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Portfolio Selection with Transaction Costs and Default Risk

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Abstract
I propose a simple consumption/investment problem with transaction costs and default risk. When default occurs, I assume the value of the risky asset drops to zero and the investor receives the terminal wealth only in the form of the other (riskless) security. I show that default risk can generate a first-order effect on the investor's asset allocation. On the contrary, the liquidity premium is one order of magnitude smaller than the transaction costs, implying that the additional source of risk determined by the possibility of default is not able to generate a first-order effect on asset pricing.

JEL Classification: C61, D11, D91, G11

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1 Introduction

A lot of academic effort has been put in understanding, modeling and managing default risk in several areas of finance, especially in corporate finance, fixed income and financial intermediation. Yet, portfolio theory says very little about how investors have to allocate optimally their wealth when some securities carry also the risk of default. Specifically, the "classical" approach, proposed by Merton (1969; 1971), consisted in determining the optimal consumption-investment policy of a representative investor allocating her wealth into several (default-free) risky assets and a riskless security. Then, over the years, researchers have mainly considered portfolio-choice problems with return predictability, labor income, transaction costs, borrowing constraints and so on and so forth, but only few explicitly considered the possibility that a security could default.[1]

In this paper I propose a very simple consumption-investment model in which an infinitely-lived investor allocates her wealth between a risky asset and a riskless security, and incurs in proportional transaction costs when exchanging one asset into the other. The novel feature of my model is that, in addition to transaction costs, the risky asset may default at some random time, thus reducing the available wealth of the agent. When default occurs, I assume, without loss of generality, that the value of the risky asset drops to zero and the investor receives the terminal wealth only in the form of the other security, i.e. the riskless one.[2]

I find that default risk generates a first-order effect on the investor's asset allocation, causing significant deviations from the classical Merton (i.e. the riskless) solution. Specifically, the presence of default risk determines an optimal investment in the risky asset, both with and without transaction costs, significantly lower than the corresponding optimal policy obtained in the absence of default risk. Moreover, as expected, the higher the probability of default, the lower the no-transaction boundaries. On the contrary, and in line with Constantinides (1986), I find that the liquidity premium stemming from my setting is one order of magnitude smaller than the transaction costs. In other words, the additional source of risk determined by the possibility of default is not enough to generate a first-order effect on asset pricing.

[1] In this regard, notable exceptions are Korn and Kraft (2003), Hou and Jin (2005), Kraft and Steffensen (2009), Bo et al. (2012), and Sbuelz (2014) among the others.

[2] In line with the purpose of the paper, the latter assumption of total default is only meant to show that default risk can generate first-order effects on the investor's asset allocation. Alternatively, one could assume some (positive) recovery value of the risky asset at default and investigate the effects of partial-default on asset allocation and the liquidity premium.

I contribute to the portfolio-choice literature solving a very simple intertemporal-consumption/investment partial equilibrium problem with transaction costs and default risk. In this sense, I extend the works of Constantinides (1986), Davis and Norman (1990), Dumas and Luciano (1991), Akian et al. (1996), Eastham and Hastings (1988), Liu (2004) who study the asset-allocation problem in case of proportional transaction costs and constant investment set, and Korn and Kraft (2003), Hou and Jin (2005), Kraft and Steffensen (2009), Bo et al. (2012), and Sbuelz (2014) who instead consider default risk but no transaction costs.[3]

Several studies — see Constantinides (1986) among the others — have shown that, in models with constant-investment opportunity set, transaction costs have only a second-order effect on the liquidity premium. On the contrary, Lynch and Tan (2011) argue that in a time-varying opportunity set characterized by predictability in stock returns, state-dependent transaction costs and wealth shocks, transaction costs might produce liquidity premia of the same order of magnitude as empirical results. A similar conclusion is obtained by Jang et al. (2007) in a stochastic opportunity set characterized by a regime-switching in the volatility of the risky asset.

Additional contributions are Shreve et al. (1991), Jiao and Pham (2011), Jang et al.(2014), and Delgado et al. (2015).

The rest of the paper is organized as follows. Section 2 describes the consumption/investment problem with transaction costs and default risk, whereas in Section 3 I shows the main results for the asset allocation and the liquidity premium. Finally, Section 4 concludes.

2 The Model

I analyze a very simple model of consumption/investment choice with default risk and transaction costs. The financial side of my economy consists of two assets, a money bank account ("the bond") and a risky asset ("the stock"). The agent incurs proportional transaction costs at the rate $1 - s$ when exchanging one asset into the other; these costs are proportional to the dollar value of the trade.

