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On Games and Equilibria with Coherent Lower Expectations

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Giuseppe De Marco[*] and Maria Romaniello[**]

Abstract

Different solution concepts for strategic form games have been introduced in order to weaken the consistency assumption that players' beliefs - about their opponents strategic choices - are correct in equilibrium. The literature has shown that ambiguous beliefs are an appropriate device to deal with this task. In this note, we introduce an equilibrium concept in which players do not know the opponents' strategies in their entirety but only the *coherent lower expectations* of some random variables that depend on the actual strategies taken by the others. This equilibrium concept generalizes the already existing concept of equilibrium with partially specified probabilities by extending the set of feasible beliefs and allowing for comparative probability judgements. We study the issue of the existence of the equilibrium points in our framework and find that equilibria exist under rather classical assumptions.

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1 Introduction

The concept of Nash equilibrium relies on two fundamental ideas. Firstly, the agents choose their optimal strategies according on the beliefs they have about the strategic choice made by the other players. Secondly, beliefs are consistent; that is, agents have correct expectations about the actual strategies chosen by their opponents. One of the main criticisms to the Nash equilibrium concept has always been the strength of such consistency condition as, in many settings, it is not clear why players must have exactly correct beliefs about each other. Therefore, different solution concepts have been introduced in order to weaken such consistency condition by taking into account *perturbed* beliefs. In a random belief equilibrium, (Friedman and Mezzetti (2005)), players' beliefs about others' strategy choices are randomly drawn from a belief distribution that is dispersed around a central strategy profile called the focus. From a different perspective, other solutions concept are based on the assumption that players have ambiguous beliefs about opponents behavior. A strand of this literature follows the *Choquet expected utility approach* (Dow and Werlang (1994), Eichberger and Kelsey (2000), Marinacci (2000), Eichberger, Kelsey and Schipper (2008)), that is, beliefs are represented by capacities (nonadditive measures) and expectations are expressed in terms of Choquet integrals. In the *maxmin expected utility approach* (Lo (1996), Klibanoff (1996), Groes et al. (1998) and Lehrer (2012)) beliefs are described by sets of probability distributions, so that ambiguity averse agents evaluate these sets by the worst feasible expectations. The approach proposed by Lehrer (2012) is based on the partial specification of players actions; that is, players' beliefs about the strategic choice of their opponents are given by partially specified probabilities. More precisely, players do not know the mixed strategy profile chosen by their opponents in their entirety but only the expectations of some (specified) random variables that depend on the actual choice taken by the others. This is the case, for instance, of players who are able to assess the probability of some subsets of pure strategies but do not know in which way these probabilities are distributed within those subsets.

We point out in this paper that partially specified probabilities are a particular case of *coherent lower expectations* which, in turn, emerge as a key concept in the literature on *imprecise probabilities*; as a consequence, equilibria with partially specified probabilities can be immediately generalized to a concept of equilibrium in which each agent knows the coherent lower expectations, instead of the expectations, of the random variables which are specified. This is done in the present paper by introducing the so-called concept of *equilibrium with coherent lower expectations*. At a first sight, this concept is similar to the Lehrer's one as ambiguity stems from the actual strategies played but, at the same time, it differs from Lehrer's concept as the true strategy profile is not necessarily contained in the belief of every player. Most importantly, it is more flexible than Lehrer's one since lower expectations can model comparative probability judgements, such as "a given event is at least as probable as ...", which cannot always be reconducted to partially specified probabilities. This is the case, for instance, of players who are able to assess that the probability of some subsets of pure strategies is at least as probable as a given value which might depend on the strategy profile.

Despite the features of the concept of equilibrium are rather general, this is not reflected in more restrictive assumptions for the existence of the corresponding equilibrium points. This is our main result: equilibria with coherent lower expectations exist, provided that a standard continuity assumption holds and a regularity condition for the beliefs, (namely, the system of linear inequalities which defines the beliefs of each player has interior points), is satisfied. The existence result is obtained rather easily once it is shown that equilibria with coherent lower expectations have an equivalent formulation as equilibria of games under ambiguous belief correspondences as defined in De Marco and Romaniello (2012). In fact, our existence result follows directly from the existence theorems for equilibria under ambiguous belief correspondences. As a final remark, the paper by Groes et al. (1998) proposes an approach similar to ours and that of Lehrer (2012) since expectation about opponents' strategic choices are described by lower probabilities. However, Walley (2000) points out at page 134 that *"coherent lower expectations are more general and more informative than coherent lower probabilities"*[3], so that our concept generalizes also the one in Groes et al. (1998).

