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On the Stability of Equilibria in Incomplete Information Games under Ambiguity

Giuseppe De Marco and Maria Romaniello

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University of Naples Federico II

University of Salerno

Bocconi University, Milan

CSEF - Centre for Studies in Economics and Finance
DEPARTMENT OF ECONOMICS – UNIVERSITY OF NAPLES
80126 NAPLES - ITALY
Tel. and fax +39 081 675372 – e-mail: csef@unisa.it

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Giuseppe De Marco* and Maria Romaniello**

Abstract

In this paper, we look at the (Kajii and Ui) *mixed equilibrium* notion, which has been recognized by previous literature as a natural solution concept for incomplete information games in which players have *multiple priors* on the space of payoff relevant states. We investigate the problem of stability of mixed equilibria with respect to perturbations on the sets of multiple priors. We find out that the (Painlevé-Kuratowski) convergence of posteriors ensures that stability holds; whereas, convergence of priors is not enough to obtain stability since it does not always implies convergence of posteriors when we consider updating rules (for multiple priors) based on the classical Bayesian approach.

Keywords: Incomplete information games, multiple priors, equilibrium stability.

* Università di Napoli Parthenope and CSEF. Address: Dipartimento di Studi Aziendali e Quantitativi, Università di Napoli Parthenope. Via Parisi 13, Napoli 80129, Italy. E-mail: giuseppe.demarco@uniparthenope.it.

** Seconda Università di Napoli. Address: Dipartimento di Economia, Seconda Università di Napoli. Corso Gran Priorato di Malta, Capua 81043, Italy. E-mail: maria.romaniello@unina2.it

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1 Introduction

Many results have been obtained in the literature showing that limits of equilibria of perturbed games are equilibria of the unperturbed game as perturbed games converge to the unperturbed one in the appropriate sense (see for instance [8] for the standard problem, [7], [12],[13],[18],[14], [15],[20],[21], [24] for recent results under relaxed or different assumptions). This *limit property* has interesting implications because it provides a useful theoretical tool for the comparative statics analysis on Nash equilibria and underlies the classical theory on Nash equilibrium refinements based on stability with respect to trembles (see [23] for a survey and a complete list of references).

Aim of this paper is to study the limit property for an equilibrium notion, called *mixed equilibrium*, defined by Kajii and Ui in [10] and therein recognized as one of the most natural solution concepts for a class of incomplete information games under *ambiguity*. In their work, Kajii and Ui partially follow the Harsanyi's approach as they assume that the source of uncertainty can be expressed by an underlying state space (i.e. *payoff relevant states*); but, at the same time, they deviate from the classical model of uncertainty, in which agents are endowed with a single *common prior* (probability distribution) over the state space, since they allow for *multiple priors* that are not necessarily common across agents[1]. In particular, the mixed equilibrium concept is an *interim equilibrium* concept in which each player chooses the best action for any realization of a private signal in equilibrium. As emphasized in [10], an *ex ante equilibrium* notion could also be defined in their framework in a natural way, but it turns out that a player with multiple priors tends to exhibit dynamic inconsistent behavior, meaning that a strategy that is optimal ex ante may specify actions that will be deemed inferior once private information is received[2]. However, the interim approach is more interesting in the applications while the ex ante one has only theoretical implications.

In this paper we look at the problem of stability of mixed equilibria with respect to perturbations on the sets of priors. The question is whether the limit of a sequence of mixed equilibria, corresponding to a family of sequences of perturbed sets of priors[3], is a mixed equilibrium of the game in which each set of priors is obtained as the *limit* of the corresponding sequence of sets of priors. A particular case is the one in which the sequences of sets of priors all converge to the same (common) single prior. The question in this case is whether the limit of the sequence of mixed equilibria is a Bayesian Nash equilibrium of the incomplete information game corresponding to the *limit common prior*.

The problem we address in this paper is also motivated by an interesting counterexample presented in [10] in which it is shown that limits of mixed equilibria of perturbed games are not necessarily equilibria in the unperturbed game, even when the perturbed sets of priors converge in the sense of Painlevé-Kuratowski to the sets of priors of the unperturbed game. In this work, we show that the limit property for mixed equilibria holds depending on how players update probabilities (evaluate posteriors) after the observation of the signal. More precisely, our limit theorem states that the (Painlevé-Kuratowski) convergence of posteriors ensures that the limit

[1]That is, each player is endowed with a set of priors (probability distributions) over the state space; these sets might different.

[2]In the Harsanyi's framework, the ex ante maximization of utility coincides with the interim notion because the expected utility model with Bayesian updating is dynamically consistent.

[3]That is, a sequence for each player.

property holds; whereas, a final counterexample[4] shows that the convergence of priors is not enough for the limit property since it does not always implies convergence of posteriors when we consider updating rules (for multiple priors) based on the classical Bayesian approach.

Related literature on ambiguous games

The recent literature on *ambiguous games* has investigated possible generalizations of the Nash equilibrium concept in presence of ambiguous beliefs, that is, beliefs which cannot be expressed as a single probability distribution over contingencies (see for instance [5], [16], [11], [6],[17], [10], [2] and [4] and references therein). In some of these papers, ambiguous beliefs concern the set of *payoff relevant states* (this is the classical ambiguity problem in the single agent case), in others, ambiguity arises specifically from the strategic interaction since it involves players' beliefs about their opponents' behavior. In [4], we propose a model in which ambiguity is summarized by *beliefs correspondences* which represent the exogenous ability of each player to put restrictions on beliefs over outcomes consistently with the strategy profile and introduce the corresponding equilibrium notion (called *equilibrium under ambiguous beliefs correspondences*). In particular, that model embodies the class of incomplete information games without private information and with multiple priors on the set of payoff relevant states, but it also includes some models in which ambiguity concerns beliefs over opponents' strategy choices.

