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Coalitional Fairness with Participation Rates

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Abstract

This paper investigates coalitional fairness in pure exchange economies with asymmetric information. We study allocations of resources which are immune from envy when comparisons take place between coalitions. The model allows negligible and non-negligible traders, only partially informed about the true state of nature at the time of consumption, to exchange any number, possibly infinite, of commodities. Our analysis is based on the Aubin approach to coalitions and cooperation, i.e. on a notion of cooperation allowing traders to take part in one or more coalitions simultaneously employing only shares of their endowments (participation rates). We introduce and study in detail the notion of coalition fairness with participation rates (or Aubin c-fairness) and show that flexibility in cooperation permits to recover the failure of fairness properties of equilibrium allocations. Our results provide applications to several market outcomes (ex-post core, fine core, ex-post competitive equilibria, rational expectations equilibria) and emphasize the consequences of the convexification effect due to participation rates for models with large traders and infinitely many commodities.

JEL Classification: C71, D51, D82

Keywords: Aubin coalitions; Fairness; Asymmetric information; Core; Rational expectations equilibria; Lyapunov convexity theorem

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1 Introduction

This paper focuses on coalition fairness in exchange economies with uncertainty and asymmetric information where small, negligible traders coexist with influential, large agents. We shall refer to such an economy as a *differential information mixed economy* (DIME, for short) and, in order to deal with infinite time horizons or commodity differentiation, we shall assume that in a DIME both types of traders may exchange any number, possibly infinite, of commodities. Our aim is to establish conditions for non envy between coalitions under an allocation of resources, when the economy is far from being perfectly competitive because of the presence of monopolies or oligopolies (large traders) and/or because agents are not symmetrically informed about the true state of nature at the time of consumption. The analysis, based on the Aubin approach to coalition formation, aims to show that flexibility in cooperation deriving from participation rates permits: 1. to solve market imperfections due to the presence of large traders and asymmetric information; 2. to provide the convexification effect necessary to deal with coalition net trades of many commodities; 3. to produce, as consequence of 1. and 2., characterizations of equilibrium allocations in terms of fairness.

In the first part of the paper, we analyze fairness of allocations at the ex-post stage. This is done requiring that, under a given resource allocation, no coalition envies the net trade of any other whatever is the realization of uncertainty at the time of consumption. Then, we impose coalition fairness at the interim stage. In this case, we presume that traders can share their information in a potentially envious (or blocking) coalition. Consequently, we require that, under a given allocation, no coalition envies the net trade of any other one, in an event which the members of the coalition can jointly discern after partial or full communication. We notice that, in both cases, due to the presence of market imperfections, not only core-like stability properties are no longer enough to characterize competitive equilibria, but also strong requests of coalition fairness stability do not suffice. Then we examine whether it is possible to enhance classical results valid for perfectly competitive economies (see [34] and [17]) considering the contributions of [13] and [14] about ex-post, fine core and rational expectations equilibria and assuming in the model cooperation by means of participation rates.

Let us recall classical results from the existing literature. The study of coalition fairness was initiated by [34] and [17] (see also [35] and [33]). In these works various concepts of fairness are introduced for complete information economies with finitely many commodities and several versions of the equivalence between fair and competitive allocations are established. In particular, the notion of envy is extended in [34] from individuals to coalitions, taking into account the possibility that a coalition may envy the aggregate resources of another one, simply because these resources would be compatible with a redistribution within the coalition making each of its members better off. It is proved in [34] that, starting from a symmetric equal sharing of the initial endowment among traders, the only way to produce coalition fair allocations is via competitive equilibrium mechanisms. This type of characterization, because of its intrinsic symmetric requirement, may not be fully satisfactory in the setting of mixed and differential information economies. Indeed, in these models, agents' positions in a competitive equilibrium may remain asymmetric even though the initial total resources are equally shared among them. This is due for example to differences in their economic weight and/or private information that, in turn, affect their budget sets. By contrast, the notion of coalition fairness introduced in [17] assesses fairness of allocations in terms of net trades of coalitions and produces a characterization of competitive equilibria even when the equal income hypothesis is dropped. Then, as it comes also from [12], it may be more profitable in our framework.

Following [17], an allocation is said to be coalitionally fair (c-fair) if no coalition can benefit, after a redistribution among its members, from the aggregate net trade of some other disjoint coalition. This notion formalizes a request of stability for the market deriving from the fact that no coalition envies the net trade of any other. It is an easy consequence of definition that c-fair allocations include competitive allocations and are included in core allocations. Hence, under perfect competition, as consequence of the core equivalence theorem, c-fair allocations are the same as core and competitive allocations, while a counterexample is provided in [17] to show that the inclusions may be strict when large traders act on the market. This latter remark makes of course the study of c-fair allocations of specific interest in a model like the one we propose, where, due to the presence of large traders, there is no guarantee that core allocations can be decentralized by prices. As a further point of interest for the analysis, we notice that in mixed markets the absence of envy does not necessarily translate from coalitions to individuals. Precisely, individual equitability of c-fair allocations may fail due to the presence of large traders as well as to the exchange of infinitely many commodities[1] so that it may be the case that envy arises at individual level though the allocation of resources is c-fair.

The key tool introduced in the paper to overcome the above mentioned difficulties is the possibility that traders may take advantage from Aubin cooperative behavior. Since the middle of seventies (see [4]), a solid literature[2] has established that the investigation of outcomes of real markets in the light of Aubin approach to cooperation, permits to extend several results concerning equilibria and optimal allocations to economies that are not necessarily perfectly competitive and/or that allow for infinitely many commodities (see [9]). By Aubin cooperative approach (personalized, or differentiated, participation) it is meant that an economic agent participates in a coalition according to a personal participation rate. Moreover, the trader's participation rate may change coalition by coalition and, as a further element of flexibility, it does not exclude the simultaneous participation of traders in envious and envied coalitions. The standard interpretation of such a rate is that it represents the coefficient (not necessarily 0 or 1) of resources that the agent wishes to invest into the coalition. Considerably, it comes out that a c-fair notion allowing participation rates permits to state also in the general case of a DIME a complete characterization of core and competitive allocation in terms of non-envy between coalitions. Moreover, it may give rise to a non-envy interpretation of coalition fairness on the individual base. Roughly speaking, from a technical point

[1] This failure is technically due to the fact that the convexifying effect of Lyapunov's theorem doesn't apply even when we are in the case of a continuum of traders.

[2] [16], [25], [20] and [22] are just few items along the most recent arc of two decades.

of view, the participation rates are capable to produce on the set of coalitions and on the range of net trades a convexification effect sufficient to compensate for both the presence of atoms and that of many commodities. The use of personalized participation is shown to be relevant only for large traders.

It should be noticed that, at the level of the analysis provided in the first part of the paper, the information structure does not affect results since they are formulated referring to the ex-post stage. This part of the paper produces, as a particular case, consequences for complete information economies. In the second part of the paper, c-fairness is studied at the interim stage with reference to (Wilson) fine allocations and rational expectations equilibria. This means that the information structure enters into the model and affects fairness and competitive equilibrium notions.

Since in our model uncertainty in resources and in signal in each state do not depend on agents' actions, it becomes relevant to reduce as much as possible uncertainty effect on agents welfare requiring ex-post coalitional (as well as individual) non envy also in the case in which trades take place at the interim stage. Under the assumption that agents can share their information in the relevant coalitions (the envious ones or the blocking ones), we see that the corresponding fine (c-fair or core) allocations do not necessarily satisfy c-fairness properties at the ex-post stage. This failure may be determined by the presence of large traders, but also by the presence of many commodities. This is clear looking at the existing corresponding literature in the case of the core: in [14] it is proved that the fine core is a subset of the ex-post core, i.e. core stability is preserved moving from the interim to the ex-post stage. Moreover, assuming that traders derive the same level of utility in those states that they are not able to distinguish (i.e. measurability of utilities), one has that the ex-post core coincides with the set of rational expectations equilibria (see [13]). Then, as consequence, it is obtained in [14] that every fine core allocation is a rational expectations equilibrium, but also that every fine core allocation is ex-post c-fair, given the core equivalence theorem valid in large economies.

We must notice at this point that the economic framework of [14] is that of a differential information economy with a finite dimensional commodity space and where only small traders act. In other words, for the validity of the previous results, the role played by the assumption of an atomless economy is twice pivotal: first, because it is an atomless space of agents that permits to apply (when the commodity space is finite dimensional) Vind's Theorem[3]; second, because it is only under non-atomicity that an equivalence theorem between rational expectations equilibria and ex-post coalition fair allocations can be obtained.

We show that the use of participation rates permits to apply a special form of Vind's Theorem, despite the fact that our commodity space is infinite dimensional, in order to generalize the approach of [14] and to preserve stability of interim choices after the realization of uncertainty. The main idea is to prove that, by means of a differentiated participation, each trader may be included in an envious (or in a blocking) coalition. This makes each envious coalition comparable, in terms of information, to an arbitrary

[3]The Vind's Theorem comes out to be necessary in [14] in order to block a non-core allocation by means of a coalition of arbitrarily big measure and, consequently, to obtain that a fine core allocation is also an ex-post core allocation.

large one and then permits to generate from each ex-post improvement a corresponding fine one.

On the other hand, participation rates permit to extend core equivalence results ensuring, also in our model, a complete characterization of competitive equilibria (ex-post or rational expectations) in terms of ex-post c-fair allocations, a result comparable with the one provided in the classical paper by [17] under perfect competition. We should notice, however, that this characterization is obtained, in some cases, at the strong cost of measurability assumptions on utility functions. This is so because, at the interim stage, the inclusion of rational expectations equilibrium allocations in the set of ex-post c-fair allocations and, a fortiori, in the set of ex-post c-fair allocations may fail[4].

To sum up, the features of our main results and proofs show that the use of personalized participation rates permits to nullify at several instances the presence of large traders, but also to apply the Lyapunov's convexity theorem (on which the Vind's result is in turn based) in a proper way.

The plan of the paper is as follows. After the presentation of differential information mixed economies in Section 2, Section 3 introduces Aubin coalitional fairness (à la Gabszewicz, [17]) in a DIME. This part of the paper covers the investigation of ex-post coalition fairness[5]. Thanks to a series of preliminary results (Subsection 3.1), we are able to introduce a definition, Definition 3.4, which is coherent with the personalized participation approach since it allows traders to participate in more coalitions, only subject to not over using their endowment. We note that in the notion of c-fairness with participation rates, one can always assume that small traders use either all or nothing of their endowment (standard participation) and we prove the following results: 1. c-fair allocations with participation rates are, under suitable assumptions, individual envy-free (Subection 3.4); 2. c-fair allocations with participation rates are robust to the embedding of the original economy into the atomless economy obtained by splitting each atom over an interval of negligible traders (Subsection 3.5). With Subsection 3.6, we begin the comparison of coalition fairness with other optimality notions like the ex-post core. Under suitable assumptions, we provide a direct proof of the equivalence between ex-post core and c-fair allocations, a proof not related to core equivalence results. This point is relevant in the case of economies with many commodities since we cover the case in which the commodity space needs not to be separable and, consequently, the range of applicability of our result is wider (see the discussion in Remark 3.23).

In Section 4, the investigation of c-fairness is extended considering the fine blocking mechanism. In this Section, the relation between the different solution concepts depends on the structure of information present in a coalition. Fine c-fairness and the corresponding fine core with participation rates (resulting to be same as fine core for atomless economies, Theorem 4.6) are analyzed. Theorem 4.3 gives conditions under which the fine c-fair allocations are also ex-post c-fair. The same conditions ensure the inclusion of the fine core in the ex-post core, despite the presence of large traders and infinitely many commodities (Theorem 4.7).

[4]See [14, Example 4.2], where, in particular, it is pointed out the relevance of measurability of utility functions in the comparison between the two notions of equilibrium.

[5]Note that in [12] the interim, finite dimensional, essentially atomless case is studied.

Section 5 investigates ex-post c-fairness of allocations emerging from competitive market mechanisms. In this Section, rational expectations equilibrium allocations are introduced and Theorem 5.6 presents conditions under which c-fairness with participation rates fully characterizes these allocations, that is the conditions under which it is still true that only the competitive equilibrium mechanism produces a fair allocation for each possible realization of uncertainty.

Concluding remarks and a brief discussion of coalition fairness with participation rates defined in the spirit of [34] are in Section 6. A short appendix concludes.

2 The economy

The framework of this paper is that of a *pure exchange mixed economy with asymmetric information*. The set of agents is denoted by T. We assume that the agents may form coalitions whose *economic weight*, or *influence*, on the market is represented by means of a measure μ defined on a σ-algebra $\mathcal{T}$ of eligible coalitions. As it is standard in the literature, according to the atomless-atomic decomposition of measures, T is partitioned into an atomless set T_0 and a set $T_1 = T \setminus T_0$ which is the union of an at most countable family $\{A_1, A_2, \ldots, A_k, \ldots\}$ of disjoint μ-atoms. The set T_0 is representative of the uninfluential (or *small*) traders; the family $\{A_1, A_2, \ldots, A_k, \ldots\}$ represents the influential or (*large*) traders. With an abuse of notation, we use the same symbol T_1 to denote the collection $\{A_1, A_2, \ldots, A_k, \ldots\}$, as well.

The assumption that the space of agents is an arbitrary complete finite measure space $(T, \mathcal{T}, \mu)$ is helpful to cover simultaneously different situations. Indeed, the standard case of a finite economy corresponds to the specification of a finite T with the measure μ equal to the counting measure. The case of a perfectly competitive economy can be considered by choosing a nonatomic measure space $(T, \mathcal{T}, \mu)$ of agents (sometimes, even more specifically, T is the interval [0,1] endowed with the Lebesgue measure). Finally the case of *mixed markets*, which allows to analyze market outcomes resulting from the interaction of "an ocean" of uninfluential agents (the price takers) with "some" influential ones (the oligopolies), corresponds to the case in which both sets T_0 and T_1 have positive μ-measure.

For what concerns the commodity space, we identify physical commodities with elements in the positive cone $\mathbb{B}_+$ of an ordered Banach space $\mathbb{B}$. The generality of $\mathbb{B}$ gives the possibility to take into account different kinds of models. For example those allowing for infinite variations of the goods' characteristics, or those considering an infinite time horizon.

The uncertainty about *"nature"* at time of consumption (or when the contracts are implemented) is, as usual, described by a measurable space $(\Omega, \mathcal{F})$. By Ω we denote the set of all possible states of nature. The algebra $\mathcal{F}$ represents the family of all possible events. We assume that $\mathcal{F}$ is generated by a partition Π of subsets of Ω. A partition is assumed to be finite, per se. Due to uncertainty, agents decisions concern random commodities $x \in {\mathbb{B}_+}^{\Omega}$. Of course, at the time of consumption, if ω is the realized state of nature, what is physically consumed is $x(\omega) \in \mathbb{B}_+$.

Agent t compares consumption under different states by means of a state-dependent utility function u_t representing his preference:

$$u_t \; : \; \Omega \times \mathbb{B}_+ \longrightarrow \mathbb{R}.$$

In the sequel, we shall refer to standard continuity, monotonicity, quasi-concavity assumptions dealing with the functions $u_t(\omega, \cdot)$ (see for example, [2]). Moreover, the function $u_t(\cdot, x)$ is $\mathcal{F}$-measurable for all $t \in T$ and for all $x \in \mathbb{B}_+$. Obviously, the $\mathcal{F}$-measurability just means that $u_t(\cdot, x)$ is constant over any single element of Π.

Throughout the paper it is assumed that all state-dependent functions are $\mathcal{F}$-measurable.

Agents do not have the same information regarding the states of the world. First, different agents t may have different prior beliefs $\mathbb{P}_t \gg 0$ on $(\Omega, \mathcal{F})$. Second, t has at his own disposal a private information represented by an information algebra $\mathcal{F}_t$ generated by a partition Π_t of Ω. Π_t consists of elements of $\mathcal{F}$ (therefore, the partition Π_t is coarser than Π) and according to the usual interpretation: if the realized state is ω, then trader t observes $\Pi_t(\omega)$, that denotes the unique element of Π_t containing ω; so the only information available to t is that the event $\Pi_t(\omega)$ prevails, without any possibility to distinguish among states belonging to the same element of the partition. When trading takes place, agent t trades with this information.

The last element to be introduced for a complete description of the economy is the initial endowment of any agent t. This is the state-dependent function

$$e_t : \Omega \longrightarrow \mathbb{B}_+$$

which is assumed to be $\mathcal{F}_t$-measurable. Moreover, for each $\omega \in \Omega$, the function $t \in T \to e_t(\omega) = e(\omega, t) \in \mathbb{B}_+$ is assumed to be μ-integrable in the sense of Bochner and such that **the total initial endowment of the economy,** $\int_T e_t(\omega)\, d\mu$**, is a strictly positive vector**[6]. We always write integrability to mean Bochner integrability. Finally, as it is standard, we assume the $(\mathcal{T} \otimes \mathcal{B})$-measurability of the mapping $(t, x) \mapsto u_t(\omega, x)$ for any $\omega \in \Omega$, where $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{B}_+$.

We denote by $\mathcal{E}$ our economy, i.e. the following collection:

$$\mathcal{E} = \{\, (T, \mathcal{T}, \mu);\; \mathbb{B};\; (\Omega, \mathcal{F});\; (u_t,\; \mathbb{P}_t,\; \mathcal{F}_t, e_t)_{t \in T} \,\}$$

to which all assumptions described above apply. This model of economy is known in the literature as *differential information economy* (DIE, for short). To stress the role of the interaction of small and large traders, we also refer to $\mathcal{E}$ as a differential information mixed economy (shortened as DIME in this case)

Together with $\mathcal{E}$ we shall consider a family of complete information economies $\{\mathcal{E}(\omega) : \omega \in \Omega\}$ obtained by fixing for each state ω the utility functions $(u_t(\omega, \cdot))_{t \in T}$ and the initial endowment $e(\omega, \cdot)$. Such a family plays a certain role in the paper since we privilege the ex-post approach. The ex-post approach is not affected by the structure of private information. On the contrary, this structure will be central in Sections 4 and 5, where all the features of our model, in particular the elements $\mathcal{F}_t$ and $\mathbb{P}_t$, will be used.

[6]See [1] for basic definitions in ordered Banach spaces.

3 Coalition fairness of allocations: the ex-post stage

In this section we shall study equilibria taking into account personalized participation of agents in coalitions or, as we can also say after [4], the Aubin approach to cooperation (see [16], [25]; several other references are available in [20]). By *personalized participation* we mean that for any agent t, a number $\gamma_t = \gamma(t) \in [0,1]$ is given in order to represent the personal proportion of resources that t wants to invest into the coalition γ. Following this line of investigation, we shall use here Aubin coalitions in order to exhibit coalition fairness properties of different classes of allocations emerging from market mechanisms.

Let $\mathcal{E}$ be a differential information economy. We shall call *assignment* a function x associating to each agent, in any state of the world, an element of the consumption set, i.e. a function

$$x \,:\, \Omega \times T \longrightarrow I\!\!B_+,$$

such that, for a. e. $t \in T$, the partial function $x_t := x(\cdot, t)$ is $\mathcal{F}$-measurable and, for each $\omega \in \Omega$, the partial function $x_\omega := x(\omega, \cdot)$ is μ-integrable on T.

