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Giuseppe De Marco and Maria Romaniello

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University of Naples Federico II

University of Salerno

Bocconi University, Milan

CSEF - Centre for Studies in Economics and Finance
DEPARTMENT OF ECONOMICS – UNIVERSITY OF NAPLES
80126 NAPLES - ITALY
Tel. and fax +39 081 675372 – e-mail: csef@unisa.it

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Abstract

In this paper we consider a model of games of incomplete information under ambiguity in which players are endowed with *variational preferences*. We provide an existence result for the corresponding mixed equilibrium notion. Then we study the limit behavior of equilibria under perturbations on the indices of ambiguity aversion.

Keywords: Incomplete information games, multiple priors, variational preferences, equilibria.

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* Università di Napoli Parthenope and CSEF. Address: Dipartimento di Studi Aziendali e Quantitativi, Università di Napoli Parthenope. Via Parisi 13, Napoli 80129, Italy. E-mail: giuseppe.demarco@uniparthenope.it.

** Seconda Università di Napoli. Address: Dipartimento di Economia, Seconda Università di Napoli. Corso Gran Priorato di Malta, Capua 81043, Italy. E-mail: maria.romaniello@unina2.it

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1 Introduction

Kajii and Ui present in [10] a model of games of incomplete information under ambiguity which represents a minor departure from the standard Bayesian approach. Nevertheless, it captures significative differences with respect to the standard model. In fact, they partially follow the Harsanyi's approach as they assume that the source of uncertainty can be expressed by an underlying state space; but, at the same time, they deviate from the classical model of uncertainty since they allow for *multiple priors* (that are not necessarily common across agents[2]) instead of a single *common prior*. Ambiguity is then resolved by assuming that players are pessimistic, that is, they evaluate an ambiguous belief by the worst expected utility possible given the set of probability distributions (multiple priors). This particular attitude towards ambiguity, known as *maxmin expected utility preferences*, has been axiomatized by Gilboa and Schmeidler in [8] and commonly used in many applications. Kajii and Ui also present in [10] an equilibrium notion for their model called *mixed equilibrium*, which is an *interim equilibrium* concept where each player chooses the best mixed action for any realization of a private signal[3]; then, they show that mixed equilibria exist under standard assumptions.

In this paper we study a generalization of the mixed equilibrium concept in which we relax the assumption imposed on the ambiguity attitudes of the players. More precisely we consider players endowed with *variational preferences* as introduced by Maccheroni, Marinacci and Rustichini in [13]. This class of preferences embodies the maxmin model and help to better understand the theoretical foundations of the works of Hansen and Sargent on model uncertainty in macroeconomics[4] (see [9]). Under variational preferences, players evaluate an ambiguous belief by the worst possible value (given the set of probability distributions) assumed by the sum of the expected utility with a nonnegative function of the probabilities called *index of ambiguity aversion*. This index has been demonstrated to play a very important role as a measure of ambiguity aversion. For instance, maximal ambiguity aversion corresponds to index ambiguity aversion identically equal to zero and gives back maxmin preferences. While, minimal ambiguity aversion corresponds to ambiguity neutrality and gives back subjective utility preferences. The index of ambiguity aversion is lower semicontinuous but discontinuous in general. Despite this drawback, our first result shows that, in the Kajii and Ui setting of incomplete information, variational preferences are represented by continuous functions of the strategy profile. As a consequence of this result, we prove in this paper that equilibria (called *MMR mixed equilibria*) exist under standard assumptions.

Maccheroni, Marinacci and Rustichini show in [13] that variational preferences become more ambiguity averse as the ambiguity indices become smaller. So they point out (pp 1459-1450) that it is natural to wonder about the limit behavior of sequences of variational preferences. In particular, it is important to determine the conditions under which a sequence of variational preferences converges to the variational preference corresponding to the limit index of ambiguity aversion. In [13], a limit result is given (Proposition 12). In this paper, we give a new result which is based on a different assumption of convergence on the sequence of indices of ambiguity aversion, namely *hypographical convergence* (see for instance [1],[17]), and guarantees the

[2]That is, each player is endowed with a set of priors (probability distributions) over the state space and those sets might different.

[3]Indeed, they also give a definition of *equilibrium in beliefs*, which, however is not directly related with the aim of this note.

[4]In fact, the model by Hansen and Sargent provides a particular class of variational preferences as well.

continuous convergence of sequences of variational preferences. This result has an immediate application to mixed equilibria since continuous convergence of preferences imply convergence of corresponding sequences of MMR-mixed equilibria. The problem of the limit behavior of the equilibria in games has been extensively studied in the literature (see, for instance, [7] for the standard problem, [6], [14],[15], [16],[18], [20] for recent results under relaxed or different assumptions and references). The question whether the limit property extends to the equilibrium concepts in ambiguous games has been studied in [3] for equilibria under ambiguous beliefs correspondences (see [2],[4]), in [19] for an equilibrium notion for ambiguous games which relies on the Beweley unanimity rule. Finally, in [5], we investigate the stability for the Kajii and Ui's notion of mixed equilibrium under perturbations on the set of multiple priors.