The question whether the limit property extends to the equilibrium concepts in ambiguous games has not been completely clarified yet in the literature. On the one hand, the nature of the definition of equilibrium makes it reasonable to expect that the extension to ambiguous games holds. On the other hand, [10] shows that the extension fails in simple examples. [3] shows that the limit property holds for equilibria under ambiguous beliefs correspondences; key for this result result is the sequential convergence assumption imposed on the sequence of beliefs correspondences. The sensitivity of equilibrium to ambiguity is also discussed by [22] in a different context. In fact, an equilibrium notion for ambiguous games which relies on the Beweley unanimity rule is therein defined. This concept is then used to construct approximations of standard Bayesian equilibria.

2 The model

We consider a finite set of players $I = \{1, \dots n\}$. For every player i, $\Psi_i = \{\psi_i^1, \dots, \psi_i^{k(i)}\}$ is the (finite) pure action set of player i, $\Psi = \prod_{i \in I} \Psi_i$ and $\Psi_{-i} = \prod_{j \neq i} \Psi_j$. Denote with X_i the set of mixed actions of player i, that is, each action $x_i \in X_i$ is a vector $x_i = (x_i(\psi_i))_{\psi_i \in \Psi_i} \in \mathbb{R}_+^{k(i)}$ such that $\sum_{\psi_i \in \Psi_i} x_i(\psi_i) = 1$. Denote also with $X = \prod_{j=1}^n X_j$ and with $X_{-i} = \prod_{j \neq i} X_j$.

Let Θ be a finite set of payoff relevant states and denote with $\Delta(\Theta)$ the set all the probability distribution over Θ. Then, player i has a payoff function $f_i : \Psi \times \Theta \to \mathbb{R}$ and a set of priors $\mathscr{P}_i \subseteq \Delta(\Theta)$ over Θ.

We follow the Kajii and Ui's setup in [10]. The incompleteness of information is summarized by a random signal $\tau = (\tau_i)_{i \in I}$. When states $\theta \in \Theta$ occurs, player i privately observes a signal $\tau_i(\theta)$ and then chooses a pure strategy $\psi_i \in \Psi_i$. Denote with T_i the range of τ_i, i.e. $\tau_i : \Theta \to T_i$ for every player i. A strategy of player i is a function $\sigma_i : T_i \to X_i$; therefore, for every $t_i \in T_i$, $\sigma_i(t_i)$

[4]In this paper, we revise the Kajii and Ui counterexample emphasizing the role played by the assumptions of our limit theorem.

is a vector in X_i where each component $\sigma_i(\psi_i|t_i)$ denotes the probability of player i choosing action ψ_i when he observes t_i. The set of all the strategies σ_i of player i is denoted by S_i; moreover, $S_{-i} = \prod_{j\neq i} S_j$ and $S = \prod_{i=1}^{n} S_i$. Finally denote with $\sigma(\psi|\tau(\theta)) = \prod_{i=1}^{n} \sigma_i(\psi_i|\tau_i(\theta)))$ and $\sigma_{-i}(\psi_{-i}|\tau_{-i}(\theta)) = \prod_{j\neq i} \sigma_j(\psi_j|\tau_j(\theta))$.

Given $P \in \mathscr{P}_i$ and $t_i \in T_i$, denote with $P(\cdot|t_i) \in \Delta(\Theta)$ the conditional probabilities over Θ, that is

$$P(E|t_i) = \frac{P(\tau_i^{-1}(t_i) \cap E)}{P(\tau_i^{-1}(t_i))} \quad \forall E \subseteq \Theta.$$

Let

$$\mathscr{P}_i(t_i) = \{P(\cdot|t_i) \in \Delta(\Theta) \,|\, P \in \mathscr{P}_i\}$$

be the set of conditional probability distributions once t_i has been observed[5]. An updating rule $\Phi_i : T_i \to 2^{\Delta(\Theta)}$ gives, for every $t_i \in T_i$, a subset of conditional probabilities $\Phi_i(t_i) \subseteq \mathscr{P}_i(t_i)$. If $\Phi_i(t_i) = \mathscr{P}_i(t_i)$ for every $t_i \in T_i$ then Φ_i is called *Full Bayesian Updating Rule*.

After t_i is observed, player i uses posteriors in $\Phi_i(t_i)$ to evaluate his actions. The interim payoff to a randomized action $x_i \in X_i$, given $\sigma_{-i} \in S_{-i}$ and $Q_i \in \Phi_i(t_i)$ is

$$U_i(x_i, \sigma_{-i}|Q_i) = \sum_{\theta\in\Theta} \sum_{\psi_i\in\Psi_i} \sum_{\psi_{-i}\in\Psi_{-i}} x_i(\psi_i)\sigma_{-i}(\psi_{-i}|\tau_{-i}(\theta))Q_i(\theta)f_i(\psi_i, \psi_{-i}|\theta). \tag{1}$$

In line with the work of [17], ambiguity is solved by considering two different kind of (extreme) attitudes towards ambiguity: pessimism and optimism. In a multiple priors setting, the pessimistic attitude towards ambiguity is modelled by preferences which evaluate an ambiguous belief by the worst expected utility possible given the set of probability distributions ([9]). Similarly, an ambiguity-loving or optimistic agent evaluates beliefs by the most optimistic expected utility possible with the given set of probability distributions.