I assume the dollar amount invested in the riskless asset, $x(t)$, and the dollar amount invested in the

[3] In particular, Hou and Jin (2005) propose a dynamic asset-allocation model in which investors face both equity risk and credit risk, whereas Sbuelz (2014) explores the implications of a dynamic-portfolio problem characterized by the joint presence of default risk and systemic risk, multiple assets and a constant-investment opportunity set.

stock, $y(t)$, evolve according to the following equations:

$$dx_t = rx_t dt - c_t dt + sdL - dU, \tag{1}$$

$$dy_t = \alpha y_t dt + \sigma y_t dZ_t - dL + sdU, \tag{2}$$

where the drift α and the volatility σ, as well as the short rate r, are assumed constant. dZ_t is a standard brownian motion and c_t is consumption. Moreover, the processes L and sU represent the cumulative dollar amount of sales and purchases of the stock, respectively. They are non-decreasing and increase only when (respectively) some amount of the risky asset or of the riskless asset is sold.

An important feature of the risky asset is that it may (exogenously) default at a random date τ in the future.[4] More precisely, when default occurs, I assume without loss of generality that there are no proceeds from the liquidation of the risky asset – i.e. its value drops immediately to zero – and the investor receives the terminal wealth only in the form of the other security, that is the riskless one.[5]

Following Merton (1971) I assume that, before the event of default, the investor derives her utility from intertemporal consumption, whereas in the event of default she derives her utility from terminal wealth only. In other words, the consumer/investor problem is

$$\max_{c_t, U_t, L_t} \; V(x,y) = E\left[\int_0^{\tau} e^{-\rho t}\frac{c_t^{1-\gamma}}{1-\gamma}dt + e^{-\rho\tau}V_\tau(x_\tau, y_\tau)\right], \tag{3}$$

subject to

$$dx_t = rx_t dt - c_t dt + sdL - dU,$$

$$dy_t = \alpha y_t dt + \sigma y_t dZ_t - dL + sdU,$$

[4] Given the purpose of this paper, i.e. to study the optimal consumption/investment policy with transaction costs and default risk, I treat the asset's default as a purely exogenous feature of the risky security, without any attempt to explain the determinants of the default event.

[5] As explained at the previous section, the present paper intends to show that (total) default risk can generate first-order effects on the investor's asset allocation but not on the liquidity premium. Alternatively, one could also investigate the asset allocation implications of different assumptions regarding the value of the risky asset at default (for example assuming some partial recovery).

where ρ is the subjective-discount rate and γ is the constant-relative risk aversion parameter. The indirect utility function $V_\tau(x_\tau, y_\tau)$ is given by

$$V_\tau(x_\tau, y_\tau) = E_\tau \left[\frac{W_\tau^{1-\gamma}}{1-\gamma} \right], \tag{4}$$

where W_τ corresponds to the terminal wealth of the agent at the time of default, τ. Specifically, under the assumption of no proceeds from the liquidation of the risky asset, the terminal wealth becomes simply equal to the amount invested in the riskless security, that is $W_\tau = x_\tau$.

Before default occurs, the solvency region is characterized by three parts, as in Davis and Norman (1990) and Jang et al. (2007): a no-trading region (NT), a buy region (B), and a sell region (S). Moreover, considering the isoelastic utility function and the linear nature of constraints (1) and (2), the value function V is homogeneous of degree $1-\gamma$. This homogeneity property implies that the transaction boundaries are straight lines in the (x, y) plane.[6] Therefore, the investor finds optimal to trade only the minimum amount to keep the fraction of wealth invested in the risky asset inside the no-trading region, defined by the thresholds

$$\underline{\lambda} \leq \frac{y}{x+y} \leq \overline{\lambda}. \tag{5}$$

In other words,

$$U \text{ increases only when } \frac{y}{x+y} = \underline{\lambda},$$

and

$$L \text{ increases only when } \frac{y}{x+y} = \overline{\lambda}.$$

In the no-trading region, the value function V satisfies the following HJB equation.

$$\rho V(x,y) = \max_c \left[\begin{array}{c} rxV_x(x,y) + \alpha y V_y(x,y) + 0.5\sigma^2 y^2 V_{yy}(x.y) + \\ \frac{c^{1-\gamma}}{1-\gamma} - cV_x(x,y) + \eta(V_\tau - V(x,y)) \end{array} \right],$$

where η is the probability that the risky asset defaults.

