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Abstract

We study exchange economies in ordered normed spaces $(X, \|\cdot\|)$ where agents have possibly different consumption sets. We define the notion of semi-interior point of the positive cone X_+ of X, a notion weaker than the one of interior point. We prove that if X_+ has semi-interior points, then the second welfare theorem holds true and a quasi equilibrium allocation exists. In both cases the supporting price is continuous with respect to a new norm $|||\cdot|||$ on X which is strongly related with the initial norm and the ordering, and in some sense can be considered as an extension of the norm adopted in classical equilibrium models. Many examples of cones in normed and Banach spaces with semi-interior points but with empty interior are provided, showing that this class of cones is a rich one. We also consider spaces ordered by strongly reflexive cones where we prove the existence of a quasi equilibrium without the closedness condition (i.e. without the condition that the utility space is closed). The results in the case of semi-interior points derive from those concerning the case of ordering cones with nonempty interior.

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1 Introduction

In this article we study equilibrium with reference to a new class of cones in normed spaces, i.e. cones with *semi-interior points*. As it is known, when a normed space $(X, \|\cdot\|)$ ordered by the cone P, i.e. $X_+ = P$, represents the commodity space of a pure exchange economy, then the existence of interior points of P is a condition with many important implications, see for example in [9], [15], [2], [16], [5].

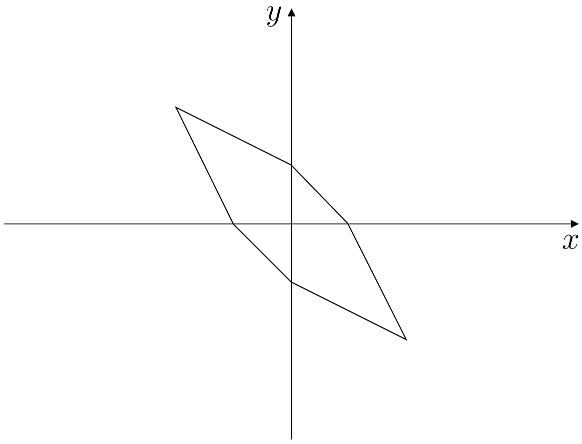
In particular, it is well known the relevance in general equilibrium theory of the assumption that the total initial endowment ω is an interior point of X_+. Many results and proofs are based on this assumption and its weakening. Indeed, the interior point condition is not satisfied in several circumstances, what induces scholars to assume suitable conditions in order to guarantee welfare and existence theorems (see for example [11], [12], [9], [15],[16],[2], [14], [17] and new results in this direction in [19] and [20]).

Here we focus on the notion of semi-interior point of a cone P in a normed space X, a notion that is weaker than the one of interior point of P since it is defined as follows. By definition, for an interior point x_0 of P, there exists a real number $\rho > 0$ so that $x_0 + \rho U \subseteq P$, where U is the unit ball of X. Therefore, we have $x_0 + \rho U_+ \subseteq P$ and $x_0 - \rho U_+ \subseteq P$, where $U_+ = U \cap P$ is the positive part of the unit ball of X defined by the cone P. While the first of the previous two inclusions is satisfied for any vector $x_0 \in P$, because the elements of ρU_+ are positive, the second one is not always true. If a point x_0 of P satisfies this condition, i.e. if $x_0 - \rho U_+ \subseteq P$ for some real number $\rho > 0$, then we say that x_0 is a **semi-interior point** of P.

When the commodity space is an ordered normed space $(X, \|\cdot\|)$ whose positive cone has semi-interior points, we prove the second welfare theorem and the existence of equilibria. In both cases, the supporting prices are continuous with respect to a new norm of the commodity space, denoted by $|||\cdot|||$. The latter is inspired by the definition of semi-interior point of P, and defined by means of the initial norm and the positive cone X_+. Indeed, $|||\cdot|||$ is the Minkowski functional defined by the convex hull, $co(U_+ \cup (-U_+))$, of the union of the sets U_+ and $-U_+$. It comes out that any semi-interior point of P is an interior point of P with respect to the norm $|||\cdot|||$ of X.

We give several examples of cones in normed spaces with semi-interior points and empty interior. For instance, consider an intertemporal two goods economy analyzed along infinitely many time periods $n = 1, 2, \ldots$. For any period n, the commodity space is $\mathbb{R}^2$ endowed with the usual topology and agents consume vectors of $\mathbb{R}^2_+$. Among the possible norms that generate the usual topology of the plane, let us fix the use of the norm $\|\cdot\|_n$ whose unit ball is the polygon (see the figure below) with vertices

$$(1,0), (0,1), (-n,n), (-1,0), (0,-1), (n,-n).$$



Unit ball of $(\mathbb{R}^2, \|\cdot\|_2)$

Consider as commodity space the minimal linear space containing bounded consumption flows $x = (x_1, x_2, \ldots)$, $x_i \geq 0$. Then the commodity space is $(X, \|\cdot\|_\infty)$, where $X = P - P$, P is the set of all bounded consumption flows and

$$\|x\|_\infty \equiv sup_{n \in \mathbf{N}} ||x_n||_n, \quad \text{for } x \in (\mathbb{R}^2)^{\mathbf{N}}.$$

Although the positive cone P of X has empty interior, under standard assumptions (in particular not invoking properness-like conditions), and thanks to the existence of semi-interior points in X_+, we show that valuation equilibria exist and are supported by linear prices. As we said before, the equilibrium price comes out to be continuous with respect to a different norm that can be adopted on X. Such a norm, in the present case, coincides with the following: $|||\cdot||| = \sup_n \|\cdot\|_{\ell_1}$[1].
Now, consider for each $y \in P$ a modified vector $y^{[k]} \in P$ defined by means of $y^{[k]} = y + (-\alpha, \alpha)\mathbf{1}_{A_k}$, where A_k is any subset of $\{i \in \mathbf{N} : i \geq k\}$. We note that the original norm in X annihilates the difference between the two flows of consumption $y, y^{[k]} \in P$. In other words, as time goes, gains/losses for one good are compensated by losses/gains of the same amount for the other. With respect to the new norm, we do not have that $y^{[k]}$ converges to y. Hence the new norm better describes problems in which differences in consumption emerging in the long run do matter.

