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Stable Sets for Asymmetric Information Economies

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Abstract

An exchange economy with asymmetrically informed agents is considered with an exogenous rule that regulates the information sharing among agents. For it, the notion of stable sets à la Von Neumann and Morgenstern is analyzed. Two different frameworks are taken into account as regards preferences: a model without expectations and a model with expected utility. For the first one, it is shown that the set V of all individually rational, Pareto optimal, symmetric allocations is the unique stable set of symmetric allocations. For the second one, an example is presented which shows that the same set *V* is not externally stable and a weaker result is proved. Finally, the coalitional incentive compatibility of allocations belonging to the unique stable set is provided.

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1 Introduction

For a game with transferable utility, formed by a set N of players and by a function ν that associates each $S \subseteq N$ with a real number $\nu(S)$, the stable set has been the very first solution concept to be considered. It was introduced by von Neumann and Morgenstern (1944) and it hinges on two requirements of self-consistency which can be interpreted as acceptable behaviors within a society. In the most general framework of an abstract game $(X, \succ)$ consisting of a set X of outcomes and a dominance relation $\succ$ over them, a stable set is defined as a subset $S \subseteq X$ that satisfies the following two conditions:

1. no outcome in the stable set S is dominated by another outcome in S [internal stability];

2. every element which lies outside the set S is dominated by some element in S [external stability].

The argument behind these conditions can be summarized as follows: the first one guarantees that, once an outcome within S is selected, there is no interest in deviating toward any other outcome y; according to the second condition, on the contrary, every outcome which is not in S is unstable because dominated by some element in S. Of course, only one of these two properties, that is the internal stability, is satisfied by the core. With regard to the external stability, a core outcome must be undominated by any outcome, including those that can, in their turn, be dominated. Stable sets can be considered as arising exactly by this conceptual deficiency of the core[1].

Despite there is not a huge literature on stable sets, mainly due to the technical difficulty to work with them [2], the seminal definition by von Neumann and Morgenstern is general enough to allow applications in a wide variety of formats which go from cooperative games (Lucas, 1994), voting theory (Anesi, 2010) and exchange economies (Einy and Shitovitz, 2003, Greenberg at al., 2002, Hart, 1974). In particular, the analysis of a pure exchange economy, as far as stability is concerned, may produce different conclusions with

[1]Some controversy regarding the very logic under the stable set concept has been arisen by Harsanyi that, during the seventies, formulated the notion of sophisticated stable sets. The flaw that the new concept aims to amend is the following: in the stable set logic a deviation cannot be considered as valid if there is a further deviation toward some stable outcome.

[2]The stable set theory has some undesirable properties. Precisely, as in the following quotation by (Lucas, 1994): *Although some lack of uniqueness for cooperative game solution concepts seems reasonable, there are clearly too many stable sets for most games. At the same time there are some games, presumably rare, for which no stable sets at all exist.*

respect to the analysis which focuses on the payoff space. This is due to the fact that, differently by core and Pareto optimal allocations, the utility levels derived from a stable set in the allocation space might not be a stable set in the payoff space and conversely (see Greenberg at al., 2002).

In this paper we analyze stable sets in a general equilibrium context. Precisely, we are interested in economies with a finite number of agents which trade finite commodities under exogenous uncertainty represented by a finite set Ω of distinct states of nature. We assume that agents exchange at the ex-ante stage, that is, before the true state of nature has been realized, and that they initially have asymmetric information modeled by partitions of Ω. Moreover, an exogenous rule regulates the information sharing process among the individuals by specifying which is the information that each agent can use in every possible coalition he can join. This approach is general enough to include usual way of modeling information sharing, like the private, the fine, the coarse rules, but also situations in which the rule may vary depending on the coalition or the case when it may be a combination of the previous ones.

In a pure exchange economy, once a dominance relation has been defined over the set of allocations, a non empty set V of individually rational allocations is said to be stable if it is internally and externally stable. Also in this framework, differently by the core, there is no general theory and there are no tools. A basic problem is due to the fact that again, differently by the core, stable sets are in general not unique. The two main theoretical questions to be investigated in exchange economies with asymmetric information are therefore the existence and uniqueness of stable sets.

Inspired by a paper of Einy and Shitovitz (2003), the primary focus of this paper is to analyze whether the set V formed by all the Pareto optimal symmetric individually rational allocations is the unique stable sets for an economy with asymmetric information. To this aim, we distinguish two different settings as concerns preferences: a model without expectations, where agents preferences are formulated without referring to subjective probabilities and an expected utility model where preferences are described by state–dependent utility functions.

As to the first model, we solve simultaneously two problems related with stable sets: the fact that they may fail to exist and that they cannot be unique. Indeed, we prove that the set V is actually the only stable set of symmetric allocations for the economy. Such a result is obtained assuming that utility functions are strictly monotone, quasi concave, continuous, that everything on the boundary is associated with a zero utility and that the initial endowment, besides being strictly positive once summed up over the traders, is such that if agent i initially owns a positive quantity of some good in a state of nature, then no other agent initially owns this good in the same state.

On the contrary, for the second model we provide an example which shows that V is not externally stable. The example basically consists of an allocation which is in the core of the economy and which is not symmetric. Further, we remark that, despite this model is less general than the previous one, not all the assumptions stated for the state-dependent utility functions translate into the corresponding ones for expected utility functions; this is the point that has to be considered in order to obtain positive results for the second model.

The negative results regarding the model with expectation are partially amended in the last part of our paper: in it, a weaker relation of dominance is introduced where agents take part in a coalition using only some shares of their strictly positive endowments. Under this notion, referred as Aubin dominance relation, the corresponding set V is proved to be stable under assumptions which can be also met in a model with expected utility functions.

In conclusion, the results proved in the paper, focusing on a way to define basic concepts based on information sharing rules, lead to well-behaved stable sets, pointing out that only the information assigned to the grand coalition matters for their non-emptiness and uniqueness.

The paper proceeds in the following order. In the first section we outline the model without expectations along with all the definitions and assumptions needed throughout the paper. This model is developed in the next two sections; in particular, Section 3 collects some preliminary results about it and Section 4 contains the main result concerning the internal and external stability of the set V. The model with expectations is illustrated in Section 5 where an example is also provided to show that the set V is not externally stable and a remark points out a difficulty in obtaining the proper assumptions on the expected utility functions. Section 6 deals with the notion of incentive compatibility for an asymmetric information economy with an information sharing rule; we show that every Pareto optimal allocation in the unique stable set V is weakly coalitional incentive compatible, provided the rule is bounded. Finally, Section 7 presents a weaker result on the stability when a weaker notion of dominance is adopted.

2 The economic model without expectations

We consider a pure exchange economy E with uncertainty and asymmetric information. The exogenous uncertainty is formulated by a measurable space $(\Omega, \mathcal{F})$ where Ω denotes a finite set of states of nature and the field $\mathcal{F}$ represents the set of all the events.

The economy is characterized by a finite population of agents, indexed by $i \in N = \{1, \dots, n\}$, and a finite number l of commodities. $\mathbb{R}^l$ is the commodity space and $\mathbb{R}^l_+$ is the consumption set of each agent in each state of nature.

Every subset of N is referred to as a coalition.

Traders are not necessarily able to distinguish which state of nature ω in a finite set Ω of s elements actually occurs. Their initial information is modeled by a measurable partition Π_i of Ω. The interpretation is that if $\omega_0 \in \Omega$ is the state of nature that is going to be realized, agent i observes the element of Π_i which contains ω_0.

The state-dependent initial endowment of physical resources for each agent i is given by:

$$e_i : \Omega \longrightarrow \mathbb{R}^l_+.$$

The function e_i is assumed to be known to trader i and constant on each element of the partition Π_i. We consider the case where trade agreements are arranged in the ex-ante stage, that is, before the state of nature occurs. At this stage, each agent $i \in N$ has to choose among plans of contingent commodities $x_i \in \mathbb{R}^{l \cdot s}_+$ according to his preferences.

With regard to preferences, we assume that agent i's preferences are described by a cardinal utility function defined over the contingent commodities, that is:

$$U_i : \mathbb{R}^{l \cdot s}_+ \longrightarrow \mathbb{R}.$$

We do not consider ex-ante agents' preferences as necessarily derived from state-dependent preferences by taking expectations with respect to some subjective probability measure over the states of the world (as, for example, in Maus 2003, Radner 1968). The case of expected utility functions can be seen, under suitable assumptions, as included in this more general approach.

