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Abstract

MinSup problems with constraints described by quasi-equilibrium problems are considered in Banach spaces. The solutions set of such problems may be empty even in very good situations, so the aim of this paper is twofold. First, we determine appropriate regularizations which allow to asymptotically reach the value of the original problem. Then, among these regularizations we identify those which allow to bypass the lack of exact solutions to these problems by a suitable concept of viscosity solution whose existence is then proved under reasonable assumptions.

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1 Introduction

Given a Hausdorff topological space (X, τ) and a real Banach space E with dual E^*, let $H \subseteq X$ be a τ-closed set, $K \subseteq E$ be a closed and convex set. If h is a real-valued function defined in $H \times E \times E$ and S is a set-valued map from $H \times K$ to K with nonempty values, we consider, for any $x \in H$, the parametric Quasi-Equilibrium Problem $(QE)(x)$ (called *quasi-variational* problem in [21]) which consists in finding $u \in K$ such that

$$u \in S(x,u) \;\text{ and }\; h(x,u,v) \;\leq\; 0 \;\;\forall\, v \in S(x,u)$$

and we denote by $\mathcal{Q}(x)$ the solution set, that is

$$u \in \mathcal{Q}(x) \quad \Longleftrightarrow \quad u \;\text{ solves }\; (QE)(x).$$

We stress that the solution map $\mathcal{Q}$ is generally set-valued even under restrictive assumptions [6]. We also observe that, by taking appropriate functions and maps, several parametric problems can be described by a Quasi-Equilibrium Problem$(QE)(x)$: Variational Inequality [6], Complementarity Problem [31], Nash Equilibrium Problem [30], Implicit Variational Problem [18], Quasi-Variational Inequality [6], Generalized Variational Inequality [18], Generalized Quasi-variational Inequality [18], Equilibrium Problem [7], Social (or Generalized) Nash Equilibrium Problem [10], Mixed Quasivariational-like Inequality [8].
In this paper, we consider the following MinSup problem which can be seen as a model of a pessimistic two-stage problem [26], [27], [14], [11], [33] with quasi-equilibrium constraints:

$$(MS) \;\text{ find }\; x_o \in H \text{ such that } \sup_{u \in \mathcal{Q}(x_o)} f(x_o,u) \;=\; \min_{x \in H} \sup_{u \in \mathcal{Q}(x)} f(x,u)$$

where f is a function from $H \times K$ to $\mathbb{R} \cup \{+\infty\}$.
The set of solutions and the infimum of the problem (MS) are denoted by $\mathcal{MS}$ and ω respectively, so we have

$$x_o \in \mathcal{MS} \Longleftrightarrow \; u_o \in \mathcal{Q}(x_o) \text{ and } \sup_{u \in \mathcal{Q}(x_o)} f(x_o,u) = \min_{x \in H} \sup_{u \in \mathcal{Q}(x)} f(x,u) \tag{1}$$

and

$$\omega = \inf_{x \in H} \sup_{u \in \mathcal{Q}(x)} f(x,u). \tag{2}$$

Unfortunately, constrained MinSup problems may fail to have a solution when the constraint map is not lower semicontinuous (see, for example, [16]) and this could be the case for the map $\mathcal{Q}$ [26], [19]. So, it is interesting to point the attention to the value ω, which is defined but hardly computable in the absence of solutions. We show that ω can be obtained as the limit of values of appropriate MinSup problems having solutions under suitable assumptions. More precisely, for any given $x \in H$ and any positive real number ε, we consider the ε-quasi-equilibrium problem

$$(QE)^{\varepsilon}(x) \quad \text{find } u_\varepsilon \in K \text{ such that } d\left(u_\varepsilon, S\left(x, u_\varepsilon\right)\right) \leq \varepsilon \text{ and } h(x,u_\varepsilon,v) \;\leq\; \varepsilon \;\;\forall\, v \in S(x,u_\varepsilon)$$

and the strict ε-quasi-equilibrium problem

$$(SQE)^{\varepsilon}(x) \quad \text{find } u_\varepsilon \in K \text{ such that } d\left(u_\varepsilon, S\left(x, u_\varepsilon\right)\right) < \varepsilon \text{ and } h(x,u_\varepsilon,v) \;<\; \varepsilon \;\;\forall\, v \in S(x,u_\varepsilon).$$

We denote by $\mathcal{Q}^\varepsilon$ and $\mathfrak{S}^\varepsilon$ the corresponding solutions maps

$$\mathcal{Q}^\varepsilon : x \in H \to \mathcal{Q}^\varepsilon(x) \;=\; \{u_\varepsilon \in K \text{ such that } d\left(u_\varepsilon, S\left(x, u_\varepsilon\right)\right) \leq \varepsilon \text{ and } h(x, u_\varepsilon, v) \;\leq\; \varepsilon \;\;\forall\, v \in S(x, u_\varepsilon)\}$$

$$\mathfrak{S}^\varepsilon : x \in H \to \mathfrak{S}^\varepsilon(x) \;=\; \{u_\varepsilon \in K \text{ such that } d\left(u_\varepsilon, S\left(x, u_\varepsilon\right)\right) < \varepsilon \text{ and } h(x, u_\varepsilon, v) \;<\; \varepsilon \;\;\forall\, v \in S(x, u_\varepsilon)\}$$

and by ω_ε and σ_ε the values

$$\omega_\varepsilon \;=\; \inf_{x \in H} \sup_{u \in \mathcal{Q}^\varepsilon(x)} f(x, u) \qquad \sigma_\varepsilon \;=\; \inf_{x \in H} \sup_{u \in \mathfrak{S}^\varepsilon(x)} f(x, u)$$

of the following MinSup problems

$$(MS)^\varepsilon \;\; \text{find } \; x_\varepsilon \in H \text{ such that } \quad \sup_{u \in \mathcal{Q}^\varepsilon(x_\varepsilon)} f(x_\varepsilon, u) \;=\; \min_{x \in H} \sup_{u \in \mathcal{Q}^\varepsilon(x)} f(x, u) \;=\; \omega_\varepsilon$$

$$(SMS)^\varepsilon \;\; \text{find } \; x_\varepsilon \in H \text{ such that } \quad \sup_{u \in \mathfrak{S}^\varepsilon(x_\varepsilon)} f(x_\varepsilon, u) \;=\; \min_{x \in H} \sup_{u \in \mathfrak{S}^\varepsilon(x)} f(x, u) \;=\; \sigma_\varepsilon.$$

