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Asymptotic Behavior of Regularized Optimization Problems with Quasi-variational Inequality Constraints

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Abstract

The great interest into hierarchical optimization problems and the increasing use of game theory in many economic or engineering applications led to investigate optimization problems with constraints described by the solutions to a quasi-variational inequality (variational problems having constraint sets depending on their own solutions, present in many applications as social and economic networks, financial derivative models, transportation network congestion and traffic equilibrium). These problems are bilevel problems such that at the lower level a parametric quasi-variational inequality is solved (by one or more followers) meanwhile at the upper level the leader solves a scalar optimization problem with constraints determined by the solutions set to the lower level problem. In this paper, mainly motivated by the use of approximation methods in infinite dimensional spaces (penalization, discretization, Moreau-Yosida regularization ...), we are interested in the asymptotic behavior of the sequence of the infimal values and of the sequence of the minimum points of the upper level when a general scheme of perturbations is considered. Unfortunately, we show that the global convergence of exact values and exact solutions of the perturbed bilevel problems cannot generally be achieved. Thus, we introduce suitable concepts of regularized optimization problems with quasi-variational inequality constraints and we investigate, in Banach spaces, the behavior of the approximate infimal values and of the approximate solutions under and without perturbations.

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1 Introduction

Let (X, τ) be a Hausdorff topological space and let E be a real Banach space with dual E^*. Given a nonempty closed subset H of X, a nonempty convex, closed subset K of E, an operator A from $H \times K$ to E^* and a set-valued map S from $H \times K$ to K with nonempty values, we consider, for every $x \in H$, the parametric quasi-variational inequality $Q(x)$:

$$\text{find } \ u \in S(x,u) \ \text{ such that } \ \langle A(x,u), u - w \rangle \ \le \ 0 \ \ \forall\, w \in S(x,u). \tag{1}$$

We denote by $\mathcal{Q}(x)$ the set of solutions to the problem $Q(x)$ and we remark that the solution map $\mathcal{Q}: \ x \in H \to \mathcal{Q}(x)$ is set-valued even under restrictive assumptions [5].
The problem $Q(x)$ is a particular case of a more general parametric problem considered in [18]

$$\text{find } u \in K \text{ such that } h(x,u,w) + \phi(x,u,u) \ \le \ \phi(x,u,w) \ \ \forall\, w \in E \tag{2}$$

where $h: H \times K \times K \to \mathbb{R}$ and $\phi: H \times K \times K \to \mathbb{R} \cup \{+\infty\}$ are single-valued maps.
Indeed, it suffices to consider the function ϕ defined by the indicator function ψ of the set $S(x,u)$ which takes the value 0 on $S(x,u)$ and the value $+\infty$ otherwise, that is $\phi(x,u,w) = \psi_{S(x,u)}(w)$, and the function h defined by $h(x,u,w) = \langle A(x,u), u - w\rangle$. Also observe that when $h(x,u,w) = \langle A(x,u), u - w\rangle$ and $\phi(x,w) = \psi_{T(x)}(w)$, where T is a set-valued map from H to K, problem (2) becomes a parametric variational inequality [14].
The great interest into hierarchical optimization problems and the increasing use of game theory in many economic or engineering applications [23], [10], [6], [9], [24] led to investigate optimization problems with constraints described by the solutions to a quasi-variational inequ! ality. Such a problem, that we denominate *Semiquasi-variational Bilevel Problem*, consists in finding $(x_o, u_o) \in H \times K$ such that

$$(SB) \qquad\qquad u_o \in \mathcal{Q}(x_o) \text{ and } \ f(x_o,u_o) \ = \ \min_{x\in H} \min_{u \in \mathcal{Q}(x)} f(x,u)$$

where f is a function from $H \times K$ to $\mathbb{R} \cup \{+\infty\}$.
Here, in analogy with the term *semivectorial* introduced in [4], the term "semiquasi-variational" means that at the lower level a parametric quasi-variational inequality is solved (by one or more followers) meanwhile at the upper level the leader solves a scalar optimization problem with constraints determined by the solutions set to the lower level problem.
The set of solutions and the infimal value of the problem (SB) are denoted by $\mathcal{M}$ and φ respectively, so we have

$$(x_o,u_o) \in \mathcal{M} \iff \ u_o \in \mathcal{Q}(x_o) \text{ and } f(x_o,u_o) = \varphi = \ \min_{x\in H} \min_{u \in \mathcal{Q}(x)} f(x,u). \tag{3}$$

In this paper, motivated by the use of approximation methods (penalization, discretization, Moreau-Yosida regularization...), we are interested in the asymptotic behavior of the infimal values and of the minimum points when a general scheme of perturbations is considered. More precisely, given a sequence of operators $(A_n)_n$, a sequence of functions $(f_n)_n$ and a sequence of set-valued maps $(S_n)_n$, we are interested in the asymptotic behavior of the sequences $(\varphi_n)_n$ of the infimal values for the problems $(SB)_n$ as well in the asymptotic behavior of the corresponding sequences $(\mathcal{M}_n)_n$ of solutions sets, where

$$(x_n,u_n) \in \mathcal{M}_n \iff \ u_n \in \mathcal{Q}_n(x_n) \text{ and } f_n(x_n,u_n) = \varphi_n = \ \min_{x\in H} \min_{u \in \mathcal{Q}_n(x)} f_n(x,u).$$

Results concerning the values behavior have been recently obtained in finite dimensional spaces:
− for the infimal value of optimization problems with variational inequality constraints [20];
− for the security value of minsup problems with quasi-variational or variational inequality constraints [21].
Here, in the setting of infinite dimensional spaces, we have two aims: one is to investigate the behavior of the values of semiquasi-variational bilevel problems, one is to study the asymptotic behavior of their solutions.
When the set-valued maps S and S_n are constant with respect to u, namely $S : H \to K$ and $S_n : H \to K$, the quasi-variational inequalities $Q(x)$ and $Q_n(x)$ amount to variational inequalities so the results on the infimal values behavior of this paper extend the corresponding results in finite dimensional spaces given in [20]. However, this extension needs different conditions on the operators A and A_n compared with the assumptions presented in [20].
In fact, we recall that in order to avoid very restrictive assumptions in the investigation of variational inequalities in infinite dimensional spaces, the *Minty* variational inequality (also called dual variational inequality in [5]) is usually considered. Therefore, due to the infinite dimension of the space E, we also introduce the parametric Minty quasi-variational inequality $\widetilde{Q}(x)$ which consists in finding $u \in K$ such that

$$u \in S(x,u) \text{ and } \langle A(x,w), u-w\rangle \le 0 \;\; \forall\, w \in S(x,u)$$

and we denote by $\widetilde{\mathcal{Q}}(x)$ the set of solutions to $\widetilde{Q}(x)$ for every $x \in H$. Then, the *Minty Semiquasi-variational Bilevel Problem* $\widetilde{(SB)}$ consists in finding $(x_o, u_o) \in H \times K$ such that

$$u_o \in \widetilde{\mathcal{Q}}(x_o) \text{ and } f(x_o,u_o) = \min_{x\in H} \min_{u \in \widetilde{\mathcal{Q}}(x)} f(x,u).$$

The solutions set and the infimal value are denoted by $\widetilde{\mathcal{M}}$ and $\widetilde{\varphi}$ respectively. The investigation of such problems under perturbations, which will involve the solution sets $\widetilde{\mathcal{M}}_n$ and the infimal values $\widetilde{\varphi}_n$, will be crucial for the investigation of the asymptotic behavior of semiquasi-variational bilevel problems in infinite dimensional spaces. We remark that partial results concerning the asymptotic behavior of solutions to optimization problems with Minty variational inequality constraints have been stated in [14].
The rest of this paper is organized as follows. The next section contains brief preliminaries on continuity, convergence and monotonicity properties for functions, set-valued mappings and operators, as well as two lemmas proven in [13] and in [20]. In the third section, we analyze the convergence of the exact solutions and of the infimal values and we show that the global convergence of exact values and solutions cannot generally be achieved. Thus, we introduce suitable concepts of approximate values and solutions for semiquasi-variational problems and we investigate in Section 4 and in Section 5 the behavior of the approximate infimal values under and without perturbations respectively. In Section 6 we study the behavior of the related approximate solutions and we also introduce a concept of *hybrid* approximate solution that turns out to be useful in the perturbed case. Section 7 contains a brief discussion on the obtained results.