In the economy E it is not ruled out the case that more agents have the same characteristics. In particular, two agents are said to be of the same type if they are identical as regards the initial endowment, the initial information and the utility function. We denote by m the number of types in the economy E and by T_j the set of all traders of type j, with $1 \leq j \leq m$[3].

[3]When necessary, the double subscript i, j will be used to name traders, where the first index $i \in \{1, \dots, m\}$ denotes the type and the second subscript j denote the j-th copy of type i.

It clearly holds that $m \leq n$, that $|T_1| + \ldots + |T_m| = n$ and that $\{T_1, \ldots, T_m\}$ forms a partition of the set N.
We assume that the initial information of each trader may change when he becomes member of a coalition and we adopt the notion of information sharing rule as introduced by Allen (2006) in order to model this phenomenon.
Given a coalition S, an information sharing rule for S is a function $\Gamma(S)$ which associates a partition $\Gamma_i(S)$ of Ω to each member $i \in S$; the partition $\Gamma_i(S)$ is intended as the information that agent i can dispose of once the coalition S has been formed. An information sharing rule Γ for the economy E is a collection $(\Gamma(S))_{S \subset N}$. This general way of modeling information sharing within coalitions includes as particular examples some concrete and obvious cases: the private, the fine, the coarse information sharing rules. In the private information sharing rule, each trader just uses his initial information in every coalition he takes part in; that is:

$$\Gamma_i(S) = \Pi_i \, , \, \forall S \subseteq N \, , \, \forall \, i \in S.$$

In the case of the fine information sharing rule, traders pool their information within every coalition; that is:

$$\Gamma_i(S) = \bigvee_{i \in S} \Pi_i \, , \, \forall S \subseteq N \, , \, \forall \, i \in S.$$

In the coarse information sharing rule, agents within a group are restricted to use their common information, that is:

$$\Gamma_i(S) = \bigwedge_{i \in S} \Pi_i \, , \, \forall S \subseteq N \, , \, \forall \, i \in S.$$

However, all arbitrary possibilities are allowed. In particular, the definition of information sharing rule does not imply any relation among the information sharing rules used by different coalitions, or a relation between the coalition's information and the agents' private information.
The exchange economy with asymmetric information is thus formalized by the collection:

$$E = \{(\Omega, \mathcal{F});\ N = \{1, \ldots, n\};\ \mathbb{R}^l_+;\ (\Pi_i, U_i, e_i)_{i \in N}, ;\ \Gamma\}.$$

The following definitions are needed for developing the paper.

Definition 2.1 (ALLOCATION) An **allocation** for the coalition S in the economy E is a vector $x = (x_i)_{i \in S}$ with $x_i \in \mathbb{R}^{l \cdot s}_+$ such that:

i) x_i is $\Gamma_i(S)$ - measurable, for every $i \in S$ (*informational feasibility*);

ii) $\sum_{i \in S} x_i = \sum_{i \in S} e_i$ (*physical feasibility*).

The term “allocation” will be the short form for “allocation for the grand coalition N”. In particular, the information $\Gamma_i(N)$ is interpreted as the final information available to trader i at the time of consumption. Hence, at the time of consumption the same communication takes place as in the grand coalition.

Definition 2.2 (Dominance) Let x and y be two allocations and $S \subseteq N$ be a non empty coalition. We say that x ex-ante dominates y via S if:

i) x is an allocation for S;

ii) $U_i(x_i) > U_i(y_i), \forall\, i \in S$.

Moreover, we say that x dominates y, denoted by $x \succ y$, if there exists a non empty coalition S such that x ex-ante dominates y via S. Along with the dominance relation, some basic equilibrium notions are defined.

Definition 2.3 (weakly Pareto optimality) An allocation x is said to be ex-ante weakly Pareto optimal (or ex-ante weakly efficient) if there does not exist an allocation y which dominates x via N.

Definition 2.4 (Pareto optimality) An allocation x is said to be ex-ante Pareto optimal (or ex-ante efficient) if there does not exist an allocation y such that:

$$U_{\underline{i}}(y_{\underline{i}}) > U_{\underline{i}}(x_{\underline{i}}) \text{ for at least one } \underline{i} \in N \text{ and } U_i(y_i) \geq U_i(x_i) \text{ for every } i \in N \setminus \{\underline{i}\}.$$

Definition 2.5 (Core) An allocation x is said to be a **core allocation** for the economy E if there do not exist a coalition $S \subseteq N$ and an allocation y for the coalition S such that $U_i(y_i) > U_i(x_i)$, $\forall\; i \in S$.

That is, a feasible allocation belongs to the core of the economy if it is not possible for agents to join a coalition, redistribute their endowment among themselves letting each member use the information prescribed by the sharing rule Γ and obtain a strictly preferred allocation for each of them.
We will denote by $C^{\Gamma}(E)$ the set of the core allocations for the economy E under the information sharing rule Γ.

Definition 2.6 (Individual rationality) An allocation x is ex-ante individually rational if it holds that:

$$U_i(x_i) \geq U_i(e_i), \forall i \in N.$$

We will denote by I the set of all the individually rational allocations for the economy E.

Definition 2.7 (Symmetry) An allocation x is said to be symmetric if it gives the same utility to traders of the same type. That is:

$$\text{for every } 1 \leq i \leq m, \text{ for every } j, k \in T_i, U_i(x_{i,j}) = U_i(x_{i,k}) \, .$$

Definition 2.8 (Internal stability) A set V of individually rational allocations is internally stable if the following condition holds:

$$\text{if } x \in V \text{ then there is no } y \in V \text{ such that } y \succ x.$$

Definition 2.9 (External stability) A set V of individually rational allocations is externally stable if the following condition holds:

$$\text{if } x \in I \setminus V \text{ then there is } y \in V \text{ such that } y \succ x.$$

Definition 2.10 (Stable set) A set V of individually rational allocations is said to be a (Von Neumann–Morgenstern) stable set if it is both internally and externally stable.

It follows immediately from the definitions that the core is internally stable and therefore it is a subset of each stable set. If it is externally stable, then it contains each stable set and therefore it is the unique stable set of the economy[4].

Remark 2.1 The previous core notion encompasses traditional ways to define the core with asymmetric information when the information of coalitions is taken as given rather than chosen strategically. This is the case of the private, the fine, the coarse, the weak fine, the strong coarse core notions among the others. More extreme cases are possible, as the one in which all information use is forbidden or the one in which the asymmetric information is removed by requiring pooling of all information (compare Allen, 2006). Similar considerations hold true for the efficiency notions (for a comparison among them we refer to Hahn and Yannelis, 1997).

[4]The core rarely constitutes a stable set. In most cases, it consists generically of a single competitive allocation, by the equivalence theorem. This implies that it is not externally stable.

2.1 The Assumptions

The Assumption which, differently combined, will be used throughout the paper are the following.

(2.1) For every $i \in N$, U_i is weakly strictly monotone.
That is, for every $x, y \in \mathbb{R}^{l \cdot s}_+$, $x \gg y$ implies $U_i(x) > U_i(y)$. [5]

(2.2) For every $i \in N$, U_i is continuous.

(2.3) For every $i \in N$, U_i is quasi–concave.

(2.4) If a is on the boundary of $\mathbb{R}^{l \cdot s}_+$, then for every $i \in N$, $U_i(a) = U_i(0)$.

(2.5) $\sum_{i \in N} e_i \gg 0$.

(2.6) For all $j \in \{1, \ldots, m\}$ there exist $k_j \in \{1, \ldots, l\}$ and $\omega_j \in \Omega$ such that for every $r \neq j$, $e_r^{k_j}(\omega_j) = 0$ (where $e_r^{k_j}(\omega_j)$ denotes the k_j component of the vector $e_r(\omega_j)$).

Assumptions (2.1) and (2.4) together imply that the utility function of every agent is strictly positive on the interior of the orthant $\mathbb{R}^{l \cdot s}_+$ and obtains the zero value on its boundary. So agents prefer ex-ante interior commodities to the boundary ones (the boundary aversion assumption). Assumption (2.6) means that, although each commodity will be present on the market in each possible state by (2.5), each type of traders has a corner on some commodity in some state of nature. In models without uncertainty, it is sometimes referred as the glove market assumption on initial endowments.
The next assumptions are relative to the information sharing rule.