We prove that, under suitable assumptions, ω_ε and σ_ε converge to ω whenever ε converges to zero, so both of the values are "good" candidates to approach the value ω.
However, for the same reasons as for the problem (MS), the problem $(MS)^\varepsilon$ may fail to have solutions (see Example 4.1), whereas we prove, under appropriate conditions of minimal character, that the problem $(SMS)^\varepsilon$ has a solution x_ε.
We also introduce two other types of approximate values for (MS), μ_ε and ν_ε, defined by the aid of different approximate solutions maps for quasi-equilibrium problems, $\mathcal{T}^\varepsilon$ and $\mathcal{S}^\varepsilon$, and which also converge to ω whenever ε converges to zero, .
Moreover, we define a regularization class, called *inner regularization*, for the lower level problem and we prove that $\mathfrak{S} = \{\mathfrak{S}^\varepsilon, \varepsilon > 0\}$ and $\mathcal{S} = \{\mathcal{S}^\varepsilon, \varepsilon > 0\}$ are inner regularizations under suitable assumptions. Then, in the spirit of [2], we introduce a concept of *viscosity solution* associated to an inner regularization and we prove existence results for viscosity solutions associated to a general inner regularization and to the families $\mathfrak{S}$ and $\mathcal{S}$.
This paper carries on the study on Optimistic (MinMin) Bilevel Problems with Variational Inequality Constraints and on Pessimistic (MinSup) Bilevel Problems with Quasi -variational Inequality constraints in finite dimensional spaces, made in [20] and in [22] respectively, and on Optimistic Bilevel Problems with Quasi-variational inequality in infinite dimensional spaces, made in [24]. Moreover, we emphasize that here we concentrate our attention on unperturbed problems having in mind to investigate in a separate paper how to approach ω in the presence of perturbations.

2 Preliminaries and auxiliary results

Let τ and σ be topologies on the set X and on the space E respectively.
We denote by s and w the strong and the weak topology on the space E, and by $cl_\tau^{seq}(Y)$ the *sequential closure* of a set $Y \subseteq X$.
A function $g : H \subseteq X \to \mathbb{R} \cup \{+\infty\}$ is *sequentially τ-coercive* on H if for every $t \in \mathbb{R}$ there exists a set $C_t \subseteq X$, sequentially compact in the topology τ, such that

$$Lev_t\, g \;=\; \{x \in H : g(x) \leq t\} \;\subseteq\; C_t.$$

A function $f : H \times K \to \mathbb{R} \cup \{+\infty\}$ is *sequentially σ-coercive with respect to u* on the set K *uniformly with respect to $x \in H$* (*coercive in u on K* for short) if for every $t \in \mathbb{R}$ there exists a set $Y_t \subseteq E$ sequentially compact in the topology σ such that

$$(Lev_t\, f)\,(x) = \{u \in K : f(x,u) \le t\} \subseteq\ Y_t$$

for every $x \in H$.
A sequence of functions $(g_n)_n$ defined on X *sequentially τ-epiconverges* to g in X if
- for every $x \in X$ and every sequence $(x_n)_n$ τ-converging to x in X

$$g(x)\ \le\ \liminf_n g_n(x_n),$$

- for every $x \in X$ there exists a sequence $(x'_n)_n$ τ-converging to x in X such that

$$\limsup_n g_n(x'_n)\ \le\ g(x).$$

For the above notion and related arguments see [1] and [9].
If $(K_n)_n$ is a sequence of nonempty subsets of E, the Painlevé-Kuratowski upper and lower limits of the sequence $(K_n)_n$, with respect to σ, are defined respectively by [5]
- $z \in \sigma\text{-}\limsup_n K_n$ if there exists a sequence $(z_k)_k$ σ-converging to z such that for a subsequence (K_{n_k}) of $(K_n)_n$ $z_k \in K_{n_k}$ for each $k \in \mathbb{N}$;
- $z \in \sigma\text{-}\liminf_n K_n$ if there exists a sequence $(z_n)_n$ σ-converging to z such that $z_n \in K_n$ for n sufficiently large.

We recall that both these sets are σ-closed and may be empty.
A set-valued map F from H to K is:
$-$ *(τ, σ)-sequentially subcontinuous* over H, *(τ, σ)-subcontinuous* for short, if for every $x \in H$, every sequence $(x_n)_n$ τ-converging to x in H, every sequence $(u_n)_n$ such that $u_n \in F(x_n)$, for every $n \in \mathbb{N}$, has a subsequence σ-converging;
$-$ *(τ, σ)-sequentially lower semicontinuous* over H, *(τ, σ)-lower semicontinuous* for short, if for every $x \in X$ and every sequence $(x_n)_n$ τ-converging to x in H

$$F(x)\ \subseteq\ \sigma\text{-}\liminf_n\ F(x_n);$$

$-$ *(τ, σ)-sequentially closed* over H, *(τ, σ)-closed* for short, if for every $x \in H$ and every sequence $(x_n)_n$ τ-converging to x in H

$$\sigma\text{-}\limsup_n\ F(x_n)\ \subseteq\ F(x).$$

A function l from $K \times K$ to $\mathbb{R}$ is:
$-$ *pseudomonotone* over K if

$$l(u,v) \le 0 \implies l(v,u) \ge 0 \quad \forall\ u, v \in K;$$

$-$ *monotone* over K if

$$l(u,v) + l(v,u) \ge 0 \quad \forall\ u, v \in K.$$

Throughout this paper, we omit the term *sequentially* for short, when there is no ambiguity, and we make the following blanket assumptions [6]:

$\mathcal{A}$) $h(x, u, u) = 0$, for every $x \in H$ and every $u \in K$;

$\mathcal{B}$) $\mathcal{Q}(x) \neq \emptyset$, for every $x \in H$.

