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Equilibria Under Passive Beliefs for Multi-leader-follower Games with Vertical Information: Existence Results

Maria Carmela Ceparano[*] and Jacqueline Morgan[**]

Abstract
We consider two-stage multi-leader-follower games, called *multi-leader-follower games with vertical information*, where leaders in the first stage and followers in the second stage choose simultaneously an action but those chosen by any leader are observed by only one "exclusive" follower. This partial unobservability leads to extensive form games that have no proper subgames but may have an infinity of Nash equilibria. So it is not possible to refine using the concept of subgame perfect Nash equilibrium and, moreover, the concept of weak perfect Bayesian equilibrium could be not useful since it does not prescribe limitations on the beliefs out of the equilibrium path. This has motivated the introduction of a selection concept for Nash equilibria based on a specific class of beliefs, called *passive beliefs*, that each follower has about the actions chosen by the leaders rivals of his own leader. In this paper, we illustrate the effectiveness of this concept and we investigate the existence of such a selection for significant classes of problems satisfying generalized concavity properties and conditions of minimal character on possibly discontinuous data.

Keywords: multi-leader-follower game; selection of equilibria; passive belief; discontinuous function; fixed point; generalized concavity.

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1 Introduction

In this paper we consider two-stage multi-leader-follower games, called *multi-leader-follower games with vertical information*, where leaders in the first stage and followers in the second stage choose simultaneously an action but an exclusivity between any leader and a single follower is embodied assuming that the actions chosen by a leader are observed only by his follower. Real-world situations can be modeled as such games, for example in the setting of multilateral vertical contracting where competing manufacturers (the leaders) delegate retail decisions to exclusive retailers (the followers) offering a private contract, that is a wholesale price for unit of good sold and a franchise fee ([23]).

The partial unobservability of the leaders' actions makes an ineffective refinement the concept of *subgame perfect Nash equilibrium* ([27]) because of the absence of proper subgames. Moreover, the concept of *weak perfect Bayesian equilibrium* ([19]) could be not useful for the selection between Nash equilibria since it does not prescribe limitations on the beliefs out of the equilibrium path. Unfortunately, in multi-leader-follower games with vertical information, the concept of *simple perfect Bayesian equilibrium* ([9]) does not allow to select among weak perfect Bayesian equilibria (see Sect. 3). So, one way to overcome multiplicity of equilibria is to restrict the attention only to specific beliefs that any follower has about the strategy chosen by the leaders rivals of his own leader. Taking into account the specificity of the structure and in line with the economic literature, we focus on the case of *passive beliefs*: in fact passive beliefs are quite a common assumption mainly in multilateral vertical contracting but they are also used in mechanism design, games of electoral competition and consumer search literature, as pointed out in [6].

The aim of this paper is to present multi-leader-follower games with vertical information in a general setting, to introduce the concept of equilibrium under passive beliefs for these games and to investigate, when the optimal reaction of any follower is single-valued, the existence of such an equilibrium under conditions of generalized concavity and minimal character for possibly discontinuous payoff functions, differently from [23] where results are given in a concave and differential setting. For the sake of simplicity the action sets are assumed to be subsets of finite Euclidean spaces; however, the results could be extended to action sets in infinite-dimensional spaces.

Note that the solution concept used in (the more investigated) multi-leader-follower games *with observed actions*, where any follower observes the actions chosen by any leader in the first stage ([28], [11], [24], [33], . . . , and more recently [12], [15], [13], [32]), is no longer applicable to games with vertical information. In fact, as we will see in Section 3, the existence of an equilibrium under passive beliefs is related to the existence of a solution to a "joint" set of Parametric Optimization Problems, differently to the previous papers in which they solve a hierarchical Nash equilibrium problem.

The outline of the paper is the following.

Section 2 is devoted to the formalization of the model and to the illustration by a simple example of the problem of multiplicity of Nash equilibria.

In Section 3 a refinement concept based on passive beliefs is defined and its effectiveness is obtained using the above-mentioned example. An existence result for possibly discontinuous payoff functions is given when the optimal reaction of any follower satisfies a linear property and explicit sufficient conditions on

the data are determined in order to obtain such a property.
In Section 4, a reinforcement of the condition of exclusivity is investigated and the condition of linearity can be eliminated when the actions of both leaders and followers are assumed to be real numbers. Finally, an economic example illustrates the applicability of our analysis when the previous existence results cannot be applied.

2 Multi-leader-follower games with vertical information

First, let us formalize the concept of general two-stage multi-leader-follower game with a finite number of players and vertical information. In the first stage k players, called leaders L_i, $i \in I = \{1, \dots, k\}$, choose simultaneously an action x_i in X_i, a nonempty subset of $\mathbb{R}^{n_i}$. In the second stage k players, called followers F_i, $i \in I$, choose simultaneously an action y_i in Y_i, a nonempty subset of $\mathbb{R}^{m_i}$. Let $\mathbf{X} := \prod_{i=1}^{k} X_i$ and $\mathbf{X}_{-i} := \prod_{\substack{r=1 \\ r\neq i}}^{k} X_r$. An element of $\mathbf{X}$ is denoted by $\mathbf{x} = (x_1, \dots, x_k)$ and an element of $\mathbf{X}_{-i}$ is denoted by $\mathbf{x}_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k)$. Analogous notations are used for the followers' variables.

Definition 2.1. *A multi-leader-follower game is said to be* with vertical information *if, for any $i \in I$, any action x_i chosen by leader L_i is observed by only one follower F_i, called exclusive follower of leader L_i. A multi-leader-follower game with vertical information will be referred to as* multi–leader/follower *game.*

So, in the case of vertical information, each follower has as many information sets as the number of actions of his corresponding leader. Then, a strategy of follower F_i is a function β_i from X_i to Y_i, that is $\beta_i \in S_i = (Y_i)^{X_i}$. Let $\mathbf{S} := \prod_{i=1}^{k} S_i$ be the set of all (pure) strategy profiles of the followers.
For the sake of generality, we assume that, for $i \in I$, the leader L_i's payoff function can depend explicitly on his follower F_i's *optimal value function* v_i defined as the optimal payoff of the follower F_i for any fixed action profile of all the other players. This can be viewed as an altruistic\spiteful behaviour, depending on the way the optimal value function of a follower affects his leader's payoff function. This assumption is compatible with the fact that, if a follower is an exclusive retailer of the good produced by a leader, then the latter can take into account the profit of his retailer (possibly in a percentage term) when he has to decide about the strategy to play. Note that, in an engineering context, problems that can be modeled in this way are the so-called *parameter design problem* and *the resource allocation problem for decentralized systems* ([29], [30]).

The objective of each player is to maximize his own payoff function, taking into account that it depends also on the choices of the other players. Let $i \in I$. The payoff function of follower F_i is assumed to be a real-valued function f_i defined on $\mathbf{X} \times \mathbf{Y}$. The optimal value function v_i of follower F_i, also called *value function* or *marginal function* (see, e.g., [16]), is defined on $\mathbf{X} \times \mathbf{Y}_{-i}$ by:

$$v_i(\mathbf{x}, \mathbf{y}_{-i}) := \sup_{y_i \in Y_i} f_i(\mathbf{x}, y_i, \mathbf{y}_{-i})$$

and, for the sake of simplicity, it is assumed to be finite for any $(\mathbf{x}, \mathbf{y}_{-i}) \in$

$\mathbf{X} \times \mathbf{Y}_{-i}$.
The payoff of leader L_i associated to an action profile $(\mathbf{x}, \mathbf{y})$ is assumed to be

$$\check{l}_i(\mathbf{x}, \mathbf{y}) := l_i(\mathbf{x}, \mathbf{y}, v_i(\mathbf{x}, \mathbf{y}_{-i})), \tag{1}$$

where l_i is a given function from $\mathbf{X} \times \mathbf{Y} \times \mathbb{R}$ to $\mathbb{R}$.