The paper is organized as follows: In Section 2 we describe the model of games under incomplete information and multiple priors, define the notion of MMR mixed equilibrium and provide the existence result. Section 3 is devoted to the stability issue; first we study the limit behavior of the variational preferences and then we apply this result to the problem of stability of MMR equilibria.

2 Model and equilibria

We consider a finite set of players $I = \{1, \dots n\}$. For every player i, $\Psi_i = \{\psi_i^1, \dots, \psi_i^{k(i)}\}$ is the (finite) pure action set of player i, $\Psi = \prod_{i \in I} \Psi_i$ and $\Psi_{-i} = \prod_{j \neq i} \Psi_j$. Denote with X_i the set of mixed actions of player i, that is, each action $x_i \in X_i$ is a vector $x_i = (x_i(\psi_i))_{\psi_i \in \Psi_i} \in \mathbb{R}_+^{k(i)}$ such that $\sum_{\psi_i \in \Psi_i} x_i(\psi_i) = 1$. Denote also with $X = \prod_{j=1}^n X_j$ and with $X_{-i} = \prod_{j \neq i} X_j$. Let Θ be a finite set of payoff relevant states then, player i has a payoff function $f_i : \Psi \times \Theta \to \mathbb{R}$. Denote with $\Delta(\Theta)$ the set all the probability distribution over Θ, then player i is endowed of a set of priors $\mathbb{P}_i \subseteq \Delta(\Theta)$.

We follow the Kajii and Ui's setup in [10]. The incompleteness of information is summarized by a random signal $\tau = (\tau_i)_{i \in I}$. When state $\theta \in \Theta$ occurs, player i privately observes a signal $\tau_i(\theta)$ and then chooses a pure strategy $\psi_i \in \Psi_i$. Denote with T_i the range of τ_i, i.e. $\tau_i : \Theta \to T_i$ for every player i. A strategy of player i is a function $\sigma_i : T_i \to X_i$; therefore, for every $t_i \in T_i$, $\sigma_i(t_i)$ is a vector in X_i where each component $\sigma_i(\psi_i | t_i)$ denotes the probability of player i choosing action ψ_i when he observes t_i. The set of all the strategies σ_i of player i is denoted by S_i; moreover, $S_{-i} = \prod_{j \neq i} S_j$ and $S = \prod_{i=1}^n S_i$. Finally denote with $\sigma(\psi | \tau(\theta)) = \prod_{i=1}^n \sigma_i(\psi_i | \tau_i(\theta)))$ and $\sigma_{-i}(\psi_{-i} | \tau_{-i}(\theta)) = \prod_{j \neq i} \sigma_j(\psi_j | \tau_j(\theta))$.

Given $P \in \mathbb{P}_i$ and $t_i \in T_i$, denote with $P(\cdot | t_i) \in \Delta(\Theta)$ the conditional probabilities over Θ, that is

$$P(E | t_i) = \frac{P(\tau_i^{-1}(t_i) \cap E)}{P(\tau_i^{-1}(t_i))} \quad \forall E \subseteq \Theta.$$

Let $\mathbb{P}_i(t_i) = \{P(\cdot | t_i) \in \Delta(\Theta) \,|\, P \in \mathbb{P}_i\}$ be the set of conditional probability distributions once t_i has been observed[5]. An updating rule $\Phi_i : T_i \to 2^{\Delta(\Theta)}$ gives, for every $t_i \in T_i$, a subset of conditional probabilities $\Phi_i(t_i) \subseteq \mathbb{P}_i(t_i)$. If $\Phi_i(t_i) = \mathbb{P}_i(t_i)$ for every $t_i \in T_i$ then Φ_i is called *Full Bayesian Updating Rule.*

[5]Degenerate probabilities are implicitly ruled out from $\mathbb{P}_i(t_i)$ in this formulation. That is, if $P \in \mathbb{P}_i$ is such that $P(\tau_i^{-1}(t_i)) = 0$, then $P(\cdot | t_i) \notin \Delta(\Theta)$ which implies that $P(\cdot | t_i) \notin \mathbb{P}_i(t_i)$.