The paper is organized as follows: In section 2, we define the model and the equilibrium concept; then, we relate it to the equilibrium concepts in De Marco and Romaniello (2012) and Lehrer (2012). Section 3 is devoted to equilibrium existence theorem.

2 Ambiguous beliefs and equilibria

The equilibrium concept

We consider a finite set on players $I = \{1, \dots n\}$; for every player i, $\Psi_i = \{\psi_i^1, \dots, \psi_i^{k_i}\}$ is the (finite) pure strategy set of player i, $\Psi = \prod_{i \in I} \Psi_i$ and $\Psi_{-i} = \prod_{j \neq i} \Psi_j$. Denote with X_i the set of mixed strategies of player i where each strategy $x_i \in X_i$ is a vector $x_i = (x_i(\psi_i))_{\psi_i \in \Psi_i} \in \mathbb{R}_+^{k_i}$ such that $\sum_{\psi_i \in \Psi_i} x_i(\psi_i) = 1$. Denote also with $X = \prod_{j=1}^n X_j$ and with $X_{-i} = \prod_{j \neq i} X_j$ and each mixed strategy profile $x \in X$ can also be denoted by $x = (x_i, x_{-i})$ where $x_i \in X_i$ and $x_{-i} \in X_{-i}$. Finally, we assume that each player i is endowed with a payoff function $f_i : \Psi \to \mathbb{R}$.

Agents have ambiguous expectations about the strategy choice of their opponents. In particular, the information available to player i about player j's strategic choice is given by the coherent lower expectations (see Walley (2000) and references therein) of specified random variables over Ψ_j. We denote with $\mathcal{Y}_{ij}$ the set of the random variables over Ψ_j which are specified to player i. More precisely, the *coherent lower expectations* of player i about player j's strategic choice are assigned by a function $P_{ij} : \mathcal{Y}_{ij} \times X_j \to \mathbb{R}$ provided that

$$P_{ij}(Y_{ij}, x_j) = \min_{y_j \in K_{ij}(x_j)} E_{y_j}[Y_{ij}]. \tag{1}$$

where $E_{y_j}[Y_{ij}]$ is the expectation of Y_{ij} under y_j, i.e. $E_{y_j}[Y_{ij}] = \sum_{d=1}^{k_j} y_j(\psi_j^d) Y_{ij}(\psi_j^d)$, and the set-valued map $K_{ij} : X_j \rightsquigarrow X_j$ is defined by

$$K_{ij}(x_j) = \left\{ y_j \in X_j \;\middle|\; E_{y_j}[Y_{ij}] \geqslant P_{ij}(Y_{ij}, x_j) \quad \forall Y_{ij} \in \mathcal{Y}_{ij} \right\} \quad \forall x_j \in X_j. \tag{2}$$

In this case, given a mixed strategy x_j of player j, player i does not know x_j in its entirety but knows the coherent lower expectations $P_{ij}(Y_{ij}, x_j)$ of each $Y_{ij} \in \mathcal{Y}_{ij}$ and the ambiguous belief of

[3]Indeed, in Walley (2000) this statement is supported by illustrative examples.

player i about player j's strategy choice is the set $K_{ij}(x_j)$ of all player j's mixed strategies that are consistent with the coherent lower probability $P_{ij}(Y_{ij}, x_j)$.

Then, the utility function of player i is defined by

$$U_i(x) = \min_{(y_j)_{j\neq i} \in \prod_{j\neq i} K_{ij}(x_j)} x_i(\psi_i) \left[\prod_{j\neq i} y_j(\psi_j) \right] f(\psi) \quad \forall x \in X. \tag{3}$$

So, we can consider the following strategic form game:

$$\Gamma = \left\{ I; (X_i)_{i\in I}, (U_i)_{i\in I} \right\}. \tag{4}$$

Hence,

Definition 2.1: Let $\mathcal{Y}_{ij}$ be the set of random variables defined over Ψ_j whose coherent lower expectations are specified to player i, for every pair (i, j). Then, any Nash equilibrium $x \in X$ of the corresponding game Γ (defined by (4)) is said to be an *equilibrium with coherent lower expectations* w.r.t the sets $(\mathcal{Y}_{ij})_{i,j}$; that is, $x \in X$ is an equilibrium with coherent lower expectations if, for every player i,

$$U_i(x_i, x_{-i}) \geqslant U_i(x'_i, x_{-i}) \quad \forall\, x'_i \in X_i. \tag{5}$$

The equivalent formulation under ambiguous belief correspondences

Now we show that equilibria with coherent lower expectations have an equivalent formulation as equilibria in games under ambiguous beliefs correspondences as firstly introduced in De Marco and Romaniello (2012)[4]. This formulation is key because it allows to apply directly the existence result in De Marco and Romaniello (2015) to obtain an existence result for equilibria with coherent lower expectations[5].