2 Preliminaries

The following notions ([3], [15]) will be used in the paper. Let τ and σ be topologies on the set X and on the space E respectively and let s, w, s^*, w^* denote the strong and the weak topology on

the spaces E and E^* respectively.
If $(K_n)_n$ is a sequence of nonempty subsets of E, the Painlevé-Kuratowski upper and lower limits of the sequence $(K_n)_n$, with respect to σ, are defined respectively by
• $z \in \sigma\text{-}\limsup_n K_n$ if there exists a sequence $(z_k)_k$ σ-converging to z such that $z_k \in K_{n_k}$ for a subsequence (K_{n_k}) of $(K_n)_n$ and for each $k \in \mathbb{N}$;
• $z \in \sigma\text{-}\liminf_n K_n$ if there exists a sequence $(z_n)_n$ σ-converging to z such that $z_n \in K_n$ for n sufficiently large.
We recall that both these sets are σ-closed and may be empty.
A function $h : H \subseteq X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is τ-*coercive* on H if for every $t \in \mathbb{R}$ there exists a set $C_t \subseteq X$, sequentially compact in the topology τ, such that

$$Lev_t\, h \;= \{x \in H : h(x) \leq t\} \;\subseteq\; C_t.$$

A function $g : H \times K \to \mathbb{R} \cup \{-\infty, +\infty\}$ is $\sigma-$*coercive with respect to* u on the set K *uniformly with respect to* $x \in H$ (*coercive in* u *on* K for short) if for every $t \in \mathbb{R}$ there exists a set $Y_t \subseteq E$ sequentially compact in the topology σ such that

$$(Lev_t\, g)\,(x) = \{u \in K : g(x,u) \leq t\} \subseteq\; Y_t$$

for every $x \in H$.
A set-valued map F from H to K is:
$-$ (τ,σ)-*sequentially subcontinuous* over H, (τ,σ)-*subcontinuous* for short, if for every $x \in H$, every sequence $(x_n)_n$ τ-converging to x in H, every sequence $(u_n)_n$ such that $u_n \in F(x_n)$, for every $n \in \mathbb{N}$, has a subsequence σ-converging;
$-$ (τ,σ)-*sequentially lower semicontinuous* over H, (τ,σ)-*lower semicontinuous* for short, if for every $x \in X$ and every sequence $(x_n)_n$ τ-converging to x in H

$$F(x) \;\subseteq\; \sigma\text{-}\liminf_n \; F(x_n);$$

$-$ (τ,σ)-*sequentially closed* over H, (τ,σ)-*closed* for short, if for every $x \in H$ and every sequence $(x_n)_n$ τ-converging to x in H

$$\sigma\text{-}\limsup_n \; F(x_n) \;\subseteq\; F(x).$$

A sequence $(F_n)_n$ of set-valued maps from H to K
$-$ (τ,σ)-*lower converges* to F in H if for every $x \in H$ and every sequence $(x_n)_n$ converging to x in H

$$F(x) \;\subseteq\; \sigma\text{-}\liminf_n \; F_n(x_n);$$

$-$ (τ,σ)-*upper converges* to F in H if for every $x \in H$ and every sequence $(x_n)_n$ τ-converging to x in H

$$\sigma\text{-}\limsup_n \; F_n(x_n) \;\subseteq\; F(x).$$

An operator T from K to E^* is:
$-$ *hemicontinuous* over K if it is continuous from every segment of K to E^* endowed with the weak topology.
$-$ *pseudomonotone* over K (see, for example, [17]) if

$$\langle Au, u-v\rangle \leq 0 \implies \langle Av, u-v\rangle \leq 0 \quad \forall\; u \text{ and } v \in K;$$

$-$ *monotone* over K (see, for example, [11]) if

$$\langle Au - Av, u - v\rangle \geq 0 \quad \forall\, u \text{ and } v \in K.$$

A sequence $(T_n)_n$ of operators from K to E^*:
$-$ (s,s^*)-G^--*converges* to T in K if for every $u \in K$ there exists a sequence $(u'_n)_n$ s-converging to u in K such that $s^*\text{-}\lim_n T_n(u'_n) = T(u)$, that is

$$Graph\, T \;\subseteq\; (s \times s^*)\text{-}\liminf_n \; Graph\, T_n;$$

$-$ is σ-*equi-coercive* on K if there exist a point $v_o \in K$ and, for every $t \in \mathbb{R}$, a set $Z_t \subseteq E$, sequentially compact in σ, such that

$$\{u \in K : \langle T_n(u), u - v_o\rangle \leq t\} \;\subseteq\; Z_t \text{ for all } n \in \mathbb{N};$$

$-$ is *uniformly bounded* on K if there exists a positive real number M such that $\sup_n ||T_n(u_n)|| \leq M$ for every weakly convergent sequence $(u_n)_n$, $u_n \in K$ for every $n \in \mathbb{N}$.
A sequence of functions $(g_n)_n$, $g_n : H \times K \to \mathbb{R} \cup \{-\infty, +\infty\}$:
$-$ *sequentially* $(\tau \times \sigma)$-*continuously converges* to a function g in $H \times K$, $c-$*converges* to g in $(\tau \times \sigma)$ for short, if for every $(x,u) \in H \times K$ and every sequence $(x_n, u_n)_n$ $(\tau \times \sigma)$-converging to (x,u) in $H \times K$, one has $\lim_n g_n(x_n, u_n) = g(x,u)$;
$-$ is $(\tau \times \sigma)$-*equi-coercive* on $H \times K$ if for every $t \in \mathbb{R}$ there exists a set $W_t \subseteq X \times E$, sequentially compact in $(\tau \times \sigma)$, such that

$$Lev_t\, g_n \;= \{(x,u) \in H \times K \;: g_n(x,u) \leq t\} \;\subseteq\; W_t \text{ for all } n \in \mathbb{N}.$$

For examples that illustrate and compare the above concepts see [3], [11], [15].

The following lemmas are basic for the next sections.

Lemma 2.1 ([13], Lemma 3.1)
Let $(H_n)_{n \in \mathbb{N} \cup \{0\}}$ be a sequence of nonempty convex subsets of E such that:

i) $H_o \subseteq s\text{-Liminf}_n\, H_n$;

ii) *there exists $m \in N$ such that $int \bigcap_{n \geq m} H_n \neq \emptyset$.*

Then, for every $u \in int\, H_o$ there exists a positive real number δ such that

$$B(u,\delta) \subseteq H_n \;\forall\, n \geq m.$$

Lemma 2.2 ([20], Lemma 2.3)
Let $(S_n)_n$ be a sequence of set-valued maps from $H \times K$ to K.
• *If $(S_n)_n$ $(\tau \times w, s)$-lower converges to S in $H \times K$, then, for every $x \in H$, every $u \in K$, every sequence $(\tau \times w)$-$(x_n, u_n)_n$ converging in $H \times K$ towards (x,u), $(x_n,u_n) \in H \times K$, one has*

$$\limsup_n d(u_n, S_n(x_n, u_n)) \;\leq\; d(u, S(x,u)).$$

• *If $(S_n)_n$ $(\tau \times w, w)$-upper converges to S in $H \times K$ and the following holds:*

C_0) *given a $(\tau \times w)$-convergent sequence $(x_n, u_n)_n$, $(x_n, u_n) \in H \times K$, every sequence $(w_n)_n$, such that $w_n \in S_n(x_n, u_n)$ for all $n \in \mathbb{N}$, has a weakly convergent subsequence,* *then, for every $x \in H$, every $u \in K$, every sequence $(\tau \times w)$-$(x_n, u_n)_n$ converging in $H \times K$ towards (x, u), $(x_n, u_n) \in H \times K$, one has*

$$d(u, S(x,u)) \;\leq\; \liminf_n \, d(u_n, S_n(x_n, u_n)).$$

3 Values and solutions: asymptotic behavior

With the notations of Section 1, given $n \in \mathbb{N}$, we consider the semiquasi-variational bilevel problem

$$(SB)_n \qquad \text{find } (x_n, u_n) \in H \times K \text{ such that } u_n \in \mathcal{Q}_n(x_n) \text{ and } \; f_n(x_n, u_n) \;=\; \min_{x \in H} \min_{u \in \mathcal{Q}_n(x)} f_n(x, u)$$

where, for any given $x \in H$,

$$\mathcal{Q}_n(x) = \{u \in K : \; u \in S_n(x,u) \text{ and } \langle A_n(x,u), u - w\rangle \leq \; 0 \;\; \forall \, w \in S_n(x,u)\}$$

is the solutions set to the quasi-variational inequality $Q_n(x)$.
As observed in the Introduction, we need to introduce also the Minty semiquasi-variational bilevel problem

$$\widetilde{(SB)}_n \qquad \text{find } (x_n, u_n) \in H \times K \text{ such that } u_n \in \widetilde{\mathcal{Q}}_n(x_n) \text{ and } \; f_n(x_n, u_n) \;=\; \min_{x \in H} \min_{u \in \widetilde{\mathcal{Q}}_n(x)} f_n(x, u)$$

where, for any given $x \in H$,

$$\widetilde{\mathcal{Q}}_n(x) = \{u \in K : u \in S_n(x,u) \text{ and } \langle A_n(x,w), u - w\rangle \leq \; 0 \;\; \forall \, w \in S_n(x,u)\}$$

is the solutions set to the Minty quasi-variational inequality $\widetilde{Q}_n(x)$.

Throughout the paper we make the following assumptions on the constraints maps:

C_1) The set-valued maps S and S_n are closed-valued and convex-valued, for every $n \in \mathbb{N}$.
C_2) The sequence $(S_n)_n$ $(\tau \times w, s)$-lower converges to S in $H \times K$.
C_3) The sequence $(S_n)_n$ $(\tau \times w, w)$-upper converges to S in $H \times K$.
C_4) The sets $\mathcal{Q}(x)$ and $\widetilde{\mathcal{Q}}(x)$, $\mathcal{Q}_n(x)$ and $\widetilde{\mathcal{Q}}_n(x)$, are nonempty for every $x \in H$ and $n \in \mathbb{N}$.