Substituting the first-order condition for the consumption policy, i.e.

$$0 < V_x(x,y) = c^{-\gamma},$$

gives the non-linear equation

[6] See Constantinides (1986) and Dumas (1992) for further details.

$$\rho V(x,y) = \left[\frac{\gamma}{1-\gamma} V_x^{\frac{\gamma-1}{\gamma}} + rxV_x(x,y) + \alpha y V_y(x,y) + 0.5\sigma^2 y^2 V_{yy}(x,y) + \eta(V_\tau - V(x,y)) \right]. \quad (6)$$

Define now θ the fraction of wealth invested in the stock, i.e. $\theta = \frac{y}{x+y}$. When trading takes place, the movement to the target position is instantaneous. Hence, the values of the discounted utility before and after the trade must be the same, that is,

$$\begin{aligned} V(x,y) &= V(x - dU, y + sdU) \qquad \text{when } \theta = \underline{\lambda}, \\ V(x,y) &= V(x + sdL, y - dL) \qquad \text{when } \theta = \overline{\lambda}. \end{aligned} \quad (7)$$

Smooth-pasting conditions have to be satisfied in order for the thresholds $\underline{\lambda}$ and $\overline{\lambda}$ to be optimal[7]. This requires that, when $\theta = \underline{\lambda}$,

$$\begin{aligned} V_x(x,y) &= V_x(x - dU, y + sdU), \\ V_y(x,y) &= V_y(x - dU, y + sdU), \end{aligned} \quad (8)$$

and, when $\theta = \overline{\lambda}$,

$$\begin{aligned} V_x(x,y) &= V_x(x + sdL, y - dL), \\ V_y(x,y) &= V_y(x + sdL, y - dL). \end{aligned} \quad (9)$$

The optimal portfolio policy is obtained by solving the differential equation (6) subject to the boundaries conditions (7-9).

Results

I solve the model described in the previous section using a numerical method and compute the optimal position of the trading boundaries. In the following analysis, as a base case, I assume parameter values similar to those used by Constantinides (1986), that is $\rho = 0.1, \alpha = 0.15, r = 0.1, \gamma = 2$ an $\sigma = 0.2$. Moreover, transaction costs are levied at the rate $(1-s) = 0.01$, whereas the default probability of the risky asset is assumed to be $\eta = 5\%/\text{year}$.

[7] See Dumas (1991) and Dixit (1991) for a discussion of the value-matching and the smooth-pasting conditions.

Figure 1 below shows the optimal position of the trading boundaries, i.e. the buy and the sell barriers $\underline{\lambda}$ and $\overline{\lambda}$, together with the optimal portfolio policies obtained in the equivalent frictionless economy, i.e. no transaction costs, both with and without default risk.

FIGURE 1 GOES HERE

Default risk generates a first-order effect on the investor's asset allocation, causing significant deviations from the classical Merton (i.e. the riskless) solution. Specifically, when the risky asset carries also the risk of default, the frictionless line, i.e. the fraction invested in the risky asset, moves down from 0.625 obtained in the canonical default-free "Merton" problem, to 0.567 obtained in the presence of default risk, whereas the thresholds $\underline{\lambda}$ and $\overline{\lambda}$ are equal to 0.397 and 0.59, respectively.

Table I below provides further information on how the optimal trading policies change as a function of the default probability η and the transaction costs $(1-s)$.

TABLE 1 GOES HERE

As expected, an increase in the transaction cost rate determines a widening of the no-trading region since it is more costly for the investor to rebalance her asset allocation, whereas the higher the probability of default, the lower the investment in the risky asset, and thus the lower the thresholds $\underline{\lambda}$ and $\overline{\lambda}$.

In order to measure the effect of transaction costs on expected returns, I compute the liquidity premium stemming from my setting. Specifically, I follow Constantinides (1986) and define the liquidity premium to be the maximum expected return an investor is willing to exchange for zero transaction costs. Obviously, the equivalent frictionless economy used to compare with would still encompass the risk of default.

TABLE 2 GOES HERE

Table 2 above reports the ratio of the liquidity premium over the transaction costs rate (LPTC) for different values of the default probability, η, and the transaction costs, $(1-s)$. Interestingly, the LPTC ratio increases with the probability of default whereas seems to be pretty independent on the transaction costs (for transaction costs higher than 1%). However, and most importantly, the liquidity premium is one order of magnitude smaller than the transaction costs, implying that the additional source of risk determined by the possibility of default is not enough to generate a first-order effect on asset pricing.

Finally, in order to measure the effect of default on expected returns, I introduce the concept of default-free premium. Specifically, and consistently with the definition of the liquidity premium, in my setting I define the default-free premium to be the maximum expected return an investor is willing to exchange to not incur in the event of default. Obviously, to isolate the role of default, the equivalent default-free economy used to compare with my setting would still encompass transaction costs.