Let $(x_1, \ldots, x_k, \beta_1, \ldots, \beta_k) \in \mathbf{X} \times \mathbf{S}$ be a strategy profile denoted by $(\mathbf{x}, \boldsymbol{\beta})$. The action profile $(\beta_1(x_1), \ldots, \beta_k(x_k))$ of the followers is denoted by $\boldsymbol{\beta}(\mathbf{x})$ and the action profile $(\beta_1(x_1), \ldots, \beta_{i-1}(x_{i-1}), \beta_{i+1}(x_{i+1}), \ldots, \beta_k(x_k))$ is denoted by $\boldsymbol{\beta}_{-i}(\mathbf{x}_{-i})$.

Now, let Γ be the normal form of the multi–leader/follower game:

$$\Gamma := \{\{L_i\}_{i\in I}, \{F_i\}_{i\in I}; (X_i)_{i\in I}, (S_i)_{i\in I}; (\bar{l}_i)_{i\in I}, (\bar{f}_i)_{i\in I}\}, \tag{2}$$

where $\bar{l}_i$ and $\bar{f}_i$ are the functions defined on $\mathbf{X} \times \mathbf{S}$, respectively, by $\bar{l}_i(\mathbf{x}, \boldsymbol{\beta}) := \check{l}_i(\mathbf{x}, \boldsymbol{\beta}(\mathbf{x}))$ and $\bar{f}_i(\mathbf{x}, \boldsymbol{\beta}) := f_i(\mathbf{x}, \boldsymbol{\beta}(\mathbf{x}))$. Applying the well-known concept of Nash equilibrium ([22]) to game Γ, a strategy profile $(\mathbf{x}^*, \boldsymbol{\beta}^*) \in \mathbf{X} \times \mathbf{S}$ is a *Nash equilibrium* of the game Γ if and only if, for any $i \in I$, $\bar{l}_i(\mathbf{x}^*, \boldsymbol{\beta}^*) \geq \bar{l}_i(x_i, \mathbf{x}^*_{-i}, \boldsymbol{\beta}^*)$, for any $x_i \in X_i$, and $\bar{f}_i(\mathbf{x}^*, \boldsymbol{\beta}^*) \geq \bar{f}_i(\mathbf{x}^*, \beta_i, \boldsymbol{\beta}^*_{-i})$ for any $\beta_i \in S_i$.

An existence result for such Nash equilibria has been given in [3] for possibly discontinuous games. However, game Γ may have an infinity of Nash equilibria as emphasized in the following example.

Example 2.1 As in [23] on multilateral vertical contracting, assume that two competing manufacturers (the *leaders*), producers of substitute goods, choose vertical separation as their organizational structure, that is they delegate the sale of the good they produce through exclusive retailers offering them a private contract. So each retailer observes only the contract offered by his corresponding manufacturer and after that he decides the retail price in a competitive setting. Here, for the sake of simplicity, we assume that the contract offered by a leader specifies a wholesale price that his follower has to pay for each unit of good sold. We assume that $i \in \{1, 2\}$, the set of the wholesale prices X_i and the set of the retail prices Y_i are both equal to $\mathbb{R}^+$ and the players face a demand function linear in both retail prices:

$$D_i(\mathbf{y}) = 1 - 2y_i + y_{-i}. \tag{3}$$

The payoff functions of leader L_i and follower F_i are defined, respectively, by $l_i(x_i, \mathbf{y}) = D_i(\mathbf{y})x_i$ and $f_i(x_i, \mathbf{y}) = D_i(\mathbf{y})(y_i - x_i)$, for any $(x_i, \mathbf{y}) \in X_i \times \mathbf{Y}$. One can verify that the strategy profiles given by

$$\check{s} = \left(\frac{2}{5}, \frac{2}{5}, \check{\beta}_1, \check{\beta}_2\right) \qquad \text{and such that, for } i \in I, \quad \check{\beta}_i(x_i) = \begin{cases} \frac{3}{5} & \text{if } x_i = \frac{2}{5}, \\ \nu & \text{otherwise,} \end{cases}$$

are Nash equilibria of game Γ for all $\nu > 4/5$. ◇

So, in order to select among Nash equilibria, in the next section we introduce a selection criterion for multi–leader/follower games based on passive beliefs.

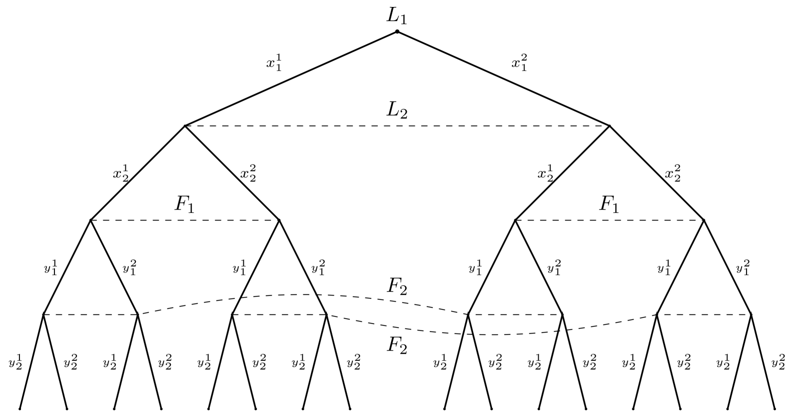

Figure 1: Representation of multi–leader/follower game in extensive form having a finite set of actions in all the information sets

3 Equilibria under passive beliefs: definition and existence

Since the unique subgame of the game presented in Section 2 is the whole game (see e.g. Fig. 1), then the set of subgame perfect Nash equilibria coincides with the set of Nash equilibria and one cannot refine using this equilibrium concept. However, another method useful for selecting among Nash equilibria of an extensive form game with a finite number of actions for any player is based on the concept of *weak perfect Bayesian equilibrium* ([19]). So, let us first extend such a concept to a multi–leader/follower game with an infinite number of actions for any player.

Remember that there exists a one to one correspondence between the information sets of follower F_i and the actions of his corresponding leader L_i. Moreover, the fact that follower F_i is in a decision node of an own information set may depend on the actions chosen by leaders L_{-i} (see e.g. Fig. 1).

So, let $\mathfrak{P}_i$ be the set of all probability measures on $\mathbf{X}_{-i}$, for any $i \in I$.

Definition 3.1. *A* system of beliefs of follower *F_i is a function μ_i that associates to any $x_i \in X_i$ a probability measure $\mu_i(x_i)(\cdot) \in \mathfrak{P}_i$, denoted by $\mu_i(\cdot|x_i)$, that represents the beliefs that F_i has about the action profile chosen by leaders L_{-i} after he has observed an action $x_i \in X_i$ of leader L_i.*

A system of beliefs *is a profile $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$, where μ_i is a system of beliefs of follower F_i, for any $i \in I$.*

Let $\boldsymbol{\beta}$ be a strategy profile of the followers, μ_i be a system of beliefs of follower F_i and $x_i \in X_i$. The corresponding expected payoff of follower F_i, if defined, is:

$$\hat{f}_i(\boldsymbol{\beta}|x_i, \mu_i) := \int_{\mathbf{X}_{-i}} f_i(x_i, \mathbf{x}_{-i}, \beta_i(x_i), \boldsymbol{\beta}_{-i}(\mathbf{x}_{-i}))\, d\,\mu_i(\mathbf{x}_{-i}|x_i), \qquad (4)$$

and we can give the following definition, suitable for multi–leader/follower games.