We point out that conditions C_1) and C_2) together imply that the sequence of sets $(S_n(x_n, u_n))_n$ Mosco converges [1] to the set $S(x,u)$ for every $(x,u) \in H \times K$ and every sequence $(x_n, u_n)_n$ $(\tau \times w)$-converging to (x,u) in $H \times K$.
In this section, we show that the infimal values φ_n of the problems $(SB)_n$ approach from above the infimal value φ of the problem (SB) under suitable assumptions and that this may fail to be true from below. Moreover, we also show that the limit of a convergent sequence of solutions to $(SB)_n$ may be not a solution to (SB) as well a solution to (SB) may be not approached by a sequence of solutions to $(SB)_n$, that is, the inclusions

$$\mathcal{M} \subseteq s - \liminf_n \, \mathcal{M}_n \qquad \text{and} \qquad w - \limsup_n \, \mathcal{M}_n \subseteq \mathcal{M}$$

may fail to be true even under very restrictive conditions. Hence, in general we cannot expect that the sequence $(\mathcal{M}_n)_n$ Mosco converges to $\mathcal{M}$ [1].

Proposition 3.1 *Assume that the following hold:*
i) for every $x \in H$, the operator $A(x,\cdot)$ is hemicontinuous on K;
ii) for every $n \in \mathbb{N}$ the operator $A_n(x,\cdot)$ is monotone on K;
iii) for every $x \in H$ and every sequence $(x_n)_n$ τ-converging to x in H, the sequence $(A_n(x_n,\cdot))_n$ is uniformly bounded over K;
iv) for every $x \in H$ and every sequence $(x_n)_n$ τ-converging to x in H, the sequence $(A_n(x_n,\cdot))_n$ (s,s^*)-G^--converges to $A(x,\cdot)$;
v) the sequence $(f_n)_n$ is $(\tau \times w)$-equicoercive on $H \times K$;
vi) for every $(x,u) \in X \times K$ and every sequence $(x_n,u_n)_n$ $(\tau \times w)$-converging to (x,u) in $H \times K$ one has

$$f(x,u) \ \le \ \liminf_n \ f_n(x_n,u_n).$$

Then,

$$\varphi \ \le \ \liminf_n \varphi_n. \tag{4}$$

Proof
Assume that (4) does not hold and let a be a real number such that $\liminf_n \varphi_n \ < \ a \ < \ \varphi$. There exist an increasing sequence of integers $(n_k)_k$ and a sequence $(x_k,u_k)_k$, $(x_k,u_k) \in H \times K$, such that

$$u_k \in \mathcal{Q}_{n_k}(x_k) \ \text{ and } \ f_{n_k}(x_k,u_k) < \ a \ \ \forall\, k \in \mathbb{N}.$$

Assumption v) implies that a subsequence of $(x_k,u_k)_k$, still denoted by $(x_k,u_k)_k$, $(\tau \times w)$-converges in $H \times K$ towards $(x_o,u_o) \in H \times K$. If we prove that $u_o \in \mathcal{Q}(x_o)$, since $\varphi \le f(x_o,u_o) \le a$ by condition vi), we obtain a contradiction.
Assumption C$_3$) implies that $u_o \in S(x_o,u_o)$, while assumption C$_2$) implies that for every $w \in S(x_o,u_o)$ there exists a sequence $(w_k)_k$ strongly converging to w such that $w_k \in S_{n_k}(x_k,u_k)$ for k sufficiently large. By assumption iv), there exists a sequence $(w'_k)_k$ strongly converging to w such that $(A_k(x_k,w'_k))_k$ strongly converges to $A(x_o,w)$. Then, by assumptions ii) and iii) we have that

$$\langle A(x_o,w), u_o - w\rangle = \lim_k \langle A_k(x_k,w'_k), u_k - w'_k\rangle \le \liminf_k \langle A_k(x_k,u_k), u_k - w'_k\rangle =$$

$$\liminf_k \left[\langle A_k(x_k,u_k), u_k - w_k\rangle + \langle A_k(x_k,u_k), w_k - w'_k\rangle\right] \le \lim_k \langle A_k(x_k,u_k), w_k - w'_k\rangle = 0.$$

Since the operator A is hemicontinuous and the set-valued map S is convex-valued and closed-valued, the Minty Lemma [5] can be applied and u_o is a solution to the quasi-variational inequality $Q(x_o)$. $\square$

Corollary 3.1 *Assume that conditions ii)-vi) of Proposition 3.1 hold, then*

$$\widetilde{\varphi} \le \liminf_n \varphi_n.$$

Moreover, if the operator $A_n(x,\cdot)$ is hemicontinuous on K for every $n \in \mathbb{N}$ and $x \in H$, then

$$\widetilde{\varphi} \le \liminf_n \widetilde{\varphi}_n.$$

Proof
From Proposition 3.1 one has that

$$\varphi \le \ \liminf_n \varphi_n$$

and $\widetilde{\varphi} \leq \varphi$ by assumptions *ii*) and *iv*) because the operator $A(x, \cdot)$ is monotone on K.
If $A_n(x, \cdot)$ is also hemicontinuous on K for every $n \in \mathbb{N}$, then $\varphi_n = \widetilde{\varphi}_n$ and

$$\widetilde{\varphi} \leq \liminf_n \widetilde{\varphi}_n. \qquad \square$$

However, a result concerning the asymptotic behavior of the values $\widetilde{\varphi}_n$ can also be obtained directly. Indeed, the next proposition proves that $\widetilde{\varphi}_n$ approach $\widetilde{\varphi}$ from above under different assumptions and without monotonicity conditions.

Proposition 3.2 *Assume that the following hold:*
i) for every $x \in H$, the operator $A(x, \cdot)$ is hemicontinuous on K;
ii) for every $x \in H$ and every sequence $(x_n)_n$ τ-converging to x in H, the sequence $(A_n(x_n, \cdot))_n$ strongly pointwise converges to $A(x, \cdot)$;
iii) for every sequence $(x_n, u_n)_n$ $(\tau \times w)$-converging in $H \times K$ there exists $m \in \mathbb{N}$ such that

$$int \bigcap_{n \geq m} S(x_n, u_n) \neq \emptyset.$$

iv) the sequence $(f_n)_n$ is $(\tau \times w)$-equicoercive in $H \times K$;
v) for every $(x, u) \in H \times K$ and every sequence $(x_n, u_n)_n$ $(\tau \times w)$-converging to (x, u) in $H \times K$ one has

$$f(x, u) \ \leq \ \liminf_n \ f_n(x_n, u_n).$$

Then,

$$\widetilde{\varphi} \ \leq \ \liminf_n \widetilde{\varphi}_n. \tag{5}$$

Proof
Assume that (5) does not hold and let a be a real number such that $\liminf_n \widetilde{\varphi}_n \ < \ a \ < \ \widetilde{\varphi}$. There exist an increasing sequence of integers $(n_k)_k$ and a sequence $(x_k, u_k)_k$, $(x_k, u_k) \in H \times K$, such that

$$u_k \in \widetilde{\mathcal{Q}}_{n_k}(x_k) \ \text{ and } \ f_{n_k}(x_k, u_k) < \ a \ \ \forall \ k \in \mathbb{N}.$$

Assumption *iv*) implies that a subsequence of $(x_k, u_k)_k$, still denoted by $(x_k, u_k)_k$, $(\tau \times w)$-converges in $H \times K$ towards $(x_o, u_o) \in H \times K$. If we prove that $u_o \in \widetilde{\mathcal{Q}}(x_o)$, since $\widetilde{\varphi} \leq f(x_o, u_o) \leq a$ by condition *v*), we get a contradiction.
Assumption C$_3$) implies that $u_o \in S(x_o, u_o)$, so we have to prove that $\langle A(x_o, w), w - u_o \rangle \leq 0$ for every $w \in S(x_o, u_o)$.
If $w \in int\, S(x_o, u_o)$, then, due to Lemma 2.1 applied to $(S(x_k, u_k))_k$ and $S(x_o, u_o)$, $w \in int\, S(x_k, u_k)$ for k sufficiently large and one has $\langle A(x_o, w), w - u_o \rangle = \lim_k \langle A(x_k, w), w - u_k \rangle \leq 0$ since $u_k \in \widetilde{\mathcal{Q}}_{n_k}(x_k)$.
If $w \notin int\, S(x_o, u_o)$, being $S(x_o, u_o)$ a convex set, there exists a sequence $(w_k)_k$ strongly converging to w along a segment such that $w_k \in intS(t_o, u_o)$, so one has $\langle A(x_o, w_k), w_k - u_o \rangle \leq 0$ for k large. The conclusion then follows from assumption *i*). $\square$

The next example, presented in [20], shows that the sequence $(\varphi_n)_n$ may fail to approach from below the value φ and that the upper limit of the solutions sets $\mathcal{M}_n$ may be not contained in $\mathcal{M}$ even in finite dimensional spaces and in very restrictive conditions.