Denote with $\Omega = \{(f_1(\psi), \ldots, f_n(\psi)) \,|\, \psi \in \Psi\}$ the set of all the outcomes of the game, $\mathcal{P}$ the set of all the probability distributions over Ω. Define the ambiguous belief correspondence over outcomes $\mathcal{B}_i : X \rightsquigarrow \mathcal{P}$ as follows:

$$\varrho \in \mathcal{B}_i(x) \iff \exists (y_j)_{j\neq i} \in \prod_{j\neq i} K_{ij}(x_j) \quad s.t. \quad \varrho(\psi) = x_i(\psi_i) \left[\prod_{j\neq i} y_j(\psi_j) \right] \quad \forall \psi \in \Psi. \tag{6}$$

Assume that each player i has payoff $F_i : X \to \mathbb{R}$ defined by

$$F_i(x) = \min_{\varrho \in \mathcal{B}_i(x)} E_i(\varrho) \quad \forall x \in X, \tag{7}$$

[4] In De Marco and Romaniello (2013) it is given the equivalent formulation of equilibria with partially specified probabilities as equilibria in games under ambiguous beliefs correspondences.

[5] De Marco and Romaniello (2015) present results of existence and stability for the equilibria in games under ambiguous beliefs correspondences under relaxed assumptions on players' preferences. These results could be useful to study generalizations of the analysis presented in this paper. However, here we focus on a minor departure from the Lehrer's analysis in order to better highlight the differences between the two approaches.

where $E_i[\varrho] = \sum_{\psi \in \Psi} \varrho(\psi) f_i(\psi)$ then we consider the following game under ambiguous beliefs correspondences $\mathcal{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_n)$ and pessimistic players:

$$\Gamma^{\mathcal{B}} = \{I; (X_i)_{i\in I}; (F_i)_{i\in I}\}.$$

This game is a classical strategic form game and we call *equilibria under ambiguous belief correspondences* $\mathcal{B}_i$ the classical Nash equilibria of $\Gamma^{\mathcal{B}}$. That is, (x_i, x_{-i}) is an equilibrium under ambiguous belief correspondences $\mathcal{B}_i$ if, for every player i

$$F_i(x_i, x_{-i}) \geqslant F_i(x'_i, x_{-i}) \quad \forall x'_i \in X_i.$$

It immediately follows from the definition that a strategy profile x is an equilibrium with coherent lower expectations if and only if it is an equilibrium under ambiguous belief correspondences defined by (6).

Equilibria with partially specified probabilities

This subsection highlights the relation of our equilibrium notion with respect to the Lehrer's concept of *equilibrium with partially specified probabilities.* Let $\mathcal{Y}_{ij}^L$ be the set of random variables defined over Ψ_j whose expectations are specified to player i, that is, given a mixed strategy x_j, player i does not know x_j in its entirety but knows the expectation of each $Y_{ij} \in \mathcal{Y}_{ij}^L$. Then, the ambiguous belief $K_{ij}^L(x_j)$ of player i about player j's strategy choice is the set of all player j's mixed strategies that are consistent with x_j and $\mathcal{Y}_{ij}^L$ in the following way:

$$K_{ij}^L(x_j) = \left\{ y_j \in X_j \;\middle|\; E_{y_j}[Y_{ij}] = E_{x_j}[Y_{ij}] \quad \forall Y_{ij} \in \mathcal{Y}_{ij}^L \right\}. \tag{8}$$

where $E_{y_j}[Y_{ij}]$ and $E_{x_j}[Y_{ij}]$ represent the expectation of Y_{ij} with respect to y_j and x_j respectively. Then, the utility function of player i is constructed as

$$U_i^L(x) = \min_{(y_j)_{j\neq i} \in \prod_{j\neq i} K_{ij}^L(x_j)} x_i(\psi_i) \left[\prod_{j\neq i} y_j(\psi_j) \right] f(\psi) \quad \forall x \in X \tag{9}$$

and the game is

$$\Gamma^L = \left\{ I; (X_i)_{i\in I}, (U_i^L)_{i\in I} \right\} \tag{10}$$

Definition 2.2 (Lehrer (2012)): Let $\mathcal{Y}_{ij}$ be the set of random variables defined over Ψ_j whose expectations are specified to player i, for every pair (i,j). Then, any Nash equilibrium $x \in X$ of the corresponding game Γ^L (defined by (10)) is said to be an *equilibrium with partially specified probabilities* w.r.t the sets $(\mathcal{Y}_{ij})_{i,j}$.