The following results will be widely used in the paper.

Proposition 2.1 [21, Lemma 3.1] *If $h(x, \cdot, v)$ is w-lower semicontinuous on the segments of K for every $(x, v) \in H \times K$, $h(x, u, \cdot)$ is concave on K for every $(x, u) \in H \times K$ and S is convex-valued and closed-valued, then the set $\mathcal{Q}$ of solutions to the quasi-equilibrium problem $(QE)(x)$ contains the set of solutions to the following problem*

$$\text{find } u \in S(x, u) \quad \text{such that} \quad h(x, v, u) \ \geq \ 0 \quad \forall\, v \in S(x, u).$$

Proposition 2.2 *Let S be a set-valued map from $H \times K$ to K.*
If S is $(\tau \times s, s)$-lower semicontinuous in $H \times K$, then, for any $(x, u) \in H \times K$ and any sequence $(x_n, u_n)_n$ $(\tau \times s)$-converging in $H \times K$ towards (x, u) one has

$$\limsup_n \, d(u_n, S(x_n, u_n)) \ \leq \ d(u, S(x, u)).$$

Proof
Omitted since it is an easy adaptation of the first part of [20, Lemma 2.3].

Proposition 2.3 *If S is $(\tau \times w, w)$-closed and $(\tau \times w)$-subcontinuous in $H \times K$, then, for any $(x, u) \in H \times K$ and any sequence $(x_n, u_n)_n$ $(\tau \times w)$-converging in $H \times K$ towards (x, u) one has*

$$d(u, S(x, u)) \ \leq \ \liminf_n \, d(u_n, S(x_n, u_n)).$$

The same result holds if the space E is reflexive and S is only $(\tau \times w, w)$-closed.

Proof
The proof of the first part is omitted since it is an easy adaptation of the second part of [24, Lemma 2.2].
Assume that the space E is reflexive and that there exists $a \in \mathbb{R}$ such that

$$\liminf_n \, d(u_n, S(x_n, u_n)) < a < d(u, S(x, u)).$$

Then, for every $k \in \mathbb{N}$ there exists a positive integer $n_k \geq k$ such that $d(u_{n_k}, S(x_{n_k}, u_{n_k})) < a$. So, we determine a sequence $(z_{n_k})_k$ such that $z_{n_k} \in S(x_{n_k}, u_{n_k})$ and $||z_{n_k} - u_{n_k}|| < a$ for every $k \in \mathbb{N}$. Since the sequence $(z_{n_k})_k$ is bounded and E is reflexive, a subsequence has to weakly converge to $z \in K$ and $z \in S(x, u)$ because S is $(\tau \times w, w)$-closed. Therefore we have

$$||z - u|| \leq \liminf_k d(u_{n_k}, S(x_{n_k}, u_{n_k})) \leq a < d(u, S(x, u))$$

and this gives a contradiction. $\square$

3 Convergence towards ω of the approximate values

We prove that the regularization method presented in the Introduction allows to asymptotically reach the value of the MinSup problem (MS).

Proposition 3.1 *Assume that the following hold:*
i) the set-valued map S is convex-valued, $(\tau \times w, w)$-subcontinuous, $(\tau \times w, s)$-lower semicontinuous and $(\tau \times w, w)$-closed on $H \times K$;
ii) the function $h(x, u, \cdot)$ is concave on K for every $(x, u) \in H \times K$;
iii) the function $h(x, \cdot, v)$ is w-lower semicontinuous over the segments of K for every $(x, v) \in H \times K$;
iv) for every $(x, u, v) \in H \times K \times K$ and every sequence $(x_n, u_n, v_n)_n$ $(\tau \times w \times s)$-converging to (x, u, v) in $H \times K \times K$ one has

$$\liminf_n h(x_n, u_n, v_n) + h(x, v, u) \geq 0;$$

v) the function $-f$ is w-coercive in u on K;
vi) for every $x \in H$ there exists a sequence $(x_n)_n$ τ-converging to x in H such that for every $u \in K$ and every sequence $(u_n)_n$ w-converging to u in K one has

$$\limsup_n f(x_n, u_n) \leq f(x, u).$$

Then,

$$\omega = \lim_{\varepsilon \to 0} \omega_\varepsilon.$$

Proof
We observe that

$$\lim_{\varepsilon \to 0} \omega_\varepsilon = \inf_{\varepsilon > 0} \omega_\varepsilon,$$

since ω_ε is increasing with respect to ε and ε converges to zero. Moreover, $\omega \leq \omega_\varepsilon$ since $\mathcal{Q} \subseteq \mathcal{Q}^\varepsilon$. So, we have to prove that

$$\inf_{\varepsilon > 0} \omega_\varepsilon \leq \omega. \tag{3}$$

Assume that (3) does not hold. There exist $c \in \mathbb{R}$ such that

$$\omega < c < \inf_{\varepsilon > 0} \omega_\varepsilon$$

and $\widetilde{x} \in H$ such that

$$f(\widetilde{x}, u) < c \quad \forall\, u \in \mathcal{Q}(\widetilde{x}). \tag{4}$$