Example 3.1 Let $E=\mathbb{R}$, $K=[0,+\infty[$, $S(x,u)=S_n(x,u)=K$, $A_n(x,u)=1/n$ and $f_n(x,u)=-u+1/n$. The sequences $(A_n)_n$ and $(f_n)_n$ uniformly converge, and therefore also continuously converge, to the functions $A(x,u)=0$ and $f(x,u)=-u$ respectively. One easily checks that $\mathcal{Q}_n(x)=\{0\}$, $\mathcal{Q}(x)=[0,+\infty[$, so that $\varphi_n=1/n$ and $\varphi=-\infty$. Moreover $\mathcal{M}=\emptyset$ and $\mathcal{M}_n=X\times\{0\}$ for every $n\in\mathbb{N}$.

The next example shows that $\mathcal{M}$ can be not included in $s-\liminf_n \mathcal{M}_n$.

Example 3.2 Let $E=\mathbb{R}$, $K=[0,+\infty[$, $S(x,u)=S_n(x,u)=K$, $A_n(x,u)=A(x,u)=0$ and $f_n(x,u)=u/n$. The sequence $(f_n)_n$ continuously converges to the function $f(x,u)=0$. One easily checks that $\mathcal{Q}_n(x)=\mathcal{Q}(x)=[0,+\infty[$, so that $\mathcal{M}_n=X\times\{0\}$ for every $n\in\mathbb{N}$ meanwhile $\mathcal{M}=X\times[0,+\infty[$.

The involved operators being monotone and continuous, both examples can be also applied for $\widetilde{\varphi}$, $\widetilde{\varphi}_n$, $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}_n$, since $\mathcal{Q}_n(x)=\widetilde{\mathcal{Q}}_n(x)$ and $\mathcal{Q}(x)=\widetilde{\mathcal{Q}}(x)$ for every x.
Nevertheless, in the next section we prove that the infimal values φ and $\widetilde{\varphi}$ can be approached from above and from below by the infimal values of suitable *regularized* problems and this allows to get results also on the solutions sets.

4 Approaching the infimal value via regularization

Given a positive real number ε, in whole the paper we consider the following *approximate solutions* maps

$$\mathcal{Q}^\varepsilon:\ x\in H\to\mathcal{Q}^\varepsilon(x)=\{u\in K: d(u,S(x,u))\le\varepsilon\ \text{ and } \langle A(x,u),u-w\rangle\ \le\varepsilon\ \forall\ w\in S(x,u)\}$$

$$\mathfrak{S}^\varepsilon:\ x\in H\to\mathfrak{S}^\varepsilon(x)=\{u\in K: d(u,S(x,u))<\varepsilon\ \text{ and } \langle A(x,u),u-w\rangle<\varepsilon\ \forall\ w\in S(x,u)\}$$

that respectively associate to every x the *approximate solutions set*, introduced in [12], and the *strict approximate solutions set* to $Q(x)$.
Then, in line with [22] [15], [20], we consider the following *regularized* semiquasi-variational bilevel problems

$$(SB)^\varepsilon \quad \text{find } (x_o,u_o)\in H\times K \text{ such that } u_o\in\mathcal{Q}^\varepsilon(x_o) \text{ and } f(x_o,u_o)=\min_{x\in H}\min_{u\in\mathcal{Q}^\varepsilon(x)} f(x,u)$$

$$(\widehat{SB})^\varepsilon \quad \text{find } (x_o,u_o)\in H\times K \text{ such that } u_o\in\mathfrak{S}^\varepsilon(x_o) \text{ and } f(x_o,u_o)=\min_{x\in H}\min_{u\in\mathfrak{S}^\varepsilon(x)} f(x,u)$$

whose corresponding *approximate infimal values* are respectively

$$\varphi^\varepsilon\ =\ \inf_{x\in H}\inf_{u\in\mathcal{Q}^\varepsilon(x)} f(x,u) \qquad \psi^\varepsilon\ =\ \inf_{x\in H}\inf_{u\in\mathfrak{S}^\varepsilon(x)} f(x,u).$$

We point out that

$$\varphi^\varepsilon\ \le\ \psi^\varepsilon\ \le\ \varphi, \tag{6}$$

since

$$\mathcal{Q}(x)\ \subseteq\ \mathfrak{S}^\varepsilon(x)\ \subseteq\ \mathcal{Q}^\varepsilon(x)$$

for every $x\in H$.
Both these approximate infimal values approach the value φ when the data belong to suitable classes.

Proposition 4.1 *Assume that the following hold:*
i) for every $x \in H$, the operator $A(x,\cdot)$ is hemicontinuous on K;
ii) for every $x \in H$, the operator $A(x,\cdot)$ is pseudomonotone on K;
iii) for every $u \in K$, the operator $A(\cdot,u)$ is (τ,s^*)-continuous on H;
iv) the set-valued map S is $(\tau\times w,w)$-subcontinuous, $(\tau\times w,s)$-lower semicontinuous and $(\tau\times w,w)$-closed over H;
v) for every sequence $(x_n,u_n)_n$ $(\tau\times w)$-converging in $H\times K$ there exists $m\in\mathbb{N}$ such that

$$int \bigcap_{n\geq m} S(x_n,u_n)\neq\emptyset;$$

vi) the function f is $(\tau\times w)$-lower semicontinuous and $(\tau\times w)$-coercive on $H\times K$.
Then,

$$\varphi=\lim_{\varepsilon\to 0}\varphi^{\varepsilon}\ =\lim_{\varepsilon\to 0}\psi^{\varepsilon}$$

Proof First we observe that, in light of inequalities (6), it is sufficient to prove that

$$\varphi\ \leq\ \sup_{\varepsilon>0}\varphi^{\varepsilon} \tag{7}$$

since $\lim_{\varepsilon\to 0}\varphi^{\varepsilon}\ =\sup_{\varepsilon>0}\varphi^{\varepsilon}$.
We start by proving that, if $(\varepsilon_n)_n$ is a sequence of positive real numbers converging to 0, for every $x\in X$ and every sequence $(x_n)_n$ τ-converging to x in X one has

$$w-\limsup_n\ \mathcal{Q}^{\varepsilon_n}(x_n)\ \subseteq\ \mathcal{Q}(x). \tag{8}$$

Let $(u_n)_n$ be a sequence weakly converging to $u\in K$ such that, for every $n\in\mathbb{N}$, $u_n\in\mathcal{Q}^{\varepsilon_n}(x_n)$, that is

$$d(u_n,S(x_n,u_n))\leq\varepsilon_n \text{ and } \langle A(x_n,u_n),u_n-v\rangle\ \leq\varepsilon_n\ \ \forall\ v\in S(x_n,u_n).$$

Lemma 2.2 applied with $S_n=S$ implies that $u\in S(x,u)$ since

$$d(u,S(x,u))\leq\liminf_n d(u_n,S(x_n,u_n))=0.$$

So, we have to prove that $\langle A(x,u),u-w\rangle\leq 0$ for every $w\in S(x,u)$.
If $w\in int\,S(x,u)$, then, due assumption $v)$ and Lemma 2.1, $w\in int\,S(x_n,u_n)$ for n sufficiently large and $\langle A(x_n,u_n),u_n-w\rangle\leq 0$ for such integers n. Condition $ii)$ implies that $\langle A(x_n,w),u_n-w\rangle\leq 0$, so one gets $\langle A(x,w),u-w\rangle=\lim_n\langle A(x_n,w),u_n-w\rangle\leq 0$ by assumption $iii)$.
If $w\notin int\,S(x,u)$, being $S(x,u)$ a convex set, there exists a sequence $(w_n)_n$ strongly converging to w along a segment such that $w_n\in intS(x,u)$. So, one has $\langle A(x,w_n),u-w_n\rangle\leq 0$ for n large and, by assumption $i)$, $\langle A(x,w),u-w\rangle\leq 0$. The operator $A(x,\cdot)$ being hemicontinuous over K, one can apply the Minty Lemma ([5]) and

$$\langle A(x,u),u-w\rangle\leq\ 0,$$

so that $u\in\mathcal{Q}(x)$.
Now, assume that inequality (7) does not hold. Then, there exists a real number a such that

$$\sup_{\varepsilon>0}\varphi^{\varepsilon}\ <\ a\ <\varphi. \tag{9}$$

Let $(\varepsilon_n)_n$ be a sequence of positive real numbers decreasing to 0 and let $(x_n, u_n)_n$ be a sequence such that, for every $n \in \mathbb{N}$,

$$u_n \in \mathcal{Q}^{\varepsilon_n}(x_n) \text{ and } \varphi^{\varepsilon_n} \le f(x_n, u_n) < a.$$

By assumption *vi)* there exists a subsequence of $(x_n, u_n)_n$, still denoted by $(x_n, u_n)_n$, $(\tau \times w)$-converging in $H \times K$ to a point $(x_o, u_o) \in H \times K$ and such that

$$f(x_o, u_o) \le \liminf_n f(x_n, u_n) \le a.$$

From the first part of the proof one infers that $u_o \in \mathcal{Q}(x_o)$, so one has $\varphi \le f(x_o, u_o) \le a$ which is in contradiction with (9). □

A similar result holds also for the value $\widetilde{\varphi}$ of the Minty semiquasi-variational problem $\widetilde{(SB)}$ that can be approached by the regularized values

$$\widetilde{\varphi}^{\,\varepsilon} = \inf_{x \in H} \inf_{u \in \widetilde{\mathcal{Q}}^{\varepsilon}(x)} f(x, u) \quad \text{and} \quad \widetilde{\psi}^{\,\varepsilon} = \inf_{x \in H} \inf_{u \in \widetilde{\mathfrak{S}}^{\varepsilon}(x)} f(x, u)$$

where, for every $x \in H$,

$$\widetilde{\mathcal{Q}}^{\varepsilon}(x) = \{u \in K \;:\; d(u, S(x, u)) \le \varepsilon \text{ and } \langle A(x, w), u - w\rangle \le \varepsilon \;\forall\, w \in S(x, u)\}$$

$$\widetilde{\mathfrak{S}}^{\varepsilon}(x) = \{u \in K \;:\; d(u, S(x, u)) < \varepsilon \text{ and } \langle A(x, w), u - w\rangle < \varepsilon \;\forall\, w \in S(x, u)\}.$$

In fact, the following proposition provides a sufficient condition for approximating the infimal value $\widetilde{\varphi}$ by $\widetilde{\varphi}^{\,\varepsilon}$ and $\widetilde{\psi}^{\,\varepsilon}$.