Concerning the norm $|||\cdot|||$, it is also worth to remark that, in general, when X is a Banach space and the cone P is closed and generating, then the initial norm of X and the new norm $|||\cdot|||$ are equivalent (see Proposition 3.3). So, in the classical equilibrium models (as for example in the finite dimensional Arrow-Debreu model and in the Banach lattice equilibrium models) the two norms are equivalent and, therefore, both of them can be used as the norm of the commodity space. If X is a normed space, this equivalence is no longer true. However in this case, from the above remarks, the norm $|||\cdot|||$ of X can be considered as a natural norm to work with.

[1]Namely, with an “usual” ℓ_∞-type norm in $(\mathbb{R}^2)^{\mathbf{N}}$.

The paper is organized as follows. After Section 2 introducing the notion of semi-interior point and presenting examples of cones with semi-interior points and empty interior, in Section 3 we define the new norm $|||\cdot|||$.

Then in Section 4 we present, in ordered normed spaces under the nonempty interior hypothesis, the classical results of general equilibrium. We also give their proofs following standard methods. We do this both for our sake of completeness and since we weaken the assumptions of closedness of the utility space and continuity of preferences by assuming them only radially [2]. Section 4 should be considered as an intermediate part of our study to be applied to the case in which the positive cone of the commodity space has semi-interior points. The case where the positive cone has nonempty interior has been studied extensively in the literature. Although some conditions are new and some improvements are provided, we do not consider the results of this section as new. The main contributions of the paper are: the notion of semi-interior point, the new topology which is defined by the positive cone of the space and, as showed in Section 5, the fact that to ensure the second welfare theorem and the existence of equilibria, we only need cones with semi-interior, rather than interior, points. The examples of this kind of cones strengthen the article.

Finally, Section 6 is devoted to the case in which P is a strongly reflexive cone of a Banach space E and $X = P - P$ is the subspace of E generated by the cone P. Note that in this case the cone P is a closed cone of the Banach space E and therefore P is complete, but the space X generated by P is not necessarily a closed subspace of E. Therefore, we cannot suppose that X is a Banach space. The notion of reflexive cone has been introduced in [10], where a detailed study of this class of cones is contained. A cone P of a normed space E is strongly reflexive if the positive part $U_+ = U \cap P$ of the unit ball of E defined by P is compact. Although in Banach spaces the unit ball need not necessarily be compact, the class of infinite dimensional, strongly reflexive cones is a rich one. The examples 2.6 and 2.7 provide strongly reflexive and normal cones P of Banach spaces E with semi-interior points so that the space $X = P - P$ generated by P is dense in E (the cone P is almost generating). In particular, example 2.6 is a general one because it describes cones in the class of Banach lattices E with a positive Schauder basis. Example 2.7 is an example of a subcone of the positive cone of $L_1[0,1]$.

The second welfare theorem and the existence of equilibrium in strongly reflexive and normal cones can be proved without the assumption of the closedness of the utility space. As an application we show that quasi equilibrium allocations exist in the case of commodity spaces as those described in Examples 2.6 and 2.7.

[2] A similar form of weak continuity for preferences is adopted in the classical paper [11]

2 Cones with semi-interior points

In this section we define the notion of semi-interior point of a cone. This notion, the pioneering example 2.5 as well as the notion of the new norm of the next section, have been defined by I. Polyrakis and have been announced in Paris 2014, during the XXII European Workshop on General Equilibrium Theory, in his talk "Cones with semi-interior points and a second welfare theorem".

Let X be a normed space and let P be a cone of X. Suppose that X is ordered by the cone P. We shall denote by $U = \{x \in X \mid ||x|| \leq 1\}$ the unit ball of X and by U_+ the set

$$U_+ = U \cap P.$$

We shall call U_+ the **positive part** of the unit ball of X or the positive part of the unit ball of X defined by P, if this clarification is needed.

Definition 2.1 (I. Polyrakis)**.** *The vector $x_0 \in P$ is a semi-interior point of P if there exists a real number $\rho > 0$ so that $x_0 - \rho U_+ \subseteq P$.*

Clearly, any interior point of P is a semi-interior point of P. The next characterization will be useful in the sequel.

Proposition 2.2. *Suppose that P is a cone of a normed space X. The vector $x_0 \in P$ is a semi-interior point of P if and only if there exists a real number $k > 0$ so that kx_0 is an upper bound of the positive part U_+ of the unit ball of X.*

Proof. Suppose that x_0 is a semi-interior point of P. Then $x_0 - \rho U_+ \subseteq P$, x_0 is an upper bound of ρU_+ and $\frac{1}{\rho}x_0$ is an upper bound of U_+.

For the converse, if we suppose that kx_0 is an upper bound of U_+ then we can show that $x_0 - \frac{1}{k}U_+ \subseteq P$ and x_0 is a semi-interior point of P. ■

In the next example, X is a normed space ordered by a closed and generating cone X_+ without semi-interior points. Note that in this example X is not complete.

Example 2.3. Let $X = c_{00}$ be the space of finite real sequences i.e. the set of sequences $a = \{a(i)\}$ with $a(i) \neq 0$ for at most a finite number of i and suppose that X is ordered by the pointwise ordering. Then $X_+ = \{a \in X \mid a(i) \geq 0 \text{ for any } i\}$ is the positive cone of X. X is a normed space with norm $||a|| = max\{|a(i)| \mid i \in \mathbb{N}\}$.