After t_i is observed, player i uses posteriors in $\Phi_i(t_i)$ to evaluate his actions. The interim payoff to a randomized action $x_i \in X_i$, given $\sigma_{-i} \in S_{-i}$ and $Q_i \in \Phi_i(t_i)$ is

$$U_i(x_i, \sigma_{-i}|Q_i) = \sum_{\theta\in\Theta} \sum_{\psi_i\in\Psi_i} \sum_{\psi_{-i}\in\Psi_{-i}} x_i(\psi_i)\sigma_{-i}(\psi_{-i}|\tau_{-i}(\theta))Q_i(\theta)f_i(\psi_i, \psi_{-i}|\theta). \tag{1}$$

As explained in the Introduction, in this work we deviate from the Kajii and Ui's approach, where only the pessimistic attitude towards ambiguity is taken into account. In fact, we consider more general attitudes towards ambiguity which are represented by the so called *variational preferences* as introduced in ([13]) by Maccheroni, Marinacci and Rustichini. More precisely, for every player i, we consider the following utility function

$$V_i(x_i, \sigma_{-i}|t_i) = \min_{Q_i\in\Phi_i(t_i)} \left[U_i(x_i, \sigma_{-i}|Q_i) + c_i(Q_i)\right],$$

where $c_i : \Delta(\Theta) \to \overline{\mathbb{R}}_+$, called *index of ambiguity aversion*, is a convex and lower semicontinuous function in its domain $\Delta(\Theta)$, and $\overline{\mathbb{R}}_+ = [0, +\infty[\cup\{+\infty\}$. Hence,

$$\Gamma = \{I; \Theta; (\mathbb{P}_i)_{i\in I}; (\Phi_i)_{i\in I}; (S_i)_{i\in I}; (V_i)_{i\in I}\}$$

is the corresponding game[6]. Then

DEFINITION 2.1: A strategy profile $\sigma^* \in S$ is a *Maccheroni Marinacci Rustichini (MMS) mixed equilibrium* for the game Γ if for every $\theta \in \Theta$ it follows that

$$V_i(\sigma_i^*(\tau_i(\theta)), \sigma^*_{-i}|\tau_i(\theta)) = \max_{x_i\in X_i} V_i(x_i, \sigma^*_{-i}|\tau_i(\theta)) \quad \forall i \in I. \tag{2}$$

Now, we prove first that utility functions V_i of the game Γ are continuous and then, as a consequence, we provide an existence result for MMS mixed equilibria of the game Γ .

PROPOSITION 2.2: *Assume that the set $\Phi_i(t_i)$ is closed. Then, the function $V_i(\cdot, \cdot|t_i)$ is continuous in $X_i \times S_{-i}$.*

Proof. Let $(\overline{x}_i, \overline{\sigma}_{-i}) \in X_i \times S_{-i}$. First we prove that $V_i(\cdot, \cdot|t_i)$ is upper semicontinuous in $(\overline{x}_i, \overline{\sigma}_{-i})$, that is, for every sequence $\{(x_{i,\nu}, \sigma_{-i,\nu})\}_{\nu\in\mathbb{N}}$ converging to $(\overline{x}_i, \overline{\sigma}_{-i})$ it follows that

$$\limsup_{\nu\to\infty} V_i(x_{i,\nu}, \sigma_{-i,\nu}|t_i) \leqslant V_i(\overline{x}_i, \overline{\sigma}_{-i}|t_i). \tag{3}$$

Denote with F_i the function defined by $F_i(x_i, \sigma_{-i}, Q_i) = U_i(x_i, \sigma_{-i}|Q_i)+c_i(Q_i)$ for all $(x_i, \sigma_{-i}, Q_i) \in X_i \times S_{-i} \times \Delta(\Theta)$. Since $c_i(\cdot)$ is a lower semicontinuous function, then the function $F_i(\overline{x}_i, \overline{\sigma}_{-i}, \cdot)$ is lower semicontinuous in its domain. So, there exists $\overline{Q}_i \in \Phi_i(t_i)$ such that $V_i(\overline{x}_i, \overline{\sigma}_{-i}|t_i) = F_i(\overline{x}_i, \overline{\sigma}_{-i}, \overline{Q}_i)$. By definition, $V_i(x_{i,\nu}, \sigma_{-i,\nu}|t_i) \leqslant F_i(x_{i,\nu}, \sigma_{-i,\nu}, \overline{Q}_i)$ for every $\nu \in \mathbb{N}$, which implies that

$$\limsup_{\nu\to\infty} V_i(x_{i,\nu}, \sigma_{-i,\nu}|t_i) \leqslant \limsup_{\nu\to\infty} F_i(x_{i,\nu}, \sigma_{-i,\nu}, \overline{Q}_i). \tag{4}$$

Since, by definition, the function U_i is continuous in its domain then the function $F(\cdot, \cdot, \overline{Q}_i)$ is obviously continuous in $X_i \times S_{-i}$. Then, it follows that

$$\limsup_{\nu\to\infty} F_i(x_{i,\nu}, \sigma_{-i,\nu}, \overline{Q}_i) = F(\overline{x}_i, \overline{\sigma}_{-i}, \overline{Q}_i). \tag{5}$$

[6]When a game Γ is considered, then it is implicitly assumed that its utility functions V_i are well posed, (i.e. $\min_{Q_i\in\Phi_i(t_i)} [U_i(x_i, \sigma_{-i}|Q_i) + c(Q_i)]$ exist for every $x \in X$, $t_i \in T_i$ and $\sigma \in S$); obviously, this latter condition is guaranteed, for instance, when posteriors $\Phi_i(t_i)$ are closed and not empty sets for every $t_i \in T_i$.