(2.7) The information sharing rule Γ is bounded; that is, for all $i \in N$ and for all coalitions S with $i \in S$, it holds that:

$$\Gamma_i(N) \succeq \Gamma_i(S)^6.$$

(2.8) Given two coalitions S, $S' \subseteq N$ such that $\{j \in \{1, \ldots, m\} : S \cap T_j \neq \emptyset\} = \{j \in \{1, \ldots, m\} : S' \cap T_j \neq \emptyset\}$, it holds that:

$$\Gamma_i(S) = \Gamma_i(S') \text{ for every } i \in S \cap S'.$$

[5]The symbol $\gg$ has the usual interpretation, that is, given $x, y \in \mathbb{R}^k$:

$$x \gg y \iff x_i > y_i, \text{ for every } i = 1, \ldots, k.$$

[6]Given two partitions P and Q of Ω, P is finer than Q, denoted $P \succeq Q$, if for every $A \in P$ there is $B \in Q$ such that $A \subseteq B$.

The intuition behind the assumption (2.7) is that membership in the grand coalition N cannot make an individual worse off from the informational viewpoint: the information can only become finer when joining the grand coalition N.
The interpretation of the last assumption is the following: if two coalitions contain the same agents' types, then the information that a trader has in each coalition is the same. Both these properties hold for the private information sharing rule, where each trader just uses his initial information in every coalition he takes part in. They also hold valid for the fine information sharing rule, where traders pool their information within every coalition. On the contrary, the coarse information sharing rule, where agents within a group are restricted to use their common information, is not bounded while it meets Assumption (2.8)[7]. Assumption (2.8) is of course satisfied by each information sharing rule depending on information types of coalitions.

3 Some preliminary results

As a first result, we want to prove that under the Assumption (2.1), (2.2) and (2.4), every weakly Pareto optimal allocation is also Pareto optimal.

Lemma 3.1 *Under the assumptions (2.1), (2.2), (2.4), every weakly Pareto optimal allocation is Pareto optimal.*

Proof. Let us consider a feasible allocation $x = (x_1, \ldots, x_n)$ which is weakly Pareto optimal.
Assume, by contradiction, that x is not Pareto optimal. This means that there exists an allocation $y = (y_1, \ldots, y_n)$ such that $U_{\underline{i}}(y_{\underline{i}}) > U_{\underline{i}}(x_{\underline{i}})$ for at least one $\underline{i} \in N$ and $U_i(y_i) \geq U_i(x_i)$ for every $i \neq \underline{i}$.
By the inequality $U_{\underline{i}}(y_{\underline{i}}) > U_{\underline{i}}(x_{\underline{i}})$ and by the assumption (2.4), it follows that:

$$y_{\underline{i}} \gg 0.$$

Indeed, let us suppose that $y_{\underline{i}}$ has a null component; then, by Assumption (2.4), it would be $U_{\underline{i}}(y_{\underline{i}}) = U_{\underline{i}}(0)$. If also $x_{\underline{i}}$ has a null component, then $U_{\underline{i}}(x_{\underline{i}}) = U_{\underline{i}}(y_{\underline{i}})$ and the inequality $U_{\underline{i}}(y_{\underline{i}}) > U_{\underline{i}}(x_{\underline{i}})$ would be contradicted. If, on the contrary, $x_{\underline{i}} \gg 0$, then by Assumption (2.1) it follows that $U_{\underline{i}}(x_{\underline{i}}) > U_{\underline{i}}(0) = U_{\underline{i}}(y_{\underline{i}})$ and the same inequality as before would be contradicted as well.

[7] A modification of the coarse information sharing rule by altering the information assigned to the grand coalition implies that the boundedness assumption is satisfied, see Allen (2006).

By $y_{\underline{i}} \gg 0$ and by Assumption (2.2), it follows that we can pick some $0 < \varepsilon < 1$ such that $U_{\underline{i}}(y_{\underline{i}} - \varepsilon\,\mathbf{1}) > U_{\underline{i}}(x_{\underline{i}})$, where $\mathbf{1}$ denotes the unit vector in $\mathbb{R}^{l \cdot s}$.
Consider the allocation z for the economy E defined by:

$$z_i = \begin{cases} y_{\underline{i}} - \varepsilon\,\mathbf{1}, & \text{if } i = \underline{i} \\ \\ y_i + \dfrac{\varepsilon}{n-1}\,\mathbf{1}, & \text{if } i \neq \underline{i} \end{cases}$$

The allocation z is clearly physically feasible for the grand coalition N. It is also informationally feasible.
Moreover, by the assumption (2.1), it holds true that:

$$U_i(z_i) > U_i(y_i),\ \forall\, i\ \in\ \mathrm{N} \setminus \{\underline{\mathrm{i}}\}$$

We can thus conclude that the allocation z dominates x via the grand coalition N, which contradicts the weakly Pareto optimality of x and establishes the validity of our claim. □

Let V denote the set of all individually rational symmetric Pareto optimal allocations of the economy E, that is:

$$V = \{x \in I : x \text{ is symmetric and Pareto optimal}\}$$

Before proceeding to show that the set V is stable à la Von Neumann and Morgenstern, we want to show that it is non empty. To accomplish this aim, we need some preliminary results.

Lemma 3.2 *Under the assumptions (2.2), (2.3) and (2.7), the core $C^{\Gamma}(E)$ is non empty.*

Proof.
The proof goes along a standard line which basically consists in three steps: a cooperative game without side payments (N, G) is associated with the economy E; the game–theoretic core of this game is proven to be non–empty by using the Scarf's theorem (1967); and, lastly, a core allocation for the economy E is derived from a core imputation for the game (N, G).
The game in question, which in our case also accounts for the information sharing rule Γ, is defined as follows; for every coalition $S \subseteq N$:

$$G(S) = \left\{ \begin{array}{l} \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n : \text{ there exists an allocation } x \text{ for the coalition S such that} \\ \xi_i \leq U_i(x_i), \forall\, i\ \in S \end{array} \right\}$$

Among the assumptions required by Scarf's result, the only one which requires some attention is the balancedness of the game.
About this point, Theorem 7.8 in Allen (2006) states that the fact that the information sharing rule is bounded is sufficient for the game to be balanced.
To conclude the proof, it suffices to consider any allocation $x = (x_1, \ldots, x_n)$ associated with $\xi \in \text{Core(G)}$; this allocation is easily proven to be feasible for the grand coalition N and unblocked by any coalition S. □

Lemma 3.3 *Let $i \in N$ and $x, y \in \mathbb{R}_+^{l \cdot s}$ such that $U_i(x) > U_i(y)$. Then, for every $0 < \alpha < 1$, it holds that $U_i(\alpha x + (1-\alpha)y) > U_i(y)$.*

Proof. By $U_i(x) > U_i(y)$, it follows that $x \gg 0$ (the proof is the same as in the previous Lemma 3.1).
By the continuity assumption (2.2), there exists $\beta \in (0,1)$ such that $U_i(\beta x) > U_i(y)$.
Let $0 < \alpha < 1$. By the quasi–concavity assumption (2.3), it follows that:

$$U_i(\alpha\beta x + (1-\alpha)y) \geq U_i(y)$$

Since $\alpha(1-\beta)x + \alpha\beta x + (1-\alpha)y \gg \alpha\beta x + (1-\alpha)y$, by the monotonicity assumption (2.1) it follows that:

$$U_i(\alpha(1-\beta)x + \alpha\beta x + (1-\alpha)y) = U_i(\alpha x + (1-\alpha)y) > U_i(\alpha\beta x + (1-\alpha)y). \ \square$$

Proposition 3.1 *Under the assumptions (2.1), (2.2), (2.3), (2.4), (2.7) and (2.8), every core allocation is symmetric.*

Proof.
Let $x = (x_{1,1}, \ldots, x_{1,n}; \ldots; x_{m,1}, \ldots, x_{m,n})$ be a core allocation [8].
By rearranging consumers of each type, we can assume without loss of generality that $(i,1)$ is the worst–off agent among those of type i, that is:

$$U_i(x_{i,j}) \geq U_i(x_{i,1}), \text{ for all } i \text{ and } j$$

Now, assume by way of contradiction that there exist some $k\,(1 \leq k \leq m)$ and some $r\,(1 < r \leq n)$ such that:

$$U_k(x_{k,r}) > U_k(x_{k,1}).$$

[8] To preserve simplicity, the proof is done for types including the same number of traders, that is $|T_1| = |T_2| = \ldots = |T_m| = n$. However, the result is valid also in a more general framework.