X_+ does not have semi-interior points. Indeed, if we suppose that $a = \{a(i)\}$ is a semi-interior point of X_+, there exists a real number $k > 0$ so that ka is an upper bound of the positive part U_+. Since $e_i \in U_+$ for any i, where $e_i(j) = 0$ for any $j \neq i$ and $e_i(i) = 1$, we have that $a(i) \geq \frac{1}{k}e_i(i) = \frac{1}{k} > 0$ for any i and a contradiction. Hence X_+ does not have semi-interior points.

For the Krein-Smulian theorem below, see for example in [13, Theorem 3.5.2].

Theorem (Krein-Smulian). *If the positive cone P of an ordered Banach space X is closed and generating, then P gives an open decomposition of X, i.e. there exists a real number $a > 0$ so that $aU \subseteq U_+ - U_+$.*

By this theorem we obtain the next proposition:

Proposition 2.4. *If X is a Banach space ordered by the closed and generating cone P, then any semi-interior point of P is an interior point of P.*

Proof. Let x_0 be a semi-interior point of P. Then $x_0 - \rho U_+ \subseteq P$, for some $\rho > 0$, therefore

$$(x_0 - \rho U_+) + \rho U_+ = x_0 + \rho(U_+ - U_+) \subseteq P.$$

By the Krein-Smulian theorem, there exists a real number $a > 0$ so that $aU \subseteq U_+ - U_+$, therefore we have $x_0 + a\rho U \subseteq x_0 + \rho(U_+ - U_+) \subseteq P$ and x_0 is an interior point of P. ∎

The next example is crucial for the development of the theory of semi-interior points. It shows the shape of the unit ball of an ordered normed space whose positive cone does not have interior points but has semi-interior points. In this example we can see that the norm takes much lower values in some directions of non positive and non negative vectors than in the direction of positive and negative ones. So the unit ball is ”compressed” to the direction of positive or negative vectors and ”flattened” to some direction of non positive and non negative vectors. This shape of the unit ball gives a geometrical intuition and explains (in some sense) how the positive cone of the space fails to have interior points but has semi-interior points.

Example 2.5. Let X_n be the space $\mathbf{R}^2$ ordered by the pointwise ordering and with norm $||.||_n$ having, as unit ball, the polygon D_n of $\mathbf{R}^2$ with vertices

$$(1,0),(0,1),(-n,n),(-1,0),(0,-1),(n,-n).$$

It is easy to show, by taking the Minkowki's functional of D_n, that this norm is given by the formula:

$$||(x,y)||_n = |x| + |y|, \text{ if } xy \geq 0,$$

and

$$||(x,y)||_n = max\{|x|,|y|\} - \frac{n-1}{n} min\{|x|,|y|\}, \text{ if } xy < 0.$$

Suppose that E is the space of sequences $x = (x_n)_{n\in\mathbf{N}}$ so that $x_n = (x_1^n, x_2^n) \in X_n$ and $||x_n||_n \leq m_x$ for any n (the real number $m_x > 0$ depending on x).

Assume that E is ordered by the cone $P = \{x = (x_n) \in E \mid x_n \in \mathbf{R}^2_+ \text{ for any } n\}$ and that it is equipped with the norm

$$||x||_\infty = sup_{n\in\mathbf{N}} ||x_n||_n.$$

Suppose also that $X = P - P$ is the subspace of E generated by the cone P and suppose that X is ordered by the cone $X_+ = P$.

Let $\mathbf{1}$ be the constant sequence $(1,1)$ of X, i.e. $x_n = (1,1)$ for any n. Then $\mathbf{1}$ is not an interior point of X_+. Indeed, if for any natural number m we take the vector $y = (y_n)$ of X with $y_m = (-2,2)$ and $y_n = (0,0)$ for any $n \neq m$. It is easy to show that $||y||_\infty = \dfrac{2}{m}$ and $\mathbf{1} + y \notin X_+$. Therefore $\mathbf{1} + \rho U \nsubseteq X_+$, for any $\rho > 0$, and $\mathbf{1}$ is not an interior point of X_+. Similarly, we can show that any vector $x \in X_+$ is not an interior point of X_+ and then that X_+ has empty interior. The positive part of the closed unit ball U of X is the set

$$U_+ = \{x \in X_+ \mid ||x||_\infty \leq 1\}$$

and it is easy to see that for any $x = (x_n) \in U_+$ we have $||x_n||_n = x_n^1 + x_n^2 \leq 1$. So,

$$\mathbf{1} - U_+ \subseteq X_+$$

and $\mathbf{1}$ is a semi-interior point of X_+.
In particular, by Proposition 2.4, the space X is not complete.

The next two examples are examples of strongly reflexive cones P with semi-interior points but with empty interior which generate a dense subspace, i.e. $\overline{P - P} = X$. Recall that a cone P of X is **strongly reflexive** if the positive part $U_+ = U \cap P$ of U defined by P is $||.||$-compact. Note that the notion of strongly reflexive cone has been defined in [10] where it is shown that the class of strongly reflexive cones is a rich one.

Example 2.6. Let E be an infinite dimensional Banach lattice with a positive Schauder basis $\{e_i\}$. Without loss of generality, we may suppose that this basis is normalized, i.e. $||e_i|| = 1$, for any i. For example, E can be one of the spaces c_0, ℓ_1 or one of the reflexive spaces ℓ_p with $1 < p < \infty$. In Theorem 5.7 of [10], it is proved that E_+ contains a strongly reflexive cone which generates a dense subspace of E. In the present example, by a similar way we construct strongly reflexive cones $P \subseteq E_+$ with semi-interior points and empty interior with $\overline{P - P} = X$.