Being $F(\overline{x}_i, \overline{\sigma}_{-i}, \overline{Q}_i) = V_i(\overline{x}_i, \overline{\sigma}_{-i}|t_i)$, from (4,5) we get (3).

Following the classical Berge's arguments, now show that for every sequence $\{(x_{i,\nu}, \sigma_{-i,\nu})\}_{\nu\in\mathbb{N}}$ converging to $(\overline{x}_i, \overline{\sigma}_{-i})$ it follows that

$$V_i(\overline{x}_i, \overline{\sigma}_{-i}|t_i) \leq \liminf_{\nu\to\infty} V_i(x_{i,\nu}, \sigma_{-i,\nu}|t_i). \tag{6}$$

Given a sequence of strictly positive numbers $\{\varepsilon_\nu\}_{\nu\in\mathbb{N}}$; then, by definition, for every $\nu \in \mathbb{N}$ there exists $Q_{i,\nu} \in \Phi_i(t_i)$ such that $F(x_{i,\nu}, \sigma_{-i,\nu}, Q_{i,\nu}) \leqslant \min_{Q_i\in\Phi_i(t_i)} F(x_{i,\nu}, \sigma_{-i,\nu}, Q_i) + \varepsilon_\nu$ for all $\nu \in \mathbb{N}$. The function $F(x_i, \sigma_{-i}, \cdot)$ is lower semicontinuous in the compact set $\Phi_i(t_i)$; therefore, $V_i(x_{i,\nu}, \sigma_{-i,\nu}|t_i) = \min_{Q_i\in\Phi_i(t_i)} F(x_{i,\nu}, \sigma_{-i,\nu}, Q_i)$ is well posed. Hence

$$\liminf_{\nu\to\infty} F(x_{i,\nu}, \sigma_{-i,\nu}, Q_{i,\nu}) \leqslant \liminf_{\nu\to\infty} [V_i(x_{i,\nu}, \sigma_{-i,\nu}|t_i) + \varepsilon_\nu] = \liminf_{\nu\to\infty} V_i(x_{i,\nu}, \sigma_{-i,\nu}|t_i) \tag{7}$$

Hence there exists a subsequence of integers $\{\nu_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} F(x_{i,\nu_k}, \sigma_{-i,\nu_k}, Q_{i,\nu_k}) = \liminf_{\nu\to\infty} F(x_{i,\nu}, \sigma_{-i,\nu}, Q_{i,\nu}) \tag{8}$$

The set $X_i \times S_{-i} \times \Phi_i(t_i)$ is compact. Then, it follows that there exists a subsequence

$$\{(x_{i,\nu_{k_h}}, \sigma_{-i,\nu_{k_h}}, Q_{i,\nu_{k_h}})\}_{h\in\mathbb{N}} \subseteq \{(x_{i,\nu_k}, \sigma_{-i,\nu_k}, Q_{i,\nu_k})\}_{k\in\mathbb{N}}$$

converging to $(\overline{x}_i, \overline{\sigma}_{-i}, \overline{Q}_i) \in X_i \times S_{-i} \times \Phi_i(t_i)$. Hence, from (7,8), we get

$$F(\overline{x}_i, \overline{\sigma}_{-i}, \overline{Q}_i) = \lim_{h\to\infty} F(x_{i,\nu_{k_h}}, \sigma_{-i,\nu_{k_h}}, Q_{i,\nu_{k_h}}) \leqslant \liminf_{\nu\to\infty} V_i(x_{i,\nu}, \sigma_{-i,\nu}|t_i).$$

Finally, by definition $V_i(\overline{x}_i, \overline{\sigma}_{-i}|t_i) \leqslant F(\overline{x}_i, \overline{\sigma}_{-i}, \overline{Q}_i)$, which implies (6) and concludes the proof. □

Remark 2.3: In the previous proof, we proved directly the upper semicontinuity of the function V_i. However, we could also apply the general result contained in Proposition 3.1.1 of [12]. The proof of the lower semicontinuity of V_i substantially follows the standard Berge's arguments (see, for instance, Proposition 4.1.1 in [12]).