More precisely, a pessimistic player i has the (interim) pessimistic payoff defined by

$$V_i^P(x_i, \sigma_{-i}|t_i) = \max_{Q_i\in\Phi_i(t_i)} U_i(x_i, \sigma_{-i}|Q_i),$$

while, an optimistic player i has the (interim) optimistic payoff defined by

$$V_i^O(x_i, \sigma_{-i}|t_i) = \min_{Q_i\in\Phi_i(t_i)} U_i(x_i, \sigma_{-i}|Q_i), .$$

Assuming that players are partitioned in optimistic and pessimistic ones, that is, $I = O \cup P$ with $O \cap P = \emptyset$; then

$$\mathscr{G}^{O,P} = \{I; \Theta; (\mathscr{P}_i)_{i\in I}; (\Phi_i)_{i\in I}; (S_i)_{i\in I}; (V_i^O)_{i\in O}, (V_i^P)_{i\in P}\}.$$

is the corresponding game[6]. Then

[5]Degenerate probabilities are implicitly ruled out from $\mathscr{P}_i(t_i)$ in this formulation. That is, if $P \in \mathscr{P}_i$ is such that $P(\tau_i^{-1}(t_i)) = 0$, then $P(\cdot|t_i) \notin \Delta(\Theta)$ which implies that $P(\cdot|t_i) \notin \mathscr{P}_i(t_i)$.

[6]When a game $\mathscr{G}^{O,P}$ is considered, then it is implicitly assumed that its utility functions V_i^O and V_i^P are well posed, (i.e. $\max_{Q_i\in\Phi_i(t_i)} U_i(x_i, \sigma_{-i}|Q_i)$ and $\min_{Q_i\in\Phi_i(t_i)} U_i(x_i, \sigma_{-i}|Q_i)$ exist for every $x \in X$, $t_i \in T_i$ and $\sigma \in S$); obviously, this latter condition is guaranteed, for instance, when posteriors $\Phi_i(t_i)$ are closed and not empty sets for every $t_i \in T_i$.

Definition 2.1: A strategy profile $\sigma^* \in S$ is a Kajii and Ui mixed equilibrium for the game $\mathscr{G}^{O,P}$ if for every $\theta \in \Theta$ it follows that

$$V_i^O(\sigma_i^*(\tau_i(\theta)), \sigma_{-i}^*|\tau_i(\theta)) = \max_{x_i \in X_i} V_i^O(x_i, \sigma_{-i}^*|\tau_i(\theta)) \quad \forall i \in O; \tag{2}$$

$$V_i^P(\sigma_i^*(\tau_i(\theta)), \sigma_{-i}^*|\tau_i(\theta)) = \max_{x_i \in X_i} V_i^P(x_i, \sigma_{-i}^*|\tau_i(\theta)) \quad \forall i \in P. \tag{3}$$

3 The limit theorem

Given, for every player i, a sequence $\{\mathscr{P}_{i,\nu}\}_{\nu\in\mathbb{N}}$ of sets of priors over Θ and a sequence $\{\Phi_{i,\nu}\}_{\nu\in\mathbb{N}}$ of updating rules, i.e. $\Phi_{i,\nu} : T_i \to 2^{\Delta(\Theta)}$ and $\Phi_{i,\nu}(\tau_i(\theta)) \subseteq \mathscr{P}_{i,\nu}(\tau_i(\theta))$ for all $\theta \in \Theta$; consider the corresponding sequences of payoffs $\{V_{i,\nu}^O\}_{\nu\in\mathbb{N}}$ for all $i \in O$ and $\{V_{i,\nu}^P\}_{\nu\in\mathbb{N}}$ for all $i \in P$ and therefore the corresponding sequence of games[7] $\{\mathcal{G}_\nu^{O,P}\}_{\nu\in\mathbb{N}}$ where

$$\mathscr{G}_\nu^{O,P} = \{I; \Theta; (\mathscr{P}_{i,\nu})_{i\in I}; (\Phi_{i,\nu})_{i\in I}; (S_i)_{i\in I}; (V_{i,\nu}^O)_{i\in O}, (V_{i,\nu}^P)_{i\in P}\}. \tag{4}$$

Recall that (see for instance [1] or [19]):

Definition 3.1: Given a sequence of sets $\{B_\nu\}_{\nu\in\mathbb{N}}$ with $B_\nu \subset \mathbb{R}^s$ for all $\nu \in \mathbb{N}$, then

$$\operatorname*{Lim\,inf}_{\nu\to\infty} B_\nu = \{x \in \mathbb{R}^s \mid \forall\varepsilon > 0\ \exists\overline{\nu}\ s.t.\ for \nu \geq \overline{\nu}\ S(x,\varepsilon) \cap B_\nu \neq \emptyset\},$$
$$\operatorname*{Lim\,sup}_{\nu\to\infty} B_\nu = \{x \in \mathbb{R}^s \mid \forall\varepsilon > 0\ \forall\overline{\nu} \in \mathbb{N}\ \exists\nu \geq \overline{\nu}\ s.t.\ S(x,\varepsilon) \cap B_\nu \neq \emptyset\}.$$

where $S(x,\varepsilon)$ is the ball in $\mathbb{R}^s$ with center x and radius ε

Now we can state the limit theorem:

Theorem 3.2: *Given the game $\mathscr{G}^{O,P}$ corresponding to the sets of priors $\mathscr{P}_i$ and the updating rules Φ_i with $i = 1, \ldots, n$. Assume that $\{\mathscr{G}_\nu^{O,P}\}_{\nu\in\mathbb{N}}$ is a sequence of games defined by (4) such that, for every player i and every $\theta \in \Theta$*

$$\operatorname*{Lim\,inf}_{\nu\to\infty} \Phi_{i,\nu}(\tau_i(\theta)) = \operatorname*{Lim\,sup}_{\nu\to\infty} \Phi_{i,\nu}(\tau_i(\theta)) = \Phi_i(\tau_i(\theta)). \tag{5}$$

Let $\{\sigma_\nu^*\}_{\nu\in\mathbb{N}}$ be a sequence of strategy profiles such that each σ_ν^* is a mixed equilibrium of $\mathscr{G}_\nu^{O,P}$. If $\{\sigma_\nu^*\}_{\nu\in\mathbb{N}}$ converges to σ^*, (i.e. $\sigma_{i,\nu}^*(\psi_i|\tau_i(\theta)) \to \sigma_{i,\nu}^*(\psi_i|\tau_i(\theta))$ as $\nu \to \infty$, for every i, ψ_i, θ), then, σ^* is a mixed equilibrium of $\mathscr{G}^{O,P}$.