A state by state feasible assignment, namely an assignment such that for each $\omega \in \Omega$, one has

$$\int_T x_\omega \, d\mu \leq \int_T e_\omega \, d\mu,$$

is called an *allocation*.

We recall that an *Aubin coalition*[7] γ is a μ-measurable function $\gamma \,:\, T \longrightarrow [0,1]$. For an Aubin coalition γ, its support, i.e. the set $\{t \in T : \gamma(t) > 0\}$, is denoted by S. **Throughout the article, γ and S are linked symbols and each inherits possible indices of the other**. We denote by $\mathcal{A}$ the set of all Aubin coalitions. Moreover, we unify integral notation by setting $\displaystyle\int_\gamma y(t)\, d\mu(t) := \int_S \gamma(t) y(t)\, d\mu(t)$, for an integrable function y. We extend the weight μ of ordinary coalitions to $\mathcal{A}$ by setting $\widetilde{\mu}(\gamma) := \displaystyle\int_\gamma 1\, d\mu$. Clearly γ is non-negligible (i.e. $\widetilde{\mu}(\gamma) > 0$) if and only if S is not μ-negligible.

3.1 Preliminary results

For γ_1, $\gamma_2 \in \mathcal{A}$, for two assignments x and y and a state of nature ω_0, we consider the following list of properties which will be useful in the sequel:

0) γ_1 is non-negligible;

0′) $\mu(S_1) = \mu(T)$, i.e. γ_1 is (essentially) of full support;

1) $\displaystyle\int_{\gamma_1} \big[y(\omega_0, t) - e(\omega_0, t)\big]\, d\mu \leq \int_{\gamma_2} \big[x(\omega_0, t) - e(\omega_0, t)\big]\, d\mu$, (net trade feasibility);

[7] *Generalized* or *fuzzy* in some literature.

2) $\gamma_1 + \gamma_2 \leq 1$ μ-a.e. on T, i.e. (γ_1, γ_2) is an admissible pair;

2′) $\mu(S_1 \cap S_2) = 0$, i.e. γ_1 and γ_2 are (essentially) disjoint[8];

3) $u_t(\omega_0, y(\omega_0, t)) > u_t(\omega_0, x(\omega_0, t))$ for a.a. $t \in S_1$, (envy of S_1).

4) $\gamma_i = 1$, μ - a.e. on the atomless part $S_i \cap T_0$ of S_i, for both $i = 1, 2$ (full participation of small agents).

When there is no ambiguity about the fixed state of nature ω_0 under consideration, we shorten conditions 1) and 3), respectively, as $\int_{\gamma_1} (y-e)\, d\mu \leq \int_{\gamma_2} (x-e)\, d\mu$ and $y \succ_{S_1} x$.

Prior to the introduction of our *ex-post* notion of coalition fairness, we show next that, given a state of nature ω_0, the following three sets of allocations

$$\chi_1 = \{x : \gamma_1\,,\ \gamma_2 \in \mathcal{A},\ \text{and an assignment } y \text{ exist such that 0), 1) and 3) are fulfilled}\}\,,$$

$$\chi_2 = \{x : \gamma_1\,,\ \gamma_2 \in \mathcal{A},\ \text{and an assignment } y \text{ exist such that 0), 1), 2) and 3) are fulfilled}\}\,,$$

$$\chi_3 = \{x : \gamma_1\,,\ \gamma_2 \in \mathcal{A},\ \text{and an assignment } y \text{ exist such that 0), 1), 2), 3) and 4) are fulfilled}\}\,,$$

coincide under suitable assumptions. Since the above sets are clearly each a superset of the subsequent, we study the inclusion $\chi_3 \supseteq \chi_1$.

The assumptions in the three results below refer to a given state of nature or, what is clearly the same, to the case of an economy without uncertainty.

Lemma 3.1. *Assume that μ-a.a. the initial endowments e_t are strictly positive and the utilities u_t are continuous and increasing. Take for $x \in \chi_1$, the coalitions γ_1, γ_2 and the assignment y such that $\int_{\gamma_1} (y - e)\, d\mu \leq \int_{\gamma_2} (x - e)\, d\mu$, $\mu(S_1) > 0$ and $y \succ_{S_1} x$. Then it is possible to find new coalitions $\widetilde{\gamma}_i$ and a new assignment $\widetilde{y}$, such that: the supports S_i are unchanged, $\widetilde{y} \succ_{S_1} x$ still holds, the assignment $\widetilde{y}$ is strictly positive on a subset of S_1 of positive measure and $\int_{\widetilde{\gamma}_1} (\widetilde{y} - e)\, d\mu \ll \int_{\widetilde{\gamma}_2} (x - e)\, d\mu$. Moreover, if $\gamma_1 + \gamma_2 \leq 1$, then also $\widetilde{\gamma}_1 + \widetilde{\gamma}_2 \leq 1$.*

PROOF: First observe that, clearly, by continuity of preferences, whenever $y \succ_A x$ over a set A of positive measure, then we can find a subset B of A of positive measure, and an $\varepsilon \in]0, 1[$ such that $\varepsilon y \succ_B x$.

Due to this remark, let us take a set $C \subseteq S_1$ and ε with $\varepsilon y \succ_C x$. Then consider the following identity:

[8]The standard fuzzy intersection $\gamma_1 \cap \gamma_2$ of the coalitions γ_i is non-negligible.

$$\int_T \gamma_1(y-e)\,d\mu = \int_{S_1\setminus C} \gamma_1 y\,d\mu + \int_C \frac{\gamma_1}{\varepsilon}\left[\varepsilon y + (1-\varepsilon)e\right]d\mu - \left(\int_{S_1\setminus C} \gamma_1 e\,d\mu + \int_C \frac{\gamma_1}{\varepsilon} e\,d\mu\right)$$

that can be written as

$$\int_T \gamma_1(y-e)\,d\mu = \int_{S_1} \hat{\gamma}_1(\hat{y}-e)\,d\mu$$

where $\hat{\gamma}_1$ and $\hat{y}$ only differ from γ_1 and y on the set C where they take, respectively, the values $\frac{\gamma_1}{\varepsilon}$ and $\varepsilon y + (1-\varepsilon)e$.

By setting $\widetilde{\gamma}_1 = \varepsilon\hat{\gamma}_1$ and $\widetilde{\gamma}_2 = \varepsilon\gamma_2$, we have $\hat{y} \succ_{S_1} x$, $\hat{y} \gg 0$ over the set C and $\int_{\widetilde{\gamma}_1} \big(\hat{y}-e\big)\,d\mu \leq \int_{\widetilde{\gamma}_2} \big(x-e\big)\,d\mu$.

Finally, again by continuity, take a set $B \subseteq C$ and $\delta \in]0,1[$ with $\delta\hat{y} \succ_B x$. By setting $\widetilde{y}$ as equal to $\hat{y}$ except over B where $\widetilde{y} = \delta\hat{y}$, we still have $\widetilde{y} \succ_{S_1} x$ but now $\int_{\widetilde{\gamma}_1} \big(\widetilde{y}-e\big)\,d\mu \ll \int_{\widetilde{\gamma}_2} \big(x-e\big)\,d\mu$. Indeed,

$$\int_{\widetilde{\gamma}_1} \big(\widetilde{y}-e\big)\,d\mu = \int_{S_1\setminus B} \widetilde{\gamma}_1(\hat{y}-e)\,d\mu + \int_B \widetilde{\gamma}_1(\delta\hat{y}-e)\,d\mu \ll \int_{S_1} \widetilde{\gamma}_1(\hat{y}-e) \leq \int_{\widetilde{\gamma}_2} \big(x-e\big)\,d\mu.$$

□

Proposition 3.2. *Let the interior of $\mathbb{B}_+$ be nonempty. Assume further that μ-a.a. the initial endowments e_t are strictly positive and the utilities u_t are continuous and increasing. Then: $\chi_1 \subseteq \chi_2$.*

PROOF: Take $x \in \chi_1$. According to Lemma 3.1, we can assume, without loss of generality, that

$$\int_{S_1} \gamma_1\big(y-e\big)\,d\mu \ll \int_{S_2} \gamma_2\big(x-e\big)\,d\mu.$$

Since γ_1 and γ_2 can be uniformly approximated by simple functions, we can find finitely many valued coalitions $\widetilde{\gamma}_i$, with support S_i, still satisfying $\int_{S_1} \widetilde{\gamma}_1\big(y-e\big)\,d\mu \ll \int_{S_2} \widetilde{\gamma}_2\big(x-e\big)\,d\mu$. By replacing $\widetilde{\gamma}_i$ with $\frac{\widetilde{\gamma}_i}{\bar{\gamma}}$ where $\bar{\gamma}$ is the maximum of $\widetilde{\gamma}_1 + \widetilde{\gamma}_2$ we see that $x \in \chi_2$. □

Proposition 3.3. *Let the interior of $\mathbb{B}_+$ be nonempty. Assume further that μ-a.a. the initial endowments e_t are strictly positive and the utilities u_t are continuous and increasing. Then: $\chi_2 \subseteq \chi_3$.*

PROOF: Take $x \in \chi_2$. According to Lemma 3.1, we can directly assume that

$$\int_{S_1} \gamma_1(y-e)\, d\mu \ll \int_{S_2} \gamma_2(x-e)\, d\mu.$$

Let T_0 be the atomless component of T and T_1 its complement. Consider a suitable radius 2ε such that the ball centered at the point

$$\left[\int_{S_1} \gamma_1(y-e)\, d\mu - \int_{S_2} \gamma_2(x-e)\, d\mu\right]$$

is contained in the interior of $I\!B_-$. By [6, Proposition 2],

$$\overline{co}\left\{\int_C (y-e)\, d\mu : C \subseteq S_1 \cap T_0\right\} = \left\{\int_{S_1 \cap T_0} \gamma(y-e)\, d\mu : \gamma \in \mathcal{A},\ supp\gamma \subseteq S_1 \cap T_0\right\},$$

$$\overline{co}\left\{\int_C (x-e)\, d\mu : C \subseteq S_2 \cap T_0\right\} = \left\{\int_{S_2 \cap T_0} \gamma(x-e)\, d\mu : \gamma \in \mathcal{A},\ supp\gamma \subseteq S_2 \cap T_0\right\},$$

and the application of the approximate Lyapunov's theorem, we find sets $G_i \subseteq S_i \cap T_0$ such that

$$\left\|\int_{S_1 \cap T_0} \gamma_1(y-e)\, d\mu - \int_{G_1} (y-e)\, d\mu\right\| < \varepsilon$$

and

$$\left\|\int_{S_2 \cap T_0} \gamma_2(x-e)\, d\mu - \int_{G_2} (x-e)\, d\mu\right\| < \varepsilon$$

from which we have

$$\int_{G_1} (y-e)\, d\mu + \int_{S_1 \cap T_1} \gamma_1(y-e)\, d\mu \ll \int_{G_2} (x-e)\, d\mu + \int_{S_2 \cap T_1} \gamma_2(x-e)\, d\mu.$$

In the case the sets G_i are essentially disjoint, we have obtained that $x \in \chi_3$. So we must consider the possibility that the above two sets are not disjoint. For this case, let ε be such that the ball centered at

$$\frac{1}{2}\left[\int_{G_1} (y-e)\, d\mu + \int_{S_1 \cap T_1} \gamma_1(y-e)\, d\mu - \int_{G_2} (x-e)\, d\mu - \int_{S_2 \cap T_1} \gamma_2(x-e)\, d\mu\right]$$

is included into the interior of $I\!B_-$. Again by Liapunov's Theorem, we can consider the following sets

$$B_1 \subseteq G_1 \setminus G_2 \text{ such that } \left\|\int_{B_1} (y-e)\, d\mu - \frac{1}{2}\int_{G_1 \setminus G_2} (y-e)\, d\mu\right\| \leq \frac{\varepsilon}{4}$$

$$B_2 \subseteq G_2 \setminus G_1 \text{ such that } \left\|\int_{B_2} (x-e)\, d\mu - \frac{1}{2}\int_{G_2 \setminus G_1} (x-e)\, d\mu\right\| \leq \frac{\varepsilon}{4}$$

$B_3 \subseteq G_1 \cap G_2$ such that

$$\left\| \int_{B_3} (y-e)\, d\mu - \frac{1}{2} \int_{G_1 \cap G_2} (y-e)\, d\mu \right\| \le \frac{\varepsilon}{4}$$

and

$$\left\| \int_{B_3} (x-e)\, d\mu - \frac{1}{2} \int_{G_1 \cap G_2} (x-e)\, d\mu \right\| \le \frac{\varepsilon}{4}$$

and then set $B_4 = (G_1 \cap G_2) \setminus B_3$, which gives

$$\left\| \int_{B_4} (x-e)\, d\mu - \frac{1}{2} \int_{G_1 \cap G_2} (x-e)\, d\mu \right\| = \left\| - \int_{B_3} (x-e)\, d\mu + \frac{1}{2} \int_{G_1 \cap G_2} (x-e)\, d\mu \right\| \le \frac{\varepsilon}{4}.$$

For the disjoint sets $\widetilde{G}_1 = B_1 \cup B_3 \subseteq G_1$ and $\widetilde{G}_2 = B_2 \cup B_4 \subseteq G_2$ we have

$$\int_{\widetilde{G}_1} (y-e)\, d\mu + \frac{1}{2} \int_{S_1 \cap T_1} \gamma_1 \big(y-e\big)\, d\mu \; - \int_{\widetilde{G}_2} (x-e)\, d\mu - \frac{1}{2} \int_{S_2 \cap T_1} \gamma_2 \big(x-e\big)\, d\mu +$$

$$-\frac{1}{2} \left[\int_{G_1} (y-e)\, d\mu + \int_{S_1 \cap T_1} \gamma_1 \big(y-e\big)\, d\mu \; - \int_{G_2} (x-e)\, d\mu - \int_{S_2 \cap T_1} \gamma_2 \big(x-e\big)\, d\mu \right] =$$

$$= \int_{\widetilde{G}_1} \big(y-e\big)\, d\mu - \int_{\widetilde{G}_2} \big(x-e\big)\, d\mu \; - \frac{1}{2} \int_{G_1} (y-e)\, d\mu + \frac{1}{2} \int_{G_2} \big(x-e\big)\, d\mu =$$

$$= \int_{B_1} (y-e)\, d\mu - \frac{1}{2} \int_{G_1 \setminus G_2} (y-e)\, d\mu + \int_{B_3} (y-e)\, d\mu - \frac{1}{2} \int_{G_1 \cap G_2} (y-e)\, d\mu +$$

$$\frac{1}{2} \int_{G_2 \setminus G_1} (x-e)\, d\mu - \int_{B_2} \big(x-e\big)\, d\mu \; + \frac{1}{2} \int_{G_1 \cap G_2} \big(x-e\big)\, d\mu - \int_{B_4} \big(x-e\big)\, d\mu,$$

and therefore we see that

$$\int_{\widetilde{G}_1} (y-e)\, d\mu + \frac{1}{2} \int_{S_1 \cap T_1} \gamma_1 \big(y-e\big)\, d\mu \ll \int_{\widetilde{G}_2} (x-e)\, d\mu + \frac{1}{2} \int_{S_2 \cap T_1} \gamma_2 \big(x-e\big)\, d\mu.$$

Since the choice of B_1 and B_3 ensures that $\mu[\widetilde{G}_1 \cup (S_1 \cap T_1)] > 0$, the above strict inequality means that $x \in \chi_3$ as desired. □

3.2 The notion of ex-post coalition fair allocation

The previous results show that, under standard assumptions on endowments and utilities, the above three sets χ_i coincide if the positive cone of the commodity space has nonempty interior. We are now ready to extend to our framework the coalition fairness concept (*c-fairness* for short) originally due to [17] .

Definition 3.4. *An allocation x is ex-post Aubin coalition fair (c-fair) or c-fair with participation rates, if there exist no coalitions γ_1 , $\gamma_2 \in \mathcal{A}$, no state of nature $\omega_0 \in \Omega$ and no assignment y, such that the conditions 0), 1), 2) and 3) are satisfied. The set of all ex-post Aubin c-fair allocations is denoted by $\mathcal{C}_{A-fair}(\mathcal{E})$.*

Clearly, if we exclude, in the above definition, the possibility that agents cooperate using participation rates, namely if we limit coalitions to be the usual μ-measurable subsets of T, then we define a larger set of allocations, denoted by $\mathcal{C}_{fair}(\mathcal{E})$. Members of $\mathcal{C}_{fair}(\mathcal{E})$ are named *ex-post c-fair allocations* and are defined in the spirit of the original Gabszewicz's notion.

It is trivial that in order to define the set $\mathcal{C}_{fair}(\mathcal{E})$ it is indifferent to use property $2'$) rather than 2), since in this case traders use standard participation. This is not the case of $\mathcal{C}_{A-fair}(\mathcal{E})$. So, if we consider the set of allocations x for which there do not exist γ_1 , $\gamma_2 \in \mathcal{A}$, an assignment y and $\omega_0 \in \Omega$, such that 0), 1), $2'$) and 3) hold true then, obviously, we have a set that lies in between the sets $\mathcal{C}_{A-fair}(\mathcal{E})$ and $\mathcal{C}_{fair}(\mathcal{E})$ (properly, in general). We use the symbol $\mathcal{C}^w_{A-fair}(\mathcal{E})$ to denote this set. Its members are referred to as *weak ex-post Aubin coalition fair allocations.*

In general cases, admitting personalized participation in coalitions, the use of condition 2) in Definition 3.4 seems to be the most coherent. When an ex-post Aubin c-fair allocation x prevails in the economy, then it is impossible, whatever will be the prevailing state "tomorrow", to find a non-trivial coalition γ_1 envious of the net-trade of another coalition γ_2 in the sense that participants in γ_1 by redistributing this net-trade can be better off. In the framework of Aubin approach to cooperation, an agent may well participate in both γ_1 and γ_2. So, rather than requiring that the supports S_1 and S_2 of γ_1 and γ_2 are essentially disjoint, only the weaker constraint 2) (agents in $S_1 \cap S_2$ cannot over use their endowments) is required. So the envious/envied positions properly refer to the net trade of a coalition as a whole, given that traders may decide to join, in principle, several coalitions simultaneously. This makes the stability request defining Aubin c-fair allocations stronger than the one based on ordinary coalitions.

It is relevant to note that, as consequence of Proposition 3.3, in the case of atomless economies the three previous notions of c-fair allocation coincide. This is stated in the next Theorem 3.5. At the same time, the implication of Proposition 3.3 in a general mixed market is that, in the notion of Aubin c-fairness, one can always assume that small traders use all their endowments[9]. This property means that the consideration of a differentiated and flexible participation to coalitions is relevant, in many situations, only when it is applied to the case of influential traders.