Building upon the previous Proposition we prove the following:

Theorem 2.4: *Assume that for every player $i \in I$ and every $t_i \in T_i$ the set $\Phi_i(t_i)$ is closed. Then, the set of MMS mixed equilibria of Γ is not empty.*

Proof. From the previous Proposition, $V_i(\cdot, \cdot|t_i)$ is continuous in $X_i \times S_{-i}$, then, from the classical Berge Theorem it follows that the correspondence $\sigma_{-i} \rightsquigarrow \arg\max_{x_i\in X_i} V_i(x_i, \sigma_{-i}|t_i)$ is upper semicontinuous with not empty, compact images. It can be easily seen that $V_i(\cdot, \sigma_{-i}|t_i)$ is concave for every σ_{-i}. In fact, for every $Q_i \in \Phi_i(t_i)$ the function $U_i(\cdot, \sigma_{-i}|Q_i) + c_i(Q_i)$ is clearly concave and the minimum of concave functions is concave. Hence the correspondence $\sigma_{-i} \rightsquigarrow \arg\max_{x_i\in X_i} V_i(x_i, \sigma_{-i}|t_i)$ has convex values. Now the proof follows the same steps of Proposition 1 in [10]. In fact, from the previous results it immediately follows that the correspondence $\sigma_{-i} \rightsquigarrow B_{i,t_i}(\sigma_{-i})$ defined by

$$B_{i,t_i}(\sigma_{-i}) = \left\{\sigma_i \in S_i \mid \sigma_i(t_i) \in \arg\max_{x_i\in X_i} V_i(x_i, \sigma_{-i}|t_i)\right\} \quad \forall \sigma_{-i} \in S_{-i}$$

is upper semicontinuous with not empty, compact and convex images. So it is the best reply correspondence $\sigma_{-i} \rightsquigarrow B_i(\sigma_{-i})$ defined by

$$B_i(\sigma_{-i}) = \bigcap_{t_i \in T_i} B_{i,t_i}(\sigma_{-i}) \quad \forall \sigma_{-i} \in S_i$$

Hence the correspondence $\sigma \rightsquigarrow \prod_{i\in I} B_i(\sigma)$ satisfies the Kakutani fixed point theorem. Since a fixed point σ for $\prod_{i\in I} B_i$ is clearly a MMR mixed equilibrium then we get the assertion.□

3 Stability

Problem statement

For every player i, consider a sequence $\{c_{i,\nu}\}_{\nu\in\mathbb{N}}$ of indices of ambiguity aversion, the corresponding sequences of utility functions $\{V_{i,\nu}\}_{\nu\in\mathbb{N}}$ and the corresponding sequence of games[7] $\{\Gamma_\nu\}_{\nu\in\mathbb{N}}$. where

$$\Gamma_\nu = \{I; \Theta; (\mathbb{P}_i)_{i\in I}; (\Phi_i)_{i\in I}; (S_i)_{i\in I}; (V_{i,\nu})_{i\in I}\}. \tag{9}$$

In this section we look for conditions of convergence of the sequences $\{c_{i,\nu}\}_{\nu\in\mathbb{N}}$ to the indices of ambiguity aversion c_i for $i = 1, \ldots, n$, which guarantee that:

i) The corresponding sequences of variational preferences $\{V_{i,\nu}\}_{\nu\in\mathbb{N}}$ converge in an appropriate way to the variational preferences V_i corresponding to c_i, for $i = 1, \ldots, n$.

ii) Converging sequences of equilibria of the perturbed games $\{\Gamma_\nu\}_{\nu\in\mathbb{N}}$ have their limits in the set of equilibria of the unperturbed game Γ corresponding to the variational preferences V_i, for $i = 1, \ldots, n$.

Technical tools

Definition 3.1: Given a sequence of functions $\{f_\nu\}_{\nu\in\mathbb{N}}$, with $f_\nu : Z \subseteq \mathbb{R}^k \to \overline{\mathbb{R}}$ for every $\nu \in \mathbb{N}$. Then $\{f_\nu\}_{\nu\in\mathbb{N}}$ *hypoconverges* to the function f if

i) For every $z \in Z$ there exists a sequence $\{z_\nu\}_{\nu\in\mathbb{N}} \subset Z$ converging to z such that

$$\limsup_{\nu\to\infty} f_\nu(z_\nu) \leqslant f(z)$$

ii) For every $z \in Z$ and for every sequence $\{z_\nu\}_{\nu\in\mathbb{N}} \subset Z$ converging to z it follows that

$$f(z) \leqslant \liminf_{\nu\to\infty} f_\nu(z_\nu).$$

Moreover, the sequence of functions $\{f_\nu\}_{\nu\in\mathbb{N}}$ *epiconverges* to the function f if the sequence of functions $\{-f_\nu\}_{\nu\in\mathbb{N}}$ hypoconverges to the function $-f$.