By definition of Schauder basis, any vector $x \in E$ has a unique expansion $x = \sum_{i=1}^{\infty} x_i e_i$, where the real numbers x_i are the coordinates of x in the basis $\{e_i\}$. Moreover, since the basis $\{e_i\}$ is positive, for any vector x of E we have: $x \geq 0$ if and only if $x_i \geq 0$ for any i. We start with a fixed real number $\alpha \in (0,1)$ and we consider the vector

$$y = \sum_{i=1}^{\infty} \alpha^{i-1} e_i,$$

of E_+. Then we have $||y|| \geq 1$ because $y \geq e_1$ and E is a Banach lattice. We consider the closed subcone

$$K = \{x \in X \mid 0 \leq x \leq x_1 y\}$$

of E_+. Then, by definition, we have that $\sum_{i=1}^{n} \alpha^{i-1} e_i \in K$, for any n. Moreover, for any $x \in K$ we have that $0 \leq x \leq x_1 y$. Therefore for any $x \in K$ we have $||x|| \leq x_1||y||$ because E is a Banach lattice. For any $x \in U_K^+ = U \cap K$ we have $||x|| \leq 1$ and $0 \leq x_1 e_1 \leq x$ and by taking norms we have $0 \leq x_1 \leq 1$, therefore $0 \leq x \leq y$. So we have that $U_K^+ \subseteq [0,\ y]$ which implies that U_K^+ is compact because each order interval of a Banach lattice with a positive basis is compact, see in [21], Theorem 16.3. Hence the cone K is strongly reflexive. The set $U_K^+ \cup (-U_K^+)$ is compact, therefore its convex hull

$$V = co(U_K^+ \cup (-U_K^+)),$$

is also compact. So the set $\Omega = 3y + V$ is compact and we consider the cone

$$P = \bigcup_{t \geq 0} t\Omega$$

generated by Ω. It is easy to show that the cone P is strongly reflexive. Indeed, for any $z = 3y + x \in \Omega$, where $x \in V$ we have $||x|| \leq 1$ therefore $2 \leq ||z|| \leq 3||y|| + 1$. So the set

$$R = \bigcup_{0 \leq t \leq 1} t\Omega$$

is compact. For any $w \in U_P^+ = P \cap U$ we have that $w = tz$ where $z \in \Omega$ therefore $||w|| = t||z||$ hence $t = \dfrac{||w||}{||z||} \leq \dfrac{||w||}{2} \leq \dfrac{1}{2}$. Hence any vector of U_P^+ is of the form $z = 3ty + tx$, where $x \in V$ and $0 \leq t \leq \dfrac{1}{2}$. This implies that $U_P^+ \subseteq R$ and the set U_P^+ as a closed subset of R is compact. To show that $3y$ is a semi-interior point of P it is enough to show that $3y$ is an upper bound of U_P^+ in the ordering of P or equivalently that for any $z \in U_P^+$ we have that $3y - z \in P$. Suppose that $z \in U_P^+$. As we have shown above it is of the form $z = 3ty + tx$, where $x \in V$ and $0 \leq t \leq \dfrac{1}{2}$, hence

$$3y - z = 3(1-t)y + t(-x) = (1-t)\left(3y + \frac{t}{1-t}(-x)\right) \in P,$$

because $\dfrac{t}{1-t}(-x) \in V$ for any $x \in V$ and any $0 \leq t \leq \dfrac{1}{2}$. Note that $0 \leq \dfrac{t}{1-t} \leq 1$ for any $0 \leq t \leq \dfrac{1}{2}$. Therefore $3y$ is a semi-interior point of P. To show that $\overline{P - P} = X$ we remark that $e_1 \in U_K^+ \subseteq V$, therefore $e_1 = (3y + 0) - (3y - e_1) \in P - P$. Also

$$e_2 = \frac{||e_1 + ae_2||}{\alpha}\left(3y - \frac{e_1}{||e_1 + ae_2||} - \left(3y - \frac{e_1 + ae_2}{||e_1 + ae_2||}\right)\right),$$

therefore

$$e_2 \in \frac{||e_1 + ae_2||}{\alpha}(P - P) = P - P.$$

By continuing this process we can show that $e_i \in P - P$ for any i, therefore $P - P$ is dense in E. If we suppose that x is an interior point of P we get a contradiction as follows: $x + \delta U$ is contained in P for some real number $\delta > 0$, therefore $x + \delta U$ is contained in a positive multiple of U_P^+, therefore the unit ball U of E is compact, contradiction. Therefore the cone P does not have interior points.

Example 2.7. In [10, Example 5.9] a strongly reflexive cone P of $L_1^+[0,1]$ is determined which generates a dense subspace in $L_1^+[0,1]$, i.e. $\overline{P-P} = L_1^+[0,1]$. Also the positive part U_+ of the unit ball of $L_1^+[0,1]$ defined by P is dominated by the vector $y = 2T(\eta)$ in the ordering of $L_1^+[0,1]$, where $T(\eta)$ is a vector of P which is defined in Example 5.9 of [10].

Here we change the notations. So we denote the cone P of [10, Example 5.9] by K and the set U_+ by U_K^+. Also we will denote by V the convex hull of the set $U_K^+ \cup (-U_K^+)$, and by P the cone of $L_1^+[0,1]$ which is generated by the set $\Omega = 3y + V$. As in the previous example we can show that P is strongly reflexive and the vector $3y$ is an upper bound of the positive part U_P^+ of the unit ball of $L_1^+[0,1]$ defined by P, in the ordering of P. Hence $3y$ is a semi-interior point of P. Moreover, the space $P - P$ generated by P and the space $K - K$ generated by K, coincide. Indeed, any vector of P is of the form $3ty + tx$ with $t \geq 0$ and $x \in V$. Also any vector of $K - K$ is of the form $t_1x_1 - t_2x_2$ with $t_1, t_2 \geq 0$ and $x_1, x_2 \in U_K^+$ or equivalently of the form tx, with $t \geq 0$ and $x \in V$ because $t_1x_1 - t_2x_2 = (t_1 + t_2)(\frac{t_1}{t_1+t_2}x_1 + \frac{t_2}{t_1+t_2}(-x_2))$.
So for any vector $z = 3t_1y + t_1x_1 - (3t_2y + t_2x_2)$ of $P - P$ we have $z = (3t_1y - 3t_2y) + (t_1x_1 - t_2x_2) \in (K - K) + (K - K) = K - K$. Also for any vector $w = tx$ of $K - K$ we have $w = (3y + tx) - 3y \in P - P$. Therefore $P - P = K - K$ and $\overline{P-P} = \overline{K-K} = L_1^+[0,1]$.

