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Variational Preferences and Equilibria in Games under Ambiguous Beliefs Correspondences

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Giuseppe De Marco[*] and Maria Romaniello[**]

Abstract

In previous papers we studied a game model in which players' uncertainty is expressed entirely in the space of probabilities (lotteries) over consequences, it depends on the entire strategy profile chosen by the agents and it is described by the so called *ambiguous beliefs correspondences*. In this paper, we extend the previous results by embodying *variational preferences* in the model. We give a general existence result that we apply to a particular example in which beliefs correspondences depend on the equilibria of specific subgames. Then, we study the limit behavior of equilibria under perturbations on the index of ambiguity aversion.

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Table of contents

1 Introduction

It is well known that decisions often involve imprecise probabilities, therefore many models have been introduced in the literature in order to deal with this kind of *ambiguity*. In the standard decision theory models, the source of uncertainty is described by an underlying state space and ambiguous beliefs are usually represented by fixed subsets of priors (probability distributions) over this set of states[3]. The decision maker's action set is a subsets of *acts* which are functions from the state space to a space of *consequences* and he is endowed with an utility function which gives a numerical outcome for every possible consequence, that is for every action choice and realization of the state. Recent paper have also investigated this kind of ambiguity in strategic interactions: Kajii and Ui (2005) first investigate the effects of uncertainty aversion in incomplete information games with multiple priors. Bade (2011) considers games à la Aumann (1997) under more general preferences. Finally, Azrieli and Teper (2011) characterize equilibrium existence in terms of the preferences of the players[4].

There is a different strand of research in decision theory which shows that an agent, facing ambiguity, is not able to understand what the relevant states are and therefore the information available can be expressed entirely in the space of probabilities (lotteries) over consequences (see Ahn (2008), Olszewski (2007) and Stinchcombe (2003)). Despite the approach *without the state space* and the classical *multiple prior* approach can be reconciled by a generalized form of probabilistic sophistication where an ambiguous act is evaluated by its induced set of distributions over consequences (see Ahn (2008) and Olszewski (2007)), game theory provides further evidence that ambiguity cannot always be reconducted to the classical approach with a state space and multiple priors. In fact, the recent literature on ambiguous games (see for instance Dow and Werlang (1994), Lo (1996), Klibanoff (1996), Eichberger and Kelsey (2000) and Marinacci (2000), Lehrer (2012), Riedel and Sass (2013)) has shown that in a game there is a specific source of ambiguity since players may have ambiguous beliefs about opponents' strategy choices. There is no evidence in the literature showing that this kind of ambiguity can be properly reconducted to incomplete information games à la Harsanyi (that is, with state space) under multiple priors which, in turn, must be generalized in order to encompass this specific game theoretical issue. In previous papers, we introduced and studied the (so called) model of *game under ambiguous beliefs correspondences*[5] which provides a rather general tool to study ambiguity in games. The key point of our approach is that, for every player, ambiguity is directly represented by a belief correspondence which maps the set of strategy profiles to the set of all subsets of probability distributions over the outcomes

[3]This is known as the *multiple prior* approach (see Gilboa and Scmeidler (1989)). A slightly different approach involves non additive probabilities (capacities) à la Schmeidler (1990) instead of multiple priors. Those models build upon the Anscombe and Aumann (1963) model of decision under uncertainty rather then the Savage's one (1954).

[4]Similar approaches and applications can be found in Xiong (2014) and Zhang, Luo and Ma (2013).

[5]De Marco and Romaniello (2012) presents the general model, an existence theorem and many motivating examples. Stability of the equilibria is studied in De Marco and Romaniello (2013,b). An application to coalition formation is the subject of De Marco and Romaniello (2011).

of the game. For each player and for every given strategy profile, the belief correspondence gives the set of probability distributions over the possible outcomes of the game that the corresponding player perceives to be feasible and consistent with the actual strategy profile. On the one hand, beliefs correspondences might represent objective (exogenous) ambiguity as done in Ahn (2008) and Olszewski (2007); on the other hand, it turns out (see the examples in De Marco and Romaniello (2013,b)) that many existing models of ambiguous game have an equivalent formulation in terms of belief correspondences. For example, a notion of equilibrium in incomplete information games with multiple priors and the partially specified equilibrium concept by Lehrer (2012) can both be regarded as particular cases of our notion of *equilibrium under ambiguous belief correspondences*.

In our previous works, we considered players endowed only with the classical maxmin prefernces[6]. In this paper we study a generalization of our equilibrium concept in which we relax the assumption imposed on the ambiguity attitudes of the players. More precisely we consider players endowed with *variational preferences* as introduced and axiomatized by Maccheroni, Marinacci and Rustichini (2006). This class of preferences generalizes maxmin preferences and embodies the approach of the works of Hansen and Sargent (2001) on uncertainty in macroeconomics since their model provides a particular class of variational preferences as well. Under variational preferences, players evaluate any ambiguous belief by the worst possible value (given the set of probability distributions) assumed by the sum of the expected utility with a nonnegative function of the probabilities called *index of ambiguity aversion*. This index plays a very important role as a measure of ambiguity aversion. For instance, maximal ambiguity aversion corresponds to index ambiguity aversion identically equal to zero and gives back maxmin preferences. While, minimal ambiguity aversion corresponds to ambiguity neutrality and gives back subjective utility preferences. Given the flexibility of the variational preferences model, it seems interesting and natural to look at this kind of preferences in games. In De Marco and Romaniello (2013,a) we extend the Kajii and Ui's notion of mixed equilibrium by allowing for variational preferences and we investigate the issue of the existence and stability of equilibria. In this paper we look at the equilibria in games under variational preferences in the more complex model with ambiguous belief correspondences; first, we present an existence theorem and then we show in an illustrative example (in which beliefs over opponents' strategy profile are given by the set of Nah equilibria of specific subgames) that the assumptions in the main existence theorem can be easily obtained in specific applications.