Proposition 4.2 *Assume that conditions*
i) for every $x \in H$, the operator $A(x, \cdot)$ is hemicontinuous on K;
ii) for every $u \in K$, the operator $A(\cdot, u)$ is (τ, s^*)-continuous on H;
iii) the set-valued map S is $(\tau \times w, w)$-subcontinuous, $(\tau \times w, s)$-lower semicontinuous and $(\tau \times w, w)$-closed over $H \times K$;
iv) for every sequence $(x_n, u_n)_n$ $(\tau \times w)$-converging in $H \times K$ there exists $m \in \mathbb{N}$ such that

$$int \bigcap_{n \ge m} S(x_n, u_n) \ne \emptyset.$$

v) the function f is $(\tau \times w)$-lower semicontinuous and $(\tau \times w)$-coercive on $H \times K$.
Then,

$$\widetilde{\varphi} = \lim_{\varepsilon \to 0} \widetilde{\varphi}^{\,\varepsilon} = \lim_{\varepsilon \to 0} \widetilde{\psi}^{\,\varepsilon}.$$

Proof Arguing as in Proposition 4.1, in order to prove that

$$\widetilde{\varphi} \le \sup_{\varepsilon > 0} \widetilde{\varphi}^{\,\varepsilon} \tag{10}$$

we first show that for every sequence $(\varepsilon_n)_n$ of positive real numbers converging to 0, for every $x \in X$ and every sequence $(x_n)_n$ τ-converging to x in X one has

$$w - \limsup_n \widetilde{\mathcal{Q}}^{\varepsilon_n}(x_n) \subseteq \widetilde{\mathcal{Q}}(x). \tag{11}$$

Let $(u_n)_n$ be a sequence weakly converging to $u \in K$ such that, for every $n \in \mathbb{N}$, $u_n \in \widetilde{\mathcal{Q}}^{\varepsilon_n}(x_n)$, that is

$$d(u_n, S(x_n, u_n)) \leq \varepsilon_n \text{ and } \langle A(x_n, v), u_n - v\rangle \leq \varepsilon_n \ \forall\, v \in S(x_n, u_n).$$

Then, Lemma 2.2 applied with $S_n = S$ implies that $u \in S(x, u)$ since

$$d(u, S(x, u)) \leq \liminf_n d(u_n, S(x_n, u_n)) = 0.$$

So, we have to prove that $\langle A(x, w), u - w\rangle \leq 0$ for every $w \in S(x, u)$.
If $w \in int\, S(x, u)$, then, due to assumption $iv)$ and Lemma 2.1, $w \in int\, S(x_n, u_n)$ for n sufficiently large and one has $\langle A(x_n, w), u_n - w\rangle \leq 0$ that implies $\langle A(x, w), u - w\rangle = \lim_n \langle A(x_n, w), u_n - w\rangle \leq 0$ by assumption $ii)$.
If $w \notin int\, S(x, u)$, being $S(x, u)$ a convex set, there exists a sequence $(w_n)_n$ strongly converging to w along a segment such that $w_n \in intS(x, u)$, so one has $\langle A(x, w_n), u - w_n\rangle \leq 0$ for n large and again $\langle A(x, w), u - w\rangle \leq 0$ by assumption $i)$
Now, assume that inequality (10) does not hold. Then, there exists a real number a such that

$$\sup_{\varepsilon > 0} \widetilde{\varphi}^{\,\varepsilon} \ < \ a \ < \widetilde{\varphi}. \tag{12}$$

Let $(\varepsilon_n)_n$ be a sequence of positive real numbers decreasing to 0 and let $(x_n, u_n)_n$ be a sequence such that, for every $n \in \mathbb{N}$,

$$u_n \in \widetilde{\mathcal{Q}}^{\varepsilon_n}(x_n) \ \text{ and } \ \widetilde{\varphi}^{\varepsilon_n} \leq f(x_n, u_n) < a.$$

By assumption $v)$, there exists a subsequence of $(x_n, u_n)_n$, still denoted by $(x_n, u_n)_n$, $(\tau \times w)$-converging to a point $(x_o, u_o) \in H \times K$ and such that

$$f(x_o, u_o) \ \leq \ \liminf_n f(x_n, u_n) \ \leq \ a.$$

From the first part of the proof one infers that $u_o \in \widetilde{\mathcal{Q}}(x_o)$, and one has $\widetilde{\varphi} \ \leq \ a$ which is in contradiction with (12). □

Remark 4.1 We point out that even if the operator $A(x, \cdot)$ is hemicontinuous and pseudomonotone and $\mathcal{Q}(x) = \widetilde{\mathcal{Q}}(x)$, the approximate solutions sets $\mathcal{Q}^\varepsilon(x)$ and $\widetilde{\mathcal{Q}}^\varepsilon(x)$ (resp. $\mathfrak{S}^\varepsilon(x)$ and $\widetilde{\mathfrak{S}}^\varepsilon(x)$) may fail to be equal, so that φ^ε (resp. ψ^ε) may be strictly larger than $\widetilde{\varphi}^{\,\varepsilon}$ (resp. $\widetilde{\psi}^{\,\varepsilon}$) (see Section 4 in [20]).

5 Approaching the infimal value via perturbation and regularization

From now on, we consider regularizations of the perturbed semiquasi-variational bilevel problems defined at the beginning of Section 3:

$$(SB)^\varepsilon_n \ \text{ find } \ (x_n, u_n) \in H \times K \text{ such that } u_n \in \mathcal{Q}^\varepsilon_n(x_n) \text{ and } f_n(x_n, u_n) = \min_{x \in H} \min_{u \in \mathcal{Q}^\varepsilon_n(x)} f_n(x, u)$$

$$(\widehat{SB})^\varepsilon_n \ \text{ find } \ (x_n, u_n) \in H \times K \text{ such that } u_n \in \mathfrak{S}^\varepsilon_n(x_n) \ \text{ and } f_n(x_n, u_n) = \min_{x \in H} \min_{u \in \mathfrak{S}^\varepsilon_n(x)} f_n(x, u)$$

whose infimal values are denoted by φ_n^ε and ψ_n^ε respectively, as well as the regularized perturbed Minty semiquasi-variational bilevel problem

$$(\widetilde{SB})_n^\varepsilon \ \text{ find } \ (x_n,u_n) \in H\times K \text{ such that } u_n\in \widetilde{\mathcal{Q}}_n^\varepsilon(x_n) \text{ and } f_n(x_n,u_n)=\min_{x\in H}\ \min_{u\in\widetilde{\mathcal{Q}}_n^\varepsilon(x)} f_n(x,u)$$

whose infimal value is denoted by $\widetilde{\varphi}_n^\varepsilon$.

First we show that the infimal value φ can be approached *from above* by the regularized infimal values φ_n^ε.