Definition 3.2. *If $\bar{\boldsymbol{\sigma}} = (\bar{x}_1, \dots, \bar{x}_k, \bar{\beta}_1, \dots, \bar{\beta}_k) \in \mathbf{X} \times \boldsymbol{S}$ is a (pure) strategy profile and $\bar{\boldsymbol{\mu}} = (\bar{\mu}_1, \dots, \bar{\mu}_k)$ is a system of beliefs, then $(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\mu}})$ is a* like-weak perfect Bayesian equilibrium of the multi–leader/follower game *if, for any $i \in I$:*

(i) follower F_i plays optimally in each situation he may face, assuming that his beliefs about the strategies chosen by leaders L_{-i} are in accordance with $\bar{\mu}_i$, that is $\hat{f}_i(\bar{\boldsymbol{\beta}}|x_i, \bar{\mu}_i) = \max_{\beta_i \in S_i} \hat{f}_i((\beta_i, \bar{\boldsymbol{\beta}}_{-i})|x_i, \bar{\mu}_i)$, for any $x_i \in X_i$;

(ii) the action $\bar{x}_i$ of leader L_i at equilibrium is such that:

$$\check{l}_i(\bar{\mathbf{x}}, \bar{\boldsymbol{\beta}}(\bar{\mathbf{x}})) = \max_{x_i \in X_i} \check{l}_i(x_i, \bar{\mathbf{x}}_{-i}, \bar{\beta}_i(x_i), \bar{\boldsymbol{\beta}}_{-i}(\bar{\mathbf{x}}_{-i})),$$

where we recall that $\check{l}_i(\mathbf{x}, \mathbf{y})$ is defined in (1);

(iii) the system of beliefs $\bar{\mu}_i$ of follower F_i satisfies the consistency hypothesis: $\bar{\mu}_i(\cdot|\bar{x}_i) = \delta_{\bar{\mathbf{x}}_{-i}}(\cdot)$, where $\delta_{\bar{\mathbf{x}}_{-i}}$ is the Dirac measure *on $\mathbf{X}_{-i}$ (also known as* unit mass *at $\bar{\mathbf{x}}_{-i}$, see e.g. [26]).*

We emphasize that the definition is independent from the order of the players chosen to arrange the simultaneous moves. For a definition of the concept of weak-perfect Bayesian equilibrium for a more general extensive game one can refer to [9].

Conditions (i)-(ii) in Definition 3.2 require that the equilibrium strategies of followers and leaders are *sequentially rational given the system of beliefs* in the sense of Definition 9.C.2 in [19] for finite extensive games. Besides, condition (iii) in Definition 3.2 extends to multi–leader/follower games the consistency hypothesis (ii) of Definition 9.C.3 in [19] and it ensures that the beliefs are compatible with the equilibrium strategy profile along the equilibrium path,[1] that is the followers have correct beliefs in equilibrium. Moreover, condition (iii) implies $\hat{f}_i(\boldsymbol{\beta}|\bar{x}_i, \bar{\mu}_i) = f_i(\bar{\mathbf{x}}, \boldsymbol{\beta}(\bar{\mathbf{x}}))$ for any $\boldsymbol{\beta} \in \mathbf{S}$.
Clearly, the sequential rationality and the consistency hypothesis in Definition 3.2 are sufficient to guarantee that a like-weak perfect Bayesian equilibrium of the multi–leader/follower game is a Nash equilibrium.[2]

However, the consistency hypothesis does not impose any restriction out of the equilibrium path. So, the concept of like-weak perfect Bayesian equilibrium of the multi–leader/follower game may be not sufficient to select a finite number of Nash equilibria, as one can see in the following example.

Example 3.1 In the same situation of Example 2.1, let $\alpha \in [0,1]$, $\bar{\boldsymbol{\sigma}}_\alpha = (\bar{x}_\alpha, \bar{x}_\alpha, \bar{\beta}_\alpha, \bar{\beta}_\alpha)$ with $\bar{x}_\alpha = (3+\alpha)/(9+\alpha)$ and $\bar{\beta}_\alpha(x_i) = (3+\alpha)/(9+\alpha) + 2x_i/(3+\alpha)$, for any $x_i \in \mathbb{R}^+$, and let $\bar{\boldsymbol{\mu}}_\alpha = (\bar{\mu}_{1,\alpha}, \bar{\mu}_{2,\alpha})$ be defined by $\bar{\mu}_{i,\alpha}(\cdot|x_i) = \alpha\delta_{\bar{x}_\alpha}(\cdot) + (1-\alpha)\delta_{x_i}(\cdot)$; that is, if F_i observes x_i, then he believes that L_{-i} plays $\bar{x}_\alpha$ with probability α and x_i with probability $1-\alpha$, for any $x_i \in X_i$. Then, for any $\alpha \in [0,1]$, $(\bar{\boldsymbol{\sigma}}_\alpha, \bar{\boldsymbol{\mu}}_\alpha)$ is a like-weak perfect Bayesian equilibrium of the multi–leader/follower game and there exists an infinite number of like-weak perfect Bayesian equilibria of the multi–leader/follower game. ◇

[1] It is worth mentioning that, differently from what may happen in a general extensive game ([9]), in our specific case the system of beliefs is inferred correctly from *Bayes' rule* along the path of the equilibrium in pure strategy. This is a consequence of the fact that, if $\boldsymbol{\sigma} = (x_1, \dots, x_k, \beta_1, \dots, \beta_k)$ is a pure strategy profile, the unique information set of, for example, follower F_i that is on the path of $\boldsymbol{\sigma}$ is the one that is associated to the action x_i; then the probability of being in that information set given that $\boldsymbol{\sigma}$ has been played is equal to 1.

[2] For a proof, one can follow the same reasoning of Proposition 9.C.1 in [19].

Note that we cannot use the idea of *simple perfect Bayesian equilibium* introduced in [9] in order to select among the like-weak perfect Bayesian equilibria of a multi–leader/follower game since in our context all the information sets of the followers are not regular while all the information sets of the leaders are regular but are on the path of any equilibrium strategy profile.
Therefore, we restrict now our attention on the equilibria that are supported by a particular system of beliefs called *passive beliefs*.
In fact, a like-weak perfect Bayesian equilibrium of the multi–leader/follower game is an equilibrium under passive beliefs if a follower of a leader has the following beliefs about the actions chosen by the other leaders: when he observes an action of his corresponding leader different from the one he expects in equilibrium, then he believes that the rival leaders are still not deviating from their equilibrium strategies; that is, a follower of a leader does not revise his beliefs about the action chosen by the other leaders even if his corresponding leader is deviating. Recall that the concept of passive beliefs is used implicitly or explicitly by many authors in an economic setting (see the Introduction for references). Formally:

Definition 3.3. *The strategy profile* $\boldsymbol{\sigma}^* = (x_1^*, \dots, x_k^*, \beta_1^*, \dots, \beta_k^*) \in \mathbf{X} \times \boldsymbol{S}$ *is an* equilibrium under passive beliefs *of a multi–leader/follower game if* $(\boldsymbol{\sigma}^*, \boldsymbol{\mu}^*)$ *is a like-weak perfect Bayesian equilibrium of the multi–leader/follower game, where* $\boldsymbol{\mu}^* = (\mu_1^*, \dots, \mu_k^*)$ *is such that* $\mu_i^*(\cdot|x_i) = \delta_{\mathbf{x}^*_{-i}}(\cdot)$*, for any* $x_i \in X_i$.