The last section is devoted to the issue of stability of equilibria. The problem of the limit behavior of the equilibria in games has been extensively studied in the literature (see, for instance, Fudenberg and Tirole (1993) for the standard problem, Friedmann and Mezzetti (2005), McKelvey and Palfrey (1995), Morgan and Scalzo (2008), Yu et al. (2007)) for recent results under relaxed or different assumptions and references. The question whether the limit property extends to the equilibrium concepts in ambiguous games has been studied in De Marco and Romaniello (2013,a) for the equilibria in the Kajii and Ui model under variational preferences, in De Marco and Romaniello (2013,b) for equilibria under ambiguous beliefs correspondences and maxmin preferences,

[6]Indeed, we considered also their counterpart: maxmax preferences.

in Stauber (2011) for an equilibrium notion in ambiguous games which relies on the Beweley unanimity rule. Maccheroni, Marinacci and Rustichini (2006) raise the question of the limit behavior of variational preferences; they show that variational preferences become more ambiguity averse as the ambiguity indices become smaller. So they point out (pp 1459-1450) that it is natural to look for conditions under which a sequence of variational preferences converges to the variational preference corresponding to the limit index of ambiguity aversion and give a limit result (Proposition 12 in their paper) based on a kind of monotone convergence assumption on the sequence of indices of ambiguity aversion. However, they suggest the use of *epiconvergence*[7] to obtain further limit results. In this paper we precisely tackle this question in our model without state space; in fact, our stability result is based on a different kind of convergence[8] on the sequence of indices of ambiguity aversion which relies on the notion of epiconvergence and which guarantees the continuous convergence of variational preferences. This result, in turn, has an immediate and relevant implication in games since it guarantees the convergence of sequences of equilibria of perturbed games to the equilibria of the unperturbed game.

The paper is organized as follows: Section 2 presents the model and the equilibrium notion then it studies the equilibrium existence issue. Section 3 shows an application to the case in which beliefs over opponents' strategy profile are given by a set of equilibria. The stability issue is studied in Section 4.

2 Model and Equilibria

2.1 The model

We consider a finite set of players $I = \{1, \dots n\}$; for every player i, $\Psi_i = \{\psi_i^1, \dots, \psi_i^{k(i)}\}$ is the (finite) pure strategy set of player i, $\Psi = \prod_{i \in I} \Psi_i$ and $\Psi_{-i} = \prod_{j \neq i} \Psi_j$. Denote with X_i the set of mixed strategies of player i and each strategy $x_i \in X_i$ is a vector $x_i = (x_i(\psi_i))_{\psi_i \in \Psi_i} \in \mathbb{R}_+^{k(i)}$ such that $\sum_{\psi_i \in \Psi_i} x_i(\psi_i) = 1$. Denote also with $X = \prod_{j=1}^n X_j$ and with $X_{-i} = \prod_{j \neq i} X_j$.

The set of all the possible outcomes of the game is denoted by $\Omega \subseteq \mathbb{R}^n$ and ω_i represents the payoff of player i when outcome $\omega \in \Omega$ is realized. We denote with $\mathcal{P}$ the set of all the probability distributions over Ω so that beliefs will be represented by subsets of $\mathcal{P}$. Beliefs are unambiguous if they are singletons, they are ambiguous otherwise. The information (about the outcomes of the game) available to each player i is summarized by an exogenous set-valued map $\mathcal{B}_i : X \rightsquigarrow \mathcal{P}$, called *beliefs correspondence*, which gives to player i and for every strategy profile $x \in X$, the ambiguous belief over outcomes $\mathcal{B}_i(x) \subseteq \mathcal{P}$. The set $\mathcal{B}_i(x)$ represents the set of probability distributions over Ω which are feasible and consistent, in view of player i, with the actual strategy profile[9] x. Note

[7]see for instance Aubin and Frankowska (1990) or Rockafellar and Wets (1998) for detailed surveys.

[8]This condition has been defined and used in a more general setting in Lignola and Morgan (1992).

[9]In this view, the strategy set X has a double use: first it represents the joint set of objects of choice of players but, at the same time, it stands for the set of variables that parameterize the beliefs of each player.

that, the standard (unambiguous) normal form games (even under incomplete information) give rise to single valued beliefs correspondences. While complete ignorance would be represented by correspondences such that $\mathcal{B}_i(x) = \mathcal{P}$ for every x.

As recalled in the Introduction, we point out that the restrictions imposed by the beliefs correspondences may come up from different sources of ambiguity; for instance, they might be given by objective and exogenous ambiguity as in the decision theory models in Ahn (2008) and Olszewski (2007) or, as shown in De Marco and Romaniello (2013,b), they can come up from multiple priors on a state space or from partial information about opponents' behavior caused by partially-specified probabilities as in the Lehrer's (2012) model of partially specified equilibrium.

Now, we introduce variational preferences. We assume that each agent is endowed with the following utility function:

$$F_i^V(x) = \min_{\varrho \in \mathcal{B}_i(x)} \left[\sum_{\omega \in \Omega} \varrho(\omega)\omega_i + c_i(\varrho) \right] \quad \forall x \in X, \tag{1}$$

$c_i : A(\mathcal{P}) \to \overline{\mathbb{R}}_+$, called *index of ambiguity aversion*, is a convex and lower semicontinuous function, where $A(\mathcal{P})$ is the affine hull of the set $\mathcal{P} \subset \mathbb{R}^{|\Omega|}$, that is, the smallest affine set containing $\mathcal{P}$:

$$A(\mathcal{P}) = \left\{ \sum_{k=1}^{K} \alpha_k \varrho_k \,|\, K \in \mathbb{N}_{>0},\ \varrho_k \in \mathcal{P},\ \alpha_k \in \mathbb{R},\ \sum_{k=1}^{K} \alpha_k = 1 \right\},$$

and $\overline{\mathbb{R}}_+ = [0, +\infty[\cup \{+\infty\}$. For the sake of simplicity we denote with $E_i[\varrho] = \sum_{\omega \in \Omega} \varrho(\omega)\omega_i$ and with

$$\varphi_i(\varrho) = E_i[\varrho] + c_i(\varrho) \tag{2}$$

so that $F_i^V(x) = \min_{\varrho \in \mathcal{B}_i(x)} \varphi_i(\varrho)$. Then we consider the game

$$\Gamma^V = \{I; (X_i)_{i \in I}; (F_i^V)_{i \in I}\}.$$

This game is a classical strategic form game[10]. So,

Definition 2.1: A Nash equilibrium of Γ^V is called *MMR*[11] *equilibrium under beliefs correspondences $\mathcal{B}_i$.*

We emphasize that the notion of equilibrium above is the natural generalization of the classical concept of Nash equilibrium for our model with beliefs correspondences. In fact, the Nash equilibrium concept assumes that rational players will choose the most preferred strategy given their beliefs about what other players will do and it imposes the consistency condition that all players' beliefs are correct. Similarly, in an equilibrium under ambiguous beliefs correspondences x, each

[10] When a game Γ^V is considered, then it is implicitly assumed that its utility functions F_i^V are well posed, (i.e. $\min_{\varrho \in \mathcal{B}_i(x)} \sum_{\omega \in \Omega} \varrho(\omega)\omega_i + c_i(\varrho)$ exists for every $x \in X$); obviously, this latter condition is guaranteed, for instance, when beliefs correspondences have closed values.

[11] Maccheroni-Marinacci-Rustichini

player chooses the most preferred strategy x_i given his information on the consequences of each strategy choice $x'_i \in X_i$ under the assumption that this information is consistent with the actual strategy profile chosen, i.e. it is provided by the ambiguous belief over outcomes $\mathcal{B}_i(x'_i, x_{-i})$.