Theorem 2.8. *If X is a normed space ordered by the generating cone P, then any semi-interior point of P is an order unit of X.*

Proof. Suppose that x_0 is a semi-interior point of P. Then by Proposition 2.2, there exists a real number $k > 0$ so that kx_0 is an upper bound of the positive part U_+ of the unit ball of X. Let $x \in X$. Then $x = x_1 - x_2$, where $x_1, x_2 \in P$ and suppose that $a = max\{||x_1||,\ ||x_2||\}$. We have

$$x = x_1 - x_2 \leq x_1 = ||x_1||\frac{x_1}{||x_1||} \leq ||x_1||kx_0 \leq akx_0$$

and similarly

$$x = x_1 - x_2 \geq -x_2 = ||x_2||\frac{-x_2}{||x_2||} \geq -||x_2||kx_0 \geq -akx_0,$$

therefore $x \in [-akx_0,\ akx_0]$ and x_0 is an order unit of X. ■

We recall the next result from the theory of ordered spaces, see for example [3, Theorem 2.8].

Theorem 2.9. *If X is a Banach space ordered by the closed and generating cone P, then any order unit of X is an interior point of P.*

Remark 2.10. Theorem 2.8 combined with Proposition 2.2, says that the set of semi-interior points of a cone P of a normed space X is exactly the set of order units of X each of which is also an upper bound of some positive multiple of the positive part U_+ of the unit ball U of X. So the set of semi-interior points of a cone P is a subclass of the class of the order units of X. From Theorem 2.9 we have that in Banach spaces these two classes coincide. However, the next example shows that in normed spaces, the set of semi-interior points of P is a proper subset of the set of the order units of X, in general.

Example 2.11. This is an example of a normed space with generating positive cone P with an order unit but without semi-interior points. So by this example we get that the converse of Theorem 2.8 is not true. Analogously to Example 2.5, suppose that X_n is the space $\mathbf{R}^2$ ordered by the pointwise ordering and with norm $||.||_n$, having as unit ball the polygon D_n of $\mathbf{R}^2$ with vertices

$$(1,0),(n,n),(0,1),(-1,0),(-n,-n),(0,-1).$$

As in Example 2.5, E is the space of sequences $x=(x_n)_{n\in\mathbf{N}}$ with $x_n=(x_1^n,x_2^n)\in X_n$ so that there exists a real number $m_x>0$ depending on x with $||x_n||_n\leq m_x$ for any n. Then it is easy that $||(n,n)||_n=1$ for any n.

Suppose that E is ordered by the cone $P=\{x=(x_n)\in E \mid x_n\in\mathbf{R}^2_+ \text{ for any } n\}$ and that E is equipped with the norm

$$||x||_\infty = sup_{n\in\mathbf{N}}||x_n||_n.$$

Suppose also that $X=P-P$ is the subspace of E generated by the cone P and suppose that X is ordered by the cone $X_+=P$.

It is easy to see that the constant sequence $\mathbf{1}$ i.e the sequence $x=(x_n)$ with $x_n=(1,1)$ for any n is an order unit of X. If we suppose that $z=(z_n)$ is a semi-interior point of X, then kz is an upper bound of U_+. Since $\mathbf{1}$ is an order unit, we have $h\mathbf{1}\geq kz$ for some real number $h>0$, therefore $h\mathbf{1}$ is an upper bound of U_+. But for any m the vector $x^m=(x_n)$ with $x_m=(m,m)$ and $x_n=(0,0)$ for any $n\neq m$ belongs to U_+ therefore $(m,m)\leq h(1,1)$ for any m, which is impossible. Hence P does not have semi-interior points.

3 The topology generated by the positive cone

Suppose that X is a normed space ordered by the positive cone P, i.e $X_+=P$, and suppose also that X_+ is generating. This can be considered as the general case because if the cone P is not generating, we shall restrict our analysis to the normed space $Y = P - P$. In this section we pass to a new norm of X. This norm is related with the initial norm.

Let us denote by V the convex hull of the union of the positive and the negative part of U, i.e.

$$V = co(U_+ \cup (-U_+)) \subseteq U.$$

The vectors of V are the convex combinations between positive and negative vectors of X with norm lower than or equal to one. In other words, the vectors of V are linear combinations of "gains" and "losses" of "size" at most one. In order to show that the set V generates a norm we need to show that it is absorbing. Indeed, for any $x \in X$ we have that $x = x_1 - x_2$ where $x_1,\ x_2 \in P$ and we have

$$x = (||x_1|| + ||x_2||)\left(\frac{||x_1||}{||x_1|| + ||x_2||}\frac{x_1}{||x_1||} + \frac{||x_2||}{||x_1|| + ||x_2||}\frac{-x_2}{||x_2||}\right),$$

therefore $x \in (||x_1|| + ||x_2||)V$ and $x \in tV$ for any $t \geq ||x_1|| + ||x_2||$. Therefore the Minkowki's functional

$$q(x) = inf\{t \in \mathbf{R}_+ \mid x \in tV\},$$

is a norm of X and we shall denote this norm by $|||\cdot|||$.