Due to assumption *vi*), there exists a sequence $(\widetilde{x}_n)_n$ τ-converging to $\widetilde{x}$ in H such that $\limsup_n f(\widetilde{x}_n, u_n) \leq f(\widetilde{x}, u)$ for every $u \in K$ and every sequence $(u_n)_n$ w-converging to u in K.
Then, for every sequence of positive real numbers $(\varepsilon_n)_n$ decreasing to 0 we have that $c < \omega_{\varepsilon_n} \leq \sup_{u \in \mathcal{Q}^{\varepsilon_n}(\widetilde{x}_n)} f(\widetilde{x}_n, u)$ for every $n \in \mathbb{N}$. So, there exists a sequence $(\widetilde{u}_n)_n$ such that

$$\widetilde{u}_n \in \mathcal{Q}^{\varepsilon_n}(\widetilde{x}_n) \;\text{ and }\; c < f(\widetilde{x}_n, \widetilde{u}_n) \quad \forall\, n \in \mathbb{N}. \tag{5}$$

Form assumption $v)$ one has that a subsequence $(\widetilde{u}_{n_k})_k$ of $(\widetilde{u}_n)_n$ w-converges to a point $\widetilde{u} \in K$ and we can prove that $\widetilde{u} \in \mathcal{Q}(\widetilde{x})$. Indeed, applying the first part of Proposition 2.3 we get that $\widetilde{u} \in S(\widetilde{x}, \widetilde{u})$ because $d(\widetilde{x}, S(\widetilde{x}, \widetilde{u})) \leq \liminf_k d(\widetilde{x}_{n_k}, S(\widetilde{x}_{n_k}, \widetilde{u}_{n_k})) \leq \lim_k \varepsilon_{n_k} = 0$. So, it remains to prove that $h(\widetilde{x}, \widetilde{u}, v) \leq 0$ for every $v \in S(\widetilde{x}, \widetilde{u})$. Since S is $(\tau \times w, s)$-lower semicontinuous on K, for any given $v \in S(\widetilde{x}, \widetilde{u})$ there exists a sequence $(v_{n_k})_k$ s-converging to $v \in K$ such that $v_{n_k} \in S(\widetilde{x}_{n_k}, \widetilde{u}_{n_k})$ for k sufficiently large, so $h(\widetilde{x}_{n_k}, \widetilde{u}_{n_k}, v_{n_k}) \leq \varepsilon_{n_k}$. Therefore, by assumption $iv)$, we have

$$-h(\widetilde{x}, v, \widetilde{u}) \leq \liminf_k h(\widetilde{x}_{n_k}, \widetilde{u}_{n_k}, v_{n_k}) \leq 0$$

and $\widetilde{u} \in \mathcal{Q}(\widetilde{x})$ by Proposition 2.1.
Then, $f(\widetilde{x}, \widetilde{u}) < c$ by (4) and this is in contradiction with (5) due to assumption $vi)$. □

Remark 3.1 We can drop the subcontinuity assumption from condition $i)$ of Proposition 3.1 whenever the space E is reflexive using the second part of Proposition 2.2.

Remark 3.2 Condition $iv)$, which implies that the function $h(x, \cdot, \cdot)$ is monotone over $K \times K$, introduced by the authors, has been used in [17], [18] and [21]. It has been proved to be satisfied when, for example, $X = \mathbb{N}$ and $h(n, u, v) = \langle A_n u, u - v \rangle$, $(A_n)_n$ being a sequence of operators from E to E^*, under suitable assumptions. It is worth mentioning that in this case the quasi-equilibrium problem $(QE)(n)$ amounts to a quasi-variational inequality for every $n \in \mathbb{N}$. Condition $vi)$ is a particular case of a convergence notion introduced by Attouch and Wets in [3] and used, for example, in [26] and in [16] for MinSup problems with optimization constraints.

Corollary 3.1 *In the same assumptions of Proposition 3.1 we also have:*

$$\omega = \lim_{\varepsilon \to 0} \sigma_\varepsilon .$$

Proof
Since $\mathcal{Q}(x) \subseteq \mathfrak{S}^\varepsilon(x) \subseteq \mathcal{Q}^\varepsilon(x)$ for every $x \in H$, we have

$$\omega \ \leq \ \sigma_\varepsilon \ \leq \ \omega_\varepsilon$$

and the result follows from Proposition 3.1. □

Remark 3.3 We have implicitely proven that if assumptions $i) - iv)$ hold, then for every $x \in H$ and every sequence $(x_n)_n$ τ-converging to x in H

$$w\text{-}\limsup_n \mathcal{Q}^{\varepsilon_n}(x_n) \subseteq \mathcal{Q}(x). \tag{6}$$

One can also consider the approximate values obtained by another type of approximate solutions maps for quasi-equilibrium problems, similar (but not exactly the same) to those introduced in [21] . Namely, we define, for every $x \in H$,

$$\mathcal{T}^\varepsilon(x) = \{u_\varepsilon \in K \text{ such that } d\left(u_\varepsilon, S\left(x, u_\varepsilon\right)\right) \leq \varepsilon \text{ and } h(x, u_\varepsilon, v) \ \leq \ \varepsilon ||u_\varepsilon - v|| \ \ \forall\, v \in S(x, u_\varepsilon)\}$$

$$\mathcal{S}^\varepsilon(x) \ = \ \{u_\varepsilon \in K \text{ such that } d\left(u_\varepsilon, S\left(x, u_\varepsilon\right)\right) < \varepsilon \text{ and } h(x, u_\varepsilon, v) \ < \ \varepsilon ||u_\varepsilon - v|| \ \ \forall\, v \in (S(x, u_\varepsilon) - \{u_\varepsilon\})\} .$$

We denote by μ_ε and ν_ε the values

$$\mu_\varepsilon \;=\; \inf_{x\in H} \sup_{u\in\mathcal{T}^\varepsilon(x)} f(x,u) \qquad \nu_\varepsilon \;=\; \inf_{x\in H} \sup_{u\in\mathcal{S}^\varepsilon(x)} f(x,u)$$

of the corresponding MinSup problems

$$\text{find } \; x_\varepsilon\in H \text{ such that } \quad \sup_{u\in\mathcal{T}^\varepsilon(x_\varepsilon)} f(x_\varepsilon,u) \;=\; \min_{x\in H} \sup_{u\in\mathcal{T}^\varepsilon(x)} f(x,u)$$

$$\text{find } \; x_\varepsilon\in H \text{ such that } \quad \sup_{u\in\mathcal{S}^\varepsilon(x_\varepsilon)} f(x_\varepsilon,u) \;=\; \min_{x\in H} \sup_{u\in\mathcal{S}^\varepsilon(x)} f(x,u)$$

and we prove a result analogous to Proposition 3.1.