2.2 Equilibrium existence

Aim of this subsection is to provide an existence result for MMR equilibria which are, by definition, equilibria of the strategic form game Γ^V. It is well known (see for instance Rosen (1965)) that the existence of equilibria depends on the properties of the best reply correspondences, i.e. the set-valued[12] maps $BR_i^V : X_{-i} \rightsquigarrow X_i$ defined, for every player i, by

$$BR_i^V(x_{-i}) = \left\{ \overline{x}_i \in X_i \mid F_i^V(\overline{x}_i, x_{-i}) = \max_{x_i \in X_i} F_i^V(x_i, x_{-i}) \right\}. \tag{3}$$

Namely, the classical Rosen's existence theorem requires that each BR_i^V has to be closed with not empty, closed and convex images. Below, we investigate those properties.

Preliminaries on set-valued maps

We start by recalling well known definitions and results on set-valued maps which we use below. Following Aubin and Frankowska (1990)[13], recall that if Z and Y are two metric spaces and $\mathcal{C} : Z \rightsquigarrow Y$ a set-valued map, then

$$i)\ \operatorname*{Lim\,inf}_{z \to z'} \mathcal{C}(z) = \left\{ y \in Y \mid \lim_{z \to z'} d(y, \mathcal{C}(z)) = 0 \right\},$$
$$ii)\ \operatorname*{Lim\,sup}_{z \to z'} \mathcal{C}(z) = \left\{ y \in Y \mid \liminf_{z \to z'} d(y, \mathcal{C}(z)) = 0 \right\}$$

and $\operatorname*{Lim\,inf}_{z \to z'} \mathcal{C}(z) \subseteq \mathcal{C}(z') \subseteq \operatorname*{Lim\,sup}_{z \to z'} \mathcal{C}(z)$. Moreover

DEFINITION 2.2: Given the set-valued map $\mathcal{C} : Z \rightsquigarrow Y$, then

i) $\mathcal{C}$ is *lower semicontinuous* in z' if $\mathcal{C}(z') \subseteq \operatorname*{Lim\,inf}_{z \to z'} \mathcal{C}(z)$; that is, $\mathcal{C}$ is lower semicontinuous in z' if for every $y \in \mathcal{C}(z')$ and every sequence $(z_\nu)_{\nu \in \mathbb{N}}$ converging to z' there exists a sequence $(y_\nu)_{\nu \in \mathbb{N}}$ converging to y such that $y_\nu \in \mathcal{C}(z_\nu)$ for every $\nu \in \mathbb{N}$. Moreover, $\mathcal{C}$ is lower semicontinuous in Z if it is lower semicontinuous for all z' in Z.

ii) $\mathcal{C}$ is *closed* in z' if $\operatorname*{Lim\,sup}_{z \to z'} \mathcal{C}(z) \subseteq \mathcal{C}(z')$; that is, $\mathcal{C}$ is closed in z' if for every sequence $(z_\nu)_{\nu \in \mathbb{N}}$ converging to z' and every sequence $(y_\nu)_{\nu \in \mathbb{N}}$ converging to y such that $y_\nu \in \mathcal{C}(z_\nu)$ for every $\nu \in \mathbb{N}$, it follows that $y \in \mathcal{C}(z')$. Moreover, $\mathcal{C}$ is closed in Z if it is closed for all z' in Z;

[12] We recall that set-valued maps are indistinctly called correspondences as well

[13] All the definitions and the propositions we use, together with the proofs can be found in this book.

iii) $\mathcal{C}$ is *upper semicontinuous* in z' if for every open set U such that $\mathcal{C}(z') \subseteq U$ there exists $\eta > 0$ such that $\mathcal{C}(z) \subseteq U$ for all $z \in B_Z(z', \eta) = \{\zeta \in Z\, |; ||\zeta - z'|| < \eta\}$;

iv) $\mathcal{C}$ is *continuous* (in the sense of Painlevé-Kuratowski) in z' if it is lower semicontinuous and upper semicontinuous in z'.

Finally, recall the following result: If Z is closed, Y is compact and the set-valued map $\mathcal{C} : Z \rightsquigarrow Y$ has closed values, then, $\mathcal{C}$ is upper semicontinuous in $z \in Z$ if and only if $\mathcal{C}$ is closed in z[14].

The next definition will also be used:

DEFINITION 2.3: Let Z a convex set, then the set-valued map $\mathcal{C} : Z \rightsquigarrow Y$ is a said to be *concave* if

$$t\mathcal{C}(\overline{z}) + (1-t)\mathcal{C}(\widehat{z}) \subseteq \mathcal{C}(t\overline{z} + (1-t)\widehat{z}) \quad \forall\, \overline{z}, \widehat{z} \in Z,\ \forall\, t \in [0,1] \tag{4}$$

while it is *convex*[15] if

$$\mathcal{C}(t\overline{z} + (1-t)\widehat{z}) \subseteq t\mathcal{C}(\overline{z}) + (1-t)\mathcal{C}(\widehat{z}) \quad \forall\, \overline{z}, \widehat{z} \in Z,\ \forall\, t \in [0,1] \tag{5}$$

Continuity

Let $Dom(c_i) = \{\varrho \in A(\mathcal{P}) \,|\, c_i(\varrho) < +\infty\}$ be the *effective domain* of c_i and c_i^D the restriction of c_i to $Dom(c_i)$. The relative interior of $Dom(c_i)$ is

$$relint(Dom(c_i)) = \{\varrho \in Dom(c_i) \,|\, \exists \epsilon > 0, B(\varrho, \epsilon) \cap A(\mathcal{P}) \subset\ Dom(c_i)\}.$$

Then

PROPOSITION 2.4: *Assume that*

i) $\mathcal{B}_i$ *is continuous with not empty and closed images for every* $x \in X$.

ii) $relint(Dom(c_i)) \cap \mathcal{B}_i(x) \neq \emptyset$ *for every* $x \in X$

iii) c_i^D *is continuous in* $Dom(c_i) \cap \mathcal{P}$.

Then, F_i^V *is continuous in* X *and* BR_i^V *is a closed set-valued map with not empty and closed images for every* $x_{-i} \in X_{-i}$.

Proof. Since c_i is lower semicontinuous then the effective domain $Dom(c_i)$ is a closed set. Consider the set-valued map $\mathcal{D}_i$ defined by

$$\mathcal{D}_i(x) = Dom(c_i) \cap \mathcal{B}_i(x) \quad \forall x \in X.$$

[14]Every set-valued map in this paper satisfies the assumptions of this result. Hence upper semicontinuity and closeness coincide in this work.

[15]Note that a set-valued map is concave if and only if its graph is a convex set. For this reason, some authors call convex set-valued maps those that here we call concave.