By the assumptions (2.1) and (2.4), it follows that $x_{k,r} \gg 0$.
Define the average consumption for each type:

$$y_i = \frac{1}{n}\sum_{j=1}^{n} x_{i,j}\ ,\ \ i \in \mathrm{N}.$$

and consider the allocation:

$$y = (y_1, \dots, y_1; \dots ; y_n, \dots, y_n)$$

The allocation y is physically and informationally feasible.
Moreover, we have that $U_k(y_k) > U_k(x_{k,1})$ and $U_i(y_i) \geq U_i(x_{i,1})$ for every $i \neq k$.
By the continuity assumption, we can pick some $0 < \varepsilon < 1$ such that $U_k(y_k - \varepsilon\,\mathbf{1}) > U_k(x_{k,1})$, where $\mathbf{1}$ denotes the unit vector of $\mathbb{R}_+^{l\cdot s}$.
The coalition of the worst-off agents, that is, $S = \{11, 21, \dots, m1\}$ can thus block the allocation x by allocating to every agent $(i,1)$ the bundle $(z_{i,1})_{i\in \mathrm{I}}$ defined as follows:

$$z_{i,1} = \begin{cases} y_k - \varepsilon\,\mathbf{1}, & \text{if } i = k \\ \\ y_i + \frac{\varepsilon}{m-1}\,\mathbf{1}, & \text{if } i \neq k \end{cases}$$

Indeed, the allocation $(z_{i,1})_{i\in \mathrm{I}}$ is clearly physically and informationally feasible for the coalition S. Moreover,

$$U_i(z_{i,1}) > U_i(y_i) \geq U_i(x_{i,1}),\ \forall\, i \in \mathrm{N}\backslash\{\mathrm{k}\}$$

where the first inequality follows from Assumption A.1.
We can thus conclude that coalition S blocks the allocation x, which is impossible.
This contradiction establishes the validity of our claim, that is, the core allocation x is symmetric. □
The following result is an easy consequence of the previous statements.

Corollary 3.1 *Under the assumptions (2.1), (2.2), (2.3), (2.4), (2.7) and (2.8), the set V is non empty.*

Finally, we state the following Lemma which, apart from being interesting in its own right, is useful in bypassing the welfare theorems which are used in Einy and Shitovitz in order to prove that the set V is externally stable.

Lemma 3.4 *Under the assumption (2.3), every individually rational symmetric allocation which is not weakly Pareto optimal can be dominated by a weakly Pareto optimal symmetric allocation.*

Proof. Let z be an allocation which is individually rational symmetric and not Pareto optimal and let us define the following sets:

$$\mathcal{A} = \{x = (x_1, \dots, x_n) : x \text{ is informationally and physically feasible}\}$$

and

$$A = \{x \in \mathcal{A} : x \text{ is individually rational, symmetric and } U_i(x_i) \geq U_i(z_i), \forall i \in N\}$$

and the function $\tilde{U}$ defined as follows:

$$\tilde{U}(x_1, \dots, x_n) = \sum_{i \in N} U_i(x_i)$$

The set A is compact and non empty (since z is not Pareto optimal) and the function $\tilde{U}$ is continuous on A.
Moreover, $\tilde{U}$ has a maximal element on the set A. Let us denote it by g. It holds that g is individually rational, symmetric and $U_i(g_i) \geq U_i(z_i), \forall i \in N$.
We want to prove that g is also Pareto optimal.
By way of contradiction, let us suppose that g is not Pareto optimal. Then, there exists an allocation γ such that:

$$U_i(\gamma_i) > U_i(g_i), \forall i \in N$$

Let us consider the average over types and let us denote it by $\overline{\gamma}$. The allocation $\overline{\gamma}$ is symmetric and it is individually rational since:

$$U_i(\overline{\gamma}_i) \geq U_i(g_i), \forall i \in N$$

Moreover, $U_i(\overline{\gamma}_i) \geq U_i(z_i), \forall i \in N$.
Then, the allocation $\overline{\gamma}$ belongs to A.
But, it holds that:

$$\tilde{U}(\overline{\gamma}) = \sum_{i \in N} U_i(\overline{\gamma}_i) > \sum_{i \in N} U_i(g_i) = \tilde{U}(g)$$

and this contradicts the fact that g is a maximal element for the function $\tilde{U}$ on the set A.
Then, the maximal element g is symmetric, individually rational, Pareto optimal and such that $U_i(g_i) \geq U_i(z_i)$, $\forall\, i \in N$. □

As a consequence of the previous Lemma, Remark 2.4 in Einy and Shitovitz (which is used in proving Theorem A in their paper) can be restated for the asymmetric information framework; that is, in order to prove that V is externally stable, it suffices that every individually rational Pareto optimal allocation which is not symmetric is dominated by

some element in V (provided Assumptions (2.1), (2.2) and (2.4) hold).
In fact, let $x \in I \setminus V$. If x is not Pareto optimal, then there exists a symmetric Pareto optimal allocation y which dominates x. Since x is individually rational, y is also individually rational and therefore it is an element of the set V.
Hence, in order to prove that V is externally stable, it suffices to prove that every individually rational, non-symmetric, Pareto optimal allocation is dominated by some element in V.

4 Main result

This section contains the main result of the paper, namely that the set V is stable.
To preserve clearness and for the sake of comparison with the economic model which will be analyzed in the next section, we consider internal and external stability separately.

Theorem 4.1 *Let the economy E satisfy assumptions (2.1),(2.4),(2.6). Then, the set:*

$$V = \{x \in I : x \textit{ is symmetric and Pareto optimal } \}$$

is internally stable.

Proof. By way of contradiction, let us assume that V is not internally stable. Then, there exist two allocations x and y in V and a nonempty coalition S such that:

$$U_i(x_i) > U_i(y_i), \ \forall i \in S \tag{1}$$

and

$$\sum_{i \in S} x_i \leq \sum_{i \in S} e_i. \tag{2}$$

By (1), it follows that $x_i \gg 0, \forall i \in S$; then, by (2):

$$\sum_{i \in S} e_i \gg 0$$

By Assumption (2.6), this implies that $S \cap T_j \neq \emptyset$, for every $1 \leq j \leq m$. Since x and y are symmetric, then $U_i(x_i) > U_i(y_i), \forall i \in N$, and this contradicts the Pareto optimality of y.
Then, the set V is internally stable. □

Moving to the external stability, the following result holds valid.

Theorem 4.2 *Let the economy E satisfy assumptions (2.1)–(2.5), (2.8) and assume that:*

$$|T_1| = |T_2| = \ldots = |T_m|$$

Then, the set:

$$V = \{x \in I : x \text{ is symmetric and Pareto optimal } \}$$

is externally stable.

Proof. Let $x \in I \setminus V$. By the previous remark, we may assume that x is an individually rational non symmetric Pareto optimal allocation for the economy E.
We have to find an allocation z which belong to V and dominates x.
Consider the average consumption for each type, that is:

$$y_j = \frac{1}{|T_j|} \sum_{k \in T_j} x_k \ , \quad j = 1, \ldots, m.$$

Since x is assumed to be non-symmetric, there exist an agents' type $j \in \{1, \ldots, m\}$ and an agent $k \in T_j$ such that:

$$U_j(y_j) \neq U_j(x_k)$$

Without loss of generality, we can assume that $j = 1$. Then, by the previous Lemma 3.3, it holds that there exists $i_1 \in T_1$ such that:

$$U_1(y_1) > U_1(x_{i_1})$$

and by the quasi concavity, for every $2 \leq j \leq m$, there exists $i_j \in T_j$ such that:

$$U_j(y_j) \geq U_j(x_{i_j})$$

By the continuity assumption, we can pick some $0 < \varepsilon < 1$ such that:

$$U_1(y_1 - \varepsilon \mathbf{1}) > U_1(x_{i_1}) \tag{3}$$

where $\mathbf{1}$ denotes the unit vector in $\mathbb{R}_+^{l \cdot s}$.
Let us consider the allocation z for the economy E defined by:

$$z_i(\omega) = \begin{cases} y_1(\omega) - \varepsilon \mathbf{1}, & \text{if } i \in T_1 \\ \\ y_j(\omega) + \dfrac{\varepsilon}{m-1} \mathbf{1}, & \text{if } i \in T_j \, , j \neq 1 \end{cases}$$