Then, since $\mu_i^*(\cdot|x_i)$ has to be a *unit mass* at $\mathbf{x}^*_{-i}$, for any $x_i \in X_i$ and any $i \in I$, in case of passive beliefs the expected payoff of follower F_i defined in (4) is equal to $\hat{f}_i(\boldsymbol{\beta}|x_i, \mu_i^*) = f_i(x_i, \mathbf{x}^*_{-i}, \beta_i(x_i), \boldsymbol{\beta}_{-i}(\mathbf{x}^*_{-i}))$, for any $\boldsymbol{\beta} \in \mathbf{S}$. Therefore, a strategy profile $\boldsymbol{\sigma}^* = (x_1^*, \dots, x_k^*, \beta_1^*, \dots, \beta_k^*)$ is an equilibrium under passive beliefs if for any $i \in I$:

$$f_i(x_i, \mathbf{x}^*_{-i}, \beta_i^*(x_i), \boldsymbol{\beta}^*_{-i}(\mathbf{x}^*_{-i})) = \max_{y_i \in Y_i} f_i(x_i, \mathbf{x}^*_{-i}, y_i, \boldsymbol{\beta}^*_{-i}(\mathbf{x}^*_{-i})) \quad \text{for any } x_i \in X_i;$$
$$\check{l}_i(\mathbf{x}^*, \boldsymbol{\beta}^*(\mathbf{x}^*)) = \max_{x_i \in X_i} \check{l}_i(x_i, \mathbf{x}^*_{-i}, \beta_i^*(x_i), \boldsymbol{\beta}^*_{-i}(\mathbf{x}^*_{-i})),$$

where $\check{l}_i(\mathbf{x}, \mathbf{y})$ is defined in (1).
Let $B_i(\mathbf{x}, \mathbf{y}_{-i})$ be the *reaction set* of follower F_i to $(\mathbf{x}, \mathbf{y}_{-i}) \in \mathbf{X} \times \mathbf{Y}_{-i}$:

$$B_i(\mathbf{x}, \mathbf{y}_{-i}) := \operatorname*{Arg\,max}_{y_i \in Y_i} f_i(\mathbf{x}, \mathbf{y}). \tag{5}$$

From now on, for any $i \in I$, we assume the following:

$$(\mathcal{U}_i) \quad \left\{ \begin{array}{l} \text{for any } (\mathbf{x}, \mathbf{y}_{-i}) \in \mathbf{X} \times \mathbf{Y}_{-i},\ B_i(\mathbf{x}, \mathbf{y}_{-i}) \text{ is the singleton } \{b_i(\mathbf{x}, \mathbf{y}_{-i})\}, \\ \text{that we call } \textit{optimal reaction} \text{ of follower } F_i. \end{array} \right.$$

Let us emphasize that such an assumption is usually required in economic models in the context of multi-stage games. It is obtained if, for example, one assumes (together with classical conditions for existence) that the function $y_i \in Y_i \to f_i(\mathbf{x}, y_i, \mathbf{y}_{-i}) \in \mathbb{R}$ (in short $f_i(\mathbf{x}, \cdot, \mathbf{y}_{-i})$) is strictly quasiconcave on Y_i (see, e.g., [1]), for any $(\mathbf{x}, \mathbf{y}_{-i}) \in \mathbf{X} \times \mathbf{Y}_{-i}$.

The payoff of leader L_i associated to an action profile $(\mathbf{x}, \mathbf{y}_{-i})$ becomes:

$$\tilde{l}_i(\mathbf{x}, \mathbf{y}_{-i}) := l_i(\mathbf{x}, b_i(\mathbf{x}, \mathbf{y}_{-i}), \mathbf{y}_{-i}, v_i(\mathbf{x}, \mathbf{y}_{-i})) \tag{6}$$

and leader L_i's problem can be described by:

$$\mathcal{S}_i(\mathbf{x}_{-i}, \mathbf{y}_{-i}) \qquad \begin{cases} \text{find } x_i \in X_i \text{ such that} \\ \tilde{l}_i(x_i, \mathbf{x}_{-i}, \mathbf{y}_{-i}) = \max_{x'_i \in X_i} \tilde{l}_i(x'_i, \mathbf{x}_{-i}, \mathbf{y}_{-i}). \end{cases}$$

Remark 3.1 A strategy profile $(\mathbf{x}^*, \boldsymbol{\beta}^*) = (x_1^*, \dots, x_k^*, \beta_1^*, \dots, \beta_k^*)$ is an equilibrium under passive beliefs of the multi–leader/follower game Γ if and only if, for any $i \in I$, x_i^* solves $\mathcal{S}_i(\mathbf{x}^*_{-i}, \mathbf{y}^*_{-i})$ and $\beta_i^*(x_i) = b_i(x_i, \mathbf{x}^*_{-i}, \mathbf{y}^*_{-i})$ for any $x_i \in X_i$, where $\mathbf{y}^*_{-i} = \boldsymbol{\beta}^*_{-i}(\mathbf{x}^*_{-i})$.

Basically, a leader L_i and the corresponding follower F_i act under passive beliefs as a team that solves a parametric Bilevel Optimization problem where the parameter will be the action profile of all L_{-i} and F_{-i} at equilibrium. In other words, the strategic interaction is between the k teams. Then, taken as given the action profile $(\mathbf{x}_{-i}, \mathbf{y}_{-i})$ in $\mathbf{X}_{-i} \times \mathbf{Y}_{-i}$, team $L_i \backslash F_i$ chooses a couple that maximizes payoff functions of both L_i and F_i taking into account the hierarchical structure between L_i and F_i.

Example 3.2 One can verify that the unique equilibrium under passive beliefs of the game in Example 2.1 is the strategy profile $(2/5, 2/5, \beta_1^*, \beta_2^*)$, where $\beta_i^*(x_i) = 2/5 + x_i/2$, for any x_i and $i \in \{1, 2\}$. $\diamond$

In the remainder of this paper we assume, for any $i \in I$, that X_i and Y_i are nonempty convex and compact sets and the payoff functions satisfy the following:

$$(\mathcal{C}_i) \begin{cases} \text{(i)} & f_i \text{ is a real-valued upper semicontinuous function on } \mathbf{X} \times \mathbf{Y}; \\ \text{(ii)} & \text{for any } (\mathbf{x}, \mathbf{y}) \in \mathbf{X} \times \mathbf{Y} \text{ and any sequence } (\mathbf{x}_n, \mathbf{y}_{-i,n})_n \text{ converging} \\ & \text{to } (\mathbf{x}, \mathbf{y}_{-i}) \text{ in } \mathbf{X} \times \mathbf{Y}_{-i}, \text{ there exists a sequence } (\hat{y}_{i,n})_n \text{ in } Y_i \text{ such} \\ & \text{that } \liminf_{n \to \infty} f_i(\mathbf{x}_n, \hat{y}_{i,n}, \mathbf{y}_{-i,n}) \geq f_i(\mathbf{x}, \mathbf{y}); \\ \text{(iii)} & l_i \text{ is a real-valued upper semicontinuous function on } \mathbf{X} \times \mathbf{Y} \times \mathbb{R}; \\ \text{(iv)} & l_i(x_i, \cdot\,) \text{ is lower semicontinuous on } \mathbf{X}_{-i} \times \mathbf{Y} \times \mathbb{R}, \text{ for any } x_i \in X_i. \end{cases}$$

Remark 3.2 If f_i and l_i are continuous functions then Assumption $(\mathcal{C}_i)$ is satisfied. The vice versa is not true, as illustrated in the following example.

Example 3.3 Let $i \in \{1, 2\}$ and $X_i = Y_i = [0, 1]$.

— Let f_i be a real-valued function defined on $[0, 1]^4$ by:

$$f_i(\mathbf{x}, \mathbf{y}) = \begin{cases} x_i & \text{if } x_i \in]0, 1], y_i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, f_i is not continuous at $(\mathbf{x}, 0, y_{-i})$, for any $(\mathbf{x}, y_{-i}) \in [0, 1]^3$ such that $x_i \neq 0$, but it satisfies assumptions (i)–(ii) in $(\mathcal{C}_i)$.

— Let l_i be a real-valued function defined on $[0, 1]^4 \times \mathbb{R}$ by:

$$l_i(\mathbf{x}, \mathbf{y}, t) = \begin{cases} (y_i + y_{-i} - x_i + t)^2 & \text{if } x_i \in]0, 1], \\ (y_i + y_{-i} + t)^2 + 1 & \text{if } x_i = 0, \end{cases}$$

Then, l_i is not continuous at $(0, x_{-i}, \mathbf{y}, t)$, for any $(x_{-i}, \mathbf{y}, t) \in [0,1]^3 \times \mathbb{R}$ but it satisfies assumptions (iii)–(iv) in $(\mathcal{C}_i)$. ◇

The next theorem gives sufficient conditions for the existence of an equilibrium under passive beliefs.