$\mathcal{D}_i$ is upper semicontinuous since if we regard $Dom(c_i)$ as a constant correspondence then $\mathcal{D}_i$ it is the intersection of two closed set-valued maps which is closed by Theorem 16.25 in Aliprantis and Border (1999) and therefore $\mathcal{D}_i$ is upper semicontinuous since $\mathcal{D}_i$ has closed images. Similarly, if we regard $relint(Dom(c_i))$ as a constant correspondence then the set-valued map $x \rightsquigarrow \widehat{\mathcal{D}}_i(x) = relint(Dom(c_i)) \cap \mathcal{B}_i(x)$ is lower semicontinuous. In fact, $\widehat{\mathcal{D}}_i$ is the intersection between the constant correspondence $x \rightsquigarrow relint(Dom(c_i))$ and $\mathcal{B}_i$, the latter being lower semicontinuous from the assumptions. Now, since $relint(Dom(c_i))$ and $\mathcal{B}_i(x)$ are subsets of $A(\mathcal{P})$ for every $x \in X$, they can be regarded as subsets of $\mathbb{R}^{|\Omega|-1}$. In this case, the constant correspondence $x \rightsquigarrow relint(Dom(c_i))$ has open graph (in $X \times \mathbb{R}^{|\Omega|-1}$) and $\mathcal{B}_i$ is obviously lower semicontinuous; hence, in light of assumption *ii)*, we can apply Proposition 11.21 in Border (1985) so that their intersection is a lower semicontinuous set-valued map. It immediately follows that $x \in X \rightsquigarrow \widehat{\mathcal{D}}_i(x) \subset \mathbb{R}^{|\Omega|}$ is lower semicontinuous. Now, the two correspondences $\mathcal{D}_i$ and $\widehat{\mathcal{D}}_i$ have the same closure $x \rightsquigarrow \mathcal{D}_i^*(x)$, that is $\mathcal{D}_i^*(x)$ is the closure of the two sets $\widehat{\mathcal{D}}_i(x)$ and $\mathcal{D}_i(x)$ for every x in X. Lemma 16.22 in Aliprantis and Border (1999) states that a set-valued map is lower semicontinuous if and only if its closure is. Therefore $\mathcal{D}_i^*$ is lower semicontinuous since $\widehat{\mathcal{D}}_i$ is lower semicontinuous. Applying again the same Lemma we get that $\mathcal{D}_i$ is lower semicontinuous.

Now, denote with

$$h_i(x) = \min_{\varrho \in \mathcal{D}_i(x)} [E_i[\varrho] + c_i^D(\varrho)] \quad \forall x \in X.$$

By applying the Berge maximum theorem (see also Aubin and Frankowska (1990), Border (1985)), it follows that h_i is continuous on the compact set X. By construction,

$$h_i(x) = \min_{\varrho \in \mathcal{D}_i(x)} E_i[\varrho] + c_i^D(\varrho) = \min_{\varrho \in \mathcal{B}_i(x)} E_i[\varrho] + c_i(\varrho) = F_i^V(x) \quad \forall x \in X$$

so F_i^V is continuous in X, which implies (for the Berge maximum theorem) that BR_i^V is closed with not empty and closed images for every $x_{-i} \in X_{-i}$. □

Convexity

In order to obtain the convexity of the images of the correspondences BR_i^V, the assumptions that must be imposed on the index of ambiguity aversion are very demanding. In fact, the initial assumption of convexity for the index of ambiguity aversion implies that the utility functions F_i^V derive from the minimization of convex functions and therefore they are concave only in very particular cases. On the other hand, the sufficient conditions - usually required to have a best reply BR_i^V having convex images - involve the quasi concavity of F_i^V. However we are able to give sufficient conditions which involve the convexity of the belief correspondence (i.e. the nature of the ambiguity the player is facing) and the ambiguity attitude of the player since they impose restrictions on the index of ambiguity aversion. The next Section 3 shows that it is not difficult to find examples satisfying all of the assumptions of Proposition 2.5 below.

PROPOSITION 2.5: *Assume that for every $x_{-i} \in X_{-i}$, $\overline{x}_i, \widehat{x}_i$ in X_i and $t \in [0,1]$ it follows that if $\varrho^* \in \mathcal{B}_i(t\overline{x}_i + (1-t)\widehat{x}_i, x_{-i})$ then there exist $\overline{\varrho} \in \mathcal{B}_i(\overline{x}_i, x_{-i})$ and $\widehat{\varrho} \in \mathcal{B}_i(\widehat{x}_i, x_{-i})$ such that*

$$\varrho^* = t\overline{\varrho} + (1-t)\widehat{\varrho} \quad and \quad c_i(\varrho^*) = tc_i(\overline{\varrho}) + (1-t)c_i(\widehat{\varrho}). \tag{6}$$

Then, $F_i^V(\cdot, x_{-i})$ is a concave function and $BR_i^V(x_{-i})$ is a convex set for every $x_{-i} \in X_{-i}$.

REMARK 2.6: The assumption in the previous proposition implicitly requires $\mathcal{B}_i(\cdot, x_{-i})$ to be a convex set-valued map in X_i, that is, for every $\overline{x}_i$ and $\widehat{x}_i$ in X_i, $t \in [0,1]$ it follows that

$$\mathcal{B}_i(t\overline{x}_i + (1-t)\widehat{x}_i, x_{-i}) \subseteq t\mathcal{B}_i(\overline{x}_i, x_{-i}) + (1-t)\mathcal{B}_i(\widehat{x}_i, x_{-i}). \tag{7}$$

Moreover, it can be easily checked that if $\mathcal{B}_i(\cdot, x_{-i})$ is a convex set-valued map in X_i and c_i is an affine function then the assumption of the previous proposition are satisfied.