Proposition 5.1 *Assume that condition C_0) in Lemma 2.2 and the following hold:*
i) for every $x\in H$ and every $u\in K$, the operator $A(x,\cdot)$ is hemicontinuous on K and the operator $A(\cdot,u)$ is (τ,s^*)-continuous on H;
ii) for every $n\in\mathbb{N}$ the operator $A_n(x,\cdot)$ is monotone on K;
iii) for every $x\in H$ and every sequence $(x_n)_n$ τ-converging to x in H, the sequence $(A_n(x_n,\cdot))_n$ is uniformly bounded over K;
iv) for every $x\in H$ and every sequence $(x_n)_n$ τ-converging to x in H, the sequence $(A_n(x_n,\cdot))_n$ (s,s^*)-G^--converges to $A(x,\cdot)$;
v) the set-valued map S is $(\tau\times w,w)$-subcontinuous, $(\tau\times w,s)$-lower semicontinuous and $(\tau\times w,w)$-closed over $H\times K$;
vi) for every sequence $(x_n,u_n)_n$ $(\tau\times w)$-converging in $H\times K$ there exists $m\in\mathbb{N}$ such that

$$int \bigcap_{n\geq m} S(x_n,u_n)\neq\emptyset.$$

vii) the function f is $(\tau\times w)$-lower semicontinuous and $(\tau\times w)$-coercive on $H\times K$;
viii) the sequence $(f_n)_n$ is $(\tau\times w)$-equicoercive on $H\times K$;
ix) for every $(x,u)\in H\times K$ and every sequence $(x_n,u_n)_n$ $(\tau\times w)$-converging to (x,u) in $H\times K$ one has

$$f(x,u)\ \leq\ \liminf_n\ f_n(x_n,u_n).$$

Then, we have

$$\varphi\ \leq\ \liminf_{\varepsilon\to0}\ \liminf_n \varphi_n^\varepsilon. \tag{13}$$

Proof
It is sufficient to prove that

$$\widetilde{\varphi}^{\,\varepsilon}\ \leq\ \liminf_n \varphi_n^\varepsilon \qquad \forall\ \varepsilon>0 \tag{14}$$

since in our assumptions

$$\varphi\ \leq\ \widetilde{\varphi}\ =\ \lim_{\varepsilon\to0}\ \widetilde{\varphi}^{\,\varepsilon}.$$

Indeed, the Minty Lemma implies that $\varphi\leq\widetilde{\varphi}$ and $\widetilde{\varphi}\ =\ \lim_{\varepsilon\to0}\ \widetilde{\varphi}^{\,\varepsilon}$ by Proposition 4.2.
Assume that inequality (14) fails to be true. Then, there exist $\varepsilon>0$ and a real number a such that

$$\liminf_n \varphi_n^\varepsilon\ <\ a\ <\ \widetilde{\varphi}^{\,\varepsilon}.$$

Let $(n_k)_k$ be an increasing sequence of positive integers and let $(x_k,u_k)_k$ be a sequence in $H\times K$ such that

$$u_k\in\mathcal{Q}_{n_k}^\varepsilon(x_k) \text{ and } f_{n_k}(x_k,u_k)\ <\ a<\ \widetilde{\varphi}^{\,\varepsilon}\quad \forall\ k\in\mathbb{N}. \tag{15}$$

The sequence of functions $(f_n)_n$ being $(\tau \times w)$-equicoercive, there exists a subsequence of $(x_k, u_k)_k$, still denoted by $(x_k, u_k)_k$, such that $(x_k)_k$ τ-converges to $x_o \in H$ and $(u_k)_k$ weakly converges to $u_o \in K$.
We prove now that $u_o \in \widetilde{\mathcal{Q}}^\varepsilon(x_o)$. Since $u_k \in \mathcal{Q}^\varepsilon_{n_k}(x_k)$ and Lemma 2.2 holds, one has

$$d(u_o, S(x_o, u_o)) \leq \liminf_k d(u_k, S_{n_k}(x_k, u_k)) \leq \varepsilon.$$

Consider $w \in S(x_o, u_o)$. Assumption C_2) implies that for every $w \in S(x_o, u_o)$ there exists a sequence $(w_k)_k$ strongly converging to w such that $w_k \in S_{n_k}(x_k, u_k)$ for k sufficiently large. By assumption *iv*), there exists a sequence $(w'_k)_k$ strongly converging to w such that $(A_k(x_k, w'_k))_k$ strongly converges to $A(x_o, w)$. So, by assumptions *ii*) and *iii*) one has

$$\langle A(x_o, w), u_o - w\rangle = \lim_k \langle A_k(x_k, w'_k), u_k - w'_k\rangle \leq \liminf_k \langle A_k(x_k, u_k), u_k - w'_k\rangle =$$

$$\liminf_k [\langle A_k(x_k, u_k), u_k - w_k\rangle + \langle A_k(x_k, u_k), w_k - w'_k\rangle] \leq \varepsilon + \lim_k \langle A_k(x_k, u_k), w_k - w'_k\rangle = \varepsilon.$$

Moreover, assumption *ix*) implies that $f(x_o, u_o) \leq \liminf_k f_k(x_k, u_k) \leq a < \widetilde{\varphi}^{\,\varepsilon}$ and that gives a contradiction since $u_o \in \widetilde{\mathcal{Q}}^\varepsilon(x_o)$. □

Remark 5.1 We have implicitly proven that $\widetilde{\varphi} \;\leq\; \liminf_{\varepsilon \to 0} \liminf_n \varphi^\varepsilon_n$. We can prove directly that $\widetilde{\varphi}$ can be approached from above by $(\widetilde{\varphi}^\varepsilon_n)_n$ under different assumptions.

Proposition 5.2 *Assume that condition C_0) in Lemma 2.2, assumptions i) and v)-ix) of Proposition 5.1 and the following hold:*
ii) for every $x \in H$ and every sequence $(x_n)_n$ τ-converging to x in H, the sequence $(A_n(x_n, \cdot))_n$ strongly pointwise converges to $A(x, \cdot)$;
iii) for every sequence $(x_n, u_n)_n$ $(\tau \times w)$-converging in $H \times K$ there exists $m \in \mathbb{N}$ such that

$$int \bigcap_{n \geq m} S_n(x_n, u_n) \neq \emptyset.$$

Then, we have

$$\widetilde{\varphi} \;\leq\; \liminf_{\varepsilon \to 0} \liminf_n \widetilde{\varphi}^\varepsilon_n. \tag{16}$$

Proof
It is sufficient to prove that

$$\widetilde{\varphi}^{\,\varepsilon} \;\leq\; \liminf_n \widetilde{\varphi}^\varepsilon_n \qquad \forall\, \varepsilon > 0 \tag{17}$$

since in our assumptions

$$\widetilde{\varphi} \;= \lim_{\varepsilon \to 0} \widetilde{\varphi}^\varepsilon$$

by Proposition 4.2.
Assume that (17) fails to be true and let ε be a positive number, let a be a real number such that $\liminf_n \widetilde{\varphi}^\varepsilon_n \;< a \;<\; \widetilde{\varphi}^{\,\varepsilon}$. There exists an increasing sequence of integers $(n_k)_k$ and a sequence $(x_k, u_k)_k$ such that $x_k \in H$, $u_k \in \widetilde{\mathcal{Q}}^\varepsilon_k(x_k)$ and $f_{n_k}(x_k, u_k) \;< a$. A subsequence of (x_k, u_k), still denoted by $(x_k, u_k)_k$, $(\tau \times w)$-converges to $(x_o, u_o) \in H \times K$ by assumption *viii*) of proposition 5.1. We can prove that $u_o \in \widetilde{\mathcal{Q}}^\varepsilon(x_o)$ similarly as in Proposition 3.2 and that gives a contradiction.
□

The approximation *from below* of the value φ can be obtained through the sequence $(\psi^\varepsilon_n)_n$.

Proposition 5.3 *Assume that condition C_0) in Lemma 2.2, assumptions i)-vi) of Proposition 4.1 and the following hold:*
vii) the sequence $(A_n)_n$ $((\tau \times s), s^*)$-G^--converges to A;
viii) for every $(x,u) \in H \times K$ and every sequence $(x_n,u_n)_n$ $(\tau \times s)$-converging to (x,u) in $H \times K$ one has

$$\limsup_n f_n(x_n,u_n) \ \leq \ f(x,u).$$

Then, we have

$$\limsup_{\varepsilon \to 0} \limsup_n \psi_n^\varepsilon \leq \varphi. \tag{18}$$

Proof
It is sufficient to prove that

$$\limsup_n \psi_n^\varepsilon \leq \psi^\varepsilon \qquad \forall \ \varepsilon > 0. \tag{19}$$

since in our assumptions

$$\lim_{\varepsilon \to 0} \psi^\varepsilon = \varphi$$

by Proposition 4.1.
Let a be a real number such that $\psi^\varepsilon \ < \ a$ and let $(x_o,u_o) \in H \times K$ such that $u_o \in \mathfrak{S}^\varepsilon(x_o)$ and $f(x_o,u_o) < a$.
Assumption *vii*) says that there exists a sequence $(x_n,u'_n)_n$ $(\tau \times s)$-converging to (x_o,u_o) such that

$$s^* - \lim_n A_n(x_n,u'_n) = A(x_o,u_o).$$

We claim that $u'_n \in \mathfrak{S}_n^\varepsilon(x_n)$ for n sufficiently large.
Indeed, if it is not true, there exists a subsequence $(u'_{n_k})_{n_k}$ of $(u'_n)_n$ such that $u'_{n_k} \notin \ \mathfrak{S}^\varepsilon_{n_k}(x_{n_k})$ for every $k \in \mathbb{N}$. From assumption C_2) and Lemma 2.2, $d(u_o, S(x_o,u_o)) < \varepsilon$ implies that $d(u'_n, S_n(x_n,u'_n)) < \varepsilon$ for n sufficiently large. So, by the definition of $\mathfrak{S}^\varepsilon_{n_k}$, for every k there exists $w_{n_k} \in S_{n_k}(x_{n_k}, u'_{n_k})$ such that

$$\langle A_{n_k}(x_{n_k}, u'_{n_k}), u'_{n_k} - w_{n_k}\rangle \ \geq \varepsilon.$$

A subsequence of $(w_{n_k})_k$ (still denoted by $(w_{n_k})_k$) has to weakly converge towards a point w_o and $w_o \in S(x_o,u_o)$ by assumptions C_o) and C$_3$). Therefore

$$\langle A(x_o,u_o), u_o - w_o\rangle \ = \lim_k \langle A_{n_k}(x_{n_k}, u'_{n_k}), u'_{n_k} - w_{n_k}\rangle \ \geq \varepsilon.$$

This leads to a contradiction, so $u'_n \in \mathfrak{S}^\varepsilon_n(x_n)$ and $\psi^\varepsilon_n \leq f_n(x_n,u'_n)$ for n sufficiently large.
Therefore, by condition *viii*) one gets $\limsup\limits_n \psi_n^\varepsilon \leq \limsup\limits_n f_n(x_n,u'_n) \leq \ f(x_o,u_o) \ < \ a$ and one can conclude that $\limsup\limits_n \psi^\varepsilon_n \ \leq \ \psi^\varepsilon$. □

Assumption *viii*) of Proposition 5.3 can be weakened if condition *vii*) is strengthened.