It is clear that equilibria with partially specified probabilities are a special case of equilibria with coherent lower expectations. More precisely, let x be an equilibrium with partially specified probabilities w.r.t the sets $\mathcal{Y}_{ij}^L$ for every pair (i,j) and let $K_{ij}^L(x_j)$ be the corresponding set of beliefs as defined in (8). Now, let $\mathcal{Y}_{ij}$ be the set of random variables defined as follows:

$$Y_{ij} \in \mathcal{Y}_{ij} \iff Y_{ij} \in \mathcal{Y}_{ij}^L \quad \text{or} \quad -Y_{ij} \in \mathcal{Y}_{ij}^L, \tag{11}$$

then, for every $x_j \in X_j$, let the coherent lower probabilities $P_{ij}(\cdot, x_j) : \mathcal{Y}_{ij} \to \mathbb{R}$ be defined by $P_{ij}(Y_{ij}, x_j) = E_{x_j}[Y_{ij}]$ for all $Y_{ij} \in \mathcal{Y}_{ij}$. If follows that the corresponding set of beliefs $K_{ij}(x_j)$ defined by (2) coincides with $K^L_{ij}(x_j)$, so that equilibria with partially specified equilibria are equilibria with coherent lower expectations.

Remark 2.3: We have shown that partially specified probabilities can be reconducted to coherent lower expectations. Conversely, lower expectations can model comparative probability judgements which cannot always be described to partially specified probabilities; as already noticed in the Introduction, this is the case, for instance, of judgements like: "... a given event is at least as probable as ...". The advantages and drawbacks of coherent lower expectations are studied extensively in the literature (see, for example, Walley (2000) and references therein), here we just point out that, in our case, coherent lower expectations can model situations in which players are able to assess that the probability of some subsets of pure strategies is at least as probable as a given value which might depend on the strategy profile. Consider the following simple example: suppose that an agent, say i, can only perceive the minimum probability under which his opponent, say j, will play each pure strategy. This means that player i has the following beliefs about player j's strategic choice:

$$K_{ij}(x_j) = \left\{ y_j \in X_j \;\middle|\; y_j(\psi_j^d) \geqslant \min_{d \in \{1,\dots,k_j\}} x_j(\psi_j^d) \quad \forall d \in \{1,\dots,k_j\} \right\} \quad \forall x_j \in X_j.$$

Now, the previous belief $K_{ij}(x_j)$ comes out from coherent lower expectations by setting, in formula (2),

$$\mathcal{Y}_{ij} = \{\chi(\psi_j^d) \mid d = 1,\dots,k_j\},$$

where each $\chi(\psi_j^d)$ is the characteristic function of the event $\{\psi_j^d\}$ and

$$P_{ij}(\chi\left(\psi_j^d\right), x_j\big) = \min_{d \in \{1,\dots,k_j\}} x_j(\psi_j^d).$$

3 Equilibrium existence

Preliminaries on set-valued maps

We start by recalling well known definitions and results on set-valued maps which we use below. Following Aubin and Frankowska (1990)[6], recall that if Z and Y are two metric spaces and $\mathcal{C} : Z \rightsquigarrow Y$ a set-valued map, then

$$i)\ \operatorname*{Lim\,inf}_{z \to z'} \mathcal{C}(z) = \left\{ y \in Y \mid \lim_{z \to z'} d(y, \mathcal{C}(z)) = 0 \right\},$$
$$ii)\ \operatorname*{Lim\,sup}_{z \to z'} \mathcal{C}(z) = \left\{ y \in Y \mid \liminf_{z \to z'} d(y, \mathcal{C}(z)) = 0 \right\}$$

and $\operatorname*{Lim\,inf}_{z \to z'} \mathcal{C}(z) \subseteq \mathcal{C}(z') \subseteq \operatorname*{Lim\,sup}_{z \to z'} \mathcal{C}(z)$. Moreover

[6] All the definitions and the propositions we use, together with the proofs can be found in this book.