Proof. Let $\{(x_{1,\nu}, \ldots, x_{n,\nu})\}_{\nu\in\mathbb{N}} \subset X$ be a sequence of mixed actions profiles converging to $(x_1, \ldots, x_n) \in X$ and $\{(\sigma_{1,\nu}, \ldots, \sigma_{n,\nu})\}_{\nu\in\mathbb{N}} \subset S$ be a sequence of strategy profiles converging to $(\sigma_1, \ldots, \sigma_n) \in S$. Fixed $\overline{\theta} \in \Theta$, by definition, we recall that

$$V_{i,\nu}^O(x_i, \sigma_{-i,\nu}|\tau_i(\overline{\theta})) = \max_{Q_i \in \Phi_{i,\nu}(\tau_i(\overline{\theta}))} U_i(x_i, \sigma_{-i,\nu}|Q_i) \tag{6}$$

$$V_{i,\nu}^P(x_i, \sigma_{-i,\nu}|\tau_i(\overline{\theta})) = \min_{Q_i \in \Phi_{i,\nu}(\tau_i(\overline{\theta}))} U_i(x_i, \sigma_{-i,\nu}|Q_i), \tag{7}$$

$$V_i^O(x_i, \sigma_{-i}|\tau_i(\overline{\theta})) = \max_{Q_i \in \Phi_i(\tau_i(\overline{\theta}))} U_i(x_i, \sigma_{-i}|Q_i), \tag{8}$$

$$V_i^P(x_i, \sigma_{-i}|\tau_i(\overline{\theta})) = \min_{Q_i \in \Phi_i(\tau_i(\overline{\theta}))} U_i(x_i, \sigma_{-i}|Q_i), \tag{9}$$

[7] Again, it is implicitly assumed that the utility functions $V_{i,\nu}^O$ and $V_{i,\nu}^P$ are well posed along the sequence, (i.e. $\max_{Q_i\in\Phi_{i,\nu}(t_i)} U_i(x_i, \sigma_{-i}|Q_i)$ and $\min_{Q_i\in\Phi_{i,\nu}(t_i)} U_i(x_i, \sigma_{-i}|Q_i)$ exist for every $x \in X$, $\sigma \in S$, $t_i \in T_i$ and $\nu \in \mathbb{N}$).

First, we prove that the following conditions hold

$$\lim_{\nu\to\infty} V^O_{i,\nu}(x_{i,\nu},\sigma_{-i,\nu}|\tau_i(\overline{\theta})) = V^O_i(x_i,\sigma_{-i}|\tau_i(\overline{\theta})) \tag{10}$$

$$\lim_{\nu\to\infty} V^P_{i,\nu}(x_{i,\nu},\sigma_{-i,\nu}|\tau_i(\overline{\theta})) = V^P_i(x_i,\sigma_{-i}|\tau_i(\overline{\theta})) \tag{11}$$

In fact, for every ν, $V^O_{i,\nu}$ and $V^P_{i,\nu}$ are well posed by definition so there exist $Q'_{\overline{\theta},\nu}$ and $Q''_{\overline{\theta},\nu}$ in $\Phi_{i,\nu}(\tau_i(\overline{\theta}))$ such that

$$V^O_{i,\nu}(x_{i,\nu},\sigma_{-i,\nu}|\tau_i(\overline{\theta})) = U_i(x_{i,\nu},\sigma_{-i,\nu}|Q'_{\overline{\theta},\nu}), \tag{12}$$

$$V^P_{i,\nu}(x_{i,\nu},\sigma_{-i,\nu}|\tau_i(\overline{\theta})) = U_i(x_{i,\nu},\sigma_{-i,\nu}|Q''_{\overline{\theta},\nu}). \tag{13}$$

For every converging subsequence

$$\left\{V^O_{i,k}(x_{i,k},\sigma_{-i,k}|\tau_i(\overline{\theta}))\right\}_{k\in\mathbb{N}} \subseteq \left\{V^O_{i,\nu}(x_{i,\nu},\sigma_{-i,\nu}|\tau_i(\overline{\theta}))\right\}_{\nu\in\mathbb{N}}$$

there exists a corresponding subsequence $\{Q'_{\overline{\theta},k}\}_{k\in\mathbb{N}} \subset \{Q'_{\overline{\theta},\nu}\}_{\nu\in\mathbb{N}}$ such that $V^O_{i,k}(x_{i,k},\sigma_{-i,k}|\tau_i(\overline{\theta})) = U_i(x_{i,k},\sigma_{-i,k}|Q'_{\overline{\theta},k})$ for every k. Let $\{Q'_{\overline{\theta},h}\}_{h\in\mathbb{N}}$ be a converging subsequence of $\{Q'_{\overline{\theta},k}\}_{k\in\mathbb{N}}$, with $Q'_{\overline{\theta},h} \to Q'_{\overline{\theta}}$. From the definition (1) of the utility U_i it immediately follows that

$$\lim_{h\to\infty} U_i(x_{i,h},\sigma_{-i,h}|Q'_{\overline{\theta},h}) = U_i(x_i,\sigma_{-i}|Q'_{\overline{\theta}}). \tag{14}$$