Definition 3.1 (I. Polyrakis)**.** *We shall refer to the norm $|||\cdot|||$ of X introduced above as the norm of X generated by the positive part of the unit ball of X or by the positive cone of X, or simply by the cone P of X.*

It is easy to see that $|||x||| \geq ||x||$ for any $x \in X$, and $|||x||| = ||x||$ for any $x \in P \cup (-P)$. Therefore, the $|||\cdot|||$-topology of X is finer than the initial topology of X defined by the norm $||\cdot||$, and the dual of the initial normed space $(X,\ ||\cdot||)$ is contained in the dual of the normed space $(X,\ |||\cdot|||)$.

We denote by W the unit ball of $|||.|||$. Then W is the $|||.|||$-closure of V, and $W \subseteq U$. Also we have

$$W_+ = W \cap P = V \cap P = U \cap P = U_+.$$

Proposition 3.2. *Suppose that X is a normed space ordered by the $|||.|||$-closed cone P. If x_0 is a semi-interior point of P, then x_0 is an interior point of P with respect to the norm $|||.|||$ of X defined by the positive cone of X.*

Proof. By the definition of semi-interior point there exists $\rho > 0$ so that $x_0 - \rho U_+ \subseteq P$. But $x_0 + \rho U_+ \subseteq P$, therefore the convex hull of the union of these sets is contained in P i.e.

$$co((x_0 + \rho U_+) \cup (x_0 - \rho U_+)) = x_0 + \rho\ co(U_+ \cup (-U_+)) = x_0 + \rho V \subseteq P.$$

Hence $x_0 + \rho W \subseteq P$ since the $|||.|||$-closure of $x_0 + \rho V$ is contained in P and the proposition is proved.

∎

One more appeal to the Krein-Smulian Theorem gives the next equivalence theorem of the two norms.

Proposition 3.3. *If X is an ordered Banach space with closed and generating positive cone X_+, then the initial norm $||.||$ of X and the norm $|||.|||$ of X generated by the positive cone X_+ of X, are equivalent.*

Remark 3.4. In the above proposition, X is a Banach space ordered by a closed cone P. If the cone P is not generating, then the subspace $Y = P - P$ of X generated by P is not a Banach space in general. Therefore, we cannot say that in Y the initial norm and the norm of Y generated by its positive cone $Y_+ = P$ are equivalent. Indeed, in Example 2.6 and 2.7, the subspace $Y = P - P$ is not a Banach space because in both cases P has semi-interior but not interior points. In these examples Y is dense in X.

Remark 3.5. Suppose that in an exchange economy the commodity space is a normed space X, the consumption set is a closed and generating cone P and X is ordered by the cone P. ¿From Proposition 3.3 we have that if X is complete, then the norm of X and the norm of X defined by the cone P are equivalent. In particular this is the case if X is a Banach lattice. So, in the classical equilibrium models (for example the finite dimensional Arrow-Debreu model, the Banach lattices equilibrium models) both norms $||.||$ and $|||.|||$ of the commodity space can be equivalently adopted.

For a normed space X (i.e. not necessarily a complete one), these two norms are not equivalent in general. On the other hand, according to what above, studying equilibrium with respect to the norm $|||.|||$ can be considered as a natural way to extend the study of classical equilibrium models.

We compare below the two norms $||.||$ and $|||.|||$ in the case of Example 2.5.

Example 2.5 continued (comparison of the two norms). We shall determine the $|||.|||$-norm of the subspace $X = P - P$ of E introduced in Example 2.5. We have $X_+ = P$ and

$$U_+ = \{x = (x_n) \in X_+ \mid x_n = (x_1^n, x_2^n) \in \mathbf{R}_+^2 \ : \ ||x_n||_n = x_1^n + x_2^n \leq 1\}.$$

For any $x = (x_n), y = (y_n) \in U_+$ and any $t \in [0, 1]$ we have

$$tx + (1-t)(-y) = (t(x_1^n, x_2^n) + (1-t)(-y_1^n, -y_2^n)).$$

Since $x, y \in U_n^+$ and $U_n^+ = U_{\ell_1}^+$ where $||(a, b)||_{\ell_1} = |a| + |b|$ for any $(a, b) \in \mathbb{R}^2$ is the ℓ_1-norm of $\mathbf{R}^2$ and $U_{\ell_1}^+$ is the ℓ_1-unit ball of $\mathbf{R}^2$. It is easy to show that U_{ℓ_1} is the convex hull of $U_{\ell_1}^+ \cup (-U_{\ell_1}^+)$, therefore we have that the unit ball W of the $|||.|||$-norm of X is the cartesian product

$$W = \Pi_{i=1}^{\infty} D_i,$$

where $D_i = U_{\ell_1}$ for any i. By the above remarks it is easy to see that

$$|||x||| = sup_{n \in \mathbf{N}} ||x_n||_{\ell_1},$$

for any $x = (x_n) \in X$.

So if we have two not comparable commodities $x = (x_n), y = (y_n) \in X_+$ (i.e. $x \not\geq y$ and $y \not\geq x$) and if we suppose that $x_i = y_i$ for any $i \neq n$ and $x_n = y_n + (-1, 1)$ then we have that $||x - y|| = \frac{1}{n}$ but $|||x - y||| = 2$. This shows that the initial norm, $||.||$, of X destroys the differences between the commodities x and y as n increases but the norm $|||.|||$ of X generated by X_+ perceives more precisely the differences.

4 Cones with nonempty interior and equilibrium

In this section we study the equilibrium in an ordered normed space X whose positive cone X_+ has nonempty interior, assuming that consumers have, as consumption sets, different subcones P_i of X_+. This is an intermediate part of our study that will be applied to the case where the cone X_+ ordering the commodity space may have empty interior but has semi-interior points. We present here the results we need and give, for the sake of completeness, their proofs following standard methods. We do not consider the results of this section as new, although some of our assumptions are weaker than or sometimes not comparable with the usual ones (see remark 4.15). Note also that in the proofs, instead of continuity we use the radial continuity of the utility functions and for the utility space we assume radial closedness instead of closedness.