Definition 3.2: Given a sequence of functions $\{f_\nu\}_{\nu\in\mathbb{N}}$, with $f_\nu : Z \subseteq \mathbb{R}^k \to \overline{\mathbb{R}}$ for every $\nu \in \mathbb{N}$. Then $\{f_\nu\}_{\nu\in\mathbb{N}}$ continuously converges to the function f if it hypoconverges and epiconverges to f, that is for every $z \in Z$ and for every sequence $\{z_\nu\}_{\nu\in\mathbb{N}} \subset Z$ converging to z it follows that

$$\limsup_{\nu\to\infty} f_\nu(z_\nu) \leqslant f(z) \leqslant \liminf_{\nu\to\infty} f_\nu(z_\nu).$$

[7] Again, it is implicitly assumed that the utility functions $V_{i,\nu}$ are well posed along the sequence, (i.e. $\min_{Q_i\in\Phi_i(t_i)} [U_i(x_i, \sigma_{-i}|Q_i) + c_{i,\nu}(Q_i)]$ exist for every $x \in X$, $\sigma \in S$, $t_i \in T_i$ and $\nu \in \mathbb{N}$).

Results

PROPOSITION 3.3: *Assume that the sequence of indices of ambiguity aversion $\{c_{i,\nu}\}_{\nu\in\mathbb{N}}$ hypoconverges to the index of ambiguity aversion c_i. Then, for every $t_i \in T_i$, the sequence of function $\{V_{i,\nu}(\cdot,\cdot|t_i)\}_{\nu\in\mathbb{N}}$ continuously converges to $V_i(\cdot,\cdot|t_i)$.*

Proof. Given $(\overline{x}_i, \overline{\sigma}_{-i})$ and let $\{(x_{i,\nu}, \sigma_{-i,\nu})\}_{\nu\in\mathbb{N}} \subset X_i \times S_{-i}$ be a sequence converging to $(\overline{x}_i, \overline{\sigma}_{-i})$, first we prove that

$$\limsup_{\nu\to+\infty} V_{i,\nu}((x_{i,\nu}, \sigma_{-i,\nu})|t_i) \leqslant V_i(\overline{x}_i, \overline{\sigma}_{-i}|t_i). \tag{10}$$

If $V_i(\overline{x}_i, \overline{\sigma}_{-i}|t_i) = +\infty$ then (10) is obviously satisfied. Suppose $V_i(\overline{x}_i, \overline{\sigma}_{-i}|t_i) < \infty$. Denote with F_i and $F_{i,\nu}$ the functions defined respectively by

$$\begin{aligned} i) \quad & F_{i,\nu}(x_i, \sigma_{-i}, Q_i) = U(x_i, \sigma_{-i}, Q_i) + c_{i,\nu}(Q_i) \\ ii) \quad & F_i(x_i, \sigma_{-i}, Q_i) = U(x_i, \sigma_{-i}, Q_i) + c_i(Q_i). \end{aligned}$$

Let $\overline{Q}_i$ be such that

$$F_i(\overline{x}_i, \overline{\sigma}_{-i}, \overline{Q}_i) = \min_{Q_i\in\Phi_i(t_i)} F_i(\overline{x}_i, \overline{\sigma}_{-i}, Q_i) = V_i(\overline{x}_i, \overline{\sigma}_{-i}|t_i)$$

Since $\{c_{i,\nu}\}_{\nu\in\mathbb{N}}$ hypoconverges to c_i there exists a sequence $\{\widetilde{Q}_{i,\nu}\}_{\nu\in\Phi_i(t_i)}$ converging to $\overline{Q}_i$ such that

$$\limsup_{\nu\to+\infty} c_{i,\nu}(\widetilde{Q}_{i,\nu}) \leqslant c_i(\overline{Q}_i);$$

U_i is continuous, then $\limsup_{\nu\to+\infty} U_i(x_{i,\nu}, \sigma_{-i,\nu}|\widetilde{Q}_{i,\nu}) = U_i(\overline{x}_i, \overline{\sigma}_{-i}|\overline{Q}_i)$. Hence

$$\begin{aligned} \limsup_{\nu\to+\infty} F_{i,\nu}(x_{i,\nu}, \sigma_{-i,\nu}, \widetilde{Q}_{i,\nu}) \leqslant \\ \limsup_{\nu\to+\infty} U_i(x_{i,\nu}, \sigma_{-i,\nu}|\widetilde{Q}_{i,\nu}) + \limsup_{\nu\to+\infty} c_{i,\nu}(\widetilde{Q}_{i,\nu}) \leqslant \\ U_i(\overline{x}_i, \overline{\sigma}_{-i}|\overline{Q}_i) + c_i(\overline{Q}_i) = V_i(\overline{x}_i, \overline{\sigma}_{-i}|t_i). \end{aligned}$$