FIGURE 2 GOES HERE

Figure 2 above shows the ratio of the default-free premium to the default probability as a function of the probability of default η.[8] Interestingly, this ratio exhibits an increasing behavior, implying that a rise in the risk of default determines an increase in the default-free premium per unit of "additional risk" involved.

3 Conclusion

I analyze a very simple consumption-investment model in which an infinitely-lived investor allocates her wealth between a risky asset subject to default risk and a riskless security, and incurs in proportional transaction costs when exchanging one asset into the other. When default occurs, there are no proceeds from the liquidation of the risky asset and the investor receives the terminal wealth only in the form of the riskless security.

I show that default risk has to be seriously taken into account in portfolio choice problems since it may generate a first-order effect on the investor's asset allocation, causing significant deviations from the classical Merton (i.e. the riskless) solution. On the contrary, the liquidity premium stemming from my setting is one order of magnitude smaller than the transaction costs implying that the additional source of risk determined by the possibility of default is not enough to generate a first-order effect on asset pricing.

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[8] More precisely, in order to compute the default-free premium, I assume that, in the equivalent default-free economy, the investor receives with probability η a terminal wealth equal to $W = x + sy$.

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Table 1: Optimal position of the trading boundaries $\underline{\lambda}$ and $\overline{\lambda}$.

	$\eta = 0.01$	$\eta = 0.05$	$\eta = 0.1$	$\eta = 0.2$	$\eta = 0.3$
$1 - s = 0.001$	$[0.557, 0.633]$	$[0.493, 0.585]$	$[0.447, 0.520]$	$[0.322, 0.385]$	$[0.200, 0.250]$
$1 - s = 0.01$	$[0.436, 0.635]$	$[0.397, 0.590]$	$[0.343, 0.520]$	$[0.232, 0.390]$	$[0.128, 0.255]$
$1 - s = 0.02$	$[0.375, 0.640]$	$[0.336, 0.595]$	$[0.286, 0.530]$	$[0.184, 0.395]$	$[0.094, 0.260]$
$1 - s = 0.05$	$[0.267, 0.645]$	$[0.231, 0.600]$	$[0.188, 0.545]$	$[0.108, 0.400]$	$[0.044, 0.265]$
$1 - s = 0.1$	$[0.168, 0.670]$	$[0.139, 0.620]$	$[0.106, 0.550]$	$[0.051, 0.410]$	$[0.015, 0.280]$

Table I reports the boundaries $\left[\underline{\lambda}, \overline{\lambda}\right]$ of the no-trading region for different values of the default probability η and the transaction costs $(1 - s)$. The time-discount factor is $\rho = 0.1$, whereas the constant relative risk aversion is $\gamma = 2$. The risk-free rate is $r = 0.1$, the expected rate of return α and the standard deviation σ of the risky asset are given by $\alpha = 0.15$ and $\sigma = 0.2$. Finally, transaction costs are levied at the rate $(1 - s) = 0.01$, whereas the default probability of the risky asset is assumed to be $\eta = 5\%/\text{year}$.

Table 2: The ratio of the liquidity premium over the transaction cost rate.

	$\eta = 0.01$	$\eta = 0.05$	$\eta = 0.1$	$\eta = 0.2$	$\eta = 0.3$
$1 - s = 0.001$	0.130	0.155	0.185	0.245	0.300
$1 - s = 0.01$	0.115	0.135	0.165	0.215	0.270
$1 - s = 0.02$	0.115	0.135	0.160	0.215	0.270
$1 - s = 0.05$	0.115	0.135	0.160	0.215	0.270
$1 - s = 0.1$	0.115	0.135	0.160	0.215	0.270

Table 2 reports the ratio of the liquidity premium over the transaction costs rate (LPTC) for different values of the default probability η and the transaction costs $(1 - s)$. The time-discount factor is $\rho = 0.1$, whereas the constant relative risk aversion is $\gamma = 2$. The risk-free rate is $r = 0.1$, the expected rate of return α and the standard deviation σ of the risky asset are given by $\alpha = 0.15$ and $\sigma = 0.2$. Finally, transaction costs are levied at the rate $(1 - s) = 0.01$, whereas the default probability of the risky asset is assumed to be $\eta = 5\%$/year.

Figure 1: Optimal investment policies.