The allocation z is both physically and informationally feasible.
Let us consider a coalition S formed by the agent i_j for each possible type T_j, $j = 1, \ldots, m$; that is:

$$S = \{i_1, \ldots, i_m\}$$

It holds that the allocation z is feasible for the coalition S; in fact, for every $\omega \in \Omega$:

$$\sum_{j=1}^{m} z_j(\omega) = \sum_{j=1}^{m} y_j(\omega) = \frac{1}{|T_j|}\sum_{i=1}^{n} x_i(\omega) = \frac{1}{|T_j|}\sum_{i=1}^{n} e_i(\omega) = \sum_{j=1}^{m} e_i(\omega)$$

Since $\{j \in \{1,\ldots,m\} : S \cap T_j \neq \emptyset\} = \{1,\ldots,m\}$, it holds that $\Gamma^S_{i_j} = \Gamma^N_{i_j}$; hence, the allocation z is also informationally feasible for the coalition S.
Moreover, the monotonicity assumption (2.1) and condition (3) imply that $U_j(z_j) > U_j(x_j)$ for every $j \in \{1,\ldots,m\}$, that is z is an allocation for the economy E which dominates the allocation x via the coalition S.
If z is Pareto optimal, then $z \in V$ and the proof is finished.
If z is not Pareto optimal, as a consequence of Lemma 3.4, we can find an individually rational, symmetric, Pareto optimal allocation which dominates z and this concludes the proof. □

As a consequence of the previous theorems, we can state what follows.

Corollary 4.1 *Let the economy E satisfy assumptions (2.1)–(2.6), (2.8) and assume that:*

$$|T_1| = |T_2| = \ldots = |T_m|$$

Then, the set:

$$V = \{x \in I : x \textit{ is symmetric and Pareto optimal }\}$$

is a von Neumann-Morgenstern stable set. Moreover, V is the unique stable set of symmetric allocations.

Remark 4.1 As the proof of Theorem 4.1 clearly shows, the glove market and the boundary aversion assumptions are essential to prove internal stability of the set V. Their combination permits to prove that whenever a coalition is able to block, then it must contain each type of trader. This conclusion, due to the assumption (2.8), implies also that only the final information that traders receive in the grand coalition matters for the set V to be stable. As a further comment, we remark that the same conclusion of Theorem 4.1 (and therefore of Corollary 4.1) could be reached replacing the glove market structure assumption with a suitable restriction on coalition formation. Assuming that the only possible coalitions in the society are those containing each type of traders, the set V is still internally stable. Restriction on coalition formation are traditional in the study of the core (see for example Hervés-Beloso and Moreno-García, 2001), and especially motivated in economies with asymmetric information where the lack of communication among traders may avoid the formation of some coalitions. The boundary aversion assumption however cannot be removed. It is also essential for the validity of Theorem 4.2.

5 The model with expected utilities

In this section we want to analyze a model which differs from the one analyzed so far in that the utility functions are state dependent.

In this model, each trader i is characterized by a strictly positive probability measure q_i on Ω, representing his prior beliefs concerning states of nature and by a state-contingent cardinal utility function representing his preferences:

$$u_i : \Omega \times \mathbb{R}^l_+ \longrightarrow \mathbb{R}$$

For any $x : \Omega \longrightarrow \mathbb{R}^l_+$, consumer i's ex-ante expected utility is denoted by h_i; it is defined by:

$$h_i(x) = \sum_{\omega \in \Omega} q_i(\omega) u_i(\omega, x(\omega))$$

Notice that in general this model is a special case of the one described in the previous sections in the sense that one can move from this one to the previous one by taking the expectations of the state-dependent utility functions.

In spite of this, it is not possible to argue straightforwardly that, under the same assumptions used in the previous sections, the same results hold valid for this model. The crucial point is that not all the assumptions stated for the state-dependent utility functions translate into the corresponding ones for expected utility functions h_i. This point is analyzed in section 5.2.

5.1 A negative result

We want to show that, in a model with state-contingent cardinal utility functions and expectations, the set:

$$V = \{x \in I : x \text{ is symmetric and Pareto optimal}\}$$

is not stable à la von Neumann–Morgenstern. In particular, we provide an example which shows that it is not externally stable.

To this end, we need to find an allocation $x \in I \setminus V$ which is not dominated by any allocation $y \in V$.

The allocation defined in the Example 11, page 408 in Maus (2004) serves this purpose. Let us analyze such example.

Consider an economy with three different types of agents (denoted by 1, 2 and 3), two agents of each type, three states of nature (denoted by a, b and c) and one good in each state. Each type has a state-dependent utility function specified as follows:

$$u_1(a,x) = 3x\,; \qquad u_1(b,x) = 3x\,; \qquad u_1(c,x) = 30x.$$

$$u_2(a,x) = 3x\,; \qquad u_2(b,x) = 30x\,; \quad u_2(c,x) = 3x.$$

$$u_3(a,x) = 30x\,; \quad u_3(b,x) = 3x\,; \qquad u_3(c,x) = 3x.$$

All agents assign equal prior probability to each state of nature. The agents' endowment and their initial information are displayed below:

Type	Information Π_i	Endowment a	b	c
1	{{a,b},{c}}	1	1	0
2	{{a,c},{b}}	1	0	1
3	{{b,c},{a}}	1	1	1

Denote by ij ($i = 1,2,3$ and $j = 1,2$) the j-th agent of type i and consider the allocation x defined as follows:

Agent	State:	a	b	c
11		0	0	2
12		0	0	2
21		0	$\frac{2}{10}$	0
22		0	$\frac{38}{10}$	0
31		3	0	0
32		3	0	0

It belongs to the private core of the economy E and, as a consequence, it is both individually rational and weakly Pareto optimal. But it is not symmetric. Therefore, $x \in I \setminus V$. The allocation x cannot be dominated by any y via a coalition S because this would contradict the fact that x is a private core allocation for the economy E.

This is enough to conclude that V is not externally stable and, hence, it is not a von Neumann–Morgenstern stable set.

Note that, in the previous example, Assumption (2.6) is not satisfied. However, as shown in the previous section, this assumption is used only in order to prove that the set V is internally stable. The example we have just provided shows that Theorem 4.2 does not hold. It is an open question whether an extension of Theorem 4.1 concerning the internal stability can be obtained for the model with expectations. The proof of the internal stability implies however a difficulty related to the Assumption (2.4) as remarked in the next section.

5.2 A remark about the boundary aversion assumption

We want to compare a model with state dependent utility functions and a model without expectations from the point of view of Assumption (2.4).

We will denote by u_i and h_i the state-contingent cardinal utility function and the ex-ante expected utility function, respectively.

Assume that the utility functions u_i are such that everything in the interior is preferred to anything on the boundary of $\mathbb{R}_+^{l\cdot s}$.

We show that this assumption does not translate into the corresponding one for the expected utility functions h_i.

To this end, consider a framework with two equally probable states of nature (denoted by a and b) and two commodities. The utility functions in each state are given by:

$$u(a,(x_1,x_2)) = x_1 x_2 \qquad\qquad u(b,(x_1,x_2)) = 2x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$$

Let us consider the bundles of commodities x and y given by:

$$\begin{array}{ll} x(a) = (1/2, 1/2) & x(b) = (1/2, 1/2) \\ y(a) = (0,1) & y(b) = (1,1) \end{array}$$

It holds that:

$$h(x) = 5/8 \qquad \text{and} \qquad h(y) = 1$$

In conclusion, the state–dependent utility functions u are such that everything in the interior is preferred to anything on the boundary while the expected utility function h does not inherit the same property.

5.3 Some positive cases for expected utility models

A specific case belonging to the expected utility realm where the boundary aversion assumption is satisfied is represented by Cobb–Douglas subjective expected utility preferences.