4.1 The model

We consider an exchange economy with l consumers. We suppose that the commodity space X is an ordered normed space with closed positive cone X_+. For each $i = 1, 2, ..., l$, suppose that consumer i has:

- as consumption set, a closed subcone P_i of X_+,
- a preference relation $\succsim_i$ which is defined on P_i, and
- as initial endowment, a nonzero vector $\omega_i \in P_i$.

Let us denote the economy by

$$\mathcal{E} \ = < X, \ (P_i, \succsim_i, \omega_i)_{i=1,2,...,l} > .$$

The total endowment is $\omega = \sum_{i=1}^{l} \omega_i$. Also, suppose that the preferences are reflexive, complete and transitive. By monotone preference relations we mean that for any i and $x, y \in P_i$, then $x \geq y$ implies $x \succsim y$. We denote by

$$\mathcal{A} = \{x \equiv (x_1, x_2, \ldots, x_l) \in P_1 \times P_2 \times ... \times P_l \mid \sum_{i=1}^{l} x_i = \omega\},$$

the set of **NFD-allocations** (non-free disposal allocations) and by

$$\mathcal{K} = \{x \equiv (x_1, x_2, \ldots, x_l) \in P_1 \times P_2 \times ... \times P_l \mid \sum_{i=1}^{l} x_i \leq \omega\},$$

the larger set of **FD-allocations** (free disposal allocations).

For any $x, y \in \mathcal{K}$, we write $x \succeq y$ if $x_i \succsim_i y_i$ for any i, $x \succ y$ if $x_i \succsim_i y_i$ for any i and $x_i \succ_i y_i$ for at least one i and we shall write $x \succ\succ y$ if $x_i \succ_i y_i$ for any i.

Let us introduce the following properties repeatedly used throughout the sequel.

(A1) for each $x \in X_+$, there exist vectors $x_i \in P_i$ so that $x = \sum_{i=1}^{l} x_i$.

(A1b) For any $i = 1, 2, ..., l$, for any $x \in P_i$ and any real number $t > 0$, there exists $y_{i,x,t} \in P_i$, depending on i, x, t so that $0 \leq y_{i,x,t} \leq t\omega$ and $x + y_{i,x,t} \succ_i x$.

(A1c) The property $(A1b)$ is satisfied; moreover, for any i, and any $x \in P_i$ the family $(y_{i,x,t})_{t>0}$ of $(A1b)$ can be chosen in such a way that it satisfies the property: $\lim_{t \longrightarrow 0} y_{i,x,t} = 0$.

The property $A(1b)$ is a generalization of the classical assumption that ω is **extremely desirable** [3] for any consumer, in the case where $P_i = X_+$ for any i. In our model we have different consumption sets and we do not assume that $\omega \in P_i$ for any i. So we cannot assume that ω is extremely desirable for any consumer but we replace this assumption by the properties $A(1b)$ and $A(1c)$.

Proposition 4.1. *If $\omega \in P_i$, for any i and ω is extremely desirable for any preference relation $\succeq_i$ (i.e. for any $x \in P_i$ we have $x + \lambda\omega \succ_i x$ for any real number $\lambda > 0$) then $A(1c)$ is satisfied.*

Proof. Let $x \in P_i$ and $t > 0$. By our assumption we have that $\omega \in P_i$ for any i and $x + t\omega \succ_i x$, for any $t > 0$, therefore $A(1b)$ is satisfied with $y_{i,x,t} = t\omega$. Also we have that $\lim_{t \longrightarrow 0} y_{i,x,t} = \lim_{t \longrightarrow 0} t\omega = 0$ and $A(1c)$ is true.

■

[3] For the notion of extremely desirable bundle see for example in [2].

Proposition 4.2. *If the positive cone X_+ of X is normal, then $(A1b)$ implies $(A1c)$.*

Proof. By $(A1b)$, for any $x \in P_i$ and any real number $t > 0$, there exists $y_{i,x,t} \in P_i$, depending on i, x, t so that $0 \leq y_{i,x,t} \leq t\omega$ and $x + y_{i,x,t} \succ_i x$. Since the cone X_+ is normal and $\lim_{t \longrightarrow 0} t\omega = 0$, we have that $\lim_{t \longrightarrow 0} y_{i,x,t} = 0$, see in [3], Theorem 2.23. ■

Proposition 4.3. *If the preferences are monotone and $(A1)$ holds, then for any $x \in \mathcal{K}$ there exists $y \in \mathcal{A}$ so that $y \succsim x$.*

Proof. Let $x \notin \mathcal{A}$. Then $\sum_{i=1}^{l} x_i < \omega$ and suppose that $z = \omega - \sum_{i=1}^{l} x_i$. Then we have $z = \sum_{i=1}^{l} \varphi_i$ with $\varphi_i \in P_i$ for any i, therefore the allocation

$$\tilde{y} = (x_1 + \varphi_1, x_2 + \varphi_2, \ldots, x_l + \varphi_l),$$

is the requested one. ■

Remark 4.4. Recall that a vector x_0 of an ordered normed space X is a **quasi-interior point** of X_+ or an **almost order unit** of X if $x_0 \in X_+$ and the solid subspace $I_{x_0} = \bigcup_{n=1}^{\infty} [-nx_0,\ nx_0]$ generated by x_0 is dense in X, i.e. $\overline{I}_{x_0} = X$. Recall also that any interior point of X_+ is an order unit of X and each order unit of X is an almost order unit of X but the converse of the above implications is not true in general. Any almost order unit x_0 of X is **strictly positive** i.e. $p(x_0) > 0$ for any nonzero, positive, continuous, linear functional p of X. Indeed, if we suppose that $p(x_0) = 0$, then for any $x \in I_{x_0}$, there exists a natural number $n_x > 0$ so that $-n_x\, x_0 \leq x \leq n_n\, x_0$. Since p is positive we have