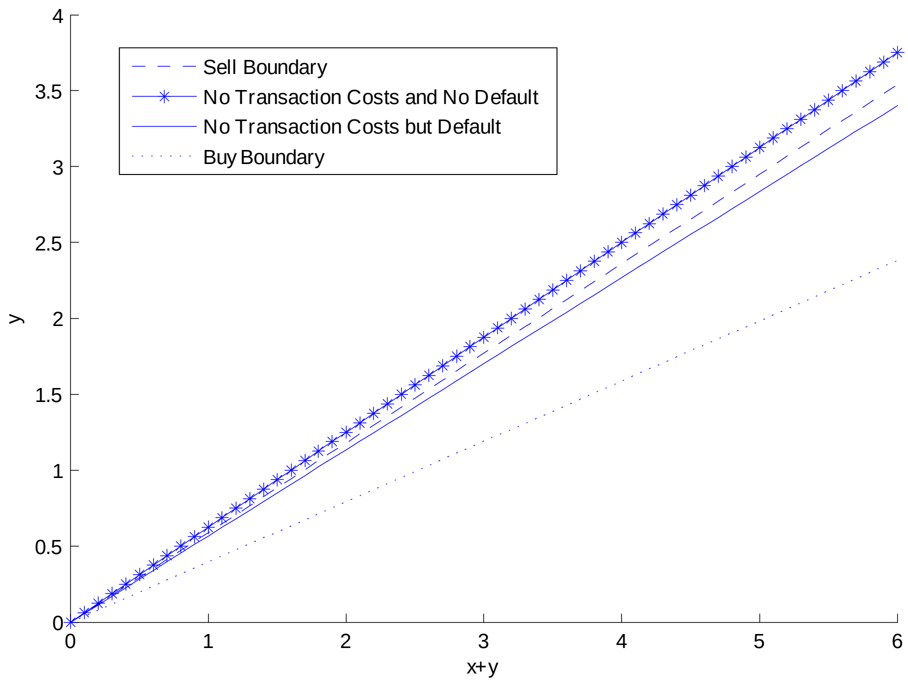

Figure 1 shows the optimal position of the trading boundaries, i.e. the buy and the sell barriers $\underline{\lambda}$ and $\overline{\lambda}$, together with the optimal portfolio policies obtained in the equivalent frictionless economy, i.e. no transaction costs, both with and without default risk. The time-discount factor is $\rho = 0.1$, whereas the constant relative risk aversion is $\gamma = 2$. The risk-free rate is $r = 0.1$, the expected rate of return α and the standard deviation σ of the risky asset are given by $\alpha = 0.15$ and $\sigma = 0.2$. Finally, transaction costs are levied at the rate $(1 - s) = 0.01$, whereas the default probability of the risky asset is assumed to be $\eta = 5\%/\text{year}$.

Figure 2: The ratio of the default-free premium to the default probability

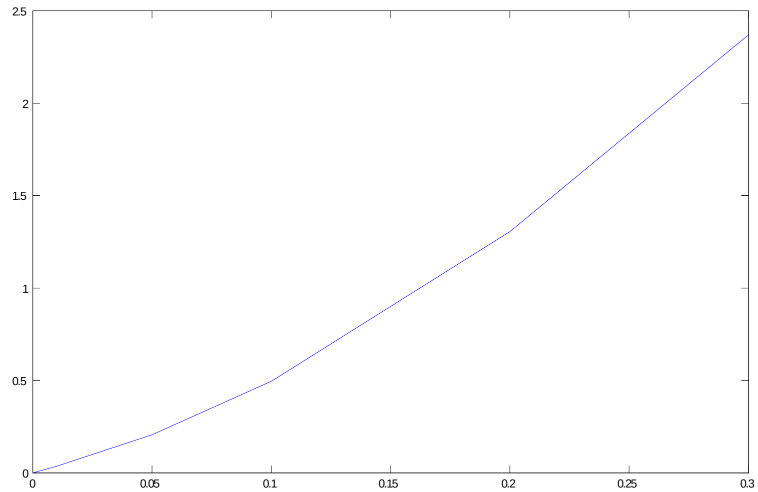

Figure 2 shows the ratio of the default-free premium to the default probability as a function of the probability of default η. The time-discount factor is $\rho = 0.1$, whereas the constant relative risk aversion is $\gamma = 2$. The risk-free rate is $r = 0.1$, the expected rate of return α and the standard deviation σ of the risky asset are given by $\alpha = 0.15$ and $\sigma = 0.2$. Finally, transaction costs are levied at the rate $(1 - s) = 0.01$, whereas the default probability of the risky asset is assumed to be $\eta = 5\%/\text{year}$.