Precisely, the axiomatic foundation of Cobb–Douglas preferences in Faro (2013) provides an (expected) utility function U which has all the properties required for the set V to be stable.
Consider the case of a finite number n of states of nature and just one commodity. Suppose that each trader i has a preference relation $\succeq$ on $\mathbb{R}^n_{++}$.
We list some relevant properties for the preference relation $\succeq$ which are needed for the axiomatization:

- $\succeq$ is homothetic if for all $x, y \in \mathbb{R}^n_{++}$ and $k > 0$ it holds that:

$$x \succeq y \Rightarrow kx \succeq ky$$

- $\succeq$ is strongly homothetic if for all $x, y, z \in \mathbb{R}^n_{++}$ it holds that:

$$x \succeq y \Rightarrow xz \succeq yz^9.$$

- $\succeq$ is log-convex if for all $x, y, \in \mathbb{R}^n_{++}$ and $k \in (0, 1)$ it holds that:

$$x \succeq y \Rightarrow x^k y^{1-k} \succeq y.$$

- $\succeq$ is power invariant if for all $x, y \in \mathbb{R}^n_{++}$ and $k > 0$ it holds that:

$$x \succeq y \Rightarrow x^k \succeq y^k$$

- $\succeq$ is indifference invariant if for all $x, y \in \mathbb{R}^n_{++}$ such that $x \succ y$, if $x' \sim x$ and $y' \sim y$ then $x' \succ y$ and $x \succ y'$;

As a consequence of his main Theorem 4, Faro (2013) provided the following result:

Proposition 5.1 *The following conditions are equivalent:*

1. *A binary relation $\succeq$ is non-trivial, reflexive, strictly monotone, locally lower continuous, indifference invariant and strongly homothetic;*

2. *A binary relation $\succeq$ is non-trivial, strictly monotone, continuous, power invariant and strongly homothetic;*

3. *A binary relation $\succeq$ is a Cobb–Douglas Subjective Expected utility preference, that is there exists a strictly positive probability $q = (q_1, \ldots, q_n)$ such that:*

$$x \succeq y \Leftrightarrow \prod_{k=1}^{n} x_k^{q_k} \geq \prod_{k=1}^{n} y_k^{q_k}$$

[9] xz denotes the coordinatewise product vector

where monotonicity, strict monotonicity and continuity properties are defined in a usual way. The representation $U(x) = \prod_{k=1}^{n} x_k^{q_k}$ can be extended to the positive orthant $\mathbb{R}_+^n$ by defining:

$$U(x) = 0 \text{ for all } x \in \partial(\mathbb{R}_+^n)$$

leading to an utility function U that clearly satisfies all the assumptions needed for the set V to be stable.
A positive conclusion holds true also for the so called Cobb–Douglas Maxmin Subjective Expected utility preferences due to the following result in Faro (2013).

Proposition 5.2 *The following conditions are equivalent:*

1. *A binary relation $\succeq$ is a non-trivial, continuous, strictly monotone, log-convex, power invariant and homothetic weak order if and only if there exists a unique non-empty closed and convex set C of strictly positive probability measures such that:*

$$x \succeq y \Leftrightarrow min_{p \in C} \prod_{k=1}^{n} x_k^{p_k} \geq min_{p \in C} \prod_{k=1}^{n} y_k^{q_k}$$

As in the previous case, the representation can be extended to the positive orthant $\mathbb{R}_+^n$.

6 Coalitional incentive compatibility of vNM stable sets

In this section we raise the question whether the allocations contained in the set V are incentive compatible. This issue is much relevant because incentive compatibility ensures somehow a form of stability: if a state contingent contract can be manipulated by an agent which does not truthfully reveals his information, then such a contract is not enforceable. In order to deal with the incentive compatibility, we need to introduce the ex-post utility functions.
For every trader $i \in N$, the ex-post utility function u_i is given by:

$$u_i : \Omega \times \mathbb{R}_+^l \to \mathbb{R}$$

We assume that $u_i(\cdot, x)$ is Γ_N^i-measurable, for every $i \in N$.
The following axiom, which is assumed to hold valid throughout the rest of the paper, relates ex-ante and ex-post utility functions; in line with the paper by de Castro, Pesce, and Yannelis (2011), we call it *Ex-ante/Ex-post Consistency Axiom.*

Ex-ante/Ex-post Consistency Axiom
For every $i \in N$, for every $\bar{\omega} \in \Omega$ and for every $x, y, z : \Omega \to \mathbb{R}^l_+$:

$$u_i(\bar{\omega}, x(\bar{\omega})) > u_i(\bar{\omega}, y(\bar{\omega})) \Rightarrow U_i(x(\bar{\omega}), z_{\Omega\setminus\{\bar{\omega}\}}) > U_i(y(\bar{\omega}), z_{\Omega\setminus\{\bar{\omega}\}})$$

where $(x(\bar{\omega}), z_{\Omega\setminus\{\bar{\omega}\}})$ denotes the function that is valued $x(\bar{\omega})$ if $\omega = \bar{\omega}$ and $z(\omega)$ otherwise.

We adopt the following definition of incentive compatibility:

Definition 6.1 An allocation $x : \Omega \to (\mathbb{R}^l_+)^n$ is weak coalitionally incentive compatible if the following does not hold:
There exist a coalition $S \subseteq N$ and two states of nature $a, b \in \Omega$ such that:

i) $\Gamma^i_S(a) \in \bigwedge_{i\in S} \Gamma^i_S, \forall i \in S$;

ii) $a \in \Gamma^j_N(b), \forall j \in N \setminus S$;

iii) $u_i(a, e_i(a) + x_i(b) - e_i(b)) > u_i(a, x_i(a)), \forall i \in S$.

Note that, if the information sharing rule Γ is bounded and such that $\Gamma^i_N = \Gamma^j_N$ for every $i, j \in N$ [10], then conditions i) and ii) cannot be met simultaneously; as a consequence, every allocation is weak coalitionally incentive compatible (see also Koutsougeras and Yannelis, 1993).

Proposition 6.1 Suppose that the information sharing rule Γ is bounded. Then, every Pareto optimal allocation is weak coalitionally incentive compatible.

Proof. Let x be a Pareto optimal allocation. By way of contradiction, let us suppose that x is not weak coalitionally incentive compatible. Hence, by definition, there exist a coalition $S \subseteq N$ and two states of nature $a, b \in \Omega$ such that:

i) $\Gamma^i_S(a) \in \bigwedge_{i\in S} \Gamma^i_S, \forall i \in S$;

ii) $a \in \Gamma^j_N(b), \forall j \in N \setminus S$;

iii) $u_i(a, e_i(a) + x_i(b) - e_i(b)) > u_i(a, x_i(a)), \forall i \in S$.

[10] This holds, for example, for the fine information sharing rule Γ defined by $\Gamma^i_S = \bigvee_{i\in S} \Pi_i$, for every $i \in S$, for every $S \subseteq N$.

By ii), it follows that:

$$a \in \bigcap_{j \notin S} \Gamma_N^j(b)$$

$\bigcap_{j \notin S} \Gamma_N^j(b)$ is a block of the partition $\bigvee_{j \notin S} \Gamma_N^j$.

Denoting by z_i the net-trade of agent i, that is, $z_i(\cdot) = x_i(\cdot) - e_i(\cdot)$, it holds then true that:

$$\sum_{i \notin S} z_i(\cdot) \text{ is } \bigvee_{i \notin S} \Gamma_N^i - \text{ measurable}$$

and thus:

$$\sum_{i \notin S} z_i(a) = \sum_{i \notin S} z_i(b)$$

By the physical feasibility of the allocation x, it follows that, for all $\omega \in \Omega$:

$$\sum_{i \in N} z_i(\omega) = \sum_{i \in S} z_i(\omega) + \sum_{i \notin S} z_i(\omega) = 0$$

and hence:

$$\sum_{i \in S} z_i(a) = -\sum_{i \notin S} z_i(a)$$

and:

$$\sum_{i \in S} z_i(b) = -\sum_{i \notin S} z_i(b)$$

Then:

$$\sum_{i \in S} z_i(a) = -\sum_{i \notin S} z_i(a) = -\sum_{i \notin S} z_i(b) = \sum_{i \in S} z_i(b)$$

Let us define, for every $i \in S$:

$$z_i^*(\omega) = \begin{cases} z_i(\omega), & \text{if } \omega \notin \Gamma_N^i(a); \\ \\ z_i(b), & \text{if } \omega \in \Gamma_N^i(a). \end{cases}$$

z_i^* is Γ_N^i-measurable, for every $i \in S$. Moreover, for every $\omega \in \Omega$ it holds that:

$$\sum_{i \in S} z_i^*(\omega) + \sum_{i \notin S} z_i(\omega) = 0$$

Indeed, if $\omega \notin \Gamma_N^i(a)$:

$$\sum_{i \in S} z_i^*(\omega) + \sum_{i \notin S} z_i(\omega) = \sum_{i \in S} z_i(\omega) + \sum_{i \notin S} z_i(\omega) = 0$$