Proposition 3.2 *Assume that the assumptions of Proposition 3.1 hold. Then*

$$\omega = \lim_{\varepsilon\to 0} \mu_\varepsilon \;=\; \lim_{\varepsilon\to 0} \nu_\varepsilon.$$

Proof
The proof of the first equality is similar to the proof of Proposition 3.1 and the difference lies in proving that the weak limit u of a sequence $(u_n)_n$, such that $u_n\in\mathcal{T}^{\varepsilon_n}(x_n)$ for every $n\in\mathbb{N}$, is a solution to $(QE)(x)$.
Let $v\in S(x,u)$ and let $(v_n)_n$ be a sequence strongly converging to v such that $v_n\in S(x_n,u_n)$ for n sufficiently large. Since $u_n\in\mathcal{T}^{\varepsilon_n}(x_n)$, we have $h(x_n,u_n,v_n)\le\varepsilon_n||u_n-v_n||$. The sequence $(||u_n-v_n||)_n$ being bounded, from condition $iv)$ we get $-h(x,v,u)\le \liminf_n h(x_n,u_n,v_n)\le 0$ and $u\in\mathcal{Q}(x)$ by Proposition 2.1.
The second equality follows because $\omega \;\le\; \nu_\varepsilon \;\le\; \mu_\varepsilon$. $\square$

4 Viscosity solutions

The strict approximate solutions maps $\mathfrak{S}^\varepsilon$ and $\mathcal{S}^\varepsilon$ have not been explicitely used in the previous results since the convergence of σ_ε and ν_ε towards ω has been easily deduced from the convergence of ω_ε and μ_ε. Nevertheless, in the following we will see that they play an important role in the construction of viscosity solutions for the problem (MS), so we start this section by a brief investigation of their properties. It is clear that both maps are not closed in general, however, the following proposition proves that they can approach the solutions map $\mathcal{Q}$ under appropriate conditions.

Proposition 4.1 *Assume that the following hold:*
i) the set-valued map S is convex-valued, $(\tau\times w,s)$-lower semicontinuous and $(\tau\times w,w)$-closed and $(\tau\times w,w)$-subcontinuous on $H\times K$;
ii) the function $h(x,u,\cdot)$ is concave on K for every $(x,u)\in H\times K$;
iii) the function $h(x,\cdot,v)$ is w-lower semicontinuous over the segments of K for every $(x,v)\in H\times K$;
iv) for every $(x,u,v)\in H\times K\times K$, for every sequence $(x_n,u_n,v_n)_n$ $(\tau\times w\times s)$-converging to (x,u,v) in $\times K\times K$,

$$\liminf_n \; h(x_n,u_n,v_n)+h(x,v,u) \;\ge\; 0.$$

Then, for every $x \in H$ and every sequence $(x_n)_n$ τ-converging to x in H we have:

$$w - \limsup_n \mathfrak{S}^{\varepsilon_n}(x_n) \subseteq \mathcal{Q}(x).$$

If, moreover, the space E is reflexive, then we also have

$$w - \limsup_n \mathcal{S}^{\varepsilon_n}(x_n) \subseteq \mathcal{Q}(x).$$

Proof
Observe that in our assumptions Proposition 3.1 applies, so the first inclusion follows from (6) because $\mathfrak{S}^{\varepsilon}(x) \subseteq \mathcal{Q}^{\varepsilon}(x)$.
Let $(u_n)_n$ be a sequence weakly converging to u in K such that $u_n \in \mathcal{S}^{\varepsilon_n}(x_n)$ for every $n \in \mathbb{N}$, i.e.

$$d(u_n, S(x_n, u_n)) < \varepsilon_n \text{ and } h(x_n, u_n, v) < \varepsilon_n ||u_n - v|| \quad \forall\, v \in (S(x, u_n) - \{u_n\})\,. \tag{7}$$

Since in our assumptions Proposition 2.1 applies, it suffices to prove that $u \in S(x, u)$ (and this follows from the second part of Proposition 2.3) and that $h(x, v, u) \geq 0$ for every $v \in S(x, u)$. Let $v \in S(x, u)$. If $v = u$ we have $h(x, u, u) = 0$ by assumption $\mathcal{A}$) in Section 2. If $v \neq u$, due to the $(\tau \times w, s)$-lower semicontinuity of the map S, there exists a sequence $(v_n)_n$ strongly converging to v such that $v_n \in (S(x, u_n) - \{u_n\})$ for n sufficiently large. Then, the result follows from *iv*) and (7). $\square$

The following proposition gives a sufficient condition for the lower semicontinuity of $\mathfrak{S}^{\varepsilon}$ and $\mathcal{S}^{\varepsilon}$.

Proposition 4.2 *Assume that the following hold:*
i) the set-valued map S is $(\tau \times w, w)$-subcontinuous, $(\tau \times w, s)$-lower semicontinuous and $(\tau \times w, w)$-closed on $H \times K$;
ii) for every $(x, u) \in H \times K$ and every sequence $(x_n)_n$ τ-converging to x in H there exists a sequence $(u_n)_n$ strongly converging to u in K such that for every $v \in K$ and every sequence $(v_n)_n$ weakly converging to w in K one has

$$h(x, u, v) \geq \limsup_n h(x_n, u_n, v_n).$$

Then, the maps $\mathfrak{S}^{\varepsilon}$ and $\mathcal{S}^{\varepsilon}$ are (τ, s)-lower semicontinuous over H.