Theorem 3.1. *Assume, for any $i \in I$, $(\mathcal{U}_i)$, $(\mathcal{C}_i)$ and for any $(\mathbf{x}, \mathbf{y}) \in \mathbf{X} \times \mathbf{Y}$, the following:*

$$(\mathcal{A}_{F_i} 1) \begin{cases} (i) & b_i(\,\cdot\,, \mathbf{x}_{-i}, \mathbf{y}_{-i}) \text{ is linear on } X_i; \\ (ii) & v_i(\,\cdot\,, \mathbf{x}_{-i}, \mathbf{y}_{-i}) \text{ is concave (resp. convex) on } X_i; \end{cases}$$

$$(\mathcal{A}_{L_i} 1) \begin{cases} (i) & l_i(\,\cdot\,, \mathbf{x}_{-i}, \,\cdot\,, \mathbf{y}_{-i}, \,\cdot\,) \text{ is quasiconcave on } X_i \times Y_i \times \mathbb{R}; \\ (ii) & l_i(\mathbf{x}, \mathbf{y}, \,\cdot\,) \text{ is increasing (resp. decreasing) on } \mathbb{R}. \end{cases}$$

Then, there exists an equilibrium under passive beliefs of game Γ.

Proof Let $i \in I$, $Z_i := X_i \times Y_i$, $\mathbf{Z}_{-i} := \mathbf{X}_{-i} \times \mathbf{Y}_{-i}$ and $\mathbf{Z} := \mathbf{X} \times \mathbf{Y}$. Define the set-valued maps M_i on $\mathbf{Z}_{-i}$ by

$$M_i(\mathbf{z}_{-i}) := \operatorname*{Arg\,max}_{x_i \in X_i} \tilde{l}_i(x_i, \mathbf{z}_{-i}), \tag{7}$$

N_i on $\mathbf{Z}_{-i}$ by $N_i(\mathbf{z}_{-i}) := \{(x_i, y_i) \in X_i \times Y_i \colon x_i \in M_i(\mathbf{z}_{-i}), y_i = b_i(x_i, \mathbf{z}_{-i})\}$ and N from $\mathbf{Z}$ to $\mathbf{Z}$ by $N(\mathbf{z}) := \prod_{i=1}^{k} N_i(\mathbf{z}_{-i})$.

Then, the set-valued map N satisfies the hypothesis of Kakutani Fixed Point Theorem [14]. Indeed, let $i \in I$:

- The optimal reaction function b_i and the optimal value function v_i of follower F_i are continuous functions on $\mathbf{X} \times \mathbf{Y}_{-i}$ in the light of assumptions (i)–(ii) in $(\mathcal{C}_i)$ and Proposition 3.1 in [17] (see also [16]).
- $M_i(\mathbf{z}_{-i})$ and $N_i(\mathbf{z}_{-i})$ are nonempty sets, for any $\mathbf{z}_{-i} \in \mathbf{Z}_{-i}$ in the light of Assumption $(\mathcal{C}_i)$ and Corollary 5.1 in [20].
- The set-valued map M_i is closed. Indeed, if l_i satisfies Assumption (iv) in $(\mathcal{C}_i)$ then it satisfies Assumption (L2) of Proposition 3.2 in [17] (taking $f_{1,n} = -l_i$, for every $n \in \mathbb{N}$).
- N_i is a closed set-valued map since b_i is continuous and M_i is closed.
- N is a convex-valued map. The assertion will be proved when $v_i(\,\cdot\,, \mathbf{z}_{-i})$ is concave on X_i and $l_i(\mathbf{x}, \mathbf{y}, \,\cdot\,)$ is increasing[3] on $\mathbb{R}$, for any $\mathbf{z}_{-i} \in \mathbf{Z}_{-i}$; in a similar way it could be proved when $v_i(\,\cdot\,, \mathbf{z}_{-i})$ is convex on X_i and $l_i(\mathbf{x}, \mathbf{y}, \,\cdot\,)$ is decreasing on $\mathbb{R}$.
  Indeed, let $\mathbf{z}_{-i} \in \mathbf{Z}_{-i}$ and $(x_i', y_i'), (x_i'', y_i'') \in N_i(\mathbf{z}_{-i})$. By definition, we have that $x_i', x_i'' \in M_i(\mathbf{z}_{-i})$ and $y_i' = b_i(x_i', \mathbf{z}_{-i})$, $y_i'' = b_i(x_i'', \mathbf{z}_{-i})$. Let $\lambda \in]0,1[$ and $\hat{x}_i = \lambda x_i' + (1-\lambda) x_i''$. From the linearity of $b_i(\,\cdot\,, \mathbf{z}_{-i})$ on X_i it follows that $\hat{y}_i = \lambda y_i' + (1-\lambda) y_i'' = b_i(\hat{x}_i, \mathbf{z}_{-i})$. Furthermore, using the concavity of $v_i(\,\cdot\,, \mathbf{z}_{-i})$, the increasingness of $l_i(\mathbf{x}, \mathbf{y}, \,\cdot\,)$ and the linearity of $b_i(\,\cdot\,, \mathbf{z}_{-i})$:

$$\begin{aligned} \tilde{l}_i(\hat{x}_i, \mathbf{z}_{-i}) &= l_i(\hat{x}_i, \mathbf{x}_{-i}, b_i(\hat{x}_i, \mathbf{z}_{-i}), \mathbf{y}_{-i}, v_i(\hat{x}_i, \mathbf{z}_{-i})) \\ &\geq l_i(\hat{x}_i, \mathbf{x}_{-i}, \hat{y}_i, \mathbf{y}_{-i}, \lambda v_i(x_i', \mathbf{z}_{-i}) + (1-\lambda) v_i(x_i'', \mathbf{z}_{-i})). \end{aligned}$$

[3] That is, if $\underline{t}_i, \bar{t}_i \in \mathbb{R}$ and $\underline{t}_i \leq \bar{t}_i$ then $l_i(\mathbf{x}, \mathbf{y}, \underline{t}_i) \leq l_i(\mathbf{x}, \mathbf{y}, \bar{t}_i)$.

Then, in the light of Assumption (i) of $(\mathcal{A}_{L_i}\,1)$, we obtain:

$$\tilde{l}_i(\lambda x_i' + (1-\lambda)x_i'', \mathbf{z}_{-i}) \geq \min\{\tilde{l}_i(x_i', \mathbf{z}_{-i}), \tilde{l}_i(x_i'', \mathbf{z}_{-i})\};$$

that is $\tilde{l}_i(\,\cdot\,, \mathbf{z}_{-i})$ is quasiconcave con X_i. Finally, from the linearity of $b_i(\,\cdot\,, \mathbf{z}_{-i})$, we get that $N_i(\mathbf{z}_{-i})$ is a convex set.

Hence, there exists a fixed point $(\mathbf{x}^*, \mathbf{y}^*)$ of N on $\mathbf{Z}$. Then $x_i^* \in M_i(\mathbf{x}_{-i}^*, \mathbf{y}_{-i}^*)$, that is x_i^* solves problem $\mathcal{S}_i(\mathbf{x}_{-i}^*, \mathbf{y}_{-i}^*)$. Furthermore, defined a strategy β_i^* for follower F_i such that $\beta_i^*(x_i) = b_i(x_i, \mathbf{x}_{-i}^*, \mathbf{y}_{-i}^*)$ for any $x_i \in X_i$, for any $i \in I$, it follows that $(\mathbf{x}^*, \boldsymbol{\beta}^*)$ is an equilibrium under passive beliefs of Γ. ◇

Remark 3.3 Assumptions (iii)–(iv) in $(\mathcal{C}_i)$ cannot be substituted for f_i pseudocontinuous (see [21]) since pseudocontinuity does not guarantee continuity of the optimal value function v_i.

We conclude this section giving sufficient conditions, explicit on the data, for Assumption $(\mathcal{A}_{F_i}\,1)$.