Proposition 5.4 *Assume that condition C_0) in Lemma 2.2, assumptions i)-vi) of Proposition 4.1 and the following hold:*
vii) for every $x \in H$ and every sequence $(x_n)_n$ converging to x in H, the sequence $(A_n(x_n,\cdot))_n$ (s,s^*)-G^--converges to $A(x,\cdot)$ in K;

viii) for every $(x,u) \in H \times K$ there exists a sequence $(x_n)_n$ τ-converging to x in K such that for every sequence $(u'_n)_n$ strongly converging to u in K one has

$$\limsup_n f_n(x_n, u'_n) \leq f(x,u).$$

Then, we have

$$\limsup_{\varepsilon \to 0} \limsup_n \psi_n^\varepsilon \leq \varphi. \tag{20}$$

Proof
It is sufficient to prove that

$$\limsup_n \psi_n^\varepsilon \leq \psi^\varepsilon \qquad \forall\ \varepsilon > 0 \tag{21}$$

since in our assumptions

$$\lim_{\varepsilon \to 0} \psi^\varepsilon = \varphi$$

by Proposition 4.1.
Let a be a real number such that $\psi^\varepsilon < a$ and let $(x_o, u_o) \in H \times K$ such that $u_o \in \mathfrak{S}^\varepsilon(x_o)$ and $f(x_o, u_o) < a$.
Assumption *viii*) says that there exists a sequence $(x_n)_n$ τ-converging to x_o in H such that for every sequence $(u'_n)_n$ strongly converging to u_o in K one has

$$\limsup_n f_n(x_n, u'_n) \ \leq \ f(x_o, u_o).$$

This is in particular true for the sequence $(u'_n)_n$ strongly converging to u_o and such that

$$s^* - \lim_n A_n(x_n, u'_n) = A(x_o, u_o)$$

which exists by assumption *vii*). Then, the rest of the proof goes as in Proposition 5.3. □

Now, we can establish a global approximation result for the infimal value φ through both the regularized perturbed values φ_n^ε and ψ_n^ε.

Theorem 5.1 *Assume that condition C_o) in Lemma 2.2 and the following hold:*
i) for every $x \in H$ and every $u \in K$, the operator $A(x, \cdot)$ is hemicontinuous on K and the operator $A(\cdot, u)$ is (τ, s^*)-continuous on H;
ii) for every $n \in \mathbb{N}$ the operator $A_n(x, \cdot)$ is monotone on K;
iii) for every $x \in H$ and every sequence $(x_n)_n$ τ-converging to x in H, the sequence $(A_n(x_n, \cdot))_n$ is uniformly bounded over K;
iv) for every $x \in H$ and every sequence $(x_n)_n$ τ-converging to x in H, the sequence $(A_n(x_n, \cdot))_n$ (s, s^*)-G^--converges to $A(x, \cdot)$;
v) the set-valued map S is $(\tau \times w, w)$-subcontinuous, $(\tau \times w, s)$-lower semicontinuous and $(\tau \times w, w)$-closed over $H \times K$;
vi) for every sequence $(x_n, u_n)_n$ $(\tau \times w, s)$-converging in $H \times K$ there exists $m \in \mathbb{N}$ such that

$$int \bigcap_{n \geq m} S(x_n, u_n) \neq \emptyset.$$

vii) the function f is $(\tau \times w)$-lower semicontinuous and $(\tau \times w)$-coercive on $H \times K$;
viii) the sequence $(f_n)_n$ is $(\tau \times w)$-equicoercive on $H \times K$;

ix) for every $(x,u) \in X \times K$ and every sequence $(x_n,u_n)_n$ $(\tau \times w)$-converging to (x,u) in $H \times K$ one has

$$f(x,u) \;\leq\; \liminf_n \, f_n(x_n,u_n);$$

x) for every $(x,u) \in H \times K$ there exists a sequence $(x_n)_n$ τ-converging to x in K such that for every sequence $(u'_n)_n$ strongly converging to u in K one has

$$\limsup_n \, f_n(x_n,u'_n) \leq f(x,u).$$

Then, we have

$$\lim_{\varepsilon\to 0} \lim_n \varphi_n^\varepsilon \;=\; \varphi \;=\; \lim_{\varepsilon\to 0} \lim_n \psi_n^\varepsilon. \tag{22}$$

Proof
The inequalities in (6) hold also for the perturbed infimal values, so, given $\varepsilon > 0$, we have

$$\varphi_n^\varepsilon \;\leq\; \psi_n^\varepsilon \;\leq\; \varphi_n \quad \forall\, n \in \mathbb{N}.$$

Therefore, by (13) and (20) in propositions 5.1 and 5.4, we can show easily that (22) is true. □

6 Solutions behavior via regularization

We first investigate the *inner approach* of the the solutions set $\mathcal{M}$ by the solutions sets $\mathcal{M}^\varepsilon$ or $\widetilde{\mathcal{M}}^\varepsilon$ of the regularized problems $(SB)^\varepsilon$ and $(\widetilde{SB})^\varepsilon$ in the absence of perturbations.

Proposition 6.1 *Assume that conditions $i) - v)$ of Proposition 4.1 and the following hold:*
vi) the function f is $(\tau \times w)$-lower semicontinuous in $H \times K$.
Then,

$$(\tau \times w)\text{-}\limsup_{\varepsilon\to 0} \, \mathcal{M}^\varepsilon \;\subseteq \mathcal{M}.$$

Proof
Let $(\varepsilon_n)_n$ be a sequence of positive real numbers converging to 0 and let $(x_n,u_n)_n$ be a sequence $(\tau \times w)$-converging in $H \times K$ to (x_o,u_o) such that $u_n \in \mathcal{M}^{\varepsilon_n}(x_n)$ for every $n \in \mathbb{N}$, that is $u_n \in \mathcal{Q}^{\varepsilon_n}(x_n)$ and

$$f(x_n,u_n) = \min_{x\in H} \min_{u\in\mathcal{Q}^{\varepsilon_n}(x)} f(x,u) \;=\; \varphi^{\varepsilon_n}.$$

Arguing as in Proposition 4.1 we get that $u_o \in \mathcal{Q}(x_o)$. So, to conclude that $(x_o,u_o) \in \mathcal{M}$, it is sufficient to prove that $f(x_o,u_o) \leq \varphi$ and this follows from inequalities (6) and condition *vi)* since

$$f(x_o,u_o) \;\leq\; \liminf_n f(x_n,u_n) \;=\; \liminf_n \, \varphi^{\varepsilon_n} \;\leq\; \varphi.$$

Proposition 6.2 *Assume that conditions $i) - iv)$ of Proposition 4.2 and the following hold:*
v) the function f is $(\tau \times w)$-lower semicontinuous in $H \times K$.
Then,

$$(\tau \times w)\text{-}\limsup_{\varepsilon\to 0} \, \widetilde{\mathcal{M}}^\varepsilon \;\subseteq \widetilde{\mathcal{M}}.$$

If, moreover,
vi) for every $x \in H$, the operator $A(x,\cdot)$ is pseudomonotone on K,
then

$$(\tau \times w)\text{-}\limsup_{\varepsilon\to 0} \, \widetilde{\mathcal{M}}^\varepsilon \;\subseteq\; \mathcal{M}.$$

Proof
Let $(\varepsilon_n)_n$ be a sequence of positive real numbers converging to 0 and let $(x_n, u_n)_n$ be a sequence $(\tau \times w)$-converging in $H \times K$ to (x_o, u_o) such that $u_n \in \widetilde{\mathcal{M}}^{\varepsilon_n}(x_n)$ for every $n \in \mathbb{N}$, that is $u_n \in \widetilde{\mathcal{Q}}^{\varepsilon_n}(x_n)$ and

$$f(x_n, u_n) = \min_{x \in H} \min_{u \in \widetilde{\mathcal{Q}}^{\varepsilon_n}(x)} f(x, u) \;=\; \widetilde{\varphi}^{\varepsilon_n}.$$

Arguing as in Proposition 4.2 we get that $u_o \in \widetilde{\mathcal{Q}}(x_o)$. So, to conclude that $(x_o, u_o) \in \widetilde{\mathcal{M}}$, it is sufficient to prove that $f(x_o, u_o) \le \widetilde{\varphi}$ and this follows from Proposition 4.2 and condition $v)$ since

$$f(x_o, u_o) \;\le\; \liminf_n f(x_n, u_n) \;=\; \lim_n \widetilde{\varphi}^{\varepsilon_n} \;\le\; \widetilde{\varphi}.$$

Finally, assumptions $i)$ of Proposition 4.2 and $vi)$ imply that $\mathcal{M} \;=\; \widetilde{\mathcal{M}}$. □

The next proposition shows that the solutions set $\mathcal{M}$ can be also approximated by the sets $\mathcal{H}^\varepsilon$ defined by

$$\mathcal{H}^\varepsilon = \{(x, u) \;:\; u \in \mathcal{Q}^\varepsilon(x) \text{ and } f(x, u) \;\le\; \psi^\varepsilon\}$$

that amount to a sort of *hybrid* approximate solutions sets since they combine both "large" and "strict" regularized problems.