Definition 3.1: Given the set-valued map $\mathcal{C}: Z \rightsquigarrow Y$, then

i) $\mathcal{C}$ is *lower semicontinuous* in z' if $\mathcal{C}(z') \subseteq \operatorname*{Lim\,inf}_{z \to z'} \mathcal{C}(z)$; that is, $\mathcal{C}$ is lower semicontinuous in z' if for every $y \in \mathcal{C}(z')$ and every sequence $(z_\nu)_{\nu \in \mathbb{N}}$ converging to z' there exists a sequence $(y_\nu)_{\nu \in \mathbb{N}}$ converging to y such that $y_\nu \in \mathcal{C}(z_\nu)$ for every $\nu \in \mathbb{N}$. Moreover, $\mathcal{C}$ is lower semicontinuous in Z if it is lower semicontinuous for all z' in Z.

ii) $\mathcal{C}$ is *closed* in z' if $\operatorname*{Lim\,sup}_{z \to z'} \mathcal{C}(z) \subseteq \mathcal{C}(z')$; that is, $\mathcal{C}$ is closed in z' if for every sequence $(z_\nu)_{\nu \in \mathbb{N}}$ converging to z' and every sequence $(y_\nu)_{\nu \in \mathbb{N}}$ converging to y such that $y_\nu \in \mathcal{C}(z_\nu)$ for every $\nu \in \mathbb{N}$, it follows that $y \in \mathcal{C}(z')$. Moreover, $\mathcal{C}$ is closed in Z if it is closed for all z' in Z;

iii) $\mathcal{C}$ is *upper semicontinuous* in z' if for every open set U such that $\mathcal{C}(z') \subseteq U$ there exists $\eta > 0$ such that $\mathcal{C}(z) \subseteq U$ for all $z \in B_Z(z', \eta) = \{\zeta \in Z \mid ||\zeta - z'|| < \eta\}$;

iv) $\mathcal{C}$ is *continuous* (in the sense of Painlevé-Kuratowski) in z' if it is lower semicontinuous and upper semicontinuous in z'.

Finally, recall the following result: If Z is closed, Y is compact and the set-valued map $\mathcal{C}: Z \rightsquigarrow Y$ has closed values, then, $\mathcal{C}$ is upper semicontinuous in $z \in Z$ if and only if $\mathcal{C}$ is closed in z[7].

The next definition will also be used:

Definition 3.2: Let Z a convex set, then the set-valued map $\mathcal{C}: Z \rightsquigarrow Y$ is a said to be *concave* if

$$t\mathcal{C}(\overline{z}) + (1-t)\mathcal{C}(\widehat{z}) \subseteq \mathcal{C}(t\overline{z} + (1-t)\widehat{z}) \quad \forall\, \overline{z}, \widehat{z} \in Z,\ \forall\, t \in [0,1] \tag{12}$$

while it is *convex*[8] if

$$\mathcal{C}(t\overline{z} + (1-t)\widehat{z}) \subseteq t\mathcal{C}(\overline{z}) + (1-t)\mathcal{C}(\widehat{z}) \quad \forall\, \overline{z}, \widehat{z} \in Z,\ \forall\, t \in [0,1] \tag{13}$$

The existence theorem

As a direct application of Theorem 3.6 in De Marco and Romaniello (2015) we have

Theorem 3.3: *Assume that, for every player i,*

i) $\mathcal{B}_i$ *is a continuous set-valued map with not empty and closed images for every* $x \in X$*.*

ii) $\mathcal{B}_i(\cdot, x_{-i})$ *is a convex set-valued map in* X_i *for every* $x_{-i} \in X_{-i}$*.*

Then, the game $\Gamma^{\mathcal{B}}$ *has at least an equilibrium.*

Proof. The proof follows directly by applying Theorem 3.6 in De Marco and Romaniello (2015), once regarded the utility function in (7) as a particular case of variational preferences in which the index of ambiguity aversion is identically equal to 0. □

[7] Every set-valued map in this paper satisfies the assumptions of this result. Hence upper semicontinuity and closedness coincide in this work.

[8] Note that a set-valued map is concave if and only if its graph is a convex set. For this reason, some authors call convex set-valued maps those that here we call concave.

In the next proposition we give conditions on the primitives of the model (coherent lower probabilities and specified random variables) which guarantee that the assumption of the previous theorem hold, so that equilibria with coherent lower expectations exist.