Proof
Assume that there is a point x in H such that $\mathfrak{S}^{\varepsilon}$ is not (τ, s)-lower semicontinuous at x. There exist a sequence $(x_n)_n$ τ-converging to x in H and a point $u \in \mathfrak{S}^{\varepsilon}(x)$ such that $u \notin s - \liminf_n \mathfrak{S}^{\varepsilon}(x_n)$. Therefore, the sequence $(u_n)_n$ in condition *ii*) has a subsequence $(u_{n_k})_k$ such that $u_{n_k} \notin \mathfrak{S}^{\varepsilon}(x_{n_k})$ for every $k \in \mathbb{N}$. Since $d(u, S(x, u)) < \varepsilon$ and S is $(\tau \times w, s)$-lower semicontinuous at x, from Proposition 2.2 we infer that $d(u_{n_k}, S(x_{n_k}, u_{n_k})) < \varepsilon$ for k sufficiently large. Then, for such indexes k there exists $v_{n_k} \in S(x_{n_k}, u_{n_k})$ such that $h(x_{n_k}, u_{n_k}, v_{n_k}) \geq \varepsilon$. The map S being $(\tau \times w, w)$-subcontinuous and $(\tau \times w, w)$-closed, there exists a subsequence of $(v_{n_k})_k$ w-converging to $v \in S(x, u)$ and $h(x, u, v) \geq \varepsilon$ by

condition ii), but this is in contradiction with $u \in \mathfrak{S}^\varepsilon(x)$.
Assume now that there is a point x in H such that $\mathcal{S}^\varepsilon$ is not (τ, s)-lower semicontinuous at x. Arguing as before, we determine $v_{n_k} \in (S(x_{n_k}, u_{n_k}) - \{u_{n_k}\})$ such that

$$h(x_{n_k}, u_{n_k}, v_{n_k}) \geq \varepsilon ||u_{n_k} - v_{n_k}||$$

and the result follows from the $(\tau \times w, w)$-subcontinuity and $(\tau \times w, w)$-closedness assumption and condition ii), due to the weak lower semicontinuity of the norm $\| \cdot \|$. $\square$

The use of strict ε-solutions has been proven to be very fruitful when $(QE)(x)$ is:

- an optimization problem [26], [16],
- a variational inequality [20],
- a quasi-variational inequality [22],
- a Nash equilibrium problem [28] [23],
- a generalized Nash equilibrium problem [29].

Now, inspired by [2], we introduce the viscosity solutions for the MinSup problem with quasi-equilibrium constraints that, roughly speaking, are the limit points of suitable approximating sequences whose corresponding values converge to the value ω.
To this end, we give the following definition.

Definition 4.1 *We say that a family of maps $\mathfrak{D} = \{\mathfrak{D}^\varepsilon, \varepsilon > 0\}$, where $\mathfrak{D}^\varepsilon : x \in H \to \mathfrak{D}^\varepsilon(x) \subseteq K$, is an inner regularization for the family of quasi-equilibrium problems $\{(QE)(x), x \in H\}$ if the following conditions are satisfied:*

R_1) *for every $x \in H$, every sequence $(x_n)_n$ τ-converging to x in H and every sequence $(\varepsilon_n)_n$ decreasing to zero, one has*

$$w - \limsup_n \mathfrak{D}^{\varepsilon_n}(x_n) \subseteq \mathcal{Q}(x);$$

R_2) *$\mathfrak{D}^\varepsilon$ is τ-lower semicontinuous on H, for every $\varepsilon > 0$.*

Corollary 4.1 *In the same assumptions of propositions 4.1 and 4.2 the family $\{\mathfrak{S}^\varepsilon, \varepsilon > 0\}$ and the family $\{\mathcal{S}^\varepsilon, \varepsilon > 0\}$ are inner regularizations for the family of quasi-equilibrium problems $\{(QE)(x), x \in H\}$.*

Then, we can define a $\mathfrak{D}$-viscosity solution for the problem (MS).

Definition 4.2 *Let $\mathfrak{D}$ be an inner regularization for the family $\{(QE)(x), x \in H\}$. A point $\widetilde{x} \in H$ is said to be a $\mathfrak{D}$-viscosity solution for the MinSup problem with quasi-equilibrium constraints (MS) if there exists a sequence $(x_{\varepsilon_n})_n$, $x_{\varepsilon_n} \in H$ for any $n \in \mathbb{N}$, such that:*

V_0) $\widetilde{x} \in cl_\tau^{seq} \{x_{\varepsilon_n}, n \in \mathbb{N}\}$;

V_1) $\displaystyle\sup_{u \in \mathfrak{D}^{\varepsilon_n}(x_{\varepsilon_n})} f(x_{\varepsilon_n}, u) = \min_{x \in H} \sup_{u \in \mathfrak{D}^{\varepsilon_n}(x)} f(x, u)$;

V_2) $\displaystyle\lim_n \sup_{u \in \mathfrak{D}^{\varepsilon_n}(x_{\varepsilon_n})} f(x_{\varepsilon_n}, u) = \omega$.

Due to Corollary 4.1, we can consider $\mathfrak{S}$-viscosity solutions and $\mathcal{S}$-viscosity solutions and, for the sake of easy readibility, in the next we use the following notations:

$$F_\varepsilon^{\mathfrak{D}}(x) = \sup_{u \in \mathfrak{D}^\varepsilon(x)} f(x,u), \quad F_\varepsilon^{\mathfrak{S}}(x) = \sup_{u \in \mathfrak{S}^\varepsilon(x)} f(x,u), \quad F_\varepsilon^{\mathcal{S}}(x) = \sup_{u \in \mathcal{S}^\varepsilon(x)} f(x,u), \quad F(x) = \sup_{u \in \mathcal{Q}(x)} f(x,u).$$

The following examples show that $\{\mathcal{Q}^\varepsilon,\ \varepsilon > 0\}$ and $\{\mathcal{T}^\varepsilon,\ \varepsilon > 0\}$ may fail to be an inner regularization even if the corresponding "strict" approximations $\{\mathfrak{S}^\varepsilon,\ \varepsilon > 0\}$ and $\{\mathcal{S}^\varepsilon,\ \varepsilon > 0\}$ are inner regularizations.