Proof of Proposition (2.5). Let $\overline{x}_i$ and $\widehat{x}_h$ be in X_i and $t \in [0,1]$. Let $\varrho^* \in \mathcal{B}_i(t\overline{x}_i + (1-t)\widehat{x}_i, x_{-i})$ be such that $F_i^V(t\overline{x}_i + (1-t)\widehat{x}_i, x_{-i}) = E_i(\varrho^*) + c_i(\varrho^*)$. From the assumptions it immediately follows that there exist $\overline{\varrho} \in \mathcal{B}_i(\overline{x}_i, x_{-i})$ and $\widehat{\varrho} \in \mathcal{B}_i(\widehat{x}_i, x_{-i})$ such that $\varrho^* = t\overline{\varrho} + (1-t)\widehat{\varrho}$ and $c_i(\varrho^*) = tc_i(\overline{\varrho}) + (1-t)c_i(\widehat{\varrho})$. Since $E_i(\varrho^*) = tE_i(\overline{\varrho}) + (1-t)E_i(\widehat{\varrho})$, then

$$\begin{aligned} F_i^V(t\overline{x}_i + (1-t)\widehat{x}_i, x_{-i}) = E_i(\varrho^*) + c_i(\varrho^*) \geq \\ t\left[\min_{\varrho \in \mathcal{B}_i(\overline{x}_i, x_{-i})} (E_i(\varrho) + c_i(\varrho))\right] + (1-t)\left[\min_{\varrho \in \mathcal{B}_i(\widetilde{x}_i, x_{-i})} (E_i(\varrho) + c_i(\varrho))\right] = \\ tF_i^V(\widetilde{x}_i, x_{-i}) + (1-t)F_i^V(\widetilde{x}_i, x_{-i}) \end{aligned}$$

and $F_i^V(\cdot, x_{-i})$ is concave for all x_{-i}. Finally, it follows that BR_i^V has convex images for every $x_{-i} \in X_{-i}$. □

From the Nash equilibrium existence theorems (see for instance Rosen (1965)), it immediately follows that

THEOREM 2.7: *Suppose that, for every player i, the assumptions of Propositions 2.4 and 2.5 are satisfied. Then, the game Γ^V has at least an MMR equilibrium.*

3 An Example: beliefs given by equilibria

As shown in De Marco and Romaniello (2013,b) a class of beliefs correspondences over outcomes can be obtained in the classical case in which players have ambiguous beliefs about opponents' strategy choices. More precisely, now we assume that each player i is endowed with a payoff function $f_i : \Psi \to \mathbb{R}$ and a belief correspondence from strategy profiles to correlated strategies, i.e. $\mathcal{K}_i : X \rightsquigarrow \Delta$, where Δ is the set of probability distributions over Ψ; when the strategy profile x is chosen by the agents, player i has the ambiguous belief $\mathcal{K}_i(x) \subseteq \Delta$ over the set of pure strategy profiles Ψ.

In this case the set of outcomes of the game is given by $\Omega = \{(f_1(\psi), \ldots, f_n(\psi)) \,|\, \psi \in \Psi\}$; then, each $\mathcal{K}_i$ induces the beliefs correspondence over outcomes $\mathcal{B}_i : X \rightsquigarrow \mathcal{P}$ in the obvious way:

$$\mathcal{B}_i(x) = \{\varrho \in \mathcal{P} \,|\, \exists \pi \in \mathcal{K}_i(x) \text{ with } \varrho(f_i(\psi)) = \pi(\psi) \; \forall \psi \in \Psi\} \quad \forall x \in X. \tag{8}$$

REMARK 3.1: If $\mathcal{K}_i(x) = \{x\}$ and the index c_i is identically equal to 0 for every player i, then the corresponding game Γ^V coincides with the mixed extension of the game $\Gamma = \{I; \Psi_1, \ldots, \Psi_n; f_1, \ldots, f_n\}$ so that the set of equilibria of Γ^V coincides with the set of Nash equilibria in mixed strategies of Γ.

A particular case of the approach described above is provided by the example given below in which beliefs to a player over his opponents' strategy profiles are given by the set of Nash equilibria of the game between the opponents once they have observed player's action[16]. The underlying idea in this example is that each player believes that his opponents will observe his action before choosing their strategies and then, they will react optimally. So that player's beliefs about opponents' behavior are naturally given by the equilibria of the game between his opponents given the player's action.

For a given player i, denote with $J_i = I \setminus \{i\}$, then, for every pure strategy $\psi_i \in \Psi_i$, consider the game

$$G(\psi_i) = \{J_i; (\Psi_j)_{j \in J_i};\; (g_j^{\psi_i})_{j \in J_i}\}$$

where Ψ_j is the pure strategy set of player j and the payoff function $g_j^{\psi_i} : \Psi_{-i} \to \mathbb{R}$ is the payoff function of player j which corresponds to the payoff of player j in the game Γ when player i chooses ψ_i, i.e., $g_j^{\psi_i}((\widehat{\psi}_h)_{h \in J_i}) = f_j(\widehat{\psi}_1, .., \widehat{\psi}_{i-1}, \psi_i, \widehat{\psi}_{i+1}, .., \widehat{\psi}_n)$ for every $\widehat{\psi}_{-i} \in \Psi_{-i}$. Denote with $\chi_i(\psi_i)$ the set of Nash equilibria in mixed strategies of the game $G(\psi_i)$. With an abuse of notation, we identify each equilibrium in $\chi_i(\psi_i)$ with the probability distribution it induces on Ψ_{-i}; in other words each element $\mu_{\psi_i} \in \chi_i(\psi_i)$ is a probability distribution on Ψ_{-i} induced by some mixed strategy equilibrium of the game $G(\psi_i)$; note that each $\mu_{\psi_i}(\psi_{-i})$ denotes the probability assigned by μ_{ψ_i} to the strategy profile ψ_{-i}. Recall that, for every player i, $\Psi_i = \{\psi_i^1, \ldots, \psi_i^{k(i)}\}$. Let $\mathcal{B}_i : X \rightsquigarrow \mathcal{P}$ be the set-valued map defined, for every $x \in X$, by

$$\varrho \in \mathcal{B}_i(x) \iff \begin{cases} \exists \mu_1 \in \chi_i(\psi_i^1), \ldots, \mu_{k(i)} \in \chi_i(\psi_i^{k(i)}) \text{ such that} \\ \\ \varrho(f_i(\psi_i^t, \psi_{-i})) = x_i(\psi_i^t)\mu_t(\psi_{-i}) \quad \forall t \in \{1, \ldots, k(i)\}, \forall \psi_{-i} \in \Psi_{-i}. \end{cases} \tag{9}$$

We emphasize that this set-valued map represents the idea that player i believes that the other players will observe his play and then they will react by choosing a Nash equilibrium.

[16]A similar idea has been firstly investigated in De Marco and Romaniello (2012) in which correlated equilibria have been used. Here we focus on Nash equilibria because we do not need the additional properties of correlated equilibria

Lemma 3.2: *The set-valued map $\mathcal{B}_i$ defined in (9) is continuous with not empty and closed values for every $x \in X$. Moreover, $\mathcal{B}_i(\cdot, x_{-i})$ is convex for every $x_{-i} \in X_{-i}$.*

Proof. For every ψ_i, $G(\psi_i)$ is a finite game so (it is well known that) the set $\chi_i(\psi_i)$ of Nash equilibria in mixed strategies of $G(\psi_i)$ is not empty and closed.