Theorem 3.5. *Let the interior of* $I\!B_+$ *be nonempty. Assume further that* μ*-a.a. the initial endowments* e_t *are strictly positive and the utilities* $u_t(\omega, \cdot)$ *are continuous and increasing for any state. Then, in an atomless economy we have* $\mathcal{C}_{A-fair}(\mathcal{E}) = \mathcal{C}^w_{A-fair}(\mathcal{E}) = \mathcal{C}_{fair}(\mathcal{E})$.

PROOF: The inclusion $\mathcal{C}_{fair}(\mathcal{E}) \subseteq \mathcal{C}_{A-fair}(\mathcal{E})$ is a straightforward consequence of Proposition 3.3, i.e. of the fact that for a certain state of nature, the set χ_2 is contained in χ_3. □

The next proposition says that in order to have that an allocation x is ex-post Aubin c-fair, it is enough to verify that the coalitions of full support are not envious.

Proposition 3.6. *Let the interior of* $I\!B_+$ *be nonempty. Assume further that* μ*-a.a. the initial endowments* e_t *are strictly positive and the utilities* $u_t(\omega, \cdot)$ *are continuous*

[9]The same conclusion holds when we consider the notion of weak Aubin c-fairness.

and strictly increasing for any state. Then, $x \in \mathcal{C}_{A-fair}(\mathcal{E})$ if and only if there do not exist γ_1 , $\gamma_2 \in \mathcal{A}$, an assignment y and $\omega_0 \in \Omega$, satisfying $0'$), 1), 2) and 3).

PROOF: We show that if in a state of nature there is an envious coalition, then also a coalition of full support is envious. To this aim, suppose that $x \notin \mathcal{C}_{A-fair}(\mathcal{E})$ and apply Lemma 3.1 in order to assume that $y \succ_{S_1} x$, the assignment y is strictly positive on a subset B of S_1 of positive measure, $\int_{\gamma_1} (y-e)\, d\mu \ll \int_{\gamma_2} (x-e)\, d\mu$, and $\gamma_1 + \gamma_2 \leq 1$.

If $\mu(T \setminus S_1) > 0$, then for a sufficiently small $\lambda > 0$ we have $\int_{S_1} \gamma_1 (y-e)\, d\mu + \lambda \int_{T\setminus S_1} (x-e)\, d\mu \ll \int_{S_2} \gamma_2 (x-e)\, d\mu$. On the other hand, continuity of preferences gives $\varepsilon \in]0,1[$ and a subset C of B where $\varepsilon y \succ_C x$ and $\mu(C) > 0$. Let us set

$$(\widetilde{y}, \widetilde{\gamma}_1) = \begin{cases} (\varepsilon y, \gamma_1), & \text{on } C \\ (y, \gamma_1), & \text{on } S_1 \setminus C \\ \left(x + \dfrac{1}{\lambda \mu(T\setminus S_1)} \displaystyle\int_C \gamma_1 (1-\varepsilon) y,\ \lambda \right), & \text{on } T \setminus S_1 \end{cases}$$

Now: the monotonicity gives $\widetilde{y} \succ_T x$ and an easy calculation [10] shows that $\int_T \widetilde{\gamma}_1 (\widetilde{y} - e)\, d\mu \ll \int_T \gamma_2 (x-e)\, d\mu$ and from Proposition 3.2 the conclusion follows. □

3.3 Existence of c-fair allocations

We shall now discuss the existence of ex-post Aubin coalition fair allocations under standard assumptions.
Consider first the case of a complete information economy $\mathcal{E}(\omega_0)$.

a) When T is a finite set of agents, [2] states the hypotheses under which the existence of competitive equilibria is guaranteed (see also [24] for a wider discussion). Similarly, for the non-atomic case we may refer to [5] for finite dimensional commodity spaces, and to [31], [26] and [28] for the infinite dimensional case. In the general case of mixed models, the existence of competitive equilibria can be reduced to the case of atomless economies. For finite dimensional commodity spaces, existence is provided by [10].

[10] $\int_T \widetilde{\gamma}_1 (\widetilde{y} - e)\, d\mu$ can be written as $\int_C \gamma_1 \varepsilon y + \int_{S_1\setminus C} \gamma_1 y + \int_{T\setminus S_1} \lambda x + \int_C \gamma_1 (1-\varepsilon) y - \int_{S_1} \gamma_1 e - \int_{T\setminus S_1} \lambda e$, but also $\int_{S_1} \gamma_1 y = \int_C \gamma_1 \varepsilon y + \int_{S_1\setminus C} \gamma_1 y + \int_C \gamma_1 (1-\varepsilon) y$, so that

$$\int_T \widetilde{\gamma}_1 (\widetilde{y} - e)\, d\mu = \int_{S_1} \gamma_1 y + \int_{T\setminus S_1} \lambda x - \int_{S_1} \gamma_1 e - \int_{T\setminus S_1} \lambda e.$$

The latter, by the choice of λ is $\ll$ than $\int_{S_2} \gamma_2 (x-e)\, d\mu$.

b) On the other hand, if x is a competitive allocation in $\mathcal{E}(\omega_0)$ (write $x \in \mathcal{W}(\mathcal{E}(\omega_0))$, and we assume that $x \notin \mathcal{C}_{A-fair}(\mathcal{E}(\omega_0))$, then there exist two coalitions γ_1 and γ_2 and an assignment y such that the conditions 0), 1), 2) and 3) are satisfied.

Since $x \in \mathcal{W}(\mathcal{E}(\omega_0))$, 3) implies that $y(t)$ does not belong to the budget set for almost all $t \in S_1$ and hence

$$\int_{\gamma_1} p \cdot y(t)\, d\mu \; > \; \int_{\gamma_1} p \cdot e(\omega_0, t)\, d\mu$$

So by 1),

$$p \cdot \int_{\gamma_2} [x(\omega_0, t) - e(\omega_0, t)]\; d\mu \; \geq p \cdot \int_{\gamma_1} [y(t) - e(\omega_0, t)]\; d\mu \; > 0$$

and a contradiction to the fact that x is competitive.

Let's move now to an arbitrary DIME $\mathcal{E}$. By combining the two above circumstances, we can promptly get that **the set $\mathcal{C}_{A-fair}(\mathcal{E})$ is nonempty**, under the assumptions on $T, I\!\!B, u.(\omega, \cdot)$ and $e.(\omega)$ that guarantee the existence of competitive equilibria in each $\mathcal{E}(\omega)$, according to a).

In fact, in this case, let $F_1, \ldots\ldots, F_m$ be all the elements of the finite partition Π generating $\mathcal{F}$ and let ω_j be a state in F_j. Select, because of a) and b), $x_j \in C_{A-fair}(\mathcal{E}(\omega_j))$, for every $j = 1, \ldots\ldots, m$ and define $x \,:\, \Omega \times T \to I\!\!B_+$ by $x(\omega, t) = x_j(t)$ whenever $\omega \in F_j$ and $t \in T$. This is a well-defined assignment in $\mathcal{E}$. Furthermore, being $e(\cdot, t)$ and $u_t(\cdot, x)$ $\mathcal{F}-$measurable, we have that $\mathcal{E}(\omega) = \mathcal{E}(\omega_j)$ for all $\omega \in F_j$. Thus, $x(\omega, \cdot) = x(\omega_j, \cdot)$ belongs to $C_{A-fair}(\mathcal{E}(\omega))$ for all $\omega \in \Omega$ and $C_{A-fair}(\mathcal{E})$ is non-empty.
It is easy to prove that the inclusion

$$\{x \,|\, x \text{ is an } \textit{assignment} \text{ and } x(\omega, \cdot) \in \mathcal{C}_{A-fair}(\mathcal{E}(\omega))\, \forall \omega \in \Omega\,\} \subseteq \mathcal{C}_{A-fair}(\mathcal{E})$$

that we have just used, is actually an equality, that is each ex-post c-fair allocation is a selection of the correspondence associated to the family of complete information economies. We show this simple result in the next proposition for the sake of completeness.

Proposition 3.7. *Let $\mathcal{E}$ be a differential information economy. Then*

$$\mathcal{C}_{A-fair}(\mathcal{E}) = \{x \,|\, x \textit{ is an assignment and } x(\omega, \cdot) \in \mathcal{C}_{A-fair}(\mathcal{E}(\omega))\, \forall \omega \in \Omega\,\}$$

PROOF: Let us denote by X the set

$$\{x \,|\, x \text{ is an } \textit{assignment} \text{ and } x(\omega, \cdot) \in \mathcal{C}_{A-fair}(\mathcal{E}(\omega)), \forall \omega \in \Omega\,\}.$$

From the definition it follows that $X \subseteq \mathcal{C}_{A-fair}(\mathcal{E})$, so it is sufficient to prove the inclusion $\mathcal{C}_{A-fair}(\mathcal{E}) \subseteq X$. Take an allocation $x \in \mathcal{C}_{A-fair}(\mathcal{E})$ and assume by contradiction that $x \notin X$. This means that for some $\omega_0 \in \Omega$ it is $x(\omega_0, \cdot) \notin \mathcal{C}_{A-fair}(\mathcal{E}(\omega_0))$, so there exist some generalized coalitions $\gamma_1,\, \gamma_2 \in \mathcal{A}$ and there exists an assignment $y : T \to I\!\!B_+$ such that the conditions 0), 1), 2) and 3) are satisfied. Let us denote by

$F(\omega_0)$ the unique element of Π containing ω_0. We define the function $z : \Omega \times T \to I\!\!B_+$ by setting

$$z(\omega, t) = \begin{cases} y(t) & \text{if } \omega \in F(\omega_0) \\ e(\omega, t) & \text{if } \omega \notin F(\omega_0) \end{cases}$$

The function z is an assignment since it is $\mathcal{F}$-measurable (it is, in fact, constant on the elements of Π). Of course

$$\int_{\gamma_1} [z(\omega_0, t) - e(\omega_0, t)]\, d\mu = \int_{\gamma_1} [y(t) - e(\omega_0, t)]\, d\mu \leq \int_{\gamma_2} [x(\omega_0, t) - e(\omega_0, t)]\, d\mu$$

moreover, for a.a. $t \in \gamma_1$, we have

$$u_t(\omega_0, z(\omega_0, t)) = u_t(\omega_0, y(t)) > u_t(\omega_0, x(\omega_0, t)).$$

The latter relations say that $x \notin \mathcal{C}_{A-fair}(\mathcal{E})$, which is a contradiction. □

3.4 Individual interpretation

In this section we explore a natural extension to our general model of Varian's definition of individual equity (see [34]). We shall see that whenever we can ensure Aubin c-fairness of an allocation arising from an equal sharing of the total initial endowment, then the allocation is also individually equitable, i.e. traders do not envy each other. It is worthwhile to point out that this result is achieved within the framework of mixed markets, where some traders may have non-negligible initial power. Naturally, it holds true also in the particular case of complete information economies. This type of interpretation deserves interest in itself for two reasons. It applies to infinite dimensional economies, with complete or asymmetric information, despite Lyapunov's Theorem does not hold: the convexifying effect comes from participation rates. It provides a response to a usual criticism against notion of fairness based on coalitions. Indeed, according to some authors (see for example [36]), this type of concept should be more properly classified as a cooperative notion rather than a notion of equity, since it requires a redistribution of resources among traders. So, it becomes important to show that c-fairness notions ensure, as consequence of cooperation, also individual non-envy.

Let us fix the attention for the moment on the case of a standard finite economy with complete information, i.e. there is no uncertainty and μ is the counting measure. Assume, for simplicity, that all traders have the same initial endowment $e_t = e$. According to Varian's definition, an allocation x is qualified as individually equitable or envy free if each trader does not prefer some other's trader commodity bundle to his own, that is, in terms of utility, it holds true that, for each $t, s \in T$

$$u_t(x_t) \geq u_t(x_s).$$

Assume that the allocation x is c-fair (with standard participation), but not individually equitable, for example because of the pair (t, s) for which it is true that $u_t(x_t) < u_t(x_s)$. Then immediately one has a contradiction choosing as pair of coalitions for which c-fairness is violated, the coalitions $S_1 = \{t\}$ and $S_2 = \{s\}$: indeed, the redistribution causing envy is possible by means of the assignment y equal to x for each trader, with the exception of trader t which receives x_s instead of x_t. In this case, one has that $y \succ_{S_1} x$ and the feasibility follows since traders have the same size. This simple argument doesn't work as soon as traders t and s are atoms of different size, in particular when the t's size is greater than the s' one. So, a c-fair allocation is not necessarily individually envy free and the argument above suggests that, in order to restore such implication, the different weights of atoms should be redistributed between coalitions S_1 and S_2 by means of differentiated participation rates.

As we shall see now, in the case of an atomless market with finitely many commodities, the proof that a c-fair allocation with standard participation is individually envy free follows a similar reasoning as in the finite case, although it is strongly based on the validity of the Lyapunov theorem and the separability of the commodity space. The previous tools permit, when used together, to aggregate envy from individual to coalitions and to reach a contradiction determining the coalitions S_1 and S_2. But again, in a mixed model, due to the presence of large traders, one needs a redistribution of weights (i.e. non-standard participation), to obtain a similar conclusion. Notice also that, since the presence of many commodities implies the use of Liapunov theorem only in its weak or approximate version, in our results below some interiority like assumptions will be adopted to overcome this problem.
Let us fix the basic notation. Given an allocation x, an agent $t \in T$ and a state of nature $\omega \in \Omega$, we define the set $A_t(\omega, x)$ of individuals that t envies at x in state ω:

$$A_t(\omega, x) = \{s \in T \ : \ u_t(\omega, x_t(\omega)) < u_t(\omega, x_s(\omega)\,)\}$$

and the set $A(\omega, x)$ of envious agents at x in state ω:

$$A(\omega, x) = \{t \in T \ : \ \mu(A_t(\omega, x)) > 0\}.$$

A natural definition of individual equitability follows. It requires that the set of envious traders has measure zero and includes, as a particular case, Varian's notion.

Definition 3.8. *An allocation x is said to be ex-post (individually) envy-free or equitable if $\mu(A(\omega, x)) = 0$ for each state ω.*

Before proving the main statements of this section, we isolate two simple useful arguments. In the first lemma, we require that the initial endowment is distributed over coalitions according to a reference bundle and their size.

Lemma 3.9. *Let $\mathcal{E}$ be a complete information economy with $e_t = e$, for μ-almost $t \in T$. Given the allocation x, suppose we find two coalitions B, C of positive μ-measure such that for almost all $t \in C$*

$$u_t(z) > u_t(x_t) \quad \textit{where} \quad z := \frac{1}{\mu(B)} \int_B x(s) d\mu(s).$$

If we find two essentially disjoint, non-negligible, coalitions $\Gamma_1 \subseteq C$, $\Gamma_2 \subseteq B$, such that the average of x on Γ_2 is still z, then $x \notin \mathcal{C}_{A-fair}(\mathcal{E})$.

PROOF: Consider the assignment y constantly equal to z over C and the coalitions γ_i with support Γ_i defined as follows: $(\gamma_1, \gamma_2) = \left(1, \frac{\mu(\Gamma_1)}{\mu(\Gamma_2)}\right)$ in case $\mu(\Gamma_1) \leq \mu(\Gamma_2)$, $(\gamma_1, \gamma_2) = \left(\frac{\mu(\Gamma_2)}{\mu(\Gamma_1)}, 1\right)$ otherwise. Then clearly

$$\int_{\gamma_1} (y - e)\, d\mu = \int_{\gamma_2} (x - e)\, d\mu.$$

and $y \succ_{\Gamma_1} x$. □

Lemma 3.10. *Let $\mathcal{E}$ be a complete information economy. Assume that the commodity space $I\!B_+$ is separable and that the functions u_t are, for μ-almost each agent $t \in T$, continuous. Given the allocation x, let $t \in T$ be such that $\mu(A_t(x)) > 0$, then a coalition $B \subseteq A_t(x)$ exists such that*

$$u_t(z) > u_t(x_t) \quad \textit{where} \quad z := \frac{1}{\mu(B)} \int_B x(s) d\mu(s).$$

PROOF: Set $A_t := A_t(x)$. Without loss of generality, we can assume that $u_t(x_s) > u_t(x_t)$, for each $s \in A_t$. Consider the set

$$H = \left\{ \frac{1}{\mu(B)} \int_B x(s) d\mu(s) : B \subseteq A_t, \mu(B) > 0 \right\}.$$

We claim that $x(s) \in cl\, H$, for μ-almost all $s \in A_t$ (see [36, Lemma 3.3, (ii)] for the argument in the finite dimensional case). For each $h \notin cl\, H$, it is true that there exists a ball $B(h)$ centered in h whose intersection with H is empty. Then we can consider a ball $B^{'}(h)$ for which $B^{'}(h) \subseteq cl\, B^{'}(h) \subseteq B(h)$ and such that $cl\, B^{'}(h)$ is also disjoint from H. The set $B = \{s \in A_t : x(s) \in cl\, B^{'}(h)\}$ has measure zero, otherwise we would have that $\frac{1}{\mu(B)} \int_B x(s) d\mu(s) \in cl\, B^{'}(h)$ and a contradiction. Consequently, $\mu\left(\left\{s \in A_t : x(s) \in B^{'}(h)\right\}\right) = 0$. From the separability assumption, it follows that for h in a countable set N, the balls $B^{'}(h)$ cover the set $I\!B_+ \setminus cl\, H$. The claim follows from the inclusion

$$\{s \in A_t : x(s) \notin cl\, H\} \subseteq \bigcup_{h \in N} \left\{s \in A_t : x(s) \in B^{'}(h)\right\}.$$

□

We first deal with the case of finitely many commodities.

Theorem 3.11. *Let $\mathcal{E}$ be a finite dimensional DIME. Assume that for each $\omega \in \Omega$ and for μ-almost each agent $t \in T$ it is true that: $e_t(\omega) = e(\omega)$; the functions $u_t(\omega, \cdot)$ are continuous on $I\!B_+$.*

Then, any ex-post Aubin c-fair allocation x is ex-post (individually) envy-free.

PROOF: By Proposition 3.7, we can reduce the proof to the case of Ω being a singleton $\{\omega\}$. Then, we simply skip ω in notations. Let us see that, under the assumption that

x exhibits envy, i.e. $\mu(A(x)) > 0$, then Aubin c-fairness of x is contradicted. We also simplify notations by setting

$$A_t := A_t(x) = \{s \in T \,:\, u_t(x_t) < u_t(x_s\,)\}; \qquad A := A(x) = \{t \in T \,:\, \mu(A_t(x)) > 0\}.$$

We shall distinguish the following two basic cases.
Case 1: the envious agents include some large trader i.e. some point of T_1.
Case 2: the envious agents are all in T_0.

In case 1, we have $\mu(A \cap T_1) > 0$ and in particular let $t \in T_1$ be a $\mu-$atom belonging to A. Since by definition the set $A_t = \{s \in T \,:\, u_t(x_t) < u_t(x_s)\}$ is of positive measure, by Lemma 3.10, we can find a subcoalition $B_t \subseteq A_t$, of positive measure for which

$$u_t(y_t) > u_t(x_t) \ \text{ where } \ y_t := \frac{1}{\mu(A_t)} \int_{A_t} x(s) d\mu(s).$$

Now it is enough to apply Lemma 3.9 where $C = \{t\}$ and $B = B_t$ to get a contradiction. So, let us move now the case 2: $A \subseteq T_0$.