If $\omega \in \Gamma^i_N(a)$:

$$\sum_{i\in S} z^*_i(\omega)+\sum_{i\notin S} z_i(\omega) = \sum_{i\in S} z_i(b)+\sum_{i\notin S} z_i(\omega) = \sum_{i\in S} z_i(b)-\sum_{i\in S} z_i(\omega) = \sum_{i\in S} z_i(b)-\sum_{i\in S} z_i(a) = 0$$

Let us consider the allocation x^* defined as follows; for every $i \in N$:

$$x^*_i = \begin{cases} e_i + z^*_i, & \text{if } i \in S; \\ \\ e_i + z_i, & \text{if } i \notin S. \end{cases}$$

It is physically feasible; in fact, for every $\omega \in \Omega$:

$$\sum_{i\in N} x^*_i(\omega) = \sum_{i\in S}[e_i(\omega) + z^*_i(\omega)] + \sum_{i\notin S}[e_i(\omega) + z_i(\omega)] = \sum_{i\in N} e_i(\omega)$$

Moreover, it is also informationally feasible; indeed, since Γ is bounded, both e_i and z^*_i are Γ^i_N-measurable.
If $i \in S$ and $\omega \in \Gamma^i_N(a)$, it holds that:

$$u_i(\omega, x^*_i(\omega)) = u_i(\omega, e_i(\omega) + z^*_i(\omega)) = u_i(\omega, e_i(\omega) + z_i(b)) =$$

(since $u_i(\omega, \cdot) = u_i(a, \cdot)$ for all $\omega \in \Gamma^i_N(a)$)

$$= u_i(a, e_i(a) + x_i(b) - e_i(b)) > u_i(a, x_i(a)) = u_i(\omega, x_i(\omega)).$$

where the inequality follows by condition iii) and the last equality is a joint consequence of the measurability of the allocation x and the function $u_i(\cdot, x)$.
On the contrary, if $i \in S$ and $\omega \notin \Gamma^i_N(a)$, it holds that:

$$u_i(\omega, x^*_i(\omega)) = u_i(\omega, x_i(\omega)).$$

By the *ex-ante/Ex-post Consistency Axiom*, it then follows that for every $i \in S$:

$$U_i(x^*_i) > U_i(x_i).$$

and this contradicts the Pareto-optimality of the allocation x.
Hence, x is weak coalitionally incentive compatible. □

7 A weaker result

It is the aim of this section to prove a weaker result about the stability of the set V in order to include the model with expected utility functions. As observed in section 5.2, the

main reason why the set V of symmetric Pareto optimal allocations may not be stable in the expected utility model is that the expectation does not inherit from state dependent utilities the boundary aversion property.
We propose a weaker notion of dominance according to which agents take part in a coalition using only same shares of their endowments. Assuming that agents have a strictly positive initial endowments in each state of nature and strict concave utility functions, we are able to show that there is a unique stable set made by symmetric allocations. This set consists of all allocations which are individually rational, symmetric and cannot be dominated by the whole society when agents use only partially the initial resources.
Precisely, we shall assume throughout the main results of this section that the physical feasibility of allocations is satisfied with free disposal and the following two conditions substitute for (2.3) and (2.5), respectively:

$(2.3)'$ For every $i \in N$, U_i is strictly concave.

$(2.5)'$ $e_i(\omega) \gg 0, \forall \omega \in \Omega, , \forall i \in N$.

Moreover, we shall dispense with assumptions (2.4) and (2.6) and requires the following assumption on the information sharing rule:

(2.9) Given two coalitions S, $S^{'} \subseteq N$ such that $S^{'} \subseteq S$, it holds that:

$$\Gamma_i(S) \succeq \Gamma_i(S^{'}) \text{ for every } i \in S^{'}.$$

The intuition behind this assumption is that the information of each trader cannot get worse when the coalition he takes part increases its size. It holds for the private and the fine information sharing rule, while it is not satisfied by the coarse. Of course assumption (2.9) implies that the information sharing rule is also bounded.

For each function $\gamma : N \to [0, 1]$, denote by S_γ the support of γ, that is, the set of traders $i \in N$ such that $\gamma(i) = \gamma_i \neq 0$. Based on this, a weaker notion of dominance over allocations can be provided.

Definition 7.1 (Weak Dominance) Let x and y be two allocations and $\gamma : N \to [0, 1]$ be such that $S_\gamma \neq \emptyset$. We say that x ex-ante w-dominates y via γ if:

i) x_i is $\Gamma_i(S_\gamma)$ - measurable, for every $i \in S_\gamma$;

ii) $U_i(x_i) > U_i(y_i), \forall\, i \in S_\gamma$;

iii) $\sum_{i\in N}\gamma_i x_i \leq \sum_{i\in N}\gamma_i e_i$.

Moreover, we say that x weakly dominates y, denoted by $x \succ^w y$, if there exists γ such that x ex-ante weakly dominates y via γ.

After rewriting condition iii) as $\sum_{i\in S_\gamma}\gamma_i x_i \leq \sum_{i\in S_\gamma}\gamma_i e_i$, i) and iii) can be interpreted, respectively, as informational and physical feasibility of x with respect to γ. The coalition γ is called an Aubin or generalized coalition and interpreted as a coalition in which a trader may employ only the share γ_i of his resources. Of course ordinary coalitions, which can be identified with their characteristic functions, represent a particular case. The weak dominance relation is also referred as the Aubin dominance. The corresponding core $C_A^\Gamma(E)$ is called the Aubin core and has been studied in the case of private information sharing rule in Graziano and Meo (2005).
Let $\widetilde{V}$ denote the set of all individually rational symmetric allocations of the economy E which cannot be weakly dominated by another allocation via the whole coalition of traders; that is, for an allocation $x \in \widetilde{V}$ there does not exists γ with full support (i.e. $S_\gamma = N$) and y such that conditions i), ii) and iii) in Definition 7.1 are satisfied. We proceed to show that the set $\widetilde{V}$ is stable à la Von Neumann and Morgenstern with respect to the weak dominance relation just defined. In what follows, we shall denote by G the set

$$G = \{\gamma : N \to [0,1] \;:\; S_\gamma = N\}.$$

We start providing a straightforward extension of lemma 3.4.

Lemma 7.1 *Under the assumption (2.3), every individually rational symmetric allocation which is not in $\widetilde{V}$ can be weakly dominated by an allocation of $\widetilde{V}$.*

Proof. Let z be an allocation which is individually rational symmetric and not in $\widetilde{V}$. Define the sets:

$$\mathcal{B} = \{x = (x_1, \ldots, x_n) : x \text{ is informationally and physically feasible with respect to } \gamma \in G\},$$

$$B = \{x \in \mathcal{B} : x \text{ is individually rational, symmetric and } U_i(x_i) \geq U_i(z_i), \forall i \in N\}$$

and the function $\tilde{U}$ as:

$$\tilde{U}(x_1, \ldots, x_n) = \sum_{i\in N} U_i(x_i)$$

The set B is compact and non empty (since z is not in $\widetilde{V}$) and the function $\tilde{U}$ is continuous on B.

Moreover, $\tilde{U}$ has a maximal element on the set B. Let us denote it by g. It holds that g is individually rational, symmetric and $U_i(g_i) \geq U_i(z_i), \forall i \in N$.
We want to prove that g belongs to $\widetilde{V}$.
By way of contradiction, let us suppose that g is not in $\widetilde{V}$. Then, there exists a function $\gamma : N \to [0,1]$ with full support and an allocation h such that:

i) h_i is $\Gamma_i(S_\gamma)$ - measurable, for every $i \in S_\gamma$;

ii) $U_i(h_i) > U_i(g_i), \forall\, i \in S_\gamma$;

iii) $\sum_{i \in N} \gamma_i h_i \leq \sum_{i \in N} \gamma_i e_i$.