Now, we show that the set-valued map $\mathcal{B}_i$ is closed for every $x \in X$. In fact, given a point $x \in X$, let $(x_\nu)_{\nu\in\mathbb{N}}$ be a sequence in X converging to x where we denote with $x_\nu = (x_{1,\nu}, \dots, x_{n,\nu})$. Let $(\varrho_\nu)_{\nu\in\mathbb{N}}$ be a sequence converging to ϱ with in $\varrho_\nu \in \mathcal{B}_i(x_\nu)$ for every $\nu \in \mathbb{N}$. It follows that $\varrho_\nu(f_i(\psi_i^t, \psi_{-i})) = x_{i,\nu}(\psi_i^t)\mu_{t,\nu}(\psi_{-i})$ with $\mu_{t,\nu} \in \chi_i(\psi_i^t)$ for every $t \in \{1, \dots, k(i)\}$, $\psi_{-i} \in \Psi_{-i}$ and every $\nu \in \mathbb{N}$. For every t, the sequence $\{\mu_{t,\nu}\}_{\nu\in\mathbb{N}}$ converges to a point μ_t; being $\chi_i(\psi_i^t)$ closed then $\mu_t \in \chi_i(\psi_i^t)$ for every t and hence $\varrho \in \mathcal{B}_i(x)$. Therefore $\mathcal{B}_i$ is closed in x. Applying the previous arguments to the constant sequence $(x_\nu)_{\nu\in\mathbb{N}}$ with $x_\nu = x$ for every $\nu \in \mathbb{N}$, it follows that the image $\mathcal{B}_i(x)$ is also closed for every $x \in X$. Being $\mathcal{P}$ compact and X closed it follows that $\mathcal{B}_i$ is upper semicontinuous in X.

The set-valued map $\mathcal{B}_i$ is also lower semicontinuous in every $x \in X$. In fact, given a point $x \in X$, consider $\varrho \in \mathcal{B}_i(x)$ and a sequence $(x_\nu)_{\nu\in\mathbb{N}}$ in X converging to x. Since for every $t \in \{1, \dots, k(i)\}$ and $\psi_{-i} \in \Psi_{-i}$ it follows that $\varrho(f_i(\psi_i^t, \psi_{-i})) = x_i(\psi_i^t)\mu_t(\psi_{-i})$ with $\mu_t \in \chi_i(\psi_i^t)$, consider ϱ_ν defined by $\varrho_\nu(f_i(\psi_i^t, \psi_{-i})) = x_{i,\nu}(\psi_i^t)\mu_t(\psi_{-i})$ for every $t \in \{1, \dots, k(i)\}$, $\psi_{-i} \in \Psi_{-i}$ and for every $\nu \in \mathbb{N}$. It immediately follows that $\varrho_\nu \to \varrho$ as $\nu \to \infty$ which implies that $\mathcal{B}_i$ is lower semicontinuous in x.

Finally, given $\alpha \in]0, 1[$, consider $\varrho \in \mathcal{B}_i(\alpha x_i' + (1-\alpha)x_i'', x_{-i})$. It follows that

$$\varrho(f_i(\psi_i^t, \psi_{-i})) = [\alpha x_i'(\psi_i^t) + (1-\alpha)x_i''(\psi_i^t)]\mu_t(\psi_{-i}) \text{ with } \mu_t \in \chi_i(\psi_i^t), \quad \forall t \in \{1, \dots, k(i)\}, \forall \psi_{-i} \in \Psi_{-i}$$

Now, if ϱ' and ϱ'' are defined respectively by $\varrho'(f_i(\psi_i^t, \psi_{-i})) = x_i'(\psi_i^t)\mu_t(\psi_{-i})$ and $\varrho''(f_i(\psi_i^t, \psi_{-i})) = x_i''(\psi_i^t)\mu_t(\psi_{-i})$ for all $t \in \{1, \dots, k(i)\}$ and $\psi_{-i} \in \Psi_{-i}$ then

$$\varrho' \in \mathcal{B}_i(x_i', x_{-i}),\ \varrho'' \in \mathcal{B}_i(x_i'', x_{-i}), \text{ and } \varrho = \alpha\varrho' + (1-\alpha)\varrho''$$

which finally implies that

$$\mathcal{B}_i(\alpha x_i' + (1-\alpha)x_i'', x_{-i}) \subseteq \alpha\mathcal{B}_i(x_i', x_{-i}) + (1-\alpha)\mathcal{B}_i(x_i'', x_{-i})$$

so that $\mathcal{B}_i(\cdot, x_{-i})$ is convex for every $x_{-i} \in X_{-i}$. □

Remark 3.3: Suppose that player i has a fixed set of ambiguous beliefs D_i over opponents' strategy profile but he believes that with probability ε his opponents will observe his action before their play and they will react optimally by playing a Nash equilibrium. Then player i has the following belief correspondence $\mathcal{B}_i' : X \rightsquigarrow \mathcal{P}$ defined by

$$\varrho \in \mathcal{B}_i'(x) \iff \begin{cases} \exists \mu_1 \in \chi_i(\psi_i^1), \dots, \mu_{k(i)} \in \chi_i(\psi_i^{k(i)}) \text{ and } d_i \in D_i \\ \text{such that} \\ \varrho(f_i(\psi_i^t, \psi_{-i})) = x_i(\psi_i^t)[(1-\varepsilon)d_i + \varepsilon\mu_t(\psi_{-i})] \quad \forall t \in \{1, \dots, k(i)\}, \forall \psi_{-i} \in \Psi_{-i}. \end{cases}$$

Following the same steps in the proof of Lemma 3.2, it follows that $\mathcal{B}_i'$ is continuous with not empty and closed values for every $x \in X$ and $\mathcal{B}_i'(\cdot, x_{-i})$ is convex for every $x_{-i} \in X_{-i}$.

Now we show that if the index of ambiguity aversion depends only on the beliefs that each player has about his opponents' behavior (as it seems natural since ambiguity concerns only opponents' strategy profiles) then existence of equilibria is readily obtained under standard assumptions.

PROPOSITION 3.4: *Assume that for every player i:*

a) The set-valued map $\mathcal{B}_i$ is defined by formula (9).

b) $\mathcal{P} \subset relint(Dom(c_i))$.

c) There exists a continuous function $\gamma_i : \Delta_{-i} \to \mathbb{R}$, where Δ_{-i} is the set of probability distributions over Ψ_{-i}, such that

$$c_i(\varrho) = \gamma_i(\varrho_{\Psi_{-i}}) \quad \forall \varrho \in \mathcal{P},$$

where $\varrho_{\Psi_{-i}}$ is the marginal of ϱ over Ψ_{-i}.

Then the game Γ^V has at least an MMR equilibrium.