From the assumptions it follows that $\operatorname*{Lim\,sup}_{\nu\to\infty} \Phi_{i,\nu}(\tau_i(\overline{\theta})) \subseteq \Phi_i(\tau_i(\overline{\theta}))$. So, the limit $Q'_{\overline{\theta}}$ belongs to $\Phi_i(\tau_i(\overline{\theta}))$ which implies that

$$U_i(x_i,\sigma_{-i}|Q'_{\overline{\theta}}) \leq V^O_i(x_i,\sigma_{-i}|\tau_i(\overline{\theta})).$$

Since the sequence $\left\{V^O_{i,k}(x_{i,k},\sigma_{-i,k}|\tau_i(\overline{\theta}))\right\}_{k\in\mathbb{N}}$ converges, then (12) implies that the sequence $\left\{U_i(x_{i,k},\sigma_{-i,k}|Q'_{\overline{\theta},k})\right\}_{k\in\mathbb{N}}$ converges and, from (14), its limit is $U_i(x_i,\sigma_{-i}|Q'_{\overline{\theta}})$. Summarizing, we get the following

$$\lim_{k\to\infty} V^O_{i,k}(x_{i,k},\sigma_{-i,k}|\tau_i(\overline{\theta})) =$$
$$\lim_{k\to\infty} U_i(x_{i,k},\sigma_{-i,k}|Q'_{\overline{\theta},k}) = U_i(x_i,\sigma_{-i}|Q'_{\overline{\theta}}) \leq V^O_i(x_i,\sigma_{-i}|\tau_i(\overline{\theta})). \tag{15}$$

Following the same steps, for every converging subsequence

$$\left\{V^P_{i,k}(x_{i,k},\sigma_{-i,k}|\tau_i(\overline{\theta}))\right\}_{k\in\mathbb{N}} \subseteq \left\{V^P_{i,\nu}(x_{i,\nu},\sigma_{-i,\nu}|\tau_i(\overline{\theta}))\right\}_{\nu\in\mathbb{N}}$$

it follows that

$$\lim_{k\to\infty} V^P_{i,k}(x_{i,k},\sigma_{-i,k}|\tau_i(\overline{\theta})) = \lim_{k\to\infty} U_i(x_{i,k},\sigma_{-i,k}|Q''_{\overline{\theta},k}) \geq V^P_i(x_i,\sigma_{-i}|\tau_i(\overline{\theta})) \tag{16}$$

Since (15) and (16) hold respectively for all converging subsequences $\left\{V^O_{i,k}(x_{i,k},\sigma_{-i,k}|\tau_i(\overline{\theta}))\right\}_{k\in\mathbb{N}}$ and $\left\{V^P_{i,k}(x_{i,k},\sigma_{-i,k}|\tau_i(\overline{\theta}))\right\}_{k\in\mathbb{N}}$, then we get the following conditions respectively for the upper and lower limits of the sequences $\left\{V^O_{i,\nu}(x_{i,\nu},\sigma_{-i,\nu}|\tau_i(\overline{\theta}))\right\}_{\nu\in\mathbb{N}}$ and $\left\{V^P_{i,\nu}(x_{i,\nu},\sigma_{-i,\nu}|\tau_i(\overline{\theta}))\right\}_{\nu\in\mathbb{N}}$:

$$\limsup_{\nu\to\infty} V^O_{i,\nu}(x_{i,\nu},\sigma_{-i,\nu}|\tau_i(\overline{\theta})) \leq V^O_i(x_i,\sigma_{-i}|\tau_i(\overline{\theta})) \tag{17}$$

$$V^P_i(x_i,\sigma_{-i}|\tau_i(\overline{\theta})) \leq \liminf_{\nu\to\infty} V^P_{i,\nu}(x_{i,\nu},\sigma_{-i,\nu}|\tau_i(\overline{\theta})). \tag{18}$$

Conversely, V_i^O and V_i^P are well posed by definition so there exist $Q'_{\overline{\theta}}$ and $Q''_{\overline{\theta}}$ in $\Phi_i(\tau_i(\overline{\theta}))$ such that

$$U_i(x_i, \sigma_{-i}|Q'_{\overline{\theta}}) = V_i^O(x_i, \sigma_{-i}|\tau_i(\overline{\theta})) \quad \text{and} \quad U_i(x_i, \sigma_{-i}|Q''_{\overline{\theta}}) = V_i^P(x_i, \sigma_{-i}|\tau_i(\overline{\theta}))$$

From the assumptions it follows that $\Phi_i(\tau_i(\overline{\theta})) \subseteq \operatorname*{Lim\,inf}_{\nu\to\infty} \Phi_{i,\nu}(\tau_i(\overline{\theta}))$. Hence there exist sequences $\{Q'_{\overline{\theta},\nu}\}_{\nu\in\mathbb{N}}$ and $\{Q''_{\overline{\theta},\nu}\}_{\nu\in\mathbb{N}}$, with $Q'_{\overline{\theta},\nu}$ and $Q''_{\overline{\theta},\nu}$ in $\Phi_{i,\nu}(\tau_i(\overline{\theta}))$ for every $\nu \in \mathbb{N}$, such that $Q'_{\overline{\theta},\nu} \to Q'_{\overline{\theta}}$ and $Q''_{\overline{\theta},\nu} \to Q''_{\overline{\theta}}$ as $\nu \to \infty$. From the definition (1) of the utility U_i it immediately follows that $U_i(x_{i,\nu}, \sigma_{-i,\nu}|Q'_{\overline{\theta},\nu}) \to U_i(x_i, \sigma_{-i}|Q'_{\overline{\theta}})$ and $U_i(x_{i,\nu}, \sigma_{-i,\nu}|Q''_{\overline{\theta},\nu}) \to U_i(x_i, \sigma_{-i}|Q''_{\overline{\theta}})$ as $\nu \to \infty$. Moreover, for every $\nu \in \mathbb{N}$, we get by definitions (6) and (7) that

$$V_{i,\nu}^O(x_{i,\nu}, \sigma_{-i,\nu}|\tau_i(\overline{\theta})) \geq U_i(x_{i,\nu}, \sigma_{-i,\nu}|Q'_{\overline{\theta},\nu}),$$
$$V_{i,\nu}^P(x_{i,\nu}, \sigma_{-i,\nu}|\tau_i(\overline{\theta})) \leq U_i(x_{i,\nu}, \sigma_{-i,\nu}|Q''_{\overline{\theta},\nu}).$$

