essentially due to the lower semicontinuity and lower convergence properties, that are not preserved by intersections ([18], [17]). First results on continuity properties of univariate set-valued maps described by inequalities can be found in [6]. Extensions to wider classes of functions, as well convergence properties, are in ([18], [17]) and in [26]. Convergence results for sequences of bivariate set-valued maps can be proven by easy adaptations of Lemma 2.1 in [26] and Lemma 2.2 in [26].

Variational inequality constraints case

Here, we consider the approximate solutions maps

$$\mathcal{V}_\varepsilon^n \;: x \in H \to \mathcal{V}_\varepsilon^n(x) = \{u \in K \;:\; d(u, S_n(x)) \le \varepsilon_2 \;\text{ and } \langle A_n(x,u), u-w\rangle \le \varepsilon_1 \;\forall\; w \in S_n(x)\}$$

$$\mathcal{S}_\varepsilon^n \;: x \in H \to \mathcal{S}_\varepsilon^n(x) = \{u \in K \;:\; d(u_n, S_n(x)) < \varepsilon_2 \;\text{ and } \langle A_n(x,u), u-w\rangle < \varepsilon_1 \;\forall\; w \in S_n(x)\}$$

and the approximate security values

$$\nu_\varepsilon^n \;=\; \inf_{x\in H} \sup_{u \in \mathcal{V}_\varepsilon^n(x)} f_n(x,u) \qquad \tau_\varepsilon^n \;=\; \inf_{x\in H} \sup_{u \in \mathcal{S}_\varepsilon^n(x)} f_n(x,u).$$

The next results can be deduced from Propositions 3.2, 4.1 and 4.3 similarly to Proposition 4.4 and Corollary 4.1.

Proposition 4.5 *Assume that assumptions $L_0) - L_4)$, $L_6)$, $U_2) - U_5)$ hold. Then*

$$\nu \;=\; \lim_{\varepsilon\to 0} \lim_n \nu_\varepsilon^n \;=\; \lim_{\varepsilon\to 0} \lim_n \tau_\varepsilon^n. \tag{14}$$

Corollary 4.2 *Assume that the set H is compact. If conditions $L_0) - L_3)$, $L_6)$, $U_2')$, $U_6)$ hold, then*

$$\nu \;=\; \lim_{\varepsilon\to 0} \lim_n \nu_\varepsilon^n \;=\; \lim_{\varepsilon\to 0} \lim_n \tau_\varepsilon^n.$$

5 Concluding remarks

We have presented a way to get lower and upper approximations of the security value of a MinSup problem with (quasi)variational inequality constraints through the security values of perturbed MinSup problems. Namely, in order to globally approach the security values ω and ν (see (13) and (14)), one has to perturb the problem, to regularize such perturbations and to pass to the limit: first with respect to the perturbation parameter, then with respect to the approximation parameter. We emphasize that Example 4.1 shows that these two final steps cannot be exchanged, nor a unique limit can be considered taking a sequence $(\varepsilon_n)_n$ converging to 0, since for $\varepsilon_n = 1/n$ $\lim_n \omega_{\varepsilon_n}^n = 1$ while $\omega = +\infty$. Although assumptions of propositions 4.4 and 4.5 are rather strong,

propositions 4.1, 4.2 and 4.3 give approximations from below and from above of the security values ω and ν that can be used whenever one of the assumptions of propositions 4.4 and 4.5 is not satisfied.
The extension to infinite dimensional spaces would require a suitable ”equilibrium” between compactness and continuity properties, [20, p. 6], and will be further investigated in a forthcoming paper.
Finally, in our approximation scheme we do not need that the set-valued maps S and S_n are convex-valued except in Proposition 4.3 where the convexity of $S_n(x, u)$ allows to weaken the assumptions on the sequence $(A_n)_n$.
In our opinion, this theoretical approach can get an insight into the inherent difficulties of the considered problem and can explain the lack of non-heuristic numerical methods in the continuous case.

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