Let $\{\overline{Q}_{i,\nu}\}_{\nu\in\Phi_i(t_i)}$ be a sequence such that

$$F_{i,\nu}(x_{i,\nu}, \sigma_{-i,\nu}, \overline{Q}_{i,\nu}) = \min_{Q_i\in\Phi_i(t_i)} F_{i,\nu}(x_{i,\nu}, \sigma_{-i,\nu}, Q_{i,\nu}) = V_{i,\nu}(x_{i,\nu}, \sigma_{-i,\nu}|t_i) \quad \forall\nu\in\mathbb{N}$$

Since $F_{i,\nu}(x_{i,\nu}, \sigma_{-i,\nu}, \overline{Q}_{i,\nu}) \leqslant F_{i,\nu}(x_{i,\nu}, \sigma_{-i,\nu}, \widetilde{Q}_{i,\nu})$ for every $\nu \in \mathbb{N}$, we get

$$\begin{aligned} \limsup_{\nu\to+\infty} V_{i,\nu}(x_{i,\nu}, \sigma_{-i,\nu}|t_i) = \limsup_{\nu\to+\infty} F_{i,\nu}(x_{i,\nu}, \sigma_{-i,\nu}, \overline{Q}_{i,\nu}) \leqslant \\ \limsup_{\nu\to+\infty} F_{i,\nu}(x_{i,\nu}, \sigma_{-i,\nu}, \widetilde{Q}_{i,\nu}) \leqslant V_i(\overline{x}_i, \overline{\sigma}_{-i}|t_i). \end{aligned}$$

Therefore (10) is satisfied.

Given $(\overline{x}_i, \overline{\sigma}_{-i})$ and let $\{(x_{i,\nu}, \sigma_{-i,\nu})\}_{\nu\in\mathbb{N}} \subset X_i \times S_{-i}$ be a sequence converging to $(\overline{x}_i, \overline{\sigma}_{-i})$, now we prove that

$$V_i(\overline{x}_i, \overline{\sigma}_{-i}|t_i) \leqslant \liminf_{\nu\to+\infty} V_{i,\nu}((x_{i,\nu}, \sigma_{-i,\nu})|t_i) \tag{11}$$

Suppose that (11) doesn't hold, that is

$$\liminf_{\nu\to+\infty} V_{i,\nu}(x_{i,\nu},\sigma_{-i,\nu}|t_i) < V_i(\overline{x}_i,\overline{\sigma}_{-i}|t_i). \tag{12}$$

Let $\overline{Q}_i$ be such that

$$F_i(\overline{x}_i,\overline{\sigma}_{-i},\overline{Q}_i) = \min_{Q_i\in\Phi_i(t_i)} F_i(\overline{x}_i,\overline{\sigma}_{-i},Q_i) = V_i(\overline{x}_i,\overline{\sigma}_{-i}|t_i)$$

and $\{\widetilde{Q}_{i,\nu}\}_{\nu\in\Phi_i(t_i)}$ be a sequence such that

$$F_{i,\nu}(x_{i,\nu},\sigma_{-i,\nu},\widetilde{Q}_{i,\nu}) = \min_{Q_i\in\Phi_i(t_i)} F_{i,\nu}(x_{i,\nu},\sigma_{-i,\nu},Q_i) = V_{i,\nu}(x_{i,\nu},\sigma_{-i,\nu}|t_i) \quad \forall\nu\in\mathbb{N}.$$

Then, there exists a subsequence $\{\nu_k\}_{k\in\mathbb{N}}$ such that $F_{i,\nu_k}(x_{i,\nu_k},\sigma_{-i,\nu_k},\widetilde{Q}_{i,\nu_k}) < V_i(\overline{x}_i,\overline{\sigma}_{-i}|t_i)$ for all k. Since $\Phi_i(t_i)$ is compact there exists a subsequence $\{\nu_{k_h}\}_{h\in\mathbb{N}}\subseteq\{\nu_k\}_{k\in\mathbb{N}}$ such that the subsequence $\{\widetilde{Q}_{\nu_{k_h}}\}_{h\in\mathbb{N}}$ converges to a point $\widetilde{Q}_i\in\Phi_i(t_i)$. Obviously it follows that

$$\liminf_{h\to+\infty} F_{i,\nu_{k_h}}(x_{i,\nu_{k_h}},\sigma_{-i,\nu_{k_h}},\widetilde{Q}_{i,\nu_{k_h}}) < V_i(\overline{x}_i,\overline{\sigma}_{-i}|t_i)$$