Hence, for every converging subsequences

$$\left\{V_{i,k}^O(x_{i,k}, \sigma_{-i,k}|\tau_i(\overline{\theta}))\right\}_{k\in\mathbb{N}} \subseteq \left\{V_{i,\nu}^O(x_{i,\nu}, \sigma_{-i,\nu}|\tau_i(\overline{\theta}))\right\}_{\nu\in\mathbb{N}}$$

and

$$\left\{V_{i,k}^P(x_{i,k}, \sigma_{-i,k}|\tau_i(\overline{\theta}))\right\}_{\nu\in\mathbb{N}} \subseteq \left\{V_{i,\nu}^P(x_{i,\nu}, \sigma_{-i,\nu}|\tau_i(\overline{\theta}))\right\}_{\nu\in\mathbb{N}},$$

we get

$$\lim_{k\to\infty} V_{i,k}^O(x_{i,k}, \sigma_{-i,k}|\tau_i(\overline{\theta})) \geq$$
$$\lim_{k\to\infty} U_i(x_{i,k}, \sigma_{-i,k}|Q'_{\overline{\theta},k}) = U_i(x_i, \sigma_{-i}|Q'_{\overline{\theta}}) = V_i^O(x_i, \sigma_{-i}|\tau_i(\overline{\theta})) \tag{19}$$

and

$$\lim_{k\to\infty} V_{i,k}^P(x_{i,k}, \sigma_{-i,k}|\tau_i(\overline{\theta})) \leq$$
$$\lim_{k\to\infty} U_i(x_{i,k}, \sigma_{-i,k}|Q''_{\overline{\theta},k}) = U_i(x_i, \sigma_{-i}|Q''_{\overline{\theta}}) = V_i^P(x_i, \sigma_{-i}|\tau_i(\overline{\theta})) \tag{20}$$

Since (19) and (20) hold respectively for the converging subsequences

$$\left\{V_{i,k}^O(x_{i,k}, \sigma_{-i,k}|\tau_i(\overline{\theta}))\right\}_{k\in\mathbb{N}} \text{ and } \left\{V_{i,k}^P(x_{i,k}, \sigma_{-i,k}|\tau_i(\overline{\theta}))\right\}_{\nu\in\mathbb{N}},$$

then we get the following conditions respectively for the lower and upper limits of the sequences $\left\{V_{i,\nu}^O(x_{i,\nu}, \sigma_{-i,\nu}|\tau_i(\overline{\theta}))\right\}_{\nu\in\mathbb{N}}$ and $\left\{V_{i,\nu}^P(x_{i,\nu}, \sigma_{-i,\nu}|\tau_i(\overline{\theta}))\right\}_{\nu\in\mathbb{N}}$:

$$V_i^O(x_i, \sigma_{-i}|\tau_i(\overline{\theta})) \leq \liminf_{\nu\to\infty} V_{i,\nu}^O(x_{i,\nu}, \sigma_{-i,\nu}|\tau_i(\overline{\theta})), \tag{21}$$
$$\limsup_{\nu\to\infty} V_{i,\nu}^P(x_{i,\nu}, \sigma_{-i,\nu}|\tau_i(\overline{\theta})) \leq V_i^P(x_i, \sigma_{-i}|\tau_i(\overline{\theta})). \tag{22}$$

Therefore, for every $(x_1, \dots, x_n) \in X$ and for every $(\sigma_1, \dots, \sigma_n) \in S$, for every sequence $\{(x_{1,\nu}, \dots, x_{n,\nu})\}_{\nu\in\mathbb{N}} \subset X$ converging to $(x_1, \dots, x_n)$ and every sequence $\{(\sigma_{1,\nu}, \dots, \sigma_{n,\nu})\}_{\nu\in\mathbb{N}} \subset S$ converging to $(\sigma_1, \dots, \sigma_n)$, conditions (17) and (21) imply that (10) holds, while, conditions (18) and (22) imply that (11) holds, for every player i.

Now, let $\{\sigma_\nu^*\}_{\nu\in\mathbb{N}}$ be the sequence of strategy profiles converging to σ^* where each σ_ν^* is an equilibrium of $\mathscr{G}_\nu^{O,P}$. Given $\overline{\theta} \in \Theta$ and a player $i \in O$, by definition it follows that, for every $\nu \in \mathbb{N}$,

$$V_{i,\nu}^O(\sigma_{i,\nu}^*(\tau_i(\overline{\theta})), \sigma_{-i,\nu}^*|\tau_i(\overline{\theta})) \geq V_{i,\nu}^O(x_i, \sigma_{-i,\nu}^*|\tau_i(\overline{\theta})) \quad \forall x_i \in X_i.$$

From (10) and taking the limit as $\nu \to \infty$, we get

$$V_i^O(\sigma_i^*(\tau_i(\overline{\theta})), \sigma_{-i}^*|\tau_i(\overline{\theta})) = \lim_{\nu\to\infty} V_{i,\nu}^O(\sigma_{i,\nu}^*(\tau_i(\overline{\theta})), \sigma_{-i,\nu}^*|\tau_i(\overline{\theta})) \geq$$
$$\lim_{\nu\to\infty} V_{i,\nu}^O(x_i, \sigma_{-i,\nu}^*|\tau_i(\overline{\theta})) = V_i^O(x_i, \sigma_{-i}^*|\tau_i(\overline{\theta})) \quad \forall x_i \in X_i$$