Set

$$\widetilde{A} := \{t \in A \,:\, \mu(A_t \cap T_1) > 0\}, \qquad z(B) := \frac{1}{\mu(B)} \int_B x_s\, d\mu(s),\ B \in \mathcal{T}, \mu(B) > 0.$$

Consider the sub-case that $\mu(\widetilde{A}) > 0$.
With reference to a dense countable subset Y of the range Z of the set function z over all subsets B of T_1 we have

$$\{t \in T : u_t(z) > u_t(x_t) \text{ for some } z \in Z\} = \bigcup_{z \in Y} \{t \in T : u_t(z) > u_t(x_t)\}.$$

Since for $t \in \widetilde{A}$ we have $\mu(A_t \cap T_1) > 0$ and for $s \in A_t \cap T_1$, $u_t(x_t) < u_t(x_s)$, we derive from Lemma 3.10 that $u_t(x_t) < u_t(z(B_t))$, with $B_t \subseteq A_t \cap T_1$. This means that

$$\widetilde{A} \subseteq \bigcup_{z \in Y} \{t \in T : u_t(z) > u_t(x_t)\},$$

and permits to find $z \in Y$, $z = z(B)$, $B \subseteq T_1$ such that the set $C := \widetilde{A} \cap \{t \in T : u_t(z) > u_t(x_t)\}$ has positive measure. Now to have a contradiction it is enough to apply Lemma 3.9 observing that C and B are necessarily disjoint.

Assume now that $\mu(\widetilde{A}) = 0$.
We have, then, that for almost all $t \in A$, the set $A_t \subseteq T_0$.
Like in the previous sub-case, let Y be a dense countable subset of the range Z of the set function z over all subsets B of T_0. Since for $t \in A$ we have $\mu(A_t) > 0$ and for $s \in A_t$, $u_t(x_t) < u_t(x_s)$, again by Lemma 3.10, $u_t(x_t) < u_t(z(B_t))$ for a coalition $B_t \subseteq A_t$. This means that

$$A \subseteq \bigcup_{z \in Y} \{t \in T : u_t(z) > u_t(x_t)\},$$

and permits to find $z \in Y$, $z = z(B)$, $B \subseteq T_0$ such that the set $C := A \cap \{t \in T : u_t(z) > u_t(x_t)\}$ has positive measure.
If the set $C \setminus B$ has positive measure, again Lemma 3.9 applies. Finally, it remains to analyze what happens when $\mu(C \setminus B) = 0$. In this case simply consider $C \subseteq B$.

By Lyapunov theorem, take a sequence B_n of subsets of B such that

$$\left(\mu(B_n), \int_{B_n} x \, d\mu\right) = \frac{1}{n}\left(\mu(B), \int_B x \, d\mu\right).$$

Since $\mu(B_n) \to 0$, for a suitable n the set $\Gamma_1 := C \setminus B_n$ has positive measure and with $\Gamma_2 = B_n$ Lemma 3.9 applies. □

While in the previous case, for the concluding argument, a straightforward application of Lyapunov's convexity theorem is sufficient, when the commodity space is infinite dimensional a more subtle argument is necessary. Moreover, we also need the assumption that x is an interior allocation.

Theorem 3.12. *Let $\mathcal{E}$ be a DIME whose commodity space $\mathbb{B}$ is separable and whose positive cone has non-empty interior. Assume that for each $\omega \in \Omega$ and for μ-almost each agent $t \in T$ it is true that: $e_t(\omega) = e(\omega)$; the functions $u_t(\omega, \cdot)$ are continuous on $\mathbb{B}_+$.*

Then, for $x \in \mathcal{C}_{A-fair}(\mathcal{E})$ we have that if x_ω is μ-a.e. a strictly positive vector, then $\mu(A(\omega, x)) = 0$.

PROOF: The infinite dimension only influences the last part of the previous proof. Indeed, we can only use an approximate Lyapunov theorem. Therefore, take any $\delta \in]0, 1[$. By Lyapunov theorem, we find (see [15]) a sequence B_n of subsets of B such that

$$\text{for all } n,\ \mu(B_n) = \delta\mu(B), \quad \text{and} \quad \int_{B_n} x \, d\mu \to \delta \int_B x \, d\mu.$$

By continuity of preferences, we get

$$C = \{t \in A : u_t(z) > u_t(x_t)\} \subseteq \bigcup_{\varepsilon \in Q \cap]0,1[} \{t \in A : u_t(\varepsilon z) > u_t(x_t)\}$$

and therefore for some ε we have that

$$\mu(A_\varepsilon) > 0, \qquad \text{where } A_\varepsilon := \{t \in A : u_t(\varepsilon z) > u_t(x_t)\}.$$

This implies that for all integers n we have $\mu(A_\varepsilon \setminus B_n) > 0$. Consider an assignment y constantly equal to εz over A_ε. Obviously $z(B_n) - \varepsilon z \to z - \varepsilon z \gg 0$ and, since $z - \varepsilon z$ belongs to the interior of $\mathbb{B}_+$, for a certain integer n we have $z(B_n) - \varepsilon z \gg 0$.

However,

$$z(B_n) - \varepsilon z = \frac{1}{\mu(B_n)} \int_{B_n} x \, d\mu - \frac{1}{\mu(A_\varepsilon \setminus B_n)} \int_{A_\varepsilon \setminus B_n} y \, d\mu.$$

So, if we take the Aubin coalitions γ_1 and γ_2 with support, respectively, $\Gamma_1 = A_\varepsilon \setminus B_n$ and $\Gamma_2 = B_n$ and constant rate of participation of agents, then clearly

$$\int_{\gamma_1} (y - e) \, d\mu \ll \int_{\gamma_2} (x - e) \, d\mu,$$

where $(\gamma_1, \gamma_2) = \left(1, \frac{\mu(\Gamma_1)}{\mu(\Gamma_2)}\right)$ in case $\mu(A_\varepsilon \setminus B_n) \leq \mu(B_n)$, and $(\gamma_1, \gamma_2) = \left(\frac{\mu(\Gamma_2)}{\mu(\Gamma_1)}, 1\right)$ otherwise. Again we contradict that x is Aubin c-fair. □

To summarize, the previous results show that, under a c-fair allocation with participation rates: large traders cannot be envious, small traders cannot be envious of large traders and, under an interiority assumption on the allocation in the presence of many commodities, there are no envious individuals at all, small or large.

Remark 3.13. Notice that, contrary to usual core equivalence assumptions, the quasi-concavity of utility functions is not required in Theorems 3.11 and 3.12, also in the presence of atoms (compare with [25]). However, the usual separability assumption on the commodity space is maintained. On the other hand, a careful look at the proof of the theorems shows that quasi-concavity of utility functions of all traders in each state of nature, joint with strict monotonicity, replaces the separability requirement.

Theorem. *If in Theorem 3.12 we do not assume separability of the commodity space but we assume that all utility functions, in any state, are also strictly monotone and quasi-concave, then its conclusion still holds true.*
PROOF: The proof closely follows the one of Theorems 3.11 and 3.12. As a first step, it is enough to apply Lemma 7.1 in order to obtain that in the conclusion of Lemma 3.10 it is $B_t = A_t$. Then, one can observe that, for a given allocation x violating individual equitability, the function

$$t \in A \to \frac{1}{\mu(A_t)} \int_{A_t} x_s \, d\mu(s)$$

has a separable range (see Lemma 7.2 in the Appendix).

Therefore, replacing in the proof of Theorem 3.11 the set Y with a countable subset, respectively, of the separable set $\left\{ \frac{1}{\mu(A_t \cap T_1)} \int_{A_t \cap T_1} x_s \, d\mu(s) : t \in \widetilde{A} \right\}$, when $\mu(\widetilde{A}) > 0$ and of the separable set $\left\{ \frac{1}{\mu(A_t)} \int_{A_t} x_s \, d\mu(s) : t \in A \right\}$, when $\mu(\widetilde{A}) = 0$, the conclusion follows. □

We also remark that, both in the case of Theorems 3.12 and the previous one, the conclusion does not follow from the joint use of core equivalence and individual equitability of equal income competitive equilibria, since the required sets of assumptions are not comparable.

Remark 3.14. Unlike ex-post c-fairness, ex-post individual envy freeness introduced in this section is a property of allocations that translates into the corresponding ex-ante one. To see this, let us denote by $h_t(x_t)$ the standard ex-ante $I\!P_t-$expected utility trader t derives from the random consumption $x_t(\omega)$.

Similarly to Definition 3.8, an *ex-ante individually envy-free allocation* x is defined by means of $\mu(A(x)) = 0$, where $A(x) = \{t \in T : \mu(A_t(x)) > 0\}$ (ex-ante envious agents), and $A_t(x) := \{s \in T : h_t(x_t) < h_t(x_s)\}$ (agents ex-ante envied by t).

Proposition. *If x is ex-post individually envy-free, then it is also ex-ante individually envy-free*

PROOF: Assume that the allocation x is ex-post individually envy free, but not ex-ante. Then the set $A(x)$ of traders that are (ex-ante) envious under x has positive measure and, for each trader $t \in A(x)$ and $s \in A_t(x)$, one can find a state ω_s in which $u_t(\omega_s, x_t(\omega_s)) < u_t(\omega_s, x_s(\omega_s))$. As consequence, for each fixed trader t which is envious ex-ante, one has that

$$A_t(x) \subseteq \bigcup_{\omega \in \Omega} \{s \in T : u_t(\omega, x_t(\omega)) < u_t(\omega, x_s(\omega))\}.$$

Since the above union is a union of finitely many sets, it has positive measure, because it includes $A_t(x)$. This permits to identify a state, ω_t, in which t is also ex-post envious. From the inclusion

$$A(x) \subseteq \bigcup_{\omega \in \Omega} \{t \in T : \mu(A_t(\omega, x)) > 0\},$$

whose right hand side is a finite union, it follows that at least one of the sets $\{t \in T : \mu(A_t(\omega, x)) > 0\}$ has positive measure and therefore a contradiction. □

Notice that the same simple argument may not work in the case of c-fair allocations, since the feasibility required for the net trade of coalitions S_1 and S_2 is not necessarily preserved moving from the ex-post to the ex-ante stage.

3.5 Intrepretation via continuum economies

According to a standard interpretation, one can think that each large trader in a mixed market model arises from a group of small identical traders that decide to join and to act on the market only together. As consequence of such agreements, one motivates the fact that no proper subcoalitions of the group are possible and then the group is identified with an atom of μ. In this section, we move back splitting the atoms into the original small traders forming them. Our aim is to show that c-fairness of an allocation, when considered on the base of participation rates, is robust to variations in traders' behavior in the sense that the property would be preserved if small traders decide to break the agreement and to act on the market independently.

Precisely, we show that the embedding of the original mixed economy $\mathcal{E}$ into the auxiliary atomless economy $\mathcal{E}^*$ obtained by splitting each large trader into a continuum of small traders of the same type, preserves c-fairness and, therefore, gives us the possibility to interpret Aubin c-fairness as c-fairness in the atomless economy $\mathcal{E}^*$.

In the proof of this type of correspondence, the possibility that a trader may join a coalition simultaneously (i.e. condition 2) introduced in section 3) will be essential.

Remind that large traders have been denoted by $A_1, A_2, \ldots$ We partition the interval $[\mu(T_0), \mu(T)]$, that we denote by T_1^*, as the disjoint union of the intervals A_i^* given by:

$$A_1^* = \Big[\mu(T_0), \mu(T_0) + \mu(A_1)\Big[, \ \ldots, \ A_i^* = \Big[\mu(T_0) + \mu\{A_1, \ldots, A_{i-1}\}, \mu(T_0) + \mu\{A_1, \ldots, A_i\}\Big[, \ \ldots$$

Now, given the economy

$$\mathcal{E} = \{ (T, \mathcal{T}, \mu);\ I\!B;\ (\Omega, \mathcal{F});\ (u_t,\ I\!P_t,\ \Pi_t, e_t)_{t \in T} \},$$

the associated atomless economy

$$\mathcal{E}^* = \{(T^*, \mathcal{T}^*, \mu^*);\ I\!\!B;\ (\Omega, \mathcal{F});\ (u_t,\ I\!\!P_t,\ \Pi_t, e_t)_{t\in T^*}\}$$

is defined as follows. The measure space $(T^*, \mathcal{T}^*, \mu^*)$ of agents is the direct sum of $(T_0, \mathcal{T}_0, \mu)$ and the interval T_1^* is endowed with the Lebesgue measure. The profile $(u_t,\ I\!\!P_t,\ \Pi_t, e_t)_{t\in T^*}$ of agents' types extends the original profile of members of T by assuming that for each trader $t \in A_i^*$, it coincides with the one of the large agent A_i. It is customary to name the interval A_i^* as *the split of the atom* A_i.

For a function $f \in L_1(\mu^*)$, we denote by f_{S^*} the average of f over the set S^*. Let us fix the following notation.

For an allocation x of the economy $\mathcal{E}$, we define over T^* the allocation $x^* = \varphi(x)$ of the economy $\mathcal{E}^*$ by setting $x_t^* = x_t$, if $t \in T_0$ and $x_t^* = x_{A_i}$, if $t \in A_i^*$.

Reciprocally, given an allocation x^* of $\mathcal{E}^*$ we define the allocation $x = \psi(x^*)$ of $\mathcal{E}$ by setting $x_t = x_t^*$, for $t \in T_0$ and $x_t = x^*_{A_i^*}$, for $t = A_i$.

Proposition 3.15. *Let x be an allocation of the original economy $\mathcal{E}$ such that $\varphi(x) \notin \mathcal{C}_{fair}(\mathcal{E}^*)$. Suppose that for any large trader A_i the utility is (in any state) continuous, quasi-concave and strictly monotone. Then we have that $x \in \chi_1$.*

PROOF: Since $\varphi(x) \notin \mathcal{C}_{fair}(\mathcal{E}^*)$, we find (with reference to a certain state of nature that we omit for brevity) two disjoint subsets S_i^* of T^*, an assignment $\overline{y}$ in the economy $\mathcal{E}^*$ such that S_1^* has positive measure, $\overline{y} \succ_{S_1^*} \varphi(x)$,[11] and $\int_{S_1^*} (\overline{y} - e^*)\ d\mu^* \leq \int_{S_2^*} (x^* - e^*)\ d\mu^*$. Once we set

$$i \in I \Leftrightarrow \mu^*(S_1^* \cap A_i^*) > 0 \qquad \text{and} \qquad j \in J \Leftrightarrow \mu^*(S_2^* \cap A_j^*) > 0,$$

the latter inequality can be written as follows

$$(+) \qquad \int_{S_1^*\cap T_0} (\overline{y} - e^*)\ d\mu^* + \sum_{i\in I} \mu^*(S_1^* \cap A_i^*)\,[\overline{y}_i - e(A_i)] \leq \int_{S_2^*\cap T_0} (x^* - e^*)\ d\mu^* +$$
$$+ \sum_{j\in J} \mu^*(S_2^* \cap A_j^*)\,[x(A_j) - e(A_j)]\,,$$

where $\overline{y}_i$ is the average of $\overline{y}$ on the set $S_1^* \cap A_i^*$.

Now we define the subsets S_1, S_2 of T, as

$$S_1 := (S_1^* \cap T_0) \cup \{A_i : i \in I\} \qquad \text{and} \qquad S_2 := (S_2^* \cap T_0) \cup \{A_j : j \in J\}$$

supporting the Aubin coalitions γ_1, γ_2 defined as

$$\gamma_1 = \begin{cases} 1, & \text{on } S_1^* \cap T_0 \\ \frac{\mu^*(S_1^* \cap A_i^*)}{\mu(A_i)}, & \text{on the point } A_i,\ \text{for } i \in I \end{cases}$$

[11]Namely, by definition, for $t \in S_1^* \cap T_0$, we have $\overline{y}(t) \succ_t x(t)$ and, for $t \in S_1^* \cap A_i^*$, $\overline{y}(t) \succ_{A_i} x(A_i)$.

$$\gamma_2 = \begin{cases} 1, & \text{on } S_2^* \cap T_0 \\ \frac{\mu^*(S_2^* \cap A_j^*)}{\mu(A_j)}, & \text{on the point } A_j, \text{ for } j \in J, \end{cases}$$

and the assignment y on T as

$$y(t) = \begin{cases} \overline{y}(t), & \text{for } t \in T_0 \\ \overline{y}_i, & \text{on the point } t = A_i, \text{ for } i \in I \\ x(t), & \text{elsewhere on } T_1 \setminus \{A_i : i \in I\}. \end{cases}$$

In this way, we see that $x \in \chi_1$. Indeed, $\mu(S_1) > 0$ since $\mu^*(S_1^*)$; the inequality $\int_{S_1} \gamma_1(y - e)\, d\mu \leq \int_{S_2} \gamma_2(x - e)\, d\mu$ corresponds to the above inequality (+) and to get $y \succ_{S_1} x$ one has only to check that $y(A_i) \succ_{A_i} x(A_i)$. The latter comes from Lemma 7.1. □

Corollary 3.16. *Assume that μ-a.a. the initial endowments e_t are strictly positive and the utilities $u_t(\omega, \cdot)$ are continuous and monotone. Also assume that $u_t(\omega, \cdot)$ is quasi-concave and strictly monotone for $t \in T_1$. Then,*

$$x \in \mathcal{C}_{A-fair}(\mathcal{E}) \Rightarrow \varphi(x) \in \mathcal{C}_{fair}(\mathcal{E}^*).$$

PROOF: If we assume $\varphi(x) \notin \mathcal{C}_{fair}(\mathcal{E}^*)$, the previous Proposition and Proposition 3.2 give the assertion. □

On the converse let us show that if for an allocation x^* of the continuum economy $\mathcal{E}^*$ we have that $\psi(x^*) \notin \mathcal{C}_{A-fair}(\mathcal{E})$, then $x^* \notin \mathcal{C}_{fair}(\mathcal{E}^*)$. For this purpose, the non emptiness of the interior of the positive cone $I\!B_+$ plays a role as well as the assumption of strict monotonicity of preferences.
Let us first notice the following.