Proposition 6.3 *Assume that $i)$-$v)$ of Proposition 4.1 and the following hold:*
$vi)$ the function f is $(\tau \times w)$-lower semicontinuous in $H \times K$.
Then,

$$(\tau \times w)\text{-}\limsup_{\varepsilon \to 0} \mathcal{H}^\varepsilon \;\subseteq \mathcal{M}.$$

Proof
Let $(\varepsilon_n)_n$ be a sequence of positive real numbers converging to 0 and let $(x_n, u_n)_n$ be a sequence $(\tau \times w)$-converging in $H \times K$ to (x_o, u_o) such that $u_n \in \mathcal{H}^{\varepsilon_n}(x_n)$ for every $n \in \mathbb{N}$, that is $u_n \in \mathcal{Q}^{\varepsilon_n}(x_n)$ and

$$f(x_n, u_n) \;\le\; \psi^{\varepsilon_n}.$$

Arguing as in Proposition 4.1 we get that $u_o \in \mathcal{Q}(x_o)$. So, to conclude that $(x_o, u_o) \in \mathcal{M}$, it is sufficient to prove that $f(x_o, u_o) \le \varphi$ and this follows from assumption $vi)$ since

$$f(x_o, u_o) \;\le\; \liminf_n f(x_n, u_n) \;\le \limsup_n \psi^{\varepsilon_n} \;\le\; \varphi. \qquad \square$$

Theorem 6.1 *Assume that condition $C_0)$ in Lemma 2.2 and the following hold:*
$i)$ for every $x \in H$ and every $u \in K$, the operator $A(x, \cdot)$ is hemicontinuous and pseudomonotone on K and the operator $A(\cdot, u)$ is (τ, s^*)-continuous on H;
$ii)$ for every $n \in \mathbb{N}$ and every $x \in H$ the operator $A_n(x, \cdot)$ is monotone on K;
$iii)$ for every $x \in H$ and every sequence $(x_n)_n$ τ-converging to x in H, the sequence $(A_n(x_n, \cdot))_n$ is uniformly bounded over K;
$iv)$ for every $x \in H$ and every sequence $(x_n)_n$ τ-converging to x in H, the sequence $(A_n(x_n, \cdot))_n$ (s, s^*)-G^--converges to $A(x, \cdot)$;
$v)$ the set-valued map S is $(\tau \times w, w)$-subcontinuous, $(\tau \times w, s)$-lower semicontinuous and $(\tau \times w, w)$-closed over $H \times K$;
$vi)$ for every sequence $(x_n, u_n)_n$ $(\tau \times w)$-converging in $H \times K$ there exists $m \in \mathbb{N}$ such that

$$int \bigcap_{n \ge m} S(x_n, u_n) \ne \emptyset;$$

vii) the function f is $(\tau \times w)$-lower semicontinuous in $H \times K$;
viii) for every $(x, u) \in X \times K$ and every sequence $(x_n, u_n)_n$ $(\tau \times w)$-converging to (x, u) in $H \times K$ one has

$$f(x, u) \;\le\; \liminf_n \, f_n(x_n, u_n);$$

ix) for every $(x, u) \in H \times K$ there exists a sequence $(x_n)_n$ τ-converging to x in K such that for every sequence $(u'_n)_n$ strongly converging to u in K one has

$$\limsup_n \, f_n(x_n, u'_n) \le f(x, u).$$

Then,

$$(\tau \times w)\text{-}\limsup_{\varepsilon \to 0} \limsup_n \, \mathcal{M}_n^\varepsilon \;\subseteq \mathcal{M}. \tag{23}$$

Proof
We first prove that

$$(\tau \times w)\text{-}\limsup_n \, \mathcal{M}_n^\varepsilon \;\subseteq \mathcal{H}^\varepsilon \;\; \forall\, \varepsilon > 0.$$

Let $(x_n, u_n)_n$ be a sequence $(\tau \times w)$-converging in $H \times K$ to (x_o, u_o) and such that $(x_n, u_n) \in \mathcal{M}_n^\varepsilon$ for every $n \in \mathbb{N}$, that is

$$u_n \in \mathcal{Q}_n^\varepsilon(x_n) \;\text{ and }\; f_n(x_n, u_n) \;=\; \varphi_n^\varepsilon.$$

We show now that $(x_o, u_o) \in \mathcal{H}^\varepsilon$, that is $u_o \in \mathcal{Q}^\varepsilon(x_o)$ and $f(x_o, u_o) \le \psi^\varepsilon = \inf_{x \in H} \inf_{u \in \mathfrak{S}^\varepsilon(x)} f(x, u)$.
In fact, arguing as in Proposition 5.1 we get that $u_o \in \mathcal{Q}^\varepsilon(x_o)$. So, it is sufficient to prove that $f(x_o, u_o) \le f(x', u')$ for any $x' \in H$ and $u' \in \mathfrak{S}^\varepsilon(x')$. Arguing as in Proposition 5.4 we get that for every $(x', u') \in H \times K$, such that $u' \in \mathfrak{S}^\varepsilon(x')$, there exists a sequence (x'_n, u'_n) $(\tau \times s)$-converging to (x', u') in $H \times K$ such that $u'_n \in \mathfrak{S}_n^\varepsilon(x'_n)$ for n sufficiently large, where the sequence $(x'_n)_n$ is the sequence existing by assumption ix). Therefore, from conditions $viii$) and ix) we infer that

$$f(x_o, u_o) \le \liminf_n \, f_n(x_n, u_n) \;\le\; \liminf_n \, \psi_n^\varepsilon \;\le\; \liminf_n \, f_n(x'_n, u'_n) \;\le\; \limsup_n \, f_n(x'_n, u'_n) \;\le\; f(x', u').$$

Then, one has

$$(\tau \times w)\text{-}\limsup_{\varepsilon \to 0} \limsup_n \, \mathcal{M}_n^\varepsilon \;\subseteq\; (\tau \times w)\text{-}\limsup_{\varepsilon \to 0} \mathcal{H}^\varepsilon$$

and the result follows from Proposition 6.3. $\square$

7 Conclusions

The semiquasi-variational bilevel problem presents two difficulties: one derives from the bilevel nature of the problem [8], one derives from the quasi-variational nature of the lower level problem [2]. Moreover, the infinite dimensional setting requires the use of Minty semiquasi-variational bilevel problems and this explains the large number of assumptions in many propositions (see also, for example, [5] and [2]).
In this paper, we are not concerned with the problem of the existence of solutions, that could be approached using a general result given in [18], and we do not concentrate our interest on problems in finite dimensional spaces having in mind to present more specific results for applications in a separate paper.
In fact, we assume that $\mathcal{M}$ is nonempty and we investigate how to determine both the infimal value

and an optimal solution. So, we consider a sequence of problems $(SB)_n$ that could arise from a perturbations scheme usually used in infinite dimensional spaces such as a discretization. In Section 3, we observe that the classical approach for optimization problems ([1], [7], [25]) is not fruitful for semiquasi-variational bilevel problems, due to the possible lack of lower convergence of the constraint maps $\mathcal{Q}_n$. Therefore, we follow the approach introduced in [22] for bilevel optimization problems and we define in Section 4 and in Section 5 suitable regularized problems with or without perturbations. In this regularization scheme, the constraint maps of the lower level, namely $\mathcal{Q}_n^\varepsilon$ and $\mathfrak{S}_n^\varepsilon$, are smaller than in the original problem but have better properties [19] which allow to reach the infimal value of the problem (SB) with or without perturbations.
Finally, in Section 6 we employ the results on the infimal values behavior to establish the conver! gence of a sequence of solutions to the problems $(SB)_n^\varepsilon$ towards a solution to the original problem (SB). At our knowledge, these are the first results concerning optimal solutions of bilevel problems obtained via the infimal values behavior.
Moreover, we emphasize that the upper limits in (23) cannot be inverted in general (see Example 3.1), so that a result in line with the classical optimization result by H. Attouch (see Proposition 2.9 in [1]) could be not obtained in general.
Finally, we note that Variational Convergence Theory ([1], [7]) and Set-valued Analysis ([2], [25]) are essential tools, respectively, in the investigation of the upper level optimization problem and of the lower level quasi-variational inequality problem respectively.

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