PROPOSITION 3.4: *Assume that, for every player $j \neq i$:*

i) For every fixed $Y_{ij} \in \mathcal{Y}_{ij}$, the coherent lower probability function $P_{ij}(Y_{ij}, \cdot)$ is continuous in X_j.

ii) For every $x_j \in X_j$ there exists $y_j \in K_{ij}(x_j)$ such that:

$$y_j(\psi_j^d) > 0 \quad \forall\, d \in \{1, \dots, k_j\} \quad \text{and} \quad E_{y_j}[Y_{ij}] > P(Y_{ij}, x_j) \quad \forall Y_{ij} \in \mathcal{Y}_{ij} \tag{14}$$

Then, the belief correspondence $\mathcal{B}_i$ defined by (6) is a continuous set-valued map with not empty and closed images for every $x \in X$. Moreover, $\mathcal{B}_i(\cdot, x_{-i})$ is a convex set-valued map in X_i, for every $x_{-i} \in X_{-i}$.

Proof. First of all recall that

$$\varrho \in \mathcal{B}_i(x) \iff \exists (y_j)_{j\neq i} \in \prod_{j\neq i} K_{ij}(x_j) \quad s.t. \quad \varrho(\psi) = x_i(\psi_i) \left[\prod_{j\neq i} y_j(\psi_j)\right] \quad \forall \psi \in \Psi. \tag{15}$$

The assumptions imply that $K_{ij}(x_j) \neq \emptyset$ so $\mathcal{B}_i(x) \neq \emptyset$ for every $x \in X$.

Step 1: $\mathcal{B}_i$ is lower semicontinuous in X.

Let $|\mathcal{Y}_{ij}| = \gamma_j$ be the cardinality of $\mathcal{Y}_{ij}$, it follows that $K_{ij}(x_j)$ is completely defined by the following system of $k_j + \gamma_j$ linear inequalities in the k_j unknowns $y_j = (y_j(\psi_j^1), \dots, y_j(\psi_j^{k_j}))$

$$\begin{cases} y_j(\psi_j^d) \geqslant 0 & \forall\, d \in \{1, \dots, k_j\} \\ E_{y_j}[Y_{ij}] \geqslant P(Y_{ij}, x_j) & \forall Y_{ij} \in \mathcal{Y}_{ij} \end{cases} \tag{16}$$

and by the linear equality

$$y_j(\psi_j^1) + \cdots + y_j(\psi_j^{k_j}) = 1. \tag{17}$$

It follows immediately that $K_{ij}(x_j)$ is closed and therefore compact.

Now, denote with C the $(k_j + \gamma_j) \times k_j$ matrix of the coefficients in the system (16) and with $\mathbf{0}_{k_j}$ the row null vector of dimension k_j. Let $b(x_j)$ the $(k_j + \gamma_j)$-dimensional row vector defined as follows

$$b(x_j) = \Big(\mathbf{0}_{k_j}, \big(P_{ij}(Y_{ij}, x_j)\big)_{Y_{ij} \in \mathcal{Y}_{ij}}\Big). \tag{18}$$

If y_j^T and $b^T(x_j)$ are the transpose of y_j and $b(x_j)$ respectively, then the system (16) can be denoted as follows $Cy_j^T \geqq b^T(x_j)$. Denote also with $\mathbf{1}_{k_j} y_j^T = 1$ the linear equation (17). Theorem 2 in Robinson (1975) tells that if a system of linear equalities and inequalities is such that the coefficient matrix of the equalities has full rank and there exists a feasible point satisfying all the strict inequalities then the system is *stable* as defined at page 755 in Robinson (1975)[9]. This is our case, because assumption *(ii)* guarantees that there exists a vector $\overline{y}_j$ such that

$$C\overline{y}_j^T > b^T(x_j) \quad \text{and} \quad \mathbf{1}_{k_j} \overline{y}_j^T = 1$$

[9]Indeed, this is more clearly explained in Daniel (1975) (Section 4, page 771).

so the system is stable which, in our case, means that for every vector y_j satisfying $Cy_j^T \geqq b^T(x_j)$ and $\mathbf{1}_{k_j} y_j^T = 1$, and every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$[10] such that for every $(k_j + \gamma_j)$-dimensional row vector $\widetilde{b}$ with $\|\widetilde{b} - b(x_j)\| < \delta(\varepsilon)$ there exists $\widetilde{y}_j$ satisfying (a) $C\widetilde{y}_j^T \geqq \widetilde{b}^T$ and $\mathbf{1}_{k_j}\widetilde{y}_j^T = 1$, (b) $\|\widetilde{y}_j - y_j\| < \varepsilon$.