Lemma 3.17. *Let the interior of $I\!B_+$ be nonempty. Assume further that μ-a.a. the initial endowments e_t are strictly positive and the utilities $u_t(\omega, \cdot)$ are continuous and strictly increasing for any state. Also assume that $u_t(\omega, \cdot)$ is quasi-concave, for $t \in T_1$.*

Then, for an allocation $x^* \in \mathcal{C}_{fair}(\mathcal{E}^*)$ it is true that, for each A_i,

$$x^*_{A_i^*} \sim_{A_i} x^*(t), \quad \text{for } \mu\text{-a.a. } t \in A_i^*.$$

PROOF: The proof follows from [9, Lemma 4.3] (interpreted in the light of [9, Remark 4.8]) since any c-fair allocation is a core allocation. □

Theorem 3.18. *Let the interior of $I\!B_+$ be nonempty. Assume further that μ-a.a. the initial endowments e_t are strictly positive and the utilities $u_t(\omega, \cdot)$ are continuous and strictly increasing for any state. Also assume that $u_t(\omega, \cdot)$ is quasi-concave for $t \in T_1$. Then,*

$$x^* \in \mathcal{C}_{fair}(\mathcal{E}^*) \Rightarrow \psi(x^*) \in \mathcal{C}_{A-fair}(\mathcal{E}).$$

PROOF: Let us suppose that $\psi(x^*) \notin \mathcal{C}_{A-fair}(\mathcal{E})$ although $x^* \in \mathcal{C}_{fair}(\mathcal{E}^*)$. By Proposition 3.3, for the allocation $x := \psi(x^*)$ we may assume (in a certain state of nature) the existence of an assignment y and coalitions γ_i with supports S_i such that

$$(++) \int_{S_1\cap T_0} (y-e)\, d\mu + \int_{S_1\cap T_1} \gamma_1(y-e)\, d\mu \leq \int_{S_2\cap T_0} (x-e)\, d\mu + \int_{S_2\cap T_1} \gamma_2(x-e)\, d\mu,$$

$\gamma_1 + \gamma_2 \leq 1$, and $S_1 \cap\ S_2 \cap T_0 = \emptyset$. Naturally one also has that S_1 is of positive measure and μ-a.e. on it y is strictly preferred to x.

Now, for every index $i \in I := \{n : A_n \in S_1 \cap T_1\}$, we choose a subset B_i^* of A_i^* with

$$\mu^*(B_i^*) = \mu^*(A_i^*)\gamma_1(A_i) = \mu(A_i)\gamma_1(A_i)$$

and then define the subset S_1^* of T^* by setting

$$S_1^* = (S_1 \cap T_0) \cup (\cup_{i\in I} B_i^*).$$

Analogously, with reference to $J := \{n : A_n \in S_2 \cap T_1\}$, we define the set

$$S_2^* = (S_2 \cap T_0) \cup (\cup_{i\in J} H_i^*)$$

by choosing the sets H_i^* as follows. For $i \in J \setminus I$, let $H_i^* \subseteq A_i^*$ be such that $\mu^*(H_i^*) = \mu^*(A_i^*)\gamma_2(A_i) = \mu(A_i)\gamma_2(A_i)$. For the remaining indices $i \in J \cap I$, let $H_i^* \subseteq A_i^* \setminus B_i^*$ be such that $\mu^*(H_i^*) = \mu^*(A_i^*)\gamma_2(A_i) = \mu(A_i)\gamma_2(A_i)$.[12]

In the economy $\mathcal{E}^*$ we have the disjoint sets S_i^*, the assignment $y^* := \varphi(y)$ and we can observe, against the assumption of $x^* \in \mathcal{C}_{fair}(\mathcal{E}^*)$, that:
$\mu^*(S_1^*) > 0$, since $\mu(S_1) > 0$;
$y^* \succ_{S_1^*} x^*$, by the definition of functions y^* and x and by appealing to the previous Lemma 3.17,
and finally $\int_{S_1^*} (y^* - e^*)\, d\mu^* \leq \int_{S_2^*} (x^* - e^*)\, d\mu^*$, since such inequality coincides with inequality (++) above.

□

3.6 C-fair allocations and the Ex-post Core

Inherent in the Definition 3.4, we find another form of coalition-proofness satisfied by ex-post Aubin c-fair allocations. Indeed, assume that, for an allocation x, there exist a generalized coalition γ_1, a state of nature ω_0 and an assignment y such that one has, together with properties 0) and 3), also property

$$1^*) \int_{\gamma_1} y_{\omega_o}\, d\mu \leq \int_{\gamma_1} e_{\omega_0}\, d\mu\, .$$

Then, it is obvious that, by setting $\gamma_2 = 0$, Definition 3.4 is violated.

According to [11], we say that x is an *ex-post Aubin core allocation* of $\mathcal{E}$ and we shortly write $x \in \mathbf{C}_A(\mathcal{E})$, if there exist no Aubin coalition γ_1, state ω_o, assignment y,

[12]Notice that $\mu^*(A_i^* \setminus B_i^*) = \mu^*(A_i^*)(1 - \gamma_1(A_i)) \geq \mu^*(A_i^*)\gamma_2(A_i)$.

such that 0), 1*) and 3) are satisfied. In particular, an *ex-post Aubin c-fair allocation is an ex-post Aubin core allocation.*

Limiting, in the above definition of $\mathbf{C}_A(\mathcal{E})$, our attention to ordinary coalitions, we have the concept of *ex-post core* ([13]) that we denote by $\mathbf{C}(\mathcal{E})$. The set of allocations that cannot be blocked ex-post by the grand coalition T is the set of *ex-post (weak) Pareto optimal allocations.* Summarizing, we have that

$$\mathcal{C}_{A-fair}(\mathcal{E}) \subseteq \mathbf{C}_A(\mathcal{E}) \subseteq \mathbf{C}(\mathcal{E})$$

as well as

$$\mathcal{C}_{fair}(\mathcal{E}) \subseteq \mathbf{C}(\mathcal{E}).$$

Remark 3.19. Assume that the interior of $I\!B_+$ is nonempty. Assume further that μ-a.a. the vectors e_t are strictly positive for any state and $u_t(\omega, \cdot)$ is continuous and strictly increasing for any state. Then, in the definition of $\mathbf{C}_A(\mathcal{E})$ we can be more demanding replacing condition 0) by 0$'$). Also, we define the same set $\mathbf{C}_A(\mathcal{E})$ if we add in its definition the further condition that the value of γ_1 is 1 on the atomless part of its support.

Consequently, as in the case of c-fair allocations with participation rates, when T is atomless, then $\mathbf{C}_A(\mathcal{E}) = \mathbf{C}(\mathcal{E})$ (for related results see also [9]).

Let us discuss now the situations in which the previous inclusions are indeed equivalences. The argument of the following Lemma follows closely the one of [15, Lemma 1] (similar arguments are also developed in [8, Lemma 1]).

Lemma 3.20. *Let T be atomless, i.e. $T_1 = \emptyset$. Assume the interior of $I\!B_+$ is nonempty. Assume further that μ-a.a. the vectors e_t are strictly positive for any state and $u_t(\omega, \cdot)$ is continuous for any state. Then, if $x \notin \mathcal{C}_{fair}(\mathcal{E})$, there exist the coalitions S_1 and S_2 with $\mu(S_1) > 0$, the assignment y and the state ω_o such that*

$$\int_{S_1} \big[y(\omega_0, t) - e(\omega_0, t)\big]\, d\mu \ll \int_{S_2} \big[x(\omega_0, t) - e(\omega_0, t)\big]\, d\mu;$$

$$u_t(\omega_0, y(\omega_0, t)) > u_t(\omega_0, x(\omega_0, t)) \;\; \text{for a.a. } t \in S_1.$$

Proof: Let x be not ex-post c-fair. Then, there exist two disjoint coalitions S_1 and S_2 with $\mu(S_1) > 0$, an assignment y and a state ω_o in which

$$\int_{S_1} \big[y(\omega_0, t) - e(\omega_0, t)\big]\, d\mu \leq \int_{S_2} \big[x(\omega_0, t) - e(\omega_0, t)\big]\, d\mu;$$

$$u_t(\omega_0, y(\omega_0, t)) > u_t(\omega_0, x(\omega_0, t)) \;\; \text{for a.a. } t \in S_1.$$

Consider the state ω_o as fixed and then omit it in notations. We can write

$$\int_{S_1} y\, d\mu \leq \int_{S_1} e\, d\mu + \int_{S_2} (x - e)\, d\mu.$$

Consider an increasing sequence of subcoalitions $S_n \subseteq S_1$ such that $lim_n\, \mu\,(S_1 \setminus S_n) = 0$ and $u_t\left(\left(1 - \frac{1}{n}\right) y(t)\right) > u_t(x(t))$ for a.a. $t \in S_n$. These coalitions can be determined following the first part of [15, Lemma 1]. Define

$$y_n(t) = \begin{cases} y(t) & \text{for } t \in S_1 \setminus S_n \\ \left(1 - \frac{1}{n}\right) y(t) & \text{for } t \in S_n. \end{cases}$$

Then we have

$$\int_{S_1} y_n\, d\mu = \int_{S_1 \setminus S_n} y\, d\mu + \int_{S_n} \left(1 - \frac{1}{n}\right) y\, d\mu = \int_{S_1 \setminus S_n} y\, d\mu - \int_{S_1 \setminus S_n} \left(1 - \frac{1}{n}\right) y\, d\mu +$$

$$+ \int_{S_1 \setminus S_n} \left(1 - \frac{1}{n}\right) y\, d\mu + \int_{S_n} \left(1 - \frac{1}{n}\right) y\, d\mu = \frac{1}{n} \int_{S_1 \setminus S_n} y\, d\mu + \int_{S_1} \left(1 - \frac{1}{n}\right) y\, d\mu \leq$$

$$\leq \frac{1}{n} \int_{S_1 \setminus S_n} y\, d\mu + \left(1 - \frac{1}{n}\right) \left[\int_{S_1} e\, d\mu + \int_{S_2} (x - e)\, d\mu\right].$$

As consequence, we can write

$$\int_{S_1} (y_n - e)\, d\mu - \left(1 - \frac{1}{n}\right) \int_{S_2} (x - e)\, d\mu \leq \frac{1}{n} \left[\int_{S_1 \setminus S_n} y\, d\mu - \int_{S_1} e\, d\mu\right].$$

Since $-\int_{S_1} e\, d\mu$ belongs to the interior of the negative cone, by absolute continuity of the integral we have that, for a sufficiently large n,

$$\int_{S_1 \setminus S_n} y\, d\mu - \int_{S_1} e\, d\mu \ll 0$$

and, consequently

$$\int_{S_1} (y_n - e)\, d\mu - \left(1 - \frac{1}{n}\right) \int_{S_2} (x - e)\, d\mu \ll 0.$$

Let us fix now one of this integer, n_o and the corresponding disk, of radius ε, centered in $\int_{S_1} (y_{n_o} - e)\, d\mu - \left(1 - \frac{1}{n_o}\right) \int_{S_2} (x - e)\, d\mu$ contained in the interior of the negative cone. By the weak Lyapunov's convexity theorem, there exists a subcoalition $H_2 \subseteq S_2$ such that

$$\left\| \int_{H_2} (x - e)\, d\mu - \left(1 - \frac{1}{n_o}\right) \int_{S_2} (x - e)\, d\mu \right\| < \varepsilon$$

from which it follows that

$$\int_{S_1} (y_{n_o} - e)\, d\mu - \int_{H_2} (x - e)\, d\mu \ll 0$$

and the conclusion, when y is replaced by y_{n_o} and S_2 by H_2.

□

Let us show now that the ex-post core of an atomless economy coincides with the set of ex-post c-fair allocation.

Theorem 3.21. *Let T be atomless, i.e. $T_1 = \emptyset$. Assume the interior of $\mathbb{B}_+$ is nonempty. Assume further that μ-a.a. the vectors e_t are strictly positive for any state and $u_t(\omega, \cdot)$ is continuous and strictly increasing for any state. Then*

$$\mathcal{C}_{fair}(\mathcal{E}) = \mathbf{C}(\mathcal{E}).$$

PROOF: It is enough to prove the statement for an economy with complete information.

Let x be a core allocation and assume that it is not c-fair. Observe that since x is a core allocation, the monotonicity gives that the feasibility actually guarantees the equality $\int_T x d\mu = \int_T e d\mu$. Moreover, by Lemma 3.20 there exist the coalitions S_1 and S_2 with $\mu(S_1) > 0$ and the assignment y such that

$$\int_{S_1} (y - e)\, d\mu \ll \int_{S_2} (x - e)\, d\mu;$$

$$u_t(y(t)) > u_t(x(t)) \text{ for a.a. } t \in S_1.$$

In particular, we can write

$$\frac{1}{2}\int_{S_1} (x + y)\, d\mu - \int_{S_1} e\, d\mu = \frac{1}{2}\int_{S_1} (x - e)\, d\mu + \frac{1}{2}\int_{S_1} (y - e)\, d\mu \ll \frac{1}{2}\int_{S_1 \cup S_2} (x - e)\, d\mu.$$

Let $\varepsilon > 0$ be the radius of a disk centered in $\frac{1}{2}\int_{S_1} (x + y)\, d\mu - \frac{1}{2}\int_{S_1 \cup S_2} (x - e)\, d\mu$, which is contained in the interior of the negative cone.

Since the closure of the set

$$\left\{\int_{S_1} z\, d\mu : u_t(y(t)) > u_t(x(t)) \text{ for a.a. } t \in S_1\right\}$$

is convex, it contains the vector $\frac{1}{2}\int_{S_1} (x + y)\, d\mu$. So there exists an assignment z such that

$$\left\| \int_{S_1} (z - e)\, d\mu - \left[\frac{1}{2}\int_{S_1} (x + y)\, d\mu - \int_{S_1} e\, d\mu\right] \right\| < \varepsilon$$

and, consequently,

$$-v := \int_{S_1} (z - e)\, d\mu - \frac{1}{2}\int_{S_1 \cup S_2} (x - e)\, d\mu \ll 0.$$

Again, let $\varepsilon > 0$ choosen in such a way that the disk centered in v and radius ε is fully contained into $\mathrm{int}(\mathbb{B}_+)$.

Now, if the set $S_1 \cup S_2$ exhausts T, i.e. $\mu\left(T \setminus (S_1 \cup S_2)\right) = 0$, then the integral $\int_{S_1 \cup S_2} (x - e)\, d\mu$ is zero and $\int_{S_1} (z - e)\, d\mu \ll 0$ gives the feasibility of z, namely we violate that x is in the core. We can therefore assume that the set $T \setminus (S_1 \cup S_2)$ is of positive μ-measure. By the relative convexity of the range of the integral function over T, we find a subcoalition $S \subseteq T \setminus (S_1 \cup S_2)$ with

$$\left\| \int_S (x-e)\, d\mu - \frac{1}{2} \int_{T\setminus(S_1\cup S_2)} (x-e)\, d\mu \right\| < \varepsilon.$$

Define the assignment s as

$$s(t) = \begin{cases} z(t) & \text{for } t \in S_1 \\ x(t) & \text{for } t \in S. \end{cases}$$

Then $u_t(s(t)) \geq u_t(x(t))$, a.e. in $S_1 \cup S$ and $u_t(s(t)) > u_t(x(t))$, a.e. in S_1. Moreover

$$\left\| \int_{S_1\cup S} (s-e)\, d\mu - \left(\int_{S_1} (z-e)\, d\mu - \frac{1}{2} \int_{S_1\cup S_2} (x-e)\, d\mu \right) \right\| =$$

$$\left\| \int_S (x-e)\, d\mu - \frac{1}{2} \int_{T\setminus(S_1\cup S_2)} (x-e)\, d\mu + \frac{1}{2} \int_T (x-e)\, d\mu) \right\| < \varepsilon.$$

It follows that $\displaystyle\int_{S_1\cup S} (s-e)\, d\mu \ll 0$, so we can take a vector $w \gg 0$ such that

$$w + \int_{S_1\cup S} s\, d\mu = \int_{S_1\cup S} e\, d\mu.$$

Finally, with the following modification of s

$$s(t) = \begin{cases} z(t) & \text{for } t \in S_1 \\ x(t) + \frac{w}{\mu(S)} & \text{for } t \in S. \end{cases}$$

we violate that x is in the core since the coalition $S_1 \cup S$ blocks x via the new assignment s. Indeed $u_t(s(t)) > u_t(x(t))$, a.e. in $S_1 \cup S$ and

$$\int_{S_1\cup S} s\, d\mu = \int_{S_1} z\, d\mu + \int_S x\, d\mu + w = \int_{S_1\cup S} e\, d\mu.$$

The contradiction concludes the proof.

□

The case of a general DIME, i.e. the case in which the set T_1 of large traders is non-empty, can be now analyzed.

Theorem 3.22. *Let the interior of* $\mathbb{B}_+$ *be nonempty. Assume further that* μ*-a.a. the initial endowments* e_t *are strictly positive and the utilities* $u_t(\omega,\cdot)$ *are continuous and strictly increasing for any state. Also assume that* $u_t(\omega,\cdot)$ *is quasi-concave for* $t \in T_1$*. Then,*

$$\mathcal{C}_{A-fair}(\mathcal{E}) = \mathbf{C}_A(\mathcal{E})$$

PROOF: The proof follows from Corollary 3.16, Theorem 3.18 and Theorem 3.21. □

Remark 3.23. The equivalence between c-fair and core allocations in a large economy proved with Theorem 3.21 is not surprising. It has been proved for finite dimensional economy in [17, Theorem 1] as consequence of the core equivalence theorem. What is interesting to point out in connection with our Theorem 3.21 is that it provides a direct proof of this equivalence, not related to core equivalence results. This difference is relevant in the case of many commodities, since the commodity space needs not to be separable and, consequently, the range of applicability is wider. It was indeed shown in [29] and [32], that the class of Banach spaces $\mathbb{B}$ such that, under standard assumptions, any large economy with the commodity space $\mathbb{B}$ exhibits the core-Walras equivalence theorem is exactly the class of Banach spaces that are separable. So, examples could be provided of large economies with many commodities for which the core is exactly made by c-fair allocations despite core-equivalence, and then perfect competition, fails (see for example the discussion in [9, Remark 3.9]).

The same remark clearly applies to Theorem 3.22. Notice that Theorem 3.22 simply confirms that, as proved in section 3.5, the mixed market behaves like a large one in terms of fairness of allocations provided that the power of large traders is nullified by means of their convexification.

4 Coalition fairness of allocations: the interim stage

In this section we analyze coalition fairness properties of allocations when traders take their decisions at the interim stage. If this is the case, then clearly one has to take into account the possibility that agents form coalitions in order to better redistribute aggregate resources, but also to take some advantage of their private information by means of a communication system allowing them to share it partially or in full. This analysis, that will lead to the so called *fine* notions, requires a detailed description of the traders' asymmetric information structure and communication system, since both elements affect now the equilibrium allocations.

The finite family $\mathbb{I} := \{\mathcal{F}_t : t \in T\}$ induces a partition $(\Theta_\mathcal{I})_{\mathcal{I}\in\mathbb{I}}$ of the space of agents made of sets $\Theta_\mathcal{I} = \{t \in T : \mathcal{F}_t = \mathcal{I}\}$. We assume that each $\Theta_\mathcal{I}$ belongs to $\mathcal{T}$ and moreover that it is of positive measure. Every agent belonging to $\Theta_\mathcal{I}$ is of information type $\mathcal{I}$ and for a coalition γ of support S, we denote by $\mathbb{I}(S)$ the set of information types present in γ, namely $\mathcal{I} \in \mathbb{I}(S) \Leftrightarrow \mu(S \cap \Theta_\mathcal{I}) > 0$.