Proof. Assumption (a) immediately implies that Lemma 3.2 holds so *(i)* in Proposition 2.4 is satisfied. Assumption (b) implies that *(ii)* in Proposition 2.4 holds and Assumption (c) implies that *(iii)* in Proposition 2.4 is also satisfied. Finally, from Lemma 3.2 we get that the belief correspondences are convex.; then it can be immediately checked that Assumption c) immediately implies that the assumptions in Proposition 2.5 are satisfied. Hence, from Theorem 2.7 we get the assertion. □

4 The stability result

As recalled in the introduction, Maccheroni, Marinacci and Rustichini (2006) first study the problem of the limit behavior of variational preferences in the standard single agent decision problem with state space. Here, we look at our model with beliefs correspondences and we consider the converge assumptions defined in Lignola and Morgan (1992). In particular, we obtain the continuous convergence of variational preferences which, in turn, guarantees convergences of sequences of equilibria to an equilibrium of the unperturbed game.

Problem statement

For every player i, consider a sequence $\{c_{i,\nu}\}_{\nu\in\mathbb{N}}$ of indices of ambiguity aversion and a sequence $\{\mathcal{B}_{i,\nu}\}_{\nu\in\mathbb{N}}$ of beliefs correspondences and let $\{F^V_{i,\nu}\}_{\nu\in\mathbb{N}}$ be the corresponding sequence of variational preferences:

$$F^V_{i,\nu}(x) = \min_{\varrho\in\mathcal{B}_{i,\nu}(x)} \left[\sum_{\omega\in\Omega} \varrho(\omega)\omega_i + c_{i,\nu}(\varrho)\right] \quad \forall x \in X,$$

and the corresponding sequence of games[17] $\{\Gamma_\nu\}_{\nu\in\mathbb{N}}$, where

$$\Gamma_\nu = \{I; (X_i)_{i\in I}; (F^V_{i,\nu})_{i\in I}\}. \tag{10}$$

In this section we look for conditions of convergence of the sequences $\{c_{i,\nu}\}_{\nu\in\mathbb{N}}$ to the indices of ambiguity aversion c_i and of $\{\mathcal{B}_{i,\nu}\}_{\nu\in\mathbb{N}}$ to the belief correspondences $\mathcal{B}_i$ for $i = 1, \ldots, n$, which guarantee that:

i) The corresponding sequences of variational preferences $\{F^V_{i,\nu}\}_{\nu\in\mathbb{N}}$ converge in an appropriate way to the variational preferences F^V_i corresponding to c_i and $\mathcal{B}_i$ for $i = 1, \ldots, n$.

ii) Converging sequences of equilibria of the perturbed games $\{\Gamma_\nu\}_{\nu\in\mathbb{N}}$ have their limits in the set of equilibria of the unperturbed game Γ corresponding to the variational preferences F^V_i, for $i = 1, \ldots, n$.

Technical tools

Definition 4.1: Given a sequence of functions $\{g_\nu\}_{\nu\in\mathbb{N}}$, with $g_\nu : Z \subseteq \mathbb{R}^k \to \overline{\mathbb{R}}$ for every $\nu \in \mathbb{N}$. Then $\{g_\nu\}_{\nu\in\mathbb{N}}$ *epiconverges* to the function g if

i) For every $z \in Z$ there exists a sequence $\{z_\nu\}_{\nu\in\mathbb{N}} \subset Z$ converging to z such that

$$\limsup_{\nu\to\infty} g_\nu(z_\nu) \leqslant g(z)$$

ii) For every $z \in Z$ and for every sequence $\{z_\nu\}_{\nu\in\mathbb{N}} \subset Z$ converging to z it follows that

$$g(z) \leqslant \liminf_{\nu\to\infty} g_\nu(z_\nu).$$

Moreover, the sequence of functions $\{g_\nu\}_{\nu\in\mathbb{N}}$ *hypoconverges* to the function g if the sequence of functions $\{-g_\nu\}_{\nu\in\mathbb{N}}$ epiconverges to the function $-g$.

Definition 4.2: Given a sequence of functions $\{g_\nu\}_{\nu\in\mathbb{N}}$, with $g_\nu : Z \subseteq \mathbb{R}^k \to \overline{\mathbb{R}}$ for every $\nu \in \mathbb{N}$. Then $\{g_\nu\}_{\nu\in\mathbb{N}}$ continuously converges to the function g if it hypoconverges and epiconverges to g, that is for every $z \in Z$ and for every sequence $\{z_\nu\}_{\nu\in\mathbb{N}} \subset Z$ converging to z it follows that

$$\limsup_{\nu\to\infty} g_\nu(z_\nu) \leqslant g(z) \leqslant \liminf_{\nu\to\infty} g_\nu(z_\nu).$$

[17] Again, it is implicitly assumed that the utility functions $F^V_{i,\nu}$ are well posed along the sequence.

4.1 Result

THEOREM 4.3: *Given the n-tuple of beliefs correspondence* $(\mathcal{B}_1, \ldots, \mathcal{B}_n)$*, the n-tuple of indexes of ambiguity aversion* $(c_1, \ldots, c_n)$ *and the corresponding game* Γ^V*. Assume that,*

i) For every player i, $(\mathcal{B}_{i,\nu})_{\nu\in\mathbb{N}}$ *is a sequence of correspondences, with* $\mathcal{B}_{i,\nu} : X \rightsquigarrow \mathcal{P}$ *for every* $\nu \in \mathbb{N}$*, which is sequentially convergent to* $\mathcal{B}_i$*, that is, for every* $x \in X$ *and every sequence* $(x_\nu)_{\nu\in\mathbb{N}}$ *converging to x,*

$$\operatorname*{Lim\,sup}_{\nu\to\infty} \mathcal{B}_{i,\nu}(x_\nu) \subseteq \mathcal{B}_i(x) \subseteq \operatorname*{Lim\,inf}_{\nu\to\infty} \mathcal{B}_{i,\nu}(x_\nu) \tag{11}$$

where

$$\operatorname*{Lim\,inf}_{\nu\to\infty} \mathcal{B}_{i,\nu}(x_\nu) = \{\varrho \in \mathcal{P} \mid \forall\varepsilon > 0\ \exists\overline{\nu}\ s.t.\ for \nu \geqslant \overline{\nu}\ S(\varrho,\varepsilon) \cap \mathcal{B}_{i,\nu}(x_\nu) \neq \emptyset\},$$
$$\operatorname*{Lim\,sup}_{\nu\to\infty} \mathcal{B}_{i,\nu}(x_\nu) = \{\varrho \in \mathcal{P} \mid \forall\varepsilon > 0\ \forall\overline{\nu} \in \mathbb{N}\ \exists\nu \geqslant \overline{\nu}\ s.t.\ S(\varrho,\varepsilon) \cap \mathcal{B}_{i,\nu}(x_\nu) \neq \emptyset\}.$$

and $S(\varrho, \varepsilon)$ *is the ball in* $\mathbb{R}^{|\Omega|}$ *with center* ϱ *and radius* ε*.*