Following the same steps and from (11), we get that for a pessimistic player $i \in P$ it follows that

$$V_i^P(\sigma_i^*(\tau_i(\overline{\theta})), \sigma_{-i}^*|\tau_i(\overline{\theta})) \geq V_i^P(x_i, \sigma_{-i}^*|\tau_i(\overline{\theta})) \quad \forall x_i \in X_i$$

Since, $\overline{\theta}$ is arbitrary then σ^* is a mixed equilibrium of $\mathscr{G}^{O,P}$ and we get the assertion. □

4 A counterexample

The key assumption in the previous limit theorem is contained in formula (5) which involves two conditions respectively on upper and lower limits of the sequences of sets of posteriors. In this section we present the Kajii and Ui counterexample and we emphasize that the limit property of the equilibria fails by removing from the the assumption (5) even only one of these two conditions. More precisely, it turns out that in this example, the limit property of the equilibria fails because the condition on the upper limits is not satisfied.

The example consists in a 2 player game. The set of states is $\Theta = \{1, 2, 3a, 3b, 4a, 4b\}$. For every $\varepsilon \in [0, 1]$, players have a common set of priors over Θ, which is:

$$\mathscr{P}_{1,\varepsilon} = \mathscr{P}_{2,\varepsilon} =$$
$$\left\{ P \in \Delta(\theta) \,|\, P(\{1\}) = P(\{2\}) = \frac{\varepsilon}{2},\, P(\{3a, 3b\}) = P(\{4a, 4b\}) = \frac{1-\varepsilon}{2} \right\}.$$

Let $E = \{1, 2\} \subset \Theta$. The game has this form

	L	R
U	1,-2	0,0
D	0,-2	1,0

if $\theta \in E$

	L	R
U	1,1	0,0
D	0,1	1,0

if $\theta \notin E$

The ranges of the two signals (τ_1, τ_2) are

$$T_1 = \{\{1, 3a, 3b\}, \{2, 4a, 4b\}\}$$
$$T_2 = \{\{1, 3a, 4a\}, \{2, 3b, 4b\}\}$$

and, for every $\theta \in \Theta$, $\tau_i(\theta) \in T_i$ is the set containing θ. Assume that both players use the Full Bayesian Updating Rule and are pessimistic. Then, for every $\varepsilon \in [0, 1]$ the posteriors of Player 1 are

$$\Phi_{1,\varepsilon}(\{1, 3a, 3b\}) = \{P \in \Delta(\Theta) \mid P(\{1\}) = \varepsilon,\ P(\{3a, 3b\}) = 1 - \varepsilon\}$$
$$\Phi_{1,\varepsilon}(\{2, 4a, 4b\}) = \{P \in \Delta(\Theta) \mid P(\{2\}) = \varepsilon,\ P(\{4a, 4b\}) = 1 - \varepsilon\}.$$

For every $\varepsilon \in]0, 1]$, the posteriors of Player 2 are

$$\Phi_{2,\varepsilon}(\{1, 3a, 4a\}) =$$

$$\left\{ P \in \Delta(\Theta) \;\middle|\; \begin{array}{lcl} P(\{1\}) & = & \frac{\varepsilon}{\varepsilon+2(\lambda+\mu)} \\ P(\{3a\}) & = & \frac{2\lambda}{\varepsilon+2(\lambda+\mu)} \\ P(\{4a\}) & = & \frac{2\mu}{\varepsilon+2(\lambda+\mu)} \end{array} \quad \text{with } \lambda,\mu \in \left[0, \tfrac{1-\varepsilon}{2}\right] \right\} \tag{23}$$

and

$$\Phi_{2,\varepsilon}(\{2, 3b, 4b\}) =$$

$$\left\{ P \in \Delta(\Theta) \;\middle|\; \begin{array}{lcl} P(\{2\}) & = & \frac{\varepsilon}{\varepsilon+2(\lambda+\mu)} \\ P(\{3b\}) & = & \frac{2\lambda}{\varepsilon+2(\lambda+\mu)} \\ P(\{4b\}) & = & \frac{2\mu}{\varepsilon+2(\lambda+\mu)} \end{array} \quad \text{with } \lambda,\mu \in \left[0, \tfrac{1-\varepsilon}{2}\right] \right\} \tag{24}$$

while, for $\varepsilon = 0$, they are:

$$\Phi_{2,0}(\{1, 3a, 4a\}) =$$

$$\left\{ P \in \Delta(\Theta) \;\middle|\; \begin{array}{lcl} P(\{1\}) & = & 0 \\ P(\{3a\}) & = & \frac{\lambda}{\lambda+\mu} \\ P(\{4a\}) & = & \frac{\mu}{\lambda+\mu} \end{array} \quad \text{with } \lambda,\mu \in \left[0, \tfrac{1}{2}\right],\ (\lambda,\mu) \neq (0,0) \right\} \tag{25}$$

and

$$\Phi_{2,0}(\{2, 3b, 4b\}) =$$

$$\left\{ P \in \Delta(\Theta) \;\middle|\; \begin{array}{lcl} P(\{2\}) & = & 0 \\ P(\{3b\}) & = & \frac{\lambda}{\lambda+\mu} \\ P(\{4b\}) & = & \frac{\mu}{\lambda+\mu} \end{array} \quad \text{with } \lambda,\mu \in \left[0, \tfrac{1}{2}\right],\ (\lambda,\mu) \neq (0,0) \right\} \tag{26}$$