Example 4.1 Let $X = E = \mathbb{R}$, $H = K = [0,1]$, $S(x,u) = K$, $h(x,u,v) = v^2 - u^2 + (1+x)(u-v)$ and $f(x,u) = x + u$ for every $(x,u,v) \in [0,1]^3$. Then one easily checks that $\mathcal{Q}(0) = \{0,1\}$ and $\mathcal{Q}(x) = \{0\}$ if $0 < x \leq 1$, so that one has $F(0) = 1$, $F(x) = x$ for $0 < x \leq 1$ and the problem (MS) does not have a solution. Let $\varepsilon \in]0, 1/4[$, then we get

$$\mathcal{Q}^\varepsilon(x) = \left[0, (x+1)/2 - 1/2\sqrt{(x+1)^2 - 4\varepsilon}\right] \cup \left[(x+1)/2 + 1/2\sqrt{(x+1)^2 - 4\varepsilon}, 1\right] \quad \text{if } x \in\ [0, \varepsilon[$$

$$\mathcal{Q}^\varepsilon(x) = [0, \varepsilon] \cup \{1\} \qquad \text{if } x = \varepsilon$$

$$\mathcal{Q}^\varepsilon(x) = \left[0, (x+1)/2 - 1/2\sqrt{(x+1)^2 - 4\varepsilon}\right] \qquad \text{if } x \in\ ]\varepsilon, 1].$$

It is easy to check that

$$\sup_{u \in \mathcal{Q}^\varepsilon(x)} f(x,u) = x+1 \ \text{ if } x \in [0,\varepsilon] \ \text{ and } \sup_{u \in \mathcal{Q}^\varepsilon(x)} f(x,u) = x + (x+1)/2 - 1/2\sqrt{(x+1)^2 - 4\varepsilon} \ \text{ if } x \in\ ]\varepsilon, 1]$$

and that the regularized problem $(MS)^\varepsilon$ does not have a solution.This is essentially due to the lack of lower semicontinuity of the map $\mathcal{Q}^\varepsilon$ at the point $x = \varepsilon$.
On the contrary, since

$$\mathfrak{S}^\varepsilon(x) = \left[0, (x+1)/2 - 1/2\sqrt{(x+1)^2 - 4\varepsilon}\right[\cup \left](x+1)/2 + 1/2\sqrt{(x+1)^2 - 4\varepsilon}, 1\right] \quad \text{if } x \in\ [0, \varepsilon[$$

$$\mathfrak{S}^\varepsilon(x) = [0, \varepsilon[\qquad \text{if } x = \varepsilon$$

$$\mathfrak{S}^\varepsilon(x) = \left[0, (x+1)/2 - 1/2\sqrt{(x+1)^2 - 4\varepsilon}\right[\qquad \text{if } x \in\ ]\varepsilon, 1],$$

we have

$$F_\varepsilon^{\mathfrak{S}}(x) = x + 1 \quad \text{if } x \in [0, \varepsilon[$$

$$F_\varepsilon^{\mathfrak{S}}(x) = x + (x+1)/2 - 1/2\sqrt{(x+1)^2 - 4\varepsilon} \quad \text{if } x \in [\varepsilon, 1].$$

Therefore, the point $x_\varepsilon = \varepsilon$ is a minimum point for the function $F_\varepsilon^{\mathfrak{S}}$ and the problem $(SMS)^\varepsilon$ has a solution. Moreover, since also condition $V_2)$ is satisfied, the point $\widetilde{x} = 0$ is a $\mathfrak{S}$-viscosity solution for the MinSup problem (MS).

Example 4.2 Let $X = E = \mathbb{R}$, $H = K = [0,1]$, $S(x,u) = K$, $h(x,u,v) = x(u-v)$ and $f(x,u) = x + u$ for every $(x,u,v) \in [0,1]^3$. Then one easily checks that $\mathcal{Q}(0) = [0,1]$ and $\mathcal{Q}(x) = \{0\}$ if $0 < x \leq 1$, so that one has $F(0) = 1$, $F(x) = x$ for $0 < x \leq 1$ and the problem

(MS) does not have a solution. For every $\varepsilon > 0$ we have $\mathcal{T}^\varepsilon(x) = [0,1]$ if $x \in [0,\varepsilon]$ and $\mathcal{T}^\varepsilon(x) = \{0\}$ if $x \in]\varepsilon, 1]$ and the function

$$\sup_{u\in\mathcal{T}^\varepsilon(x)} f(x,u) = x+1 \ \text{ if } x \in [0,\varepsilon] \ \text{ and } \sup_{u\in\mathcal{T}^\varepsilon(x)} f(x,u) = x \ \text{ if } x \in\,]\varepsilon, 1]$$

is not lower semicontinuous at $x = \varepsilon$. On the contrary, since $\mathcal{S}^\varepsilon(x) = [0,1]$ if $x \in [0,\varepsilon[$ and $\mathcal{S}^\varepsilon(x) = \{0\}$ if $x \in [\varepsilon,1]$, we have that $F_\varepsilon^{\mathcal{S}}(x) = x+1$ if $x \in [0,\varepsilon[$ and $F_\varepsilon^{\mathcal{S}}(x) = x$ if $x \in [\varepsilon,1]$. So, the function $F_\varepsilon^{\mathcal{S}}$ is lower semicontinuous and it is easy to check that $x = 0$ is a $\mathfrak{S}$-viscosity solution for (MS).

Now, we prove the existence of a $\mathfrak{D}$-viscosity solution under quite general assumptions.