Lemma 3.1. *Let $i \in I$ and $(\mathbf{x}_{-i}, \mathbf{y}_{-i}) \in \mathbf{X}_{-i} \times \mathbf{Y}_{-i}$. Assume that X_i and Y_i are strictly convex sets [1] (resp. convex). Then, we have:*

- *The function $v_i(\,\cdot\,, \mathbf{x}_{-i}, \mathbf{y}_{-i})$ is concave on X_i if*

  $(\mathcal{A}_{F_i}\,2)$ $\quad f_i(\,\cdot\,, \mathbf{x}_{-i}, \,\cdot\,, \mathbf{y}_{-i})$ is concave on int$(X_i) \times$ int(Y_i) (resp. $X_i \times Y_i$).

- *The function $v_i(\,\cdot\,, \mathbf{x}_{-i}, \mathbf{y}_{-i})$ is convex on X_i if*

  $(\mathcal{A}_{F_i}\,3)$ $\quad f_i(\,\cdot\,, \mathbf{x}_{-i}, \mathbf{y})$ is convex on int(X_i) (resp. X_i), for any $y_i \in Y_i$.

Proof The first assertion can be easily proved by applying Theorem 29.1 in [25]. For the second assertion, apply Proposition 3.5 in [7]. ◇

Proposition 3.1. *Let $i \in I$ and $(\mathbf{x}_{-i}, \mathbf{y}_{-i}) \in \mathbf{X}_{-i} \times \mathbf{Y}_{-i}$. If X_i and Y_i are strictly convex (resp. convex) and $(\mathcal{U}_i)$, $(\mathcal{A}_{F_i}\,2)$ and $(\mathcal{A}_{F_i}\,3)$ hold, then $b_i(\,\cdot\,, \mathbf{x}_{-i}, \mathbf{y}_{-i})$ is linear on X_i and $(\mathcal{A}_{F_i}\,1)$ is satisfied.*

Proof We prove the assertion when $f_i(\,\cdot\,, \mathbf{x}_{-i}, \,\cdot\,, \mathbf{y}_{-i})$ is concave on $\mathrm{int}(X_i) \times \mathrm{int}(Y_i)$, $f_i(\,\cdot\,, \mathbf{x}_{-i}, y_i, \mathbf{y}_{-i})$ is linear on $\mathrm{int}(X_i)$ and X_i, Y_i are strictly convex. The other case is similar.
Let $\mathbf{y}_{-i} \in \mathbf{Y}_{-i}$, $x_i', x_i'' \in X_i$, $\lambda \in]0,1[$, $y_i' = b_i(x_i', \mathbf{x}_{-i}, \mathbf{y}_{-i})$, $y_i'' = b_i(x_i'', \mathbf{x}_{-i}, \mathbf{y}_{-i})$. Then $\lambda x_i' + (1-\lambda)\, x_i'' \in \mathrm{int}(X_i)$ and $\lambda y_i' + (1-\lambda) y_i' \in \mathrm{int}(Y_i)$, for any $\lambda \in]0,1[$ and in the light of $(\mathcal{A}_{F_i}\,2)$ we have:

$$\begin{aligned} &f_i(\lambda x_i' + (1-\lambda)x_i'', \mathbf{x}_{-i}, \lambda y_i' + (1-\lambda)y_i'', \mathbf{y}_{-i}) \\ &\quad \geq \lambda f_i(x_i', \mathbf{x}_{-i}, y_i', \mathbf{y}_{-i}) + (1-\lambda) f_i(x_i'', \mathbf{x}_{-i}, y_i'', \mathbf{y}_{-i}) \\ &\quad \geq \lambda f_i(x_i', \mathbf{x}_{-i}, \mathbf{y}) + (1-\lambda) f_i(x_i'', \mathbf{x}_{-i}, \mathbf{y}) \quad \text{for any } y_i \in Y_i, \end{aligned}$$

where the last inequality is a direct consequence of the definition of b_i, being $f_i(x_i', \mathbf{x}_{-i}, y_i', \mathbf{y}_{-i}) = f_i(x_i', \mathbf{x}_{-i}, b_i(x_i', \mathbf{x}_{-i}, \mathbf{y}_{-i}), \mathbf{y}_i) \geq f_i(x_i', \mathbf{x}_{-i}, \mathbf{y})$ for any $y_i \in Y_i$ and $f_i(x_i'', \mathbf{x}_{-i}, y_i'', \mathbf{y}_{-i}) = f_i(x_i'', \mathbf{x}_{-i}, b_i(x_i'', \mathbf{x}_{-i}, \mathbf{y}_{-i}), \mathbf{y}_i) \geq f_i(x_i'', \mathbf{x}_{-i}, \mathbf{y})$ for any $y_i \in Y_i$. That is, in the light of the convexity of $f_i(\,\cdot\,, \mathbf{x}_{-i}, \mathbf{y})$ on X_i:

$$\begin{aligned} &f_i(\lambda x_i' + (1-\lambda)x_i'', \mathbf{x}_{-i}, \lambda y_i' + (1-\lambda)y_i'', \mathbf{y}_{-i}) \\ &\quad \geq f_i(\lambda x_i' + (1-\lambda)x_i'', \mathbf{x}_{-i}, y_i, \mathbf{y}_{-i}) \quad \text{for any } y_i \in Y_i; \end{aligned}$$

and then $\lambda b_i(x_i', \mathbf{x}_{-i}, \mathbf{y}_{-i}) + (1-\lambda) b_i(x_i'', \mathbf{x}_{-i}, \mathbf{y}_{-i}) = b_i(\lambda x_i' + (1-\lambda) x_i'', \mathbf{x}_{-i}, \mathbf{y}_{-i})$, that is $b_i(\cdot, \mathbf{x}_{-i}, \mathbf{y}_{-i})$ is linear on X_i. $\diamond$

Remark 3.4 According to Proposition 3.1, an elementary example of functions f_i which satisfies Assumption $(\mathcal{A}_{F_i}\,1)$ in Theorem 3.1 is: $f_i(\mathbf{x}, \mathbf{y}) = \alpha_i(\mathbf{x}_{-i}, \mathbf{y}_{-i}) x_i + \gamma_i(\mathbf{x}_{-i}, \mathbf{y}_{-i}) g_i(y_i)$, where γ_i is nonnegative and g_i is concave on Y_i.

Remark 3.5 One can have $b_i(\cdot, \mathbf{x}_{-i}, \mathbf{y}_{-i})$ linear on X_i and $v_i(\cdot, \mathbf{x}_{-i}, \mathbf{y}_{-i})$ concave but not linear on X_i as shown by the function f_i defined on $[0,1]^4$ by $f_i(\mathbf{x}, \mathbf{y}) = y_i - y_i^2 - x_i^2 x_{-i} y_{-i}$. Then $b_i(\mathbf{x}, y_{-i}) = 1/2$ and $v_i(\mathbf{x}, y_{-i}) = 1/2 - x_i^2 x_{-i} y_{-i}$.

An example where both leaders' and followers' payoff functions are discontinuous but satisfy all the conditions required in Theorem 3.1 for the existence of an equilibrium under passive beliefs is the following.

Example 3.4 Assume $i \in \{1, 2\}$ and $X_i = Y_i = [0, 1]$.

- Let f_i be a real-valued function defined, for any $(\mathbf{x}, \mathbf{y}) \in [0, 1]^4$, by:

$$f_i(\mathbf{x}, \mathbf{y}) = \begin{cases} (1 - \frac{x_{-i} y_{-i}}{2})(x_i - y_i^2) + 2 & \text{if } y_i \neq 0 \\ 3 + x_i x_{-i} y_{-i} & \text{otherwise.} \end{cases}$$

  Then, f_i is not continuous at $(\mathbf{x}, 0, y_{-i})$, for any $(\mathbf{x}, y_{-i}) \in [0, 1]^3$, but it satisfies assumptions (i)–(ii) in $(\mathcal{C}_i)$ and $(\mathcal{A}_{F_i}\,2)$–$(\mathcal{A}_{F_i}\,3)$.