So K_{ij} is a lower semicontinuous set-valued map. In fact, let $(x_{j,v})_{\nu\in\mathbb{N}}$ be a sequence converging to x_j and $y_j \in K_{ij}(x_j)$. For every $\nu \in \mathbb{N}$, $\underset{z\in K_{ij}(x_{j,\nu})}{\arg\min} \|z - y_j\|$ is not empty since $K_{ij}(x_{j,\nu})$ is not empty and compact. Let $(y_{j,\nu})_{\nu\in\mathbb{N}}$ be a sequence such that $y_{j,\nu} \in \underset{z\in K_{ij}(x_{j,\nu})}{\arg\min} \|z - y_j\|$ for every $\nu \in \mathbb{N}$, we show that $y_{j,\nu} \to y_j$. Since $P_{ij}(Y_{i,j}, \cdot)$ is continuous for every $Y_{ij} \in \mathcal{Y}_{ij}$ then the vector valued function $b(\cdot)$ defined by (18) is continuous. So, given $\delta(\varepsilon) > 0$ there exists $\overline{\nu}$ such that for every $\nu \geqslant \overline{\nu}$ it follows that $\|b(x_{j,\nu}) - b(x_j)\| < \delta(\varepsilon)$. Therefore, since Robinson's (1975) stability holds, then, for every $\varepsilon > 0$ there exists $\overline{\nu}$ such that for every $\nu \geqslant \overline{\nu}$ there exists $\widetilde{y}_{j,\nu} \in K_{ij}(x_{j,\nu})$ satisfying $\|\widetilde{y}_{j,\nu} - y_j\| < \varepsilon$. Being $\|y_{j,\nu} - y_j\| \leqslant \|\widetilde{y}_{j,\nu} - y_j\| < \varepsilon$, we get that for every $\varepsilon > 0$ there exists $\overline{\nu}$ such that for every $\nu \geqslant \overline{\nu}$ it follows that $\|y_{j,\nu} - y_j\| < \varepsilon$. So $y_{j,\nu} \to y_j$ and K_{ij} is lower semicontinuous in x_j. Since x_j is arbitrary then K_{ij} is lower semicontinuous in X_j.

Now, let $\varrho \in \mathcal{B}_i(x)$ be the probability distribution defined as in (15) by

$$\varrho(\psi) = x_i(\psi_i)\left[\prod_{j\neq i} y_j(\psi_j)\right] \quad \forall\psi \in \Psi, \quad \text{with}\, y_j \in K_{ij}(x_j)\; j \neq i$$

let $(x_\nu)_{\nu\in\mathbb{N}}$ be a sequence converging to x. Since for every $j \neq i$, K_{ij} is lower semicontinuous in x_j then there exists a sequence $(y_{j,\nu})_{\nu\in\mathbb{N}}$ converging to y_j with $y_{j,\nu} \in K_{ij}(x_{j,\nu})$ for every $\nu \in \mathbb{N}$. Define ϱ_ν as follows:

$$\varrho_\nu(\psi) = x_{i,\nu}(\psi_i)\left[\prod_{j\neq i} y_{j,\nu}(\psi_j)\right] \quad \forall\psi \in \Psi.$$

If follows that $\varrho_\nu \to \varrho$ and $\varrho_\nu \in \mathcal{B}_i(x_\nu)$ for every $\nu \in \mathbb{N}$. So $\mathcal{B}_i$ is lower semicontinuous in x. Since x is arbitrary $\mathcal{B}_i$ is lower semicontinuous in X.

Step 2: $\mathcal{B}_i$ is upper semicontinuous in X with closed images.

Let $(x_{j,v})_{\nu\in\mathbb{N}}$ be a sequence converging to x_j and $(y_{j,\nu})_{\nu\in\mathbb{N}}$ be a sequence converging to y_j, with $y_{j,\nu} \in K_{ij}(x_{j,\nu})$ for every $\nu \in \mathbb{N}$. We show that $y_j \in K_{ij}(x_j)$. In fact, from $Cy_{j,\nu}^T \geqq b^T(x_{j,\nu})$ and $\mathbf{1}_{k_j} y_{j,\nu}^T = 1$ and the continuity of $b(\cdot)$ we get $Cy_j^T \geqq b^T(x_j)$ and $\mathbf{1}_{k_j} y_j^T = 1$ by taking the limit as $\nu \to \infty$. So, $y_j \in K_{ij}(x_j)$ and K_{ij} is closed in x_j. Since x_j is arbitrary then K_{ij} is closed in X_j.