Proposition 4.3 *Assume that $\mathfrak{D}$ is an inner regularization map. If H is sequentially compact, conditions $v) - vi)$ in Proposition 3.1 and the following hold:*
$vii)$ the function f is $(\tau \times w)$-lower semicontinuous on $H \times K$.
Then, there exists a $\mathfrak{D}$-viscosity solution for the problem (MS).

Proof
From classical results on semicontinuity properties of marginal functions [15] we get that the function $F_\varepsilon^{\mathfrak{D}}$ is τ-lower semicontinuous on H, since the function f is $(\tau \times w)$-lower semicontinuous on $H \times K$ and the map $\mathfrak{D}^\varepsilon$ is (τ, s)-lower semicontinuous on H. Therefore, for every $\varepsilon > 0$ there exists $x_\varepsilon \in H$ such that

$$F_\varepsilon^{\mathfrak{D}}(x_\varepsilon) = \min_{x\in H} \sup_{u\in\mathfrak{D}_\varepsilon(x)} f(x,u)$$

since $F_\varepsilon^{\mathfrak{D}}$ is τ-lower semicontinuous on H and H is sequentially compact.
Then, for any given sequence of positive numbers decreasing to zero, $(\varepsilon_n)_n$, the sequence $(x_{\varepsilon_n})_n$ has a limit point $\widetilde{x}$ and arguing as in Proposition 3.1 we get $\lim_n F_{\varepsilon_n}^{\mathfrak{D}}(x_{\varepsilon_n}) = \omega$, so all conditions of Definition 4.2 are satisfied. $\square$

Corollary 4.2 *If H is sequentially compact, all assumptions in propositions 3.1 and 4.2 and the following hold:*
$vii)$ the function f is $(\tau \times w)$-lower semicontinuous on $H \times K$;
Then, there exists a $\mathfrak{S}$-viscosity solution for the problem (MS).
If, moreover, the space E is reflexive, then there exists an $\mathcal{S}$-viscosity solution for the problem (MS).

Proof
The existence of a $\mathfrak{S}$-viscosity solution and of a $\mathcal{S}$- viscosity solution follows from Corollary 4.1 and Proposition 4.3. $\square$

Remark 4.1 We point out that Definition 4.2 could have been given even if the class $\mathfrak{D}$ was not an inner regularization for (MS), but an existence result would not be guaranteed in this case. Indeed, even if the map $\mathcal{Q}^\varepsilon$ in Example 4.1 satisfies condition R_1), it lacks condition R_2) and it is easy to see that condition V_1) in Definition 4.2 is not fulfilled.

Remark 4.2 In a recent paper we have defined viscosity solutions for bilevel problems with constraints described only by Nash equilibria in finite dimensional spaces [23]. Here, in order to take into account the difficulties inherent infinite dimensional spaces, in defining viscosity solutions we do not require that the approximating sequence $(\sup_{u\in\mathfrak{D}^{\varepsilon_n}(\cdot)} f(\cdot,u))_n$ τ-epiconverges to $cl^\tau(F)$, the τ-lower semicontinuous regularization, because this is guaranteed under a too strong condition (see assumption *vii*) in the proposition below). Nevertheless, we explicitely give the following epi-convergence result for the specific regularizations $\mathfrak{S}^\varepsilon$ and $\mathcal{S}^\varepsilon$, since it could be useful in finite dimensional spaces.

Proposition 4.4 *Assume that conditions $i) - v)$ in Proposition 3.1 and the following hold:*
vii) for every $x \in H$ the function $f(x,\cdot)$ is w-upper semicontinuous on K;
Then, for every sequence $(\varepsilon_n)_n$ of positive numbers decreasing to zero, the sequence of functions $(g_n)_n$, defined by $g_n(x) = F^{\mathfrak{S}}_{\varepsilon_n}(x)$ for any $n \in \mathbb{N}$, τ-epiconverges to $cl^\tau(F)$.
If, moreover, the space E is reflexive, the same occurs for the sequence defined by $g_n(x) = F^{\mathcal{S}}_{\varepsilon_n}(x)$.

Proof
It suffices to prove that $(g_n)_n$ pointwise converges to F in H since the sequence $(g_n)_n$ is monotone [1]. Given any $x \in H$, $F(x) \le \liminf_n F^{\mathfrak{S}}_{\varepsilon_n}(x)$ because $\mathcal{Q}(x) \subseteq \mathcal{Q}^\varepsilon_n(x)$. Assume that there exists $a \in \mathbb{R}$ such that $F(x) < a < \limsup_n F^{\mathfrak{S}}_{\varepsilon_n}(x)$. Then, there exist an increasing sequence of positive integers $(n_k)_k$ and a sequence $(u_{n_k})_k$ such that $u_{n_k} \in \mathfrak{S}^{\varepsilon_{n_k}}(x)$ and $a < f(x, u_{n_k})$ for every $k \in \mathbb{N}$. Due to assumption *vi*), the sequence $(u_{n_k})_k$ has a subsequence, still denoted by $(u_{n_k})_k$, weakly converging to a point $u \in K$ and, by Proposition 4.1, $u \in \mathcal{Q}(x)$. Therefore, $f(x,u) \ge a$ by assumption *vii*), so $F(x) = \sup_{u\in\mathcal{Q}(x)} f(x,u) \ge a$ and we get a contradiction. The proof for $F^{\mathcal{S}}_{\varepsilon_n}$ is similar and we omit it.

5 Conclusions

We considered a class of two-stage problems in Banach spaces which may fail to have a solution even in very restrictive assumptions. This is a wide class since it contains, for example, pessimistic (weak) bilevel optimization problems and minsup problems with constraints described by Nash equilibria or by variational and quasi-variational inequalities. We proposed different types of regularization for these problems and we showed that all of them allow to approach the value of such a problem. Then, we defined a concept of viscosity solution whose existence is guaranteed under reasonable assumptions when associated to a class of maps which are regularizing for the lower stage problem. We showed that two of the introduced regularizations are suitable to get the existence of a viscosity solution under mild assumptions.

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