- Let l_i be a real-valued function defined for any $(\mathbf{x}, \mathbf{y}, t) \in [0, 1]^4 \times \mathbb{R}$ by:

$$l_i(\mathbf{x}, \mathbf{y}, t) = \begin{cases} x_{-i} y_{-i}^2 (1 - x_i^2) & \text{if } x_i \neq 0, \\ 1 + x_{-i} y_{-i} + y_i + t & \text{if } x_i = 0,\ t \geq 0, \\ 1 + x_{-i} y_{-i} + y_i & \text{if } x_i = 0,\ t < 0, \end{cases}$$

  Then, l_i is not continuous at $(0, x_{-i}, \mathbf{y}, t)$, for any $(x_{-i}, \mathbf{y}, t) \in [0, 1]^3 \times \mathbb{R}$ but it satisfies assumptions (iii)–(iv) in $(\mathcal{C}_i)$ and $(\mathcal{A}_{L_i}\,1)$. $\diamond$

In the next section we investigate a situation where the assumption of linearity of the optimal reaction function $b_i(\cdot, \mathbf{x}_{-i}, \mathbf{y}_{-i})$ can be relaxed.

4 Reinforcement of the vertical information structure

Now we assume to be in a situation which could be interpreted as a reinforcement of exclusivity between a leader and a corresponding follower: for any $i \in I$ the payoff function f_i does not depend on variable $\mathbf{x}_{-i}$ and the payoff function l_i does not depend on variable $\mathbf{x}$. Moreover, the actions sets of followers and leaders are assumed to be subsets of $\mathbb{R}$. That is, we assume the following:

$(\mathcal{K}_i)$ $\left\{ \text{the payoff function } f_i \text{ is defined on } X_i \times \mathbf{Y} \subset \mathbb{R}^{k+1} \text{ and the payoff function } l_i \text{ is defined on } \mathbf{Y} \subset \mathbb{R}^k. \right.$

In this case, weaker assumptions guarantee that an equilibrium exists.

Theorem 4.1. *Assume, for any $i \in I$, $(\mathcal{U}_i)$, $(\mathcal{K}_i)$, $(\mathcal{C}_i)$ and, for any $\mathbf{y}_{-i} \in \mathbf{Y}_{-i}$, the following:*
$(\mathcal{A}_{F_i} 4)$ $\quad b_i(\,\cdot\,, \mathbf{y}_{-i})$ is monotone (increasing or decreasing) on X_i;
$(\mathcal{A}_{L_i} 2)$ $\quad l_i(\,\cdot\,, \mathbf{y}_{-i})$ is quasiconcave on Y_i.
Then, there exists an equilibrium under passive beliefs of game Γ.

Proof Note that assumptions (iii)–(iv) in $(\mathcal{C}_i)$ now imply that l_i is a continuous function on $\mathbf{Y}$.
For $i \in I$, let E_i be the set-valued map from $\mathbf{Y}_{-i}$ to Y_i defined by $E_i(\mathbf{y}_{-i}) := \{b_i(x_i, \mathbf{y}_{-i}) \mid x_i \in M_i(\mathbf{y}_{-i})\}$, where M_i is given in (7). Define the set-valued map E from $\mathbf{Y}$ to $\mathbf{Y}$ by:

$$E(\mathbf{y}) := \prod_{i=1}^{k} E_i(\mathbf{y}_{-i}) = \{\mathbf{b}(\mathbf{x}, \mathbf{y})\colon \mathbf{x} \in M(\mathbf{y})\},$$

where $\mathbf{b}\colon \mathbf{X} \times \mathbf{Y} \to \mathbf{Y}$ is defined by $\mathbf{b}(\mathbf{x}, \mathbf{y}) := (b_1(x_1, \mathbf{y}_{-1}), \ldots, b_k(x_k, \mathbf{y}_{-k}))$ and M is the set-valued map from $\mathbf{Y}$ to $\mathbf{X}$ such that $M(\mathbf{y}) := \prod_{i=1}^{k} M_i(\mathbf{y}_{-i})$.
As in the proof of Theorem 3.1, the function b_i is continuous on $\mathbf{X} \times \mathbf{Y}_{-i}$ and the set-valued map M_i is nonempty-valued and closed and, being X_i compact, it is also compact-valued, for any $i \in I$. So, $\mathbf{b}$ is continuous on $\mathbf{X} \times \mathbf{Y}$ and $\mathbf{M}$ is a closed set-valued map with nonempty compact values.
Now, let $i \in I$. Taken $\mathbf{y}_{-i} \in \mathbf{Y}_{-i}$ and $\bar{x}_i, \bar{\bar{x}}_i \in X_i$, assume without loss of generality that $\bar{x}_i < \bar{\bar{x}}_i$. Let $\lambda \in]0,1[$ and let $\bar{y}_i = b_i(\bar{x}_i, \mathbf{y}_{-i})$ and $\bar{\bar{y}}_i = b_i(\bar{\bar{x}}_i, \mathbf{y}_{-i})$. Then, in the light of $(\mathcal{A}_{F_i} 4)$, $b_i(\lambda\bar{x}_i + (1-\lambda)\bar{\bar{x}}_i, \mathbf{y}_{-i})$ belongs to the segment between $\bar{y}_i$ and $\bar{\bar{y}}_i$, that is there exists $\delta \in [0,1]$ such that $b_i(\lambda\bar{x}_i + (1-\lambda)\bar{\bar{x}}_i, \mathbf{y}_{-i}) = \delta\bar{y}_i + (1-\delta)\bar{\bar{y}}_i$. From the quasiconcavity of the function $l_i(\,\cdot\,, \mathbf{y}_{-i})$ it follows that:

$$\begin{aligned}
\tilde{l}_i(\lambda\bar{x}_i + (1-\lambda)\bar{\bar{x}}_i, \mathbf{y}_{-i}) &= l_i(b_i(\lambda\bar{x}_i + (1-\lambda)\bar{\bar{x}}_i, \mathbf{y}_{-i}), \mathbf{y}_{-i}) \\
&= l_i(\delta\bar{y}_i + (1-\delta)\bar{\bar{y}}_i, \mathbf{y}_{-i}) \geq \min\{l_i(\bar{y}_i, \mathbf{y}_{-i}), l_i(\bar{\bar{y}}_i, \mathbf{y}_{-i})\} \\
&= \min\{\tilde{l}_i(\bar{x}_i, \mathbf{y}_{-i}), \tilde{l}_i(\bar{\bar{x}}_i, \mathbf{y}_{-i})\},
\end{aligned}$$

that is the function $\tilde{l}_i(\,\cdot\,, \mathbf{y}_{-i})$ is quasiconcave on X_i. So, the set-valued maps M_i and M are convex-valued. Hence, in the light of Theorem 2 in [2], the set–valued map E has a fixed point on $\mathbf{Y}$. Then, there exists an equilibrium under passive beliefs of Γ. Indeed, let $\mathbf{y}^* = (y_1^*, \ldots, y_k^*)$ be a fixed point of E on $\mathbf{Y}$. According to the definition of E, there exists $x_i^* \in M_i(\mathbf{y}_{-i}^*)$ such that $y_i^* = b_i(x_i, \mathbf{y}_{-i}^*)$, for any $i \in I$. Then, the strategy profile $(x_1^*, \ldots, x_k^*, \beta_1^*, \ldots, \beta_k^*)$, where $\beta_i^*(x_i) = b_i(x_i, \mathbf{y}_{-i}^*)$, for any $x_i \in X_i$ and for any $i \in I$, is an equilibrium under passive beliefs. ◇

Now, we look for sufficient conditions, explicit on the data, for $(\mathcal{A}_{F_i} 4)$. Starting from [31], the issue of the monotonicity of the optimal reaction functions has been investigated in the literature in many works. In the following, we use a result due to [4] that provides a characterization for the monotonicity of $b_i(\,\cdot\,, \mathbf{y}_{-i})$ on X_i when the payoff functions of the followers are in the class of upper semicontinuous strictly pseudoconcave real-valued functions. This result fits our purposes since the concept of strictly pseudoconcavity is considered in a generalized version which uses the Dini derivatives ([5]) and, therefore, is compatible with discontinuous payoff functions.