Let us fix the following notation: $\mathcal{F}(S) := \bigvee\{\mathcal{I} : \mathcal{I} \in \mathbb{I}(S)\}$ and assume throughout the sequel of the section that $\mathcal{F}(T) = \mathcal{F}$. Then clearly $\mathcal{F}(S)$ represents the joint information which is present in a coalition S, that is the finer information available to the members of S when they share their private information. The condition $\mathcal{F}(T) = \mathcal{F}$ means that $\mathcal{F}$ does not contain additional events with respect to which no trader has information. We assume that the information sharing in a coalition does not necessarily reduce to the full sharing, but may take place by means of more general information structures. In the spirit of [37], an *information structure* for a generalized coalition γ is a family $(\mathcal{H}_t)_{t\in S}$ of subalgebras of $\mathcal{F}$ such that the set $\{t \in S : \mathcal{H}_t = \mathcal{G}\} \in \mathcal{T}$, for any subalgebra $\mathcal{G}$ of $\mathcal{F}$. If for any agent we also have that $\mathcal{H}_t$ lies in between $\mathcal{F}_t$ and $\mathcal{F}(S)$, that is $\mathcal{H}_t$ represents an information that may be available in γ and it is finer than the private information $\mathcal{F}_t$ and coarser than the pooled information $\mathcal{F}(S)$, then we say that $(\mathcal{H}_t)_{t\in S}$ is a *communication system* for γ (see [37]).

A communication system is full when for any t we have $\mathcal{H}_t = \mathcal{F}(S)$.

An event $F \in \mathcal{F}$ is *common knowledge* for γ with respect to a communication system $(\mathcal{H}_t)_{t\in S}$ if it belongs to $\wedge_{t\in S}\mathcal{H}_t$, i.e. an event that all the members of S are able to discern according to the given communication system. As it is usual, a communication system enlarges the field of events that the members of the coalition can distinguish, but does not produce new information.

Let us denote by $E_t(f \mid \mathcal{G})$ the $I\!P_t$-conditional expectation of f given the subalgebra $\mathcal{G}$ of $\mathcal{F}$ (i.e. the $\mathcal{G}$-measurable random variable having the same mean of f on the elements of the partition that generates $\mathcal{G}$).

4.1 The notion of fine coalition fair allocation

We introduce now the notion of *(Aubin) fine coalition fair allocations* namely allocations for which the absence of envy among coalitions is robust even with respect to the possibility of communication. A coalition can be envious only in an event that all members can distinguish using some of the possible communication system. So a kind of coordination in order to define joint envy is required. Moreover, the evaluation criterion of envious traders takes into account the conditional expectation with respect to the new information deriving from their communication.

Definition 4.1. *We say that x is an Aubin fine c-fair allocation, we write $x \in \mathcal{C}^{A-fine}_{A-fair}(\mathcal{E})$ for short, if: no Aubin coalitions γ_1, γ_2, no communication system $(\mathcal{H}_t)_{t\in S_1}$, no common knowledge event $F \in \wedge_{t\in S_1}\mathcal{H}_t$, and no assignment y can be found such that the conditions*

0) γ_1 *is non-negligible;*

1) $\displaystyle\int_{\gamma_1}(y_\omega - e_\omega)\,d\mu \le \int_{\gamma_2}(x_\omega - e_\omega)\,d\mu,\ \forall\omega \in F;$

2) $\gamma_1 + \gamma_2 \le 1$ μ *- a.e. on* T,;

3)* *for almost all* $t \in S_1$, $E_t(u_t(\cdot, y(\cdot,t)) \mid \mathcal{H}_t) > E_t(u_t(\cdot, x(\cdot)) \mid \mathcal{H}_t)$ *holds pointwise on* F*;*

are all fulfilled.

In the case of ordinary participation, the set of *fine c-fair allocations* will be denoted by $\mathcal{C}^{fine}_{fair}$. As in the ex-post case, differentiated participation do really matter in the presence of atoms.

Theorem 4.2. *Assume the interior of $I\!B_+$ is nonempty. Assume further that μ-a.a. the vectors e_t are strictly positive for any state and $u_t(\omega,\cdot)$ is continuous and strictly increasing for any state. Then, $\mathcal{C}^{A-fine}_{A-fair}(\mathcal{E}) = \mathcal{C}^{fine}_{fair}$ if T is atomless.*

PROOF: One inclusion is obvious. Let x be a fine c-fair allocation. Assume, by contradiction, that it is not Aubin fine c-fair and therefore take the coalitions γ_1, γ_2, an

assignment y, a communication system $(\mathcal{H}_t)_{t\in S}$ and a common knowledge event F for which the conditions 0), 1), 2), 3*) of Definition 4.1 are all satisfied.
We claim that it is possible to find new coalitions $\widetilde{\gamma}_i$ and a new assignment $\widetilde{y}$, such that: the supports S_i are unchanged, the assignment $\widetilde{y}$ is strictly positive on a subset of S_1 of positive measure and all the previous conditions are satisfied by $\widetilde{\gamma}_i$ and $\widetilde{y}$.

First observe that there exists a subset C, with positive $\mu-$measure, of the support of the Aubin coalition γ_1 such that we can further assume $y(\omega,t) \gg 0$, $\forall(\omega,t) \in F \times C$, without loss of generality. To prove such claim, by continuity of the expectation operator, find a part C of the support S_1 of γ_1 and $\varepsilon \in]0,1[$ such that

for almost all $t \in C$, $E_t(u_t(\cdot,\varepsilon y(\cdot,t)) \mid \mathcal{H}_t) > E_t(u_t(\cdot,x(\cdot)) \mid \mathcal{H}_t)$ pointwise on F.

Then consider for each $\omega \in F$, the following identity:

$$\int_T \gamma_1(y_\omega - e_\omega)\,d\mu = \int_{S_1\setminus C} \gamma_1 y_\omega\,d\mu + \int_C \frac{\gamma_1}{\varepsilon}\left[\varepsilon y_\omega + (1-\varepsilon)e_\omega\right] d\mu - \left(\int_{S_1\setminus C} \gamma_1 e_\omega\,d\mu + \int_C \frac{\gamma_1}{\varepsilon} e_\omega\,d\mu\right)$$

that can be written as

$$\int_T \gamma_1(y_\omega - e_\omega)\,d\mu = \int_{S_1} \hat{\gamma}_1(\hat{y}_\omega - e_\omega)\,d\mu$$

where $\hat{\gamma}_1$ and $\hat{y}$ only differ from γ_1 and y_ω on the set C where they take, respectively, the values $\dfrac{\gamma_1}{\varepsilon}$ and $\varepsilon y_\omega + (1-\varepsilon)e_\omega$.
By setting $\widetilde{\gamma}_1 = \hat{\gamma}_1$ and $\widetilde{\gamma}_2 = \gamma_2$, we have that for almost all $t \in S_1$, $E_t(u_t(\cdot,\hat{y}(\cdot,t)) \mid \mathcal{H}_t) > E_t(u_t(\cdot,x(\cdot)) \mid \mathcal{H}_t)$ holds pointwise on F, $\hat{y} \gg 0$ over the set C and $\int_{\widetilde{\gamma}_1} (\hat{y}_\omega - e_\omega)\,d\mu \le \int_{\widetilde{\gamma}_2} (x_\omega - e_\omega)\,d\mu$.
Take now, again by continuity, a set $B \subseteq C$ and $\delta \in]0,1[$ with the following property

for almost all $t \in B$, $E_t(u_t(\cdot,\delta\hat{y}(\cdot,t)) \mid \mathcal{H}_t) > E_t(u_t(\cdot,x(\cdot)) \mid \mathcal{H}_t)$ pointwise on F.

By setting $\widetilde{y}$ as equal to $\hat{y}$ except over B where $\widetilde{y} = \delta\hat{y}$, we still have that all the previous conditions are satisfied, but now $\int_{\widetilde{\gamma}_1} (\widetilde{y}_\omega - e_\omega)\,d\mu \ll \int_{\widetilde{\gamma}_2} (x_\omega - e_\omega)\,d\mu$. Indeed,

$$\int_{\widetilde{\gamma}_1} (\widetilde{y}_\omega - e_\omega)\,d\mu = \int_{S_1\setminus B} \widetilde{\gamma}_1(\hat{y}_\omega - e_\omega)\,d\mu + \int_B \widetilde{\gamma}_1(\delta\hat{y}_\omega - e_\omega)\,d\mu \ll$$

$$\ll \int_{S_1} \widetilde{\gamma}_1(\hat{y}_\omega - e_\omega) \le \int_{\widetilde{\gamma}_2} (x_\omega - e_\omega)\,d\mu.$$

Finally, since $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ can be uniformly approximated by simple functions, we can find finitely many valued coalitions still denoted by $\widetilde{\gamma}_i$, with support S_i, satisfying $\int_{S_1} \widetilde{\gamma}_1(\widetilde{y}_\omega - e_\omega)\,d\mu \ll \int_{S_2} \widetilde{\gamma}_2(x_\omega - e_\omega)\,d\mu$, pointwise on F. By replacing $\widetilde{\gamma}_i$ with $\frac{\widetilde{\gamma}_i}{\bar{\gamma}}$ where $\bar{\gamma}$ is the maximum of $\widetilde{\gamma}_1 + \widetilde{\gamma}_2$ we see that condition 2) is also satisfied.

Now, like in the proof of Proposition 3.3, by means of the weak infinite dimensional version of the Lyapunov's theorem, we can find two essentially disjoint subcoalitions $G_i \subseteq S_i$ such that $\int_{G_1} (\widetilde{y}_\omega - e_\omega)\, d\mu \ll \int_{G_2} (x_\omega - e_\omega)\, d\mu$, holds for each $\omega \in F$. Moreover,

for almost all $t \in G_1$, $E_t(u_t(\cdot, \widetilde{y}(\cdot,t)) \mid \mathcal{H}_t) > E_t(u_t(\cdot, x(\cdot)) \mid \mathcal{H}_t)$ pointwise on F.

Consider now a state $\omega_o \in F$ and the corresponding event $F(\omega_o)$ in the field $\wedge_{t \in G_1} \mathcal{H}_t$. Then clearly $F(\omega_o) \subseteq F$ and so the previous conditions imply a contradiction to the fact that the allocation x is fine c-fair $\square$

Under a suitable set of assumptions, every allocation which is c-fair at the interim stage is also c-fair at the ex-post stage. In particular, under the assumptions formulated in Theorems 3.11 and 3.12, it is also individually equitable for each possible realization of uncertainty.

Theorem 4.3. *Assume the interior of $I\!B_+$ is nonempty. Assume further that μ-a.a. the vectors e_t are strictly positive for any state and $u_t(\omega, \cdot)$ is continuous and strictly increasing for any state. Then, $\mathcal{C}^{A-fine}_{A-fair}(\mathcal{E}) \subseteq \mathcal{C}_{A-fair}(\mathcal{E})$.*

PROOF: Let x be an Aubin fine c-fair allocation and assume that it is not ex-post Aubin c-fair. Then, in a certain state of nature ω_0, by Proposition 3.6, we can assume that the envious coalition γ_1 has full support and therefore we have: $\int_T \gamma_1(\bar{y} - e)\, d\mu \leq \int_{S_2} \gamma_2(x - e)\, d\mu$ and $\bar{y} \succ_T x$.

Define now the function $z : \Omega \times T \to I\!B_+$ by setting

$$z(\omega, t) = \begin{cases} \bar{y}(t) & \text{if } \omega \in \Pi(\omega_0) \\ e(\omega, t) & \text{if } \omega \notin \Pi(\omega_0) \end{cases}$$

Then any z_t is $\mathcal{F}$-measurable and, for $\omega \in F := \Pi(\omega_0)$, $\int_T \gamma_1(z_\omega - e_\omega)\, d\mu \leq \int_{S_2} \gamma_2(x_\omega - e_\omega)\, d\mu$.

Since γ_1 has full support, the set F is a common knowledge event for γ_1. Also note that the corresponding pooled information is $\mathcal{F}$ because of our assumptions. For $\omega \in F = \Pi(\omega_0)$ we have:

$$E_t(u_t(\cdot, z(\cdot,t)) \mid \mathcal{H}_t)(\omega) = u_t(\omega, z(\omega,t)) = u_t(\omega, \bar{y}(t)) = u_t(\omega_0, \bar{y}(t)) >$$

$$> u_t(\omega_0, x(\omega_0,t)) = u_t(\omega, x(\omega,t)) = E_t(u_t(\cdot, x(\cdot)) \mid \mathcal{H}_t)(\omega)$$

and a contradiction. $\square$

4.2 Fine c-fair allocations and the fine core

As in the case of ex-post coalition fairness, Definition 4.1 implicitly detects allocations of the Aubin fine core as defined below. Indeed, if for an allocation x there exist a generalized coalition γ_1, a communication system $(\mathcal{H}_t)_{t\in S_1}$, a corresponding common knowledge event $F \in \wedge_{t\in S_1} \mathcal{H}_t$, and an assignment y such that one has, together with properties 0) and 3*), also property

$$1^{**})\ \int_{\gamma_1} y_\omega \, d\mu \leq \int_{\gamma_1} e_\omega \, d\mu, \ \forall \omega \in F;$$

then, by setting $\gamma_2 = 0$, Definition 4.1 is violated.

Definition 4.4. *We say that an allocation x is an Aubin fine core allocation, shortly $x \in \mathbf{C}^{A-fine}(\mathcal{E})$, if there exist no Aubin coalition γ_1, a communication system $(\mathcal{H}_t)_{t\in S_1}$, a corresponding common knowledge event $F \in \wedge_{t\in S_1} \mathcal{H}_t$, and an assignment y such that 0), 1**) and 3*) are satisfied.*

The previous onservation says that Aubin fine c-fair allocations form a subset of the Aubin fine core: $\mathcal{C}^{A-fine}_{A-fair}(\mathcal{E}) \subseteq \mathbf{C}^{A-fine}(\mathcal{E})$. Moreover, if we exclude in the definition above the possibility of Aubin cooperative behavior and limit coalitions to be only the ordinary ones, we obtain the concept of *fine core* defined according to [37], which is trivially a superset of $\mathbf{C}^{A-fine}(\mathcal{E})$ that we denote by $\mathbf{C}^{fine}(\mathcal{E})$. Clearly, for complete information economies, the Aubin fine core is the same as the Aubin core, i.e. $\mathbf{C}^{A-fine}(\mathcal{E}(\omega)) = \mathbf{C}_A(\mathcal{E}(\omega))$, and the fine core is simply the core, i.e. $\mathbf{C}^{fine}(\mathcal{E}(\omega)) = \mathbf{C}(\mathcal{E}(\omega))$.

Remark 4.5. Assume that the interior of $I\!B_+$ is nonempty. Assume further that μ-a.a. the vectors e_t are strictly positive for any state and $u_t(\omega, \cdot)$ is continuous and strictly increasing for any state. Then, in the definition of $\mathbf{C}^{A-fine}(\mathcal{E})$ we can be more demanding replacing condition 0) by $0'$). Also, we define the same set $\mathbf{C}^{A-fine}(\mathcal{E})$ if we add in its definition the further condition that the value of γ_1 is 1 on the atomless part of its support.
Consequently, when T is atomless, then $\mathbf{C}^{A-fine}(\mathcal{E}) = \mathbf{C}^{fine}(\mathcal{E})$. The proof of this result, similar to the one given for Theorem 4.2 is presented below for completeness.

Theorem 4.6. *Assume that the interior of $I\!B_+$ is nonempty. Assume further that μ-a.a. the vectors e_t are strictly positive for any state and $u_t(\omega, \cdot)$ is continuous and strictly increasing for any state. Then, $\mathbf{C}^{A-fine}(\mathcal{E}) = \mathbf{C}^{fine}(\mathcal{E})$ if T is atomless.*

PROOF: Let x be a fine core allocation. Assume, by contradiction, that it is not in the Aubin fine core and therefore take an Aubin coalition γ, an assignment y, a communication system $(\mathcal{H}_t)_{t\in S}$ and a common knowledge event F for which the blocking conditions: $\mu(S) > 0$; for almost all $t \in S$, $E_t(u_t(\cdot, y(\cdot, t)) \mid \mathcal{H}_t) > E_t(u_t(\cdot, x(\cdot)) \mid \mathcal{H}_t)$ holds pointwise on F;

$$(+)\ \int_{\gamma} y_\omega \, d\mu \leq \int_{\gamma} e_\omega \, d\mu, \ \ \forall \omega \in F,$$

are satisfied. Then there exists a subset C, with positive $\mu-$measure, of the support of γ such that we can further assume $y(\omega,t) \gg 0$, $\forall(\omega,t) \in F \times C$, without loss of generality.
To prove such claim, by continuity of the expectation operator, find a part C of the support S of γ and $\varepsilon \in]0,1[$ such that

$$\text{for almost all } t \in C,\ E_t(u_t(\cdot, \varepsilon y(\cdot,t)) \mid \mathcal{H}_t) > E_t(u_t(\cdot, x(\cdot)) \mid \mathcal{H}_t) \text{ pointwise on } F.$$

Then, because of condition (+) above, multiplying both terms by ε, for $\omega \in F$ it is possible to write that

$$\int_{S\backslash C} \gamma\, \varepsilon y_\omega \, d\mu + \int_C \gamma[\varepsilon y_\omega + (1-\varepsilon)e_\omega]\, d\mu \le \int_{S\backslash C} \gamma \varepsilon e_\omega \, d\mu + \int_C \gamma e_\omega \, d\mu$$

or, by setting, $\bar{\gamma} := \gamma\varepsilon$ on $S\backslash C$, $\bar{\gamma} := \gamma$ on C and $\bar{y}_\omega := y_\omega$ on $S\backslash C$, $\bar{y}_\omega := \varepsilon y_\omega + (1-\varepsilon)e_\omega$ on C,

$$\int_{\bar{\gamma}} \bar{y}_\omega \, d\mu \le \int_{\bar{\gamma}} e_\omega \, d\mu, \qquad \text{with } \bar{y}(\omega,t) \gg 0\, \forall(\omega,t) \in F \times C.$$

So, certainly we assume also that $y(\omega,t) \gg 0\, \forall(\omega,t) \in F \times C$ with $C \subseteq S$. By continuity, we find $\delta \in]0,1[$ and $D \subseteq C$ with, for almost all $t \in D$,

$$E_t(u_t(\cdot, \delta y(\cdot,t)) \mid \mathcal{H}_t) > E_t(u_t(\cdot, x(\cdot)) \mid \mathcal{H}_t) \text{ pointwise on } F$$

and defining $z_\omega := y_\omega$ on $S \setminus D$, $z_\omega := \delta y_\omega$ on D, we have that for any state in F,

$$\int_\gamma z_\omega \, d\mu = \int_D \gamma \delta y_\omega \, d\mu + \int_{S\backslash D} \gamma y_\omega \, d\mu \ll \int_D \gamma y_\omega \, d\mu + \int_{S\backslash D} \gamma y_\omega \, d\mu \le \int_\gamma e_\omega \, d\mu.$$

This means that, possibly replacing y by z, we can further assume that for the inequalities (+) the strict inequality sign $\ll$ holds true. Now the weak infinite dimensional version of the Lyapunov theorem, namely

$$\left\{ \left(\int_S \Lambda(y_\omega - e_\omega) \right)_{\omega \in F} : \Lambda\ \mu\text{−measurable, } [0,1]\text{− valued function} \right\} =$$

$$= \text{ cl } \left\{ \left(\int_X (y_\omega - e_\omega) \right)_{\omega \in F} : X\ \mu\text{−measurable, } X \subseteq S \right\},$$

applies and, with reference to the point $\left(\int_\gamma (y_\omega - e_\omega) \right)_{\omega \in F}$ of $\text{int}(\mathbb{B}_-^F)$, there exists a subset X of S such that

$$\left(\int_X (y_\omega - e_\omega) \right)_{\omega \in F} \in \text{int}(\mathbb{B}_-^F)$$

and therefore we get a contradiction to the assumption that x is a fine core allocation.
□

Going back to the general case of a DIME, we have the following result concerning fine core notions. In [14, Theorem 3.1], whose setting is that of an atomless measure space of agents with a finite dimensional space of commodities, it is proved that the fine core is another subset of the ex-post core $\mathbf{C}(\mathcal{E})$. For the validity of the result, the mentioned setting plays a pivotal role. Indeed, it is the assumption of an atomless space of agents, as well as the fact that the commodity space is finite dimensional, that permits to apply Vind's Theorem in the framework of the deterministic economies $\mathcal{E}(\omega)$ to block an allocation not in the core by means of a coalition of arbitrarily big measure; consequently a fine core allocation will also be an ex-post core allocation.