On the other hand, since $\{c_{i,\nu}\}_{\nu\in\mathbb{N}}$ hypoconverges to c_i, it follows that

$$c_i(\widetilde{Q}_i) \leqslant \liminf_{h\to+\infty} c_{i,\nu_{k_h}}(\widetilde{Q}_{i,\nu_{k_h}}),$$

U_i is continuous, then $\liminf_{h\to+\infty} U_i(x_{i,\nu_{k_h}},\sigma_{-i,\nu_{k_h}}|\widetilde{Q}_{i,\nu_{k_h}}) = U_i(\overline{x}_i,\overline{\sigma}_{-i}|\widetilde{Q}_i)$. Hence

$$\begin{aligned}
\liminf_{h\to+\infty} F_{i,\nu_{k_h}}(x_{i,\nu_{k_h}},\sigma_{-i,\nu_{k_h}},\widetilde{Q}_{i,\nu_{k_h}}) \geqslant \\
\liminf_{\nu\to+\infty} U_i(x_{i,\nu_{k_h}},\sigma_{-i,\nu_{k_h}}|\widetilde{Q}_{i,\nu_{k_h}}) + \liminf_{h\to+\infty} c_{i,\nu_{k_h}}(\widetilde{Q}_{i,\nu_{k_h}}) \geqslant \\
U_i(\overline{x}_i,\overline{\sigma}_{-i}|\widetilde{Q}_i) + c_i(\widetilde{Q}_i) \geqslant U_i(\overline{x}_i,\overline{\sigma}_{-i}|\overline{Q}_i) + c_i(\overline{Q}_i) = V_i(\overline{x}_i,\overline{\sigma}_{-i}|t_i)
\end{aligned}$$

So we get a contradiction and

$$\liminf_{\nu\to+\infty} V_{i,\nu}(x_{i,\nu},\sigma_{-i,\nu}|t_i) \geqslant V_i(\overline{x}_i,\overline{\sigma}_{-i}|t_i)$$

which completes the proof. □

REMARK 3.4: Note that we provide a direct proof of the previous Proposition. An alternative proof can be obtained by applying Propositions 3.1.2 and 4.1.2 in [11].

The following result follows easily from the previous Proposition:

PROPOSITION 3.5: *Given the game* Γ *corresponding to the vector of ambiguity indices* $(c_1,\ldots,c_n)$. *Assume that* $\{\Gamma_\nu\}_{\nu\in\mathbb{N}}$ *is a sequence of games defined by (9) such that, for every player i the sequence* $\{c_{i,\nu}\}_{\nu\in\mathbb{N}}$ *hypoconverges to* c_i. *Let* $\{\sigma^*_\nu\}_{\nu\in\mathbb{N}}$ *be a sequence of strategy profiles such that each* σ^*_ν *is a MMR mixed equilibrium of* Γ_ν. *If* $\{\sigma^*_\nu\}_{\nu\in\mathbb{N}}$ *converges to* σ^*, *(i.e.* $\sigma^*_{i,\nu}(\psi_i|\tau_i(\theta)) \to \sigma^*_{i,\nu}(\psi_i|\tau_i(\theta))$ *as* $\nu\to\infty$, *for every* i,ψ_i,θ*), then,* σ^* *is a MMR mixed equilibrium of* Γ.

Proof. Let $\{\sigma_\nu^*\}_{\nu\in\mathbb{N}}$ be the sequence of strategy profiles converging to σ^* where each σ_ν^* is a MMR mixed equilibrium of Γ_ν. Given $\overline{\theta} \in \Theta$ and a player $i \in I$, by definition it follows that, for every $\nu \in \mathbb{N}$,

$$V_{i,\nu}(\sigma_{i,\nu}^*(\tau_i(\overline{\theta})), \sigma_{-i,\nu}^*|\tau_i(\overline{\theta})) \geqslant V_{i,\nu}(x_i, \sigma_{-i,\nu}^*|\tau_i(\overline{\theta})) \quad \forall x_i \in X_i.$$

From the previous Proposition and taking the limit as $\nu \to \infty$, we get

$$V_i(\sigma_i^*(\tau_i(\overline{\theta})), \sigma_{-i}^*|\tau_i(\overline{\theta})) = \lim_{\nu\to\infty} V_{i,\nu}(\sigma_{i,\nu}^*(\tau_i(\overline{\theta})), \sigma_{-i,\nu}^*|\tau_i(\overline{\theta})) \geqslant$$
$$\lim_{\nu\to\infty} V_{i,\nu}(x_i, \sigma_{-i,\nu}^*|\tau_i(\overline{\theta})) = V_i(x_i, \sigma_{-i}^*|\tau_i(\overline{\theta})) \quad \forall x_i \in X_i$$

and the assertion follows. □

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