Now, let $(x_\nu)_{\nu\in\mathbb{N}}$ be a sequence converging to x and $(\varrho_\nu)_{\nu\in\mathbb{N}}$ be a sequence converging to ϱ, with $\varrho_\nu \in \mathcal{B}_i(x_\nu)$ for every $\nu \in \mathbb{N}$. By definition (15), $\varrho_\nu(\psi) = x_{i,\nu}(\psi_i)\left[\prod_{j\neq i} y_{j,\nu}(\psi_j)\right]$ for all $\psi \in \Psi$, where $y_{j,\nu} \in K_{ij}(x_{j,\nu})$ for all $j \neq i$. Since $y_{j,\nu} \to y_j$ and K_{ij} is closed, then $y_j \in K_{ij}(x_j)$ for every $j \neq i$. So $\varrho_\nu \to \varrho$ where $\varrho(\psi) = x_i(\psi_i)\left[\prod_{j\neq i} y_j(\psi_j)\right]$ for all $\psi \in \Psi$. Therefore, (15) implies that $\varrho \in \mathcal{B}_i(x)$ and $\mathcal{B}_i$ is closed in x. Since x is arbitrary then $\mathcal{B}_i$ is closed in X. So it is upper semicontinuous in X. Moreover, if $(x_\nu)_{\nu\in\mathbb{N}}$ is the constant sequence, with $x_\nu = x$ for every $\nu \in \mathbb{N}$, then the closedness of $\mathcal{B}_i$ in x implies that the set $\mathcal{B}_i(x)$ is closed.

[10] It is clear from Robinson's (1975) definition of stability at page 755 that δ is uniform (it does not depend on y_j) because the perturbation does not involve the matrix C.

Step 3: $\mathcal{B}_i(\cdot, x_{-i})$ is a convex set-valued map in X_i, for every $x_{-i} \in X_{-i}$.
By definition

$$\varrho \in \mathcal{B}_i(tx_i' + (1-t)x_i'', x_{-i}) \iff \tag{19}$$

$$\exists (y_j)_{j \neq i} \in \prod_{j \neq i} K_{ij}(x_j) \quad s.t. \quad \varrho(\psi) = (tx_i'(\psi_i) + (1-t)x_i''(\psi_i)) \left[\prod_{j \neq i} y_j(\psi_j)\right] \quad \forall \psi \in \Psi. \tag{20}$$

It clearly follows that

$$\varrho(\psi) = tx_i'(\psi_i) \left[\prod_{j \neq i} y_j(\psi_j)\right] + (1-t)x_i''(\psi_i)) \left[\prod_{j \neq i} y_j(\psi_j)\right] \quad \forall \psi \in \Psi.$$

Let ϱ' and ϱ'' be defined by

$$\varrho'(\psi) = x_i'(\psi_i) \left[\prod_{j \neq i} y_j(\psi_j)\right] \quad \text{and} \quad \varrho''(\psi) = x_i''(\psi_i) \left[\prod_{j \neq i} y_j(\psi_j)\right] \quad \forall \psi \in \Psi. \tag{21}$$

Then $\varrho = t\varrho' + (1-t)\varrho''$, $\varrho' \in \mathcal{B}_i(x_i', x_{-i})$ and $\varrho'' \in \mathcal{B}_i(x_i'', x_{-i})$. Since ϱ is arbitrary then

$$\mathcal{B}_i(tx_i' + (1-t)x_i'', x_{-i}) \subseteq t\mathcal{B}_i(x_i', x_{-i}) + (1-t)\mathcal{B}_i(x_i'', x_{-i})$$

It follows easily that the previous inclusion holds for every $t \in [0,1]$ and every x_i' and x_i'' in X_i. Hence, $\mathcal{B}_i(\cdot, x_{-i})$ is a convex set-valued map in X_i, for every $x_{-i} \in X_{-i}$. $\square$

Remark 3.5: In order to keep the presentation of the previous result more simple, we did not consider the case in which there exist implicit linear equalities in the system of linear inequalities. However the generalization to this case is rather straightforward. It consists in adding the assumption that the coefficient matrix of the system of the implicit equalities together with the equation in (17) has full rank. The proof is substantially identical to the one given above.

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