Dealing with a DIME, and in the same spirit of previous Theorem 4.3, the use of participation rates allows us to overcome difficulties arising from the presence of atoms and of infinitely many commodities. In particular, participation rates permit to apply a special form of Vind's Theorem consisting in the possibility of including, as we are going to see, each trader in the blocking coalition γ.

Theorem 4.7. *Assume the interior of $\mathbb{B}_+$ is nonempty. Assume further that μ-a.a. the vectors e_t are strictly positive for any state and $u_t(\omega, \cdot)$ is continuous and strictly increasing for any state. Then,* $\mathbf{C}^{A-fine}(\mathcal{E}) \subseteq \mathbf{C}_A(\mathcal{E})$.

PROOF: Let x be in the Aubin fine core and assume x is not in the Aubin ex-post core. Then there is a state of nature ω_0 for which an Aubin coalition γ and an assignment exist with $\mu(S) > 0$, $\int_\gamma y_{\omega_0}\, d\mu \leq \int_\gamma e_{\omega_0}\, d\mu$ and $u_t(\omega_0, y(\omega_0, t)) > u_t(\omega_0, x(\omega_0, t))$ for almost all $t \in S$. Since x_{ω_0} is not an Aubin core allocation of the deterministic economy $\mathcal{E}(\omega_0)$, it can be blocked by an Aubin coalition α with full support[13], namely we have $\mu(\alpha) = \mu(T)$, $\int_\alpha \bar{y}\, d\mu \leq \int_\alpha e_{\omega_0}\, d\mu$ and $u_t(\omega_0, \bar{y}(t) > u_t(\omega_0, x(\omega_0, t))$ for almost all $t \in S$.

Define now the function $z : \Omega \times T \to \mathbb{B}_+$ by setting

$$z(\omega, t) = \begin{cases} \bar{y}(t) & \text{if } \omega \in \Pi(\omega_0) \\ e(\omega, t) & \text{if } \omega \notin \Pi(\omega_0) \end{cases}$$

Then any z_t is $\mathcal{F}$-measurable and $\int_\alpha z_\omega\, d\mu \leq \int_\alpha e_\omega\, d\mu$. Moreover, since α is of full support, note that the corresponding pooled information is $\mathcal{F}$ because of our assumptions.

Because of the $\mathcal{F}$-measurability of u_t, $u_t(\cdot, x(\cdot))$ and $u_t(\cdot, \bar{y}(\cdot, t))$, at this point we recognize that z is an Aubin fine improvement of x that it is therefore blocked by α. Since the communication system is the full one, for $\omega \in F = \Pi(\omega_0)$ we have:

$$E_t(u_t(\cdot, z(\cdot, t)) \mid \mathcal{H}_t)(\omega) = u_t(\omega, z(\omega, t)) = u_t(\omega, \bar{y}(t)) = u_t(\omega_0, \bar{y}(t)) >$$

$$> u_t(\omega_0, x(\omega_0, t)) = u_t(\omega, x(\omega, t) = E_t(u_t(\cdot, x(\cdot)) \mid \mathcal{H}_t)(\omega).$$

This is a contradiction. □

[13]Such result is proved in [7, Theorem 4.13] for finite economies and in [27, Theorem 3.1] for general mixed markets with finitely many goods. It can also be extended to our present setting.

The proof of Theorem 4.7 shows that the ex-post Aubin core of a mixed market includes also the larger set of allocations that cannot be blocked by coalitions of full support via their full communication system, the one that actually coincides with the whole information $\mathcal{F}$.

5 Some consequences for competitive allocations

In this section we present c-fairness properties of competitive outcomes of the economy $\mathcal{E}$. In particular, we study conditions under which it is still true that the only way to generate allocations which coalitions perceive as fair, for each possible realization of uncertainty, is by means of competitive market mechanisms. As in the previous sections, we shall refer to the ex-post stage as well as to the interim stage. Hence, we shall focus on the notions of *ex-post competitive equilibria* and *rational expectations equilibria*. As it is known from [13] (see also [11]), under some hypotheses these two concepts coincide; a circumstance that it is not always true.

First of all, let us state a general equivalence theorem between Walrasian and Aubin Core allocations of the economy with complete information $\mathcal{E}(\omega_0)$ (compare with [9, Remark 5.1]).

Theorem 5.1. *Assume that $I\!B$ is separable and the interior of $I\!B_+$ is nonempty. Let $\omega_0 \in \Omega$ be such that: μ-a.a. $e_t(\omega_0) \gg 0$; μ-a.a. $u_t(\omega_0, \cdot)$ are continuous, strictly increasing; for each $t \in T_1$, $u_t(\omega_0, \cdot)$ are strictly quasi-concave. Then*

$$WE(\mathcal{E}(\omega_0)) = \mathbf{C}_A(\mathcal{E}(\omega_0)).$$

Let us introduce the set of ex-post competitive equilibria

$$\mathcal{W}(\mathcal{E}) := \{x \,|\, x \text{ is an } assignment \text{ and } x(\omega, \cdot) \in W(\mathcal{E}(\omega)),\ \forall \omega \in \Omega\}.$$

For a general differential information economy, by point b) of subsection 3.3 according to which

$$\mathcal{W}(\mathcal{E}) \subseteq \{x : x \text{ is an assignment and } x_\omega \in \mathcal{C}_{A-fair}(\mathcal{E}(\omega)),\ \forall \omega \in \Omega\},$$

and Proposition 3.7, it is clear that

Proposition 5.2. *In any DIME $\mathcal{E}$, it is true that $\mathcal{W}(\mathcal{E}) \subseteq \mathcal{C}_{A-fair}(\mathcal{E})$.*

Therefore, from Theorem 5.1 it follows that

Corollary 5.3. *Let $\mathcal{E}$ be a DIME such that $I\!B$ is separable and the interior of $I\!B_+$ is nonempty. Assume that for each state ω: μ-a.a. $e_t(\omega) \gg 0$; μ-a.a. $u_t(\omega, \cdot)$ are continuous, strictly increasing; for each $t \in T_1$, $u_t(\omega_0, \cdot)$ are strictly quasi-concave. Then*

$$\mathcal{W}(\mathcal{E}) = \mathcal{C}_{A-fair}(\mathcal{E}) = \mathbf{C}_A(\mathcal{E})^{14}.$$

[14] This equivalence result for the set $\mathcal{W}$ could also be provided when quasi-concavity is removed from large traders' utilities. In this case, the fairness and core notions should be defined with respect to a different notion of generalized coalition, allowing for non-convexity (see [19]).

Let us now recall the notion of rational expectations equilibrium. In this context agents t restrict their consumption choices to (ex-post) budget sets defined as follows:

$$B_t(\omega, p) = \{a \in \mathbb{B}_+ : p(\omega) \cdot a \leq p(\omega) \cdot e(\omega, t)\},$$

in the state ω and with respect to a prevailing system of prices p belonging to $(\mathbb{B}'_+)^\Omega$ [15]. Moreover, they evaluate choices through a state by state comparison of the conditional expectation of their utility, taking into account both private information and the information revealed by prices.

We denote by $\sigma(p)$ the smallest σ-algebra of $\mathcal{F}$ that makes the function p measurable ($\sigma(p)$ represents the information contained in p).

Definition 5.4. *A rational expectations equilibrium is a pair (p, x) where p is a price system, and x is an allocation such that:*

i) *$x(\cdot, t)$ is $(\sigma(p) \vee \mathcal{F}_t)$-measurable, for μ-a.a. $t \in T$;*

ii) *$x(\omega, t) \in B_t(\omega, p)$ for each $\omega \in \Omega$ and for μ-a.a. $t \in T$;*

iii) *μ-a.e. in T, if $y : \Omega \to \mathbb{B}_+$ is $(\sigma(p) \vee \mathcal{F}_t)$-measurable and satisfies $y(\omega) \in B_t(\omega, p)$ for each $\omega \in \Omega$, then*

$$E_t(u_t(\cdot, x(\cdot, t)) \mid \sigma(p) \vee \mathcal{F}_t) \geq E_t(u_t(\cdot, y(\cdot)) \mid \sigma(p) \vee \mathcal{F}_t)$$

pointwise on Ω.

The set of allocations that are rational expectations equilibria for a suitable price is denoted by $RE(\mathcal{E})$.

Clearly, when Ω reduces to a singleton, i.e. in a complete information framework, the previous definition gives back the set of Walrasian allocations; we know from [17] that, in this case, Walrasian, c-fair and Core allocations coincide for perfectly competitive economies (T atomless). With asymmetric information we remind that the following results are known:

- For mixed markets:
  - $RE(\mathcal{E})$ may be empty ([23], [3]) even when, according to [13, Theorem 3.1], $\mathbf{C}(\mathcal{E})$ is nonempty;
  - $RE(\mathcal{E}) \not\subseteq \mathbf{C}(\mathcal{E})$, according to [13, Example 4.2],
  - $\mathbf{C}^{fine}(\mathcal{E}) \not\subseteq \mathbf{C}(\mathcal{E})$, according to [14].
- For atomless markets:
  - $RE(\mathcal{E}) = \mathbf{C}(\mathcal{E})$ according to [13, Theorem 4.5];
  - $\mathbf{C}^{fine}(\mathcal{E}) \subset \mathbf{C}(\mathcal{E})$, according to [14, Theorem 3.1];

[15] Here we use standard notation for the topological dual of $\mathbb{B}$, its positive cone and the duality mapping. It is assumed that any contingent price has norm equal to one. We also recall the hypothesis that all state dependent functions are assumed to be $\mathcal{F}$-measurable.

- $\mathbf{C}^{fine}(\mathcal{E}) \subset RE(\mathcal{E})$, according to [14, Corollary 3.4].

The above results hold true, each under its suitable (standard) assumptions concerning continuity, monotonicity, convexity of preferences, for finite dimensional commodity spaces. The assumption of measurability with respect to agents private information, in some of the previous cases, is also required.

The final part of the section is devoted to analyze c-fairness of rational expectations equilibria, extending some of the mentioned results from atomless to mixed economies with infinitely many commodities.

Given the inclusions that we have already seen, it is clear that:

$$RE(\mathcal{E}) \nsubseteq \mathcal{C}_{fair}(\mathcal{E}) \ , \quad RE(\mathcal{E}) \nsubseteq \mathcal{C}_{A-fair}(\mathcal{E}).$$

This is so because a rational expectations equilibrium allocation, which is not necessarily in the ex-post core, a fortiori is not necessarily fair at the ex-post stage. Moreover, c-fairness at the ex-post stage cannot be guaranteed for rational expectations equilibria also in the case in which traders use participation rates. Note that, in interim models, constrained market equilibria are c-fair ([12]) and, in the ex-ante framework, Walrasian expectations equilibria are c-fair ([21]). Results of this type are usually driven by the corresponding inclusions valid for complete information economies. Then, it is a simple remark that rational expectations equilibria which are fully revealing are ex-post coalition fair. Indeed, in this case, the price generates full information, i.e. $\sigma(p) = \mathcal{F}$ and, consequently, the expectations reduces to ex-post utility. A similar reasoning can be applied assuming that the functions $u(\cdot, t)$ are $\mathcal{F}_t$-measurable, to prove that any rational expectations allocation is ex-post Aubin coalition fair. This is a consequence of the inclusion $RE(\mathcal{E}) \subseteq \mathcal{W}(\mathcal{E})$ proved in [13, Theorem 4.3] and Proposition 5.2.

Proposition 5.5. *Under the assumption that μ-a.a. $u_t(\cdot, x)$ are $\mathcal{F}_t$-measurable,*

$$RE(\mathcal{E}) \subseteq \mathcal{C}_{A-fair}(\mathcal{E})\,.$$

As noticed in [11, Theorem 13] the inclusion $\mathbf{C}_A(\mathcal{E}) \subseteq RE(\mathcal{E})$ can be obtained under a suitable set of assumptions. Therefore we get the next result that gives back the original simple idea proposed by [17] in the case of complete information economy. The equivalence shows the capability of participation rates to overcome difficulties related to the presence of large traders as well as to the presence of many commodities.

Theorem 5.6. *Let $\mathcal{E}$ be a DIME such that $I\!B$ is separable, the interior of $I\!B_+$ is nonempty and μ-a.a. $u_t(\cdot, b)$ are $\mathcal{F}_t$-measurable. Assume that for any state ω: μ-a.a. $e_t(\omega) \gg 0$; μ-a.a. $u_t(\omega, \cdot)$ are continuous, strictly increasing and strictly quasi-concave. Then*

$$RE(\mathcal{E}) = \mathcal{C}_{A-fair}(\mathcal{E}) = \mathbf{C}_A(\mathcal{E}).$$

When T is also atomless, then

$$RE(\mathcal{E}) = \mathcal{C}_{fair}(\mathcal{E}) = \mathbf{C}(\mathcal{E}).$$

Theorem 5.6, for the complete information case, does not require convexity of preferences for the negligible agents. However in a DIE, such hypothesis plays a key role in justifying the inclusion $\big(\mathbf{C}_A(\mathcal{E}) =\big)\mathcal{W}(\mathcal{E}) \subseteq RE(\mathcal{E})$, and for this reason it appears also in Corollary 5.7 as well as in the atomless correspondents of Theorem 5.6 and Corollary 5.7. Without the strict quasi concavity, an ex-post competitive allocation can be proved to admit a price under which conditions *ii*) and *iii*) of the Definition 5.4 of rational expectations equilibrium are satisfied, but the measurability with respect to $(\sigma(p) \vee \mathcal{F}_t)$ is not necessarily true. The role played by strict concavity is mentioned by Allen in [3][16] and discussed by Kreps in [23, Section 5].

Finally, combining Theorem 4.7 and Theorem 5.6 we have the following.

Corollary 5.7. *Let $\mathcal{E}$ be a DIME such that $\mathbb{B}$ is separable, the interior of $\mathbb{B}_+$ is nonempty and μ-a.a. $u_t(\cdot, b)$ are $\mathcal{F}_t$-measurable. Assume that for any state ω: μ-a.a. $e_t(\omega) \gg 0$; μ-a.a. $u_t(\omega, \cdot)$ are continuous, strictly increasing and strictly quasi-concave. Then*

$$\mathcal{C}^{A-fine}_{A-fair}(\mathcal{E}) \subseteq \mathbf{C}^{A-fine}(\mathcal{E}) \subseteq RE(\mathcal{E}) = \mathcal{C}_{A-fair}(\mathcal{E}) = \mathbf{C}_A(\mathcal{E}).$$

6 Concluding remarks and Varian coalition fairness

We further comment below on the main results of the paper and their assumptions. Moreover, we introduce and briefly discuss an alternative notion of coalition fairness given in the spirit of Varian's definition.

We explictly notice that Theorem 4.7 and Corollary 5.7 extend results in [14] to the case of mixed markets with infinitely many commodities. It is also worthwhile noting that, since complete information economies represent a particular case of our model, Theorem 5.6 provides a new characterization of Walrasian equilibria of mixed (and finite) exchange economies. In particular, the remarks below, dealing with possible interpretations and/or weakening of assumptions of Theorem 5.6, also cover the special case of deterministic models.

The first remark just recall the usual interpretation of Aubin blocking mechanisms given in terms of replica economies. One can introduce, for each positive integer r, the set $\mathcal{A}^r = \{\gamma \in \mathcal{A} :\ r\gamma(t) \in \{0, \dots r\},\ \ \forall t \in T\}$, which is formed by all coalitions of the r-fold replica of the economy $\mathcal{E}$, in which agents of the same type are considered identical (cfr [25]) and the set $\mathcal{A}^{\mathbb{Q}} = \{\gamma \in \mathcal{A} : \gamma(t) \in [0,1] \cap \mathbb{Q},\ \ \forall t \in T\}$ of Aubin coalitions with rational values. Denote respectively by $\mathcal{C}^r_{fair}(\mathcal{E})$ and by $\mathcal{C}^E_{fair}(\mathcal{E})$, the set of feasible allocations that are ex-post c-fair with respect to the class $\mathcal{A}^r$ and $\mathcal{A}^{\mathbb{Q}}$. The ex-post core notion defined by $\mathcal{A}^{\mathbb{Q}}$ is denoted by $\mathbf{C}^E(\mathcal{E})$ and called the Edgeworth ex-post core of the economy $\mathcal{E}$ (see [11]). Then, clearly, the inclusions $\mathcal{C}_{A-fair}(\mathcal{E}) \subseteq \mathcal{C}^E_{fair}(\mathcal{E}) \subseteq \mathbf{C}^E(\mathcal{E})$ hold true.

It is easy to show that $\mathcal{C}^E_{fair}(\mathcal{E}) = \bigcap_{r\geq 1} \mathcal{C}^r_{fair}(\mathcal{E})$. Hence, the following inclusions

[16] Strict concavity avoids "the phenomenon that the selection of a consumption vector from an agent's demand correspondence may reveal additional information to the agent beyond the agent's private information and the information conveyed by prices". This is exactly the way in which strict concavity is used in the proof of $\mathcal{W}(\mathcal{E}) \subseteq RE(\mathcal{E})$ (see [13, Theorem 4.3]), to get a unique selection of demand correspondence in states ω_1, ω_2 that are indistinguishable with respect to $(\sigma(p) \vee \mathcal{F}_t)$.

hold true (see Proposition 5.2):

$$W(\mathcal{E}) \subseteq \mathcal{C}_{A-fair}(\mathcal{E}) \subseteq \mathcal{C}^E_{fair}(\mathcal{E}) = \bigcap_{r\geq 1} \mathcal{C}^r_{fair}(\mathcal{E}) \subseteq \mathbf{C}^E(\mathcal{E}).$$

Under the assumptions of Corollary 5.3, in each complete information economy it is true that $\mathbf{C}_A(\mathcal{E}(\omega_0)) = \mathbf{C}^E(\mathcal{E}(\omega_0))$ (see [25]). Then the previous inclusions are actually equalities and provide a characterization of ex-post competitive equilibria. Similarly, under the assumption that μ-a.a. the utilities $u_t(\cdot, x)$ are $\mathcal{F}_t$-measurable, one obtains the inclusions (see Proposition 5.5):

$$RE(\mathcal{E}) \subseteq \mathcal{C}_{A-fair}(\mathcal{E}) \subseteq \mathcal{C}^E_{fair}(\mathcal{E}) = \bigcap_{r\geq 1} \mathcal{C}^r_{fair}(\mathcal{E}) \subseteq \mathbf{C}^E(\mathcal{E})$$

and, consequently, a corresponding characterization of rational expectation equilibria when the stronger assumptions of Theorem 5.6 are satisfied. Therefore, summing up, Aubin c-fair allocations and rational expectations equilibria, can be interpreted as those allocations that are c-fair in each r-fold replica of the economy $\mathcal{E}$. This interpretation is close to the one of c-fair allocations of complete information economies given in ([34, Theorem 4.2]).