Definition 4.1. *A real-valued function g defined on an interval $I \subseteq \mathbb{R}$ is said* *to be strictly* pseudoconcave (in terms of Dini derivatives) *(in short strictly* D–pseudoconcave*) if:*

- $x, y \in I,\ x < y,\ g(x) \leq g(y)$ *implies* $D^+g(x) := \limsup_{h\to 0^+} \frac{g(x+h)-g(x)}{h} > 0$;
- $x, y \in I,\ x < y,\ g(x) \geq g(y)$ *implies* $D_-g(y) := \liminf_{h\to 0^-} \frac{g(y+h)-g(y)}{h} < 0$.

We will denote $\liminf_{h\to 0^-} \frac{f_i(x_i,y_i+h,\mathbf{y}_{-i})-f_i(x_i,y_i,\mathbf{y}_{-i})}{h}$ with $D_-f_i(x_i,\boldsymbol{\cdot},\mathbf{y}_{-i})(y_i)$, and $\limsup_{h\to 0^+} \frac{f_i(x_i,y_i+h,\mathbf{y}_{-i})-f_i(x_i,y_i,\mathbf{y}_{-i})}{h}$ with $D^+f_i(x_i,\boldsymbol{\cdot},\mathbf{y}_{-i})(y_i)$.

Furthermore, we make use of the following definition: it appears in this form in [4] though it is standard for real-valued functions (see, e.g., [10] in which the authors call it *quasimonotonicity*).

Definition 4.2. *An extended real-valued function g defined on $I \subseteq \mathbb{R}$ is* quasiincreasing *if $x, y \in I$, $x < y$ and $g(x) > 0$ implies $g(y) \geq 0$. An extended real-valued function g defined on $I \subseteq \mathbb{R}$ is* quasidecreasing *if $-g$ is quasiincreasing.*

Obviously an increasing function is also quasiincreasing but the vice versa may not be true.

Proposition 4.1. *Let $i \in I$, $\mathbf{y}_{-i} \in \mathbf{Y}_{-i}$ and assume:*

$(\mathcal{A}_{F_i}\,5)$
- *(i)* $f_i(x_i,\boldsymbol{\cdot},\mathbf{y}_{-i})$ *is strictly D–pseudoconcave and upper semicontinuous on Y_i, for any $x_i \in X_i$;*
- *(ii) the function: $x_i \in X_i \to D_-f_i(x_i,\boldsymbol{\cdot},\mathbf{y}_{-i})(y_i)$ is quasiincreasing (resp. quasidecreasing) on X_i, for any $y_i \in int(Y_i)$.*

Then, $b_i(\boldsymbol{\cdot},\mathbf{y}_{-i})$ is increasing (resp. decreasing) on X_i.

Proof Note that under Assumption (i) in $(\mathcal{A}_{F_i}\,5)$ the set-valued map B_i defined in (5) is single-valued since $f_i(x_i,\boldsymbol{\cdot},\mathbf{y}_{-i})$ is strictly quasiconcave on Y_i (see, e.g., Theorem 3.5 in [8]), so Assumption $(\mathcal{U}_i)$ is guaranteed.
If $D_-f_i(x_i,\boldsymbol{\cdot},\mathbf{y}_{-i})(y_i)$ is quasiincreasing on X_i the result follows directly from Theorem 2 in [4]. If $D_-f_i(x_i,\boldsymbol{\cdot},\mathbf{y}_{-i})(y_i)$ is quasidecreasing on X_i it is sufficient to reverse the product order of X_i and apply Theorem 2 in [4]. ⋄

When the payoff function of a follower is differentiable in his own action, the strict pseudoconcavity in terms of Dini derivatives coincides with the known concept of strict pseudoconcavity introduced in [18] and Proposition 4.1 can be expressed as follows.

Corollary 4.1. *Let $i \in I$ and $\mathbf{y}_{-i} \in \mathbf{Y}_{-i}$. Assume $f_i(x_i,\boldsymbol{\cdot},\mathbf{y}_{-i})$ differentiable on $int(Y_i)$, for any $x_i \in X_i$, and*

$(\mathcal{A}_{F_i}\,5)'$
- *(i)* $f_i(x_i,\boldsymbol{\cdot},\mathbf{y}_{-i})$ *is strictly pseudoconcave on Y_i, for any $x_i \in X_i$;*
- *(ii) the function $\frac{\partial f_i}{\partial y_i}(\boldsymbol{\cdot},y_i,\mathbf{y}_{-i})$ is quasiincreasing (resp. quasidecreasing) on X_i, for any $y_i \in int(Y_i)$.*

Then, $b_i(\boldsymbol{\cdot},\mathbf{y}_{-i})$ is increasing (resp. decreasing) on X_i.

Remark 4.1 According to Corollary 4.1, an elementary example of functions f_i which satisfies Assumption $(\mathcal{A}_{F_i}\,4)$ in Theorem 4.1 is: $f_i(x_i,\mathbf{y}) = \alpha_i(x_i,\mathbf{y}_{-i})g_i(\mathbf{y})$, where α_i is a positive function on $X_i \times \mathbf{Y}_{-i}$ and $g_i(\boldsymbol{\cdot},\mathbf{y}_{-i})$ is differentiable on $\text{int}(Y_i)$ and strictly pseudoconcave on Y_i, for any $\mathbf{y}_{-i} \in \mathbf{Y}_{-i}$.

We conclude this section with an example of a game for which we can infer the existence of an equilibrium under passive beliefs by applying Theorem 4.1 and Proposition 4.1.

Example 4.1 Consider, as in [23], a modified version of Example 2.1: now the leaders offer a two-part tariff contract that specifies a wholesale price and a franchise fee to their exclusive followers that decide in a competitive setting the retail price after observing the contract. The leaders are assumed to be able to extract the whole surplus from their followers through the franchise fee (i.e. the participation constraints of the retailers are binding). The multi–leader/follower problem now to solve is such that: $X_i = Y_i = [0, 3]$, $l_i(\mathbf{y}) = D_i(\mathbf{y})y_i$ and $f_i(x_i, \mathbf{y}) = D_i(\mathbf{y})(y_i - x_i)$, where the demand function is assumed to be a *kinked demand* defined by:

$$D_i(\mathbf{y}) = \begin{cases} 2 - 2y_i + y_{-i}, & \text{if } y_i \in [0, \frac{1}{3}] \\ \frac{3}{2} - \frac{y_i}{2} + y_{-i}, & \text{if } y_i \in]\frac{1}{3}, 3]. \end{cases}$$

The profit function f_i is continuous on its domain. Since $D_- f_i(x_i, \cdot\,, y_{-i})(1/3) = 2x_i + y_{-i} + 2/3$ and $D^+ f_i(x_i, \cdot\,, y_{-i})(1/3) = \frac{x_i}{2} + y_{-i} + 7/6$, $f_i(x_i, \cdot\,, y_{-i})$ is not differentiable at $y_i = 1/3$ for all $x_i \neq 1/3$. Furthermore, it can be proven that $f_i(x_i, \cdot\,, y_{-i})$ is strictly D–pseudoconcave (but not strictly concave for $x_i < 1/3$) and the function $x_i \in [0, 1] \to D_- f_i(x_i, \cdot\,, y_{-i})(y_i)$ is quasiincreasing on X_i. Analogously, $l_i(\cdot\,, y_{-i})$ is strictly D–pseudoconcave, so $l_i(\cdot\,, y_{-i})$ is quasiconcave on Y_i by Theorem 3.5 in [8]. Therefore, the assumptions in Theorem 4.1 and Proposition 4.1 are satisfied.
Note that we cannot apply results in [23] since the demand function is neither differentiable nor strictly concave. ◇

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