Let us rename γ_i^j and h_i^j the elements relative to type j in the feasibility condition $iii)$ and denote by $\overline{\gamma}^j$ the sum $(\sum_{i \in T_j} \gamma_i^j)$ for each type j. Then by the concavity assumption we have that:

$$U_i(\overline{h}_j) > U_i(g_j), \forall\, i \in T_j, \forall\, j = 1 \ldots m$$

where:

$$\overline{h}_j = \overline{\gamma}^j \sum_{i \in T_j} \overline{\gamma}_i^j h_i^j$$

and

$$\overline{\gamma}_i^j = \frac{\gamma_i^j}{\overline{\gamma}^j}.$$

The allocation $\overline{h}$ is symmetric and it is individually rational since:

$$U_i(\overline{h}_i) \geq U_i(g_i), \forall i \in N.$$

Moreover, it belongs to B.
But, it holds that:

$$\tilde{U}(\overline{h}) = \sum_{i \in N} U_i(\overline{h}_i) > \sum_{i \in N} U_i(g_i) = \tilde{U}(g)$$

and this contradicts the fact that g is a maximal element for the function $\tilde{U}$ on the set B.

Theorem 7.1 *Let the economy E satisfy the assumptions (2.1),(2.2), $(2.3)'$, $(2.5)'$, (2.9). Then, the set $\widetilde{V}$ is internally stable.*

Proof. By way of contradiction, let us assume that $\widetilde{V}$ is not internally stable. Then, there exist two allocations x and y in $\widetilde{V}$ and $\gamma : I \to [0,1]$ such that:

i) y_i is $\Gamma_i(S_\gamma)$ - measurable, for every $i \in S_\gamma$;

ii) $U_i(y_i) > U_i(x_i), \forall\, i \in S_\gamma$;

iii) $\sum_{i \in N} \gamma_i y_i \leq \sum_{i \in N} \gamma_i e_i$.

Since $\varepsilon y_i + (1-\varepsilon)e_i$ tends to y_i as ε tends to 1, by continuity assumption, we can find $\varepsilon \in]0,1[$ with $U_i(\varepsilon y_i + (1-\varepsilon)e_i) > U_i(x_i)$ for all $i \in S_\gamma$.
The inequality $\sum_{i \in S_\gamma} \gamma_i y_i \leq \sum_{i \in S_\gamma} \gamma_i e_i$ can be written as $\sum_{i \in S_\gamma} \frac{\gamma_i}{\varepsilon}(\varepsilon y_i) \leq \sum_{i \in S_\gamma} \frac{\gamma_i}{\varepsilon}(\varepsilon e_i)$ and, consequently, as $\sum_{i \in S_\gamma} \frac{\gamma_i}{\varepsilon}[\varepsilon y_i + (1-\varepsilon)e_i] \leq \sum_{i \in S_\gamma} \frac{\gamma_i}{\varepsilon} e_i$, where $\varepsilon y_i + (1-\varepsilon)e_i$ is strictly positive and $\Gamma_i(S_\gamma)$-measurable.
Now, we prove that $w = \sum_{i \in N} \gamma_i(e_i - y_i)$ is strictly positive. By the arguments above, we can assume in iii) that y_i is strictly positive. It is then enough to take a positive ε with $U_i(\varepsilon y_i) > U_i(x_i)$ for all $i \in S_\gamma$ to have that

$$\sum_{i \in S_\gamma} \gamma_i y_i - \sum_{i \in S_\gamma} \gamma_i(\varepsilon y_i) = \sum_{i \in S_\gamma} \gamma_i y_i (1-\varepsilon)$$

is strictly positive and therefore that even

$$\sum_{i \in S_\gamma} \gamma_i y_i - \sum_{i \in S_\gamma} \gamma_i(\varepsilon y_i) + \sum_{i \in N} \gamma_i(e_i - y_i) = \sum_{i \in N} \gamma_i(e_i - \varepsilon y_i)$$

is strictly positive.
Assume that $\gamma \notin G$. Then for each $i \in N \setminus S_\gamma$ monotonicity ensures that $U_i(x_i + e_i) > U(x_i)$. Choose $\lambda > 0$ with $w - \lambda \sum_{i \notin S_\gamma} x_i \geq 0$. Now modify γ and y replacing, for $i \notin S_\gamma$, γ_i by λ and y_i by $x_i + e_i$. The modified γ belongs to G and y weakly dominates x since we have

$$\sum_{i \in S_\gamma} \gamma_i y_i + \sum_{i \notin S_\gamma} \lambda(x_i + e_i) = -w + \sum_{i \in S_\gamma} \gamma_i e_i + \sum_{i \notin S_\gamma} \lambda(x_i + e_i) =$$

$$\sum_{i \in S_\gamma} \gamma_i e_i - w + (\lambda \sum_{i \notin S_\gamma} x_i) + \sum_{i \notin S_\gamma} \lambda e_i \leq \sum_{i \in S_\gamma} \gamma_i e_i + \sum_{i \notin S_\gamma} \lambda e_i. \quad \square$$

Theorem 7.2 *Let the economy E satisfy assumptions (2.1), (2.2), (2.3)′, (2.5)′, (2.8) and assume that:*

$$|T_1| = |T_2| = \ldots = |T_m|$$

Then, the set $\widetilde{V}$ is externally stable.

Proof. Let $x \in I \setminus \widetilde{V}$. By the previous Lemma, we may assume that x is an individually rational non symmetric allocation for the economy E which cannot be blocked by $\gamma \in G$. Consider the average consumption for each type, that is:

$$y_j = \frac{1}{|T_j|} \sum_{k \in T_j} x_k \ , \quad j = 1, \ldots, m.$$

Since x is assumed to be non-symmetric, there exist an agents' type $j \in \{1, \ldots, m\}$ and an agent $k \in T_j$ such that:

$$U_j(y_j) \neq U_j(x_k)$$

Without loss of generality, we can assume that $j = 1$. Then, by the strict concavity assumption, it holds that there exists $i_1 \in T_1$ such that:

$$U_1(y_1) > U_1(x_{i_1})$$

and , for every $2 \leq j \leq m$, there exists $i_j \in T_j$ such that:

$$U_j(y_j) \geq U_j(x_{i_j})$$

By the continuity assumption, there exists $\varepsilon > 0$ such that $U_1(\varepsilon y_1) > U_1(x_{i_1})$.
Then from the inequality

$$\sum_{j=1}^{m} y_j \leq \sum_{j=1}^{m} e_j$$

we derive

$$\sum_{j=2}^{m} y_j + \frac{1}{\varepsilon}[\varepsilon y_1 + (1-\varepsilon)e_1] \leq \sum_{j=2}^{m} e_j + \frac{1}{\varepsilon} e_1$$

where $\widetilde{y}_1 = [\varepsilon y_1 + (1-\varepsilon)e_1]$ is strictly positive and $U_1(\widetilde{y}_1) > U_1(x_{i_1})$.
Again by the continuity assumption, we can pick some $0 < \delta < 1$ such that:

$$U_1(\delta \widetilde{y}_1) > U_1(x_{i_1}). \tag{4}$$

Let us consider the allocation z for the economy E defined by:

$$z_i(\omega) = \begin{cases} \delta \widetilde{y}_1(\omega), & \text{if } i \in T_1 \\ \\ y_j(\omega) + \frac{(1-\delta)}{\varepsilon(m-1)} \widetilde{y}_1, & \text{if } i \in T_j \, , j \neq 1 \end{cases}$$

The allocation z is symmetric, informationally feasible and dominates x with full support. Since it is also individually rational, if $z \in \widetilde{V}$ the proof is finished.

If $z \notin \widetilde{V}$, then there exists by previous Lemma, an allocation h in $\widetilde{V}$ which dominates z with full support. $\square$

As a consequence of the previous theorems, we can state what follows.

Corollary 7.1 *Let the economy E satisfy assumptions (2.1),(2.2), (2.3)$'$, (2.5)$'$, (2.8), (2.9) and assume that:*

$$|T_1| = |T_2| = \ldots = |T_m|$$

Then, the set: $\widetilde{V}$ is a von Neumann-Morgenstern stable set. Moreover, $\widetilde{V}$ is the unique stable set of symmetric allocations.

Remark 7.1 We remark that the free disposal assumption made in this section does not guarantee that the allocations in the set $\widetilde{V}$ are weakly coalitional incentive compatible. Moreover, under the assumption of the present section, the Aubin core $C_A^{\Gamma}(E)$ is made by symmetric allocations since it is a subset of the stable set $\widetilde{V}$. The study of Aubin core and stable sets in the case of games with generalized coalitions is provided by Muto and al. (2006).

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