ii) For every ϱ *and for every sequence* $\{\varrho_\nu\}_{\nu\in\mathbb{N}} \subset \mathcal{P}$ *converging to* ϱ *it follows*

$$c_i(\varrho) \leqslant \liminf_{\nu\to\infty} c_{i,\nu}(\varrho_\nu) \tag{12}$$

iii) For every $x \in X$*, for every sequence* $\{x_\nu\}_{\nu\in\mathbb{N}} \subset X$ *converging to x and for every* $\varrho \in \mathcal{B}_i(x)$ *there exists a sequence* $\{\varrho_\nu\}_{\nu\in\mathbb{N}}$ *converging to* ϱ*, with* $\varrho_\nu \in \mathcal{B}_{i,\nu}(x_\nu)$ *for every* $\nu \in \mathbb{N}$*, such that*

$$\limsup_{\nu\to\infty} c_{i,\nu}(\varrho_\nu) \leq c_i(\varrho) \tag{13}$$

Then,

a) For every player i, the sequence $\{F^V_{i,\nu}\}_{\nu\in\mathbb{N}}$ *continuously converges to* F^V_i

b) If the sequence $(x^*_\nu)_{\nu\in\mathbb{N}} \subset X$ *converges to* $x^* \in X$ *and, for every* $\nu \in \mathbb{N}$*,* x^*_ν *is an equilibrium of the game* Γ^V_ν*. Then,* x^* *is an equilibrium of the game* Γ^V*.*

Proof. a) First we prove that for every every $x \in X$ and for every sequence $\{x_\nu\}_{\nu\in\mathbb{N}} \subset X$ converging to x

$$F^V_i(x) \leqslant \liminf_{\nu\to\infty} F^V_{i,\nu}(x_\nu).$$

For every $\varrho \in \mathcal{P}$, denote with $\varphi_i(\varrho) = E_i[\varrho] + c_i(\varrho)$ and $\varphi_{i,\nu}(\varrho) = E_i[\varrho] + c_{i,\nu}(\varrho)$. Let $\overline{\varrho}$ and ϱ_ν be such that $\varphi_i(\overline{\varrho}) = F^V_i(x)$ and $\varphi_{i,\nu}(\varrho_\nu) = F^V_{i,\nu}(x_\nu)$. Suppose that

$$\liminf_{\nu\to\infty} F^V_{i,\nu}(x_\nu) < F^V_i(x), \tag{14}$$

then there exists a subsequence $\{x_{\nu_k}\}_{k\in\mathbb{N}}$ such that $\varphi_{i,\nu_k}(\varrho_{\nu_k}) = F^V_{i,\nu_k}(x_{\nu_k}) < F^V_i(x)$ for every $k \in \mathbb{N}$. The corresponding sequence $\{\varrho_{\nu_k}\}_{k\in\mathbb{N}}$ has a converging subsequence $\{\varrho_{\nu_h}\}_{h\in\mathbb{N}}$. Let $\widehat{\varrho}$ be its limit, then from assumption *(i)* it follows that $\widehat{\varrho} \in \mathcal{B}_i(x)$ and so $\varphi_i(\varrho) \leqslant \varphi_i(\widehat{\varrho})$. From assumption *(ii)* we get

$$F^V_i(x) = \varphi_i(\varrho) \leqslant \varphi_i(\widehat{\varrho}) \leqslant \liminf_{h\to\infty} \varphi_{i,\nu_h}(\varrho_{\nu_h}) = \liminf_{h\to\infty} F^V_{i,\nu_h}(x_{\nu_h})$$

and this contradicts (14). Then

$$F^V_i(x) \leqslant \liminf_{\nu\to\infty} F^V_{i,\nu}(x_\nu). \tag{15}$$

Now, we prove that for every every $x \in X$ and for every sequence $\{x_\nu\}_{\nu\in\mathbb{N}} \subset X$ converging to x

$$\limsup_{\nu\to\infty} F^V_{i,\nu}(x_\nu) \leqslant F^V_i(x). \tag{16}$$

Let $\varrho \in \mathcal{B}_i(x)$ be such that $\varphi_i(\varrho) = F^V_i(x)$. From assumption *(iii)* there exists a sequence $\{\varrho_\nu\}_{\nu\in\mathbb{N}}$ converging to ϱ such that $\varrho_\nu \in \mathcal{B}_{i,\nu}(x_\nu)$ for all $\nu \in \mathbb{N}$ and $\limsup_{\nu\to\infty} c_{i,\nu}(\varrho_\nu) \leqslant c_i(\varrho)$. Being, $F^V_{i,\nu}(x_\nu) \leqslant \varphi_{i,\nu}(\varrho_\nu)$ for every $\nu \in \mathbb{N}$ then immediately follows that

$$\limsup_{\nu\to\infty} F^V_{i,\nu}(x_\nu) \leqslant \limsup_{\nu\to\infty} \varphi_{i,\nu}(\varrho_\nu) \leqslant \varphi_i(\varrho) = F^V_i(x)$$

.

Summarizing, for every every $x \in X$ and for every sequence $\{x_\nu\}_{\nu\in\mathbb{N}} \subset X$ converging to x

$$\limsup_{\nu\to\infty} F^V_{i,\nu}(x_\nu) \leqslant F^V_i(x) \leqslant \liminf_{\nu\to\infty} F^V_{i,\nu}(x_\nu),$$

which means that the sequence $\{F^V_{i,\nu}\}_{\nu\in\mathbb{N}}$ continuously converges to F^V_i.

b) Let $(x^*_\nu)_{\nu\in\mathbb{N}} \subset X$ be a sequence converging to $x^* \in X$ such that, for every $\nu \in \mathbb{N}$, x^*_ν is an equilibrium of the game Γ^V_ν then, for every ν, it follows that

$$F^V_{i,\nu}(x^*_{i,\nu}, x^*_{-i,\nu}) \geq F^V_{i,\nu}(x'_i, x^*_{-i,\nu}) \quad \forall x'_i \in X_i$$

taking the limit as $\nu \to \infty$ we get

$$F^V_i(x^*_i, x^*_{-i}) = \lim_{\nu\to\infty} F^V_{i,\nu}(x^*_{i,\nu}, x^*_{-i,\nu}) \geq \lim_{\nu\to\infty} F^V_{i,\nu}(x'_i, x^*_{-i,\nu}) = F^V_i(x'_i, x^*_{-i}) \quad \forall x'_i \in X_i$$

which implies that x^*_i is a best reply to x^*_{-i} and we get the assertion.

□

Remark 4.4: The previous theorem could also be proved by applying the different stability results for marginal functions under constraints given in Lignola and Morgan (1992). We prefer to give a direct proof in order to better clarify the role of the assumptions in our model. However, the arguments we use are naturally similar to the ones contained in the proofs by Lignola and Morgan (1992).

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