The previous formulas show that the updated probabilities of the set E do not depend on the signal for both the players:

$$\{P(E) \,|\, P \in \Phi_{1,\varepsilon}(t_1)\} = \mathbb{P}_1(\varepsilon) \quad \forall t_1 \in T_1$$
$$\{P(E) \,|\, P \in \Phi_{2,\varepsilon}(t_2)\} = \mathbb{P}_2(\varepsilon) \quad \forall t_2 \in T_2$$

where

$$\mathbb{P}_1(\varepsilon) = \{\varepsilon\} \quad \forall \varepsilon \in [0,1]; \quad \mathbb{P}_2(\varepsilon) = \begin{cases} 0 & \text{if } \varepsilon = 0 \\ \left[\frac{\varepsilon}{2-\varepsilon}, 1\right] & \text{if } \varepsilon \in]0,1] \end{cases}$$

Then, for every $\varepsilon > 0$ and for every pair $(t_1, t_2) \in T_1 \times T_2$, the game is

	L	R
U	1,1-3$\mathbb{P}_2(\varepsilon)$	0,0
D	0,1-3$\mathbb{P}_2(\varepsilon)$	1,0

where $\mathbb{P}_2(\varepsilon) \in \left[\frac{\varepsilon}{2-\varepsilon}, 1\right]$.

We denote the mixed actions as follows: $x_1 = prob(U)$, $1 - x_1 = prob(D)$, $x_2 = prob(L)$ and $1 - x_2 = prob(R)$. Therefore, with an abuse of notation, x_1 identifies a mixed action of Player 1 and x_2 a mixed action of Player 2. The utility of Player 2 does not depend on Player 1's action choice so

$$V^P_{2,\varepsilon}(x_2, \sigma_1) = \min_{\mathbb{P}_2(\varepsilon) \in \left[\frac{\varepsilon}{2-\varepsilon}, 1\right]} x_2(1 - 3\mathbb{P}_2(\varepsilon)) = -2x_2$$

Hence, action $x_2 = 0$ is strictly dominant for Player 2 regardless of the signal. So, the mixed strategy $\sigma^*_{2,\varepsilon}$, defined by $\sigma^*_{2,\varepsilon}(t_2) = 0$ for all $t_2 \in T_2$, is a strictly dominant strategy for Player 2. It is trivial to check that there exists a unique best reply $\sigma^*_{1,\varepsilon}$ of Player 1 to Player 2's strategy $\sigma^*_{2,\varepsilon}$. This is defined by $\sigma^*_{1,\varepsilon}(t_1) = 0$ for all $t_1 \in T_1$. This implies that, for every $\varepsilon > 0$, the set of equilibria is given by $N(\varepsilon) = \{(\sigma^*_{1,\varepsilon}, \sigma^*_{2,\varepsilon})\}$. Now, for $\varepsilon = 0$ and for every pair $(t_1, t_2) \in T_1 \times T_2$, the game is

	L	R
U	1,1	0,0
D	0,1	1,0

It follows directly that the mixed strategy $\widehat{\sigma_2}$, defined by $\widehat{\sigma_2}(t_2) = 1$ for all $t_2 \in T_2$, is a strictly dominant strategy for Player 2. Since there exists a unique best reply $\widehat{\sigma_1}$ of Player 1 to Player 2's strategy $\widehat{\sigma_2}$, defined by $\widehat{\sigma_2}(t_1) = 1$ for all $t_1 \in T_1$, then the set of equilibria for $\varepsilon = 0$ is given by $N(0) = \{(\widehat{\sigma_1}, \widehat{\sigma_2})\}$. It immediately follows that for every sequence $\varepsilon_\nu \to 0$, the corresponding sequence of equilibria $\{(\sigma^*_{1,\varepsilon_\nu}, \sigma^*_{2,\varepsilon_\nu})\}_{\nu \in \mathbb{N}}$ do not converge to the unique equilibrium $(\widehat{\sigma_1}, \widehat{\sigma_2})$ in $N(0)$.

Now we show that the failure of the limit property in this example depends on a lack of the sequential upper convergence property of the sequence of posteriors of Player 2. This can be easily seen: consider the probability distribution $\overline{P} \in \Delta(\Theta)$ defined by $\overline{P}(a) = 1$ and $\overline{P}(\theta) = 0$ for all $\theta \in \Theta \setminus \{a\}$. It easily follows that $\overline{P} \in \Phi_{2,\varepsilon_\nu}(\{1, 3a, 4a\})$ for every $\nu \in \mathbb{N}$ since $\overline{P}$ can be obtained from (23) when $\lambda = \mu = 0$ for every ε_ν. Hence $\overline{P} \in \operatorname*{Lim\,sup}_{\nu \to \infty} \Phi_{2,\varepsilon_\nu}(\{1, 3a, 4a\})$. However, from (25), we immediately get that $\overline{P} \notin \Phi_{2,0}(\{1, 3a, 4a\})$. So

$$\operatorname*{Lim\,sup}_{\nu \to \infty} \Phi_{2,\varepsilon_\nu}(\{1, 3a, 4a\}) \nsubseteq \Phi_{2,0}(\{1, 3a, 4a\})$$

meaning that sequential upper convergence property of the sequence of posteriors does not hold. Conversely, it can be also checked that the sequential lower convergence property of the sequence of posteriors holds in this example. In fact, it follows easily from (23,24,25,26) that the following conditions hold

$$\Phi_{2,0}(\{1, 3a, 4a\}) \subseteq \operatorname*{Lim\,inf}_{\nu \to \infty} \Phi_{2,\varepsilon_\nu}(\{1, 3a, 4a\})$$
$$\Phi_{2,0}(\{2, 3b, 4b\}) \subseteq \operatorname*{Lim\,inf}_{\nu \to \infty} \Phi_{2,\varepsilon_\nu}(\{1, 3b, 4b\}).$$

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