Looking at the assumption of a non-empty interior for the positive cone of the commodity space required in our main results, it appears that many spaces usually adopted in applications do not satisfy this assumption. It is then desirable to relax the hypothesis of non emptiness of the interior of $I\!\!B_+$, by making some alternative assumptions on the economy itself. In [25] the equality between Walrasian allocations and Aubin core is obtained in a complete information economy with a space of agents not necessarily nonatomic and a commodity space represented by a separable Banach lattice. This is done by making use of two assumptions, introduced in [30] for the case of nonatomic economies: the existence of an extremely desirable commodity and the additivity condition. For the special case of an economy with finitely many agents, it is also possible to use, as an alternative hypothesis, the uniform properness of the functions $u_t(\omega, \cdot)$ since this is sufficient to establish the Aubin core-Walras equivalence state by state as shown in [6]. Summing up, we can obtain two alternative formulations of the main results of the paper, stating the corresponding characterizations in a slightly different setting.

Our final remarks deal with an alternative notion of c-fair allocation given in the spirit of Varian's approach. In Definition 3.4, what coalition γ_1 envies to γ_2 is the net trade. If the underlying concept of envy concerns directly the resources of γ_2, we fall in a concept of (ex-post) coalition envy-freeness inspired by Varian [34].

Definition 6.1. *An allocation x is ex-post Aubin coalition envy-free (or equitable) in the sense of Varian if there exist no Aubin coalitions γ_1, $\gamma_2 \in \mathcal{A}$, no state of nature $\omega_0 \in \Omega$ and no assignment y, such that the conditions 0), $\widetilde{\mu}(\gamma_1) \geq \widetilde{\mu}(\gamma_2)$, $\int_{\gamma_1} y_{\omega_0}\, d\mu \leq \int_{\gamma_2} x_{\omega_0}\, d\mu$ and 3) are satisfied. The set of all ex-post Aubin coalition envy-free allocations is denoted by $\mathcal{C}_{A-V-ef}(\mathcal{E})$.*

Assume that for each $\omega \in \Omega$ it is true that: $e_t(\omega) = e(\omega)$, that is the total initial endowment is equally divided among traders in each state of nature. By applying Proposition 3.2 we see that $\mathcal{C}_{A-fair}(\mathcal{E}) \subseteq \mathcal{C}_{A-V-ef}(\mathcal{E})$.

Proposition 6.2. *Let the interior of $I\!B_+$ be nonempty. Assume that $e(\omega)$ is a strictly positive vector and that the utilities $u_t(\omega, \cdot)$ are continuous and increasing, for each state $\omega \in \Omega$. Then: $\mathcal{C}_{A-fair}(\mathcal{E}) \subseteq \mathcal{C}_{A-V-ef}(\mathcal{E})$.*

PROOF: Indeed, any x that does not belong to $\mathcal{C}_{A-V-ef}(\mathcal{E})$, for a certain state of nature, due to the equal income hypothesis, is an allocation that belongs to the set χ_1. Proposition 3.2 says that $x \notin \mathcal{C}_{A-fair}(\mathcal{E})$. □

Notice also that while Aubin c-fair allocations are automatically weakly Pareto optimal, this is not the case for the allocations in $\mathcal{C}_{A-V-ef}(\mathcal{E})$. To more adhere to Varian approach, we should define ex-post Aubin c-fair in the sense of Varian, an allocation which is both weakly Pareto and coalition envy-free in the sense of Varian. In other words, we should use the following definition.

Definition 6.3. *An allocation x is ex-post Aubin c-fair in the sense of Varian (then we shortly write $x \in \mathcal{C}_{A-V-fair}(\mathcal{E})$) if $x \in \mathcal{C}_{A-V-ef}(\mathcal{E})$ and it is also ex-post weakly Pareto optimal.*

Given the above definition, Proposition 6.2 says that with constant initial endowment $\mathcal{C}_{A-fair}(\mathcal{E}) \subseteq \mathcal{C}_{A-V-fair}(\mathcal{E})$.

Proposition 6.4. *Assume that $e(\omega)$ is a strictly positive vector, for each state $\omega \in \Omega$. Then:*

$$\left\{ x \in \mathcal{C}_{A-V-ef}(\mathcal{E}) \, : \, \int_T x \, d\mu = e\mu(T) \right\} \subseteq \mathbf{C}_A(\mathcal{E})$$

PROOF: Let us suppose, on the contrary, that for a certain state of nature we have a generalized coalition γ_1 and an assignment y such that $\displaystyle\int_{\gamma_1} y \leq \int_{\gamma_1} e = \widetilde{\mu}(\gamma_1) e$ and $y \succ_{S_1} x$. Since the set $\left\{ \displaystyle\int_\gamma h \, d\mu \, : \gamma \in \mathcal{A} \right\}$ is convex, let h be the function $(x, 1)$ and select γ_2 such that the point $\dfrac{\widetilde{\mu}(\gamma_1)}{\mu(T)} \left(\displaystyle\int_T x \, d\mu, \, \mu(T) \right)$ of the set $\left\{ \int_\gamma h \, d\mu \, : \gamma \in \mathcal{A} \right\}$ can be written as $\int_{\gamma_2} h \, d\mu$. Observe that $\widetilde{\mu}(\gamma_1) = \widetilde{\mu}(\gamma_2)$ and that $\displaystyle\int_{\gamma_1} y \, d\mu \leq \int_{\gamma_2} x \, d\mu$, i.e. that we violate $x \in \mathcal{C}_{A-V-ef}(\mathcal{E})$. □

If preferences are strictly monotone, trivially a weakly Pareto allocation uses all of the initial resources, so the previous Proposition implies the conclusion

$$\mathcal{C}_{A-V-fair}(\mathcal{E}) \subseteq \mathbf{C}_A(\mathcal{E}).$$

The remark above lead to a natural characterization of equal income ex-post competitive equilibria. An allocation x is an equal-income Walrasian equilibrium allocation of the complete information economy $\mathcal{E}(\omega_0)$, if there exists a price p such that $x(t)$ maximizes $u_t(\omega_0, \cdot)$ on the set $\{x : p \cdot x = p \cdot e\}$, for μ-almost every $t \in T$. When x is a competitive allocation with equal income of the complete information economy $\mathcal{E}(\omega_0)$, then it must lie in $\mathcal{C}_{A-V-ef}(\mathcal{E}(\omega_0))$. For, if not, by taking the generalized coalitions γ_1 and γ_2 and the assignment y according to Definition 6.1, equal income assumption, condition $\widetilde{\mu}(\gamma_1) \geq \widetilde{\mu}(\gamma_2)$, and $\int_{\gamma_1} y_{\omega_0}\, d\mu \leq \int_{\gamma_2} x_{\omega_0}\, d\mu$ allow to replicate the argument in subsection 3.3 b). Consequently, by means of Aubin core equivalence theorem, we can get what follows.

Proposition 6.5. *Let $\mathcal{E}$ be a DIME such that $I\!B$ is separable and the interior of $I\!B_+$ is nonempty. Assume that $e(\omega)$ is a strictly positive vector and that the utilities $u_t(\omega, \cdot)$ are:*
- continuous and strictly monotone in each state $\omega \in \Omega$,
- quasi-concave for each $t \in T_1$ and in each state $\omega \in \Omega$.
Then:

$$\mathcal{W}(\mathcal{E}) = \mathcal{C}_{A-V-fair}(\mathcal{E}).$$

7 Appendix

We report in this last section the proof of two useful results. The first Lemma is well known.

Lemma 7.1. *Suppose the utility function u is continuous, quasi-concave and strictly monotone. Suppose that the function $g \geq 0$ is integrable over the set of positive measure S. Then the following is true:*

$$u(g(t)) > u(x), \text{ for a. a. } t \in S,\ x \geq 0 \Rightarrow u(g_S) > u(x)$$

where g_S is the average of g over the set S.

PROOF: The Diestel Uhl mean value theorem says that $g_S \in \overline{co}[g(S)]$; continuity and quasi-concavity give the convexity and closedness of the upper contour set $\{z : u(z) \geq u(x)\}$. So, $g(S) \subseteq \{x : u(x) \geq u(x)\}$ gives $\overline{co}[g(S)] \subseteq \{x : u(x) \geq u(x)\}$ and then $u(g_S) \geq u(x)$.

By continuity of the preference, one finds a subset B of S of positive measure and an $\varepsilon \in]0, 1[\cap Q$ such that $u(\varepsilon g(t)) > u(x)$, for $t \in B$. Then, as before, $u(\varepsilon g_B) \geq u(x)$ and $u(g_{S \setminus B}) \geq u(x)$.

Suppose $\mu(S \setminus B) > 0$, so $\theta = \frac{\mu(B)}{\mu(S)} < 1$. Trivially

$$\theta g_B + (1 - \theta) g_{S \setminus B} = g_S.$$

If g_B and $g_{S \setminus B}$ are equal, then $g_B = g_S$ and the assertion follows from the strict monotonicity that guarantees g_B positive and then that $u(g_B) > u(\varepsilon g_{S \setminus B}) \geq u(x)$. So suppose they are different. Then, by quasi-concavity, we have $u(\theta \varepsilon g_B + (1 - \theta) g_{S \setminus B}) \geq$

$\min\{u(\varepsilon g_B), u(g_{S\setminus B})\} \geq u(x)$. On the other hand, by strict monotonicity, we can write

$$u(g_S) = u(\theta(1-\varepsilon)g_B + \theta\varepsilon g_B + (1-\theta)g_{S\setminus B}) > u(\theta\varepsilon g_B + (1-\theta)g_{S\setminus B}) \geq u(x)$$

and the conclusion follows. □

The second one deals with the separability of the consumption averages evaluated over envied traders.

Lemma 7.2. *Let $\mathcal{E}$ be a complete information economy and x be an allocation. Let*

$$A_t = \{s \in T \,:\, u_t(x_t) < u_t(x_s)\}.$$

Then the function

$$t \in A \to \frac{1}{\mu(A_t)} \int_{A_t} x_s \, d\mu(s)$$

has a separable range R.

PROOF: Since the function x is strongly measurable, one can assume that its range $x(T)$ is separable and clearly

$$R \subseteq \overline{co}(x(T)).$$

The conclusion then follows from the following claim.

Claim If $X \subseteq \mathbb{B}$ is separable, so is $co(X)$.

PROOF: Suppose $X \subseteq \overline{E}$, where E is countable.
First observe that for a subset $E \subseteq \mathbb{B}$, we can write

$$co(E) = \bigcup_{n \in \mathbb{N}} \bigcup_{\lambda \in S_{n-1}} \left\{ b \in \mathbb{B} : b = \sum_{i=1}^{n} \lambda_i x_i,\ x_i \in E \right\}.$$

Denote by $co_{\mathbb{Q}}(E)$ the same set where λ is limited to be in $S_{n-1} \cap \mathbb{Q}$, i.e.

$$co_{\mathbb{Q}}(E) = \bigcup_{n \in \mathbb{N}} \bigcup_{\lambda \in S_{n-1} \cap \mathbb{Q}} \left\{ b \in \mathbb{B} : b = \sum_{i=1}^{n} \lambda_i x_i,\ x_i \in E \right\}.$$

For each subset $E \subseteq \mathbb{B}$, we have

$$co(E) \subseteq \overline{co_{\mathbb{Q}}(E)}.$$

Indeed, given $b = \sum_{i=1}^{n} \lambda_i x_i$, it is enough to take $q \in S_{n-1} \cap \mathbb{Q}$ such that, for any i,

$$|q_i - \lambda_i| \leq \frac{\varepsilon}{\sum_j \|x_j\|}$$

to have that $\|b - z\| \leq \varepsilon$, with $z \in co_{\mathbb{Q}}(E)$, $z = \sum_{i=1}^{n} q_i x_i$.

Now, to obtain the claim just observe that, if E is countable, then each of the sets

$$\left\{ b \in I\!B : b = \sum_{i=1}^{n} \lambda_i x_i,\ x_i \in E \right\}$$

is countable and then $co_{\mathbb{Q}}(E)$ is countable by definition. On the other hand, by the previous remarks, we have the inclusions

$$co(X) \subseteq co(\overline{E}) \subseteq \overline{co}(E) \subseteq \overline{co_{\mathbb{Q}}(E)}.$$

□

References

[1] C.D. Aliprantis, K.C. Border, Infinite Dimensional Analysis, Springer- Verlag, (2006).

[2] C.D. Aliprantis, D.J. Brown, O. Burkinshaw: Existence and optimality of competitive equilibria. Springer-Verlag, New York (1990).

[3] B. Allen: Generic existence of completely revealing equilibria for economies with uncertainty when prices convey information. Econometrica **49**, 1773–1199 (1981).

[4] J.P. Aubin: Mathematical methods of game and economic theory. North-Holland, (1979).

[5] R.J. Aumann: Existence of competitive equilibria in markets with a continuum of traders. Econometrica **34**, 39–50 (1966).

[6] A. Basile, A. De Simone, M.G. Graziano: On the Aubin-like characterization of competitive equilibria in infinite dimensional economies. Rivista di Matematica per le Scienze Economiche e Sociali, **19** 187-213 (1996).

[7] A. Basile, M.G. Graziano: Core equivalences for equilibria supported by non-linear prices. Positivity, **17** 621-653 (2013).

[8] A. Bhowmik: Core and coalitional fairness: the case of information sharing rules. Economic Theory, **60** 461-494 (2015).

[9] A. Bhowmik, M.G. Graziano: On Vind's theorem for an economy with atoms and infinitely many commodities. Journal of Mathematical Economics, **56** 26-36 (2015).

[10] A. D'Agata: Star-shapedness of Richter-Aumann integral on a measure space with atoms: theory and economic applications. Journal of Economic Theory, **120(1)** 108-128 (2005).

[11] A. De Simone, C. Tarantino: Some new characterization of rational expectation equilibria in economies with asymmetric information. Decisions in Economics and Finance **33**, 7-21 (2010).

[12] C. Donnini, M.G. Graziano, M. Pesce: Coalitional fairness in interim differential information economies. Journal of Economics, **111** 55-68 (2014).

[13] E. Einy, D. Moreno, B. Shitovitz: Rational expectations equilibria and the ex-post core of an economy with asymmetric information. Journal of Mathematical Economics, **34** 527-535 (2000).

[14] E. Einy, D. Moreno, B. Shitovitz: On the core of an economy with differential information. Journal of Economic Theory, **94** 262-270 (2000).

[15] O. Evren, F. Husseinov: Theorems on the core of an economy with infinitely many commodities and consumers. Journal of Mathematical Economics **44**, 1180-1196 (2008).

[16] M. Florenzano: Edgeworth equilibria, fuzzy core and equilibria of a production economy without ordered preferences. Journal of Mathematical Analysis and Applications, **153** 18-36 (1990).

[17] J.J. Gabszewicz: Coalitional Fairness of Allocations in Pure Exchange Economies. Econometrica **43**, 661-668 (1975).

[18] J. Garcia-Cutrin, C. Herves-Beloso: A discrete approach to continuum economies. Economic Theory **3**, 577-583 (1993).

[19] M.G. Graziano: Equilibria in infinite-dimensional production economies with convexified coalitions. Economic Theory **17**, 121-139 (2001).

[20] M.G. Graziano: Fuzzy cooperative behavior in response to market imperfections. International Journal of Intelligent Systems **27**, 108-131 (2012).

[21] M.G. Graziano, M. Pesce: A note on the private core and coalitional fairness under asymmetric information. Mediterranean Journal of Mathematics **7**, 573-601 (2010).

[22] F. Husseinov, N. Sagara: Concave measures and the fuzzy core of exchange economies with heterogeneous divisible commodities. Fuzzy Sets and Systems **198**, 70-82 (2012).

[23] D.M. Kreps: A note on Fulfilled Expectations equilibria. Journal of Economic Theory **14**, 32-43 (1977).

[24] A. Mas Colell, W. Zame: Equilibrium Theory in Infinite Dimensional Spaces, in Handbook of Mathematical Economics Vol.4 (W. Hildenbrand and Sonnenschein, eds.), North Holland, p. 1835-1898, 1991.

[25] M. Noguchi: A fuzzy core equivalence theorem. Journal of Mathematical Economics, **34**, 143-158 (2000).

[26] M. Noguchi: Economies with a measure space of agents and a separable commodity space Mathematical Social Science, **40(2)**, 157-173 (2000).

[27] M. Pesce: The veto mechanism in atomic differential information economies. Journal of Mathematical Economics, **53**, 33-45 (2014).

[28] K. Podczeck: Markets with infinitely many commodities and a continuum of agents with non-convex preferences. Economic Theory **9**, 385-426 (1997).

[29] K. Podczeck: Core and Walrasian Equilibria when agents' characteristics are extremely dispersed, Econ Theory, **22**, 699-725 (2003).

[30] A. Rustichini, N.C. Yannelis: Edgeworth's conjecture in economies with a continuum of agents and commodities. J. Math. Econ **20** 307-326 (1991).

[31] A. Rustichini, N.C. Yannelis: What is perfect competition?, in: M. Ali Khan and N.C. Yannelis, eds. Equilibrium theory in infinite dimensional spaces. Springer-Verlag (1991).

[32] R. Tourky, N.C. Yannelis: Markets with many more agents than commodities: Aumann's "hidden" assumption, J Econ Theory, **101**, 189-221 (2001).

[33] B. Shitovitz: Coalitional fair allocations in smooth mixed markets with an atomless sector. Mathematical Social Sciences, **25**, 27-40 (1992).

[34] H.R. Varian: Equity, Envy and Efficiency. Journal of Economic Theory, **9**, 63-91 (1974).

[35] K. Vind: Lecture notes for economics. Stanford University, Spring, **288** (1971).

[36] L. Zhou: Strictly Fair Allocations in Large Exchange Economies. Journal of Economic Theory, **57**, 158-175 (1992).

[37] R. Wilson: Information, efficiency, and the core of an economy. Econometrica, **46**, 807-816 (1978).