$$-n_x\, p(x_0) \leq p(x) \leq n_x\, p(x_0),$$

therefore $p(x) = 0$, for any $x \in I_{x_0}$. Hence $p = 0$ on I_{x_0} and therefore $p = 0$ on X because I_{x_0} is dense in X and p is continuous. But this is a contradiction because we have supposed that $p \neq 0$, therefore $p(x_0) > 0$ because p is positive. This conclusion holds true in particular for the $|||.|||$-almost order units of X and for any positive and $|||.|||$-continuous linear functional of X. Note also that any $|||.|||$-almost order unit x_0 of X is an almost order unit of X because $\overline{I}_{x_0}^{|||\cdot|||} \subseteq \overline{I}_{x_0}^{||\cdot||}$.

Let us now recall that for $x = (x_1, x_2, \ldots, x_l) \in \mathcal{A}$, one says that:

- x is **Pareto optimal** if there does not exist an allocation y so that $y \succ x$;
- x is **weakly Pareto optimal** if there does not exist an allocation y so that $y \succ\succ x$.

For $x = (x_1, x_2, \ldots, x_l) \in P_1 \times P_2 \times \ldots \times P_l$, one says that x **is supported** by the linear functional p of X if p is nonzero and for any i and any $z \in P_i$ we have $z \succeq_i x_i \Longrightarrow p(z) \geq p(x_i)$.

In the next section we shall prove the supportability of any weakly Pareto optimal allocation in cones with nonempty interior, without the assumption that the total endowment ω is itself an interior point of the cone. The supporting price vector p is nonzero on X, but we do not ensure that $p(\omega) > 0$. If moreover ω is an almost order unit of X_+, then the condition that $p(\omega) > 0$ is ensured.

4.2 A second welfare theorem

Theorem 4.5. *Suppose that in the economy $\mathcal{E}$ the conditions (A1) and (A1b) are satisfied and that the preferences are monotone and convex. If x_0 is an interior point of X_+, then any weakly Pareto optimal allocation is supported by a positive, continuous linear functional p of X with $p(x_0) = 1$.*
If moreover ω is an almost order unit of X, then we have that $p(\omega) > 0$.

Proof. Suppose that

$$x = (x_1, ..., x_l)$$

is a weakly Pareto optimal allocation and

$$G = \{y = y_1 + y_2 + ... + y_l \mid y_i \in P_i,\ y_i \succsim_i x_i \text{ for any } i\}.$$

If F is the closure of G, then F is a closed and convex subset of X_+ with $\omega \in F$. Suppose that ω is an interior point of F. Then $\omega + \rho U \subseteq F$ where U is the unit ball of X and $\rho > 0$. We get a contradiction as follows: Since ω is also an interior point of X_+, then $t\omega$ is an interior point of ρU_+ for any real number $0 < t < \frac{\rho}{||\omega||}$. Hence $z_1 = \omega(1 - \frac{\rho}{2||\omega||})$ is an interior point of the set $W = \omega - \rho U_+$ and similarly we have that $z_2 = \omega(1 - \frac{3\rho}{4||\omega||})$ is an interior point of the set $K = z_1 - \frac{\rho}{2}U_+$. Note that $||\omega - z_1|| = \frac{\rho}{2}$, $||\omega - z_2|| = \frac{3\rho}{4}$, $z_1 - z_2 = \frac{\rho\omega}{4||\omega||}$ and $||z_1 - z_2|| = \frac{\rho}{4}$ with $z_1 > z_2$.

But $K \subseteq W \subseteq F$, therefore K contains vectors of G, because G is dense in F and K has interior points. So we can suppose that there exists $y \in K \cap G$.

Since $y \in K \subseteq z_1 - X_+$ we have that $y \leq z_1 < \omega$, therefore $w = \omega - y > 0$. Suppose that $y = y_1 + y_2 + ... + y_l$ where $y_i \in P_i$ with $y_i \succsim_i x_i$ for each i. We have

$$w = \omega - y \geq \omega - z_1 = \frac{\rho}{2||\omega||}\omega.$$

Then by (A1b), for any i, any y_i and $t = \frac{\rho}{2l||\omega||}$, there exists $v_i = y_{i,y_i,t} \in P_i$ so that $0 \leq v_i \leq \frac{\rho}{2l||\omega||}\omega$ and $y_i + v_i \succ_i y_i$. Then it is easy to see that

$$m = (m_i = y_i + v_i) \in \mathcal{K}$$

because

$$\sum_{i=1}^{l} m_i = y + \sum_{i=1}^{l} v_i \leq y + \frac{\rho}{2||\omega||}\omega = y + (\omega - z_1) \leq y + (\omega - y) = \omega.$$

Also we have

$$m \succ\succ y \succcurlyeq x.$$

So, by Proposition 4.3, there exists $u \in \mathcal{A}$ so that $u \succsim m \succ\succ x$. This is a contradiction because x is weakly Pareto optimal. Therefore, ω cannot be an interior point of F.

Because of this, there exists a sequence $\{w_n\}$ of X so that $w_n \notin F$ which converges to ω. Therefore, by the separation theorem, for any n, there exists $p_n \in X^*\setminus\{0\}$ which separates w_n and F, i.e. $p_n(w_n) < p_n(w)$ for any $w \in F$. We shall show that p_n is positive, i.e. that $p_n(x) \geq 0$ for any $x \in X_+$. So we take a vector $x \in X_+$. Then, by $(A1)$, there exist vectors $\phi_i \in P_i$ so that $x = \sum_{i=1}^{l} \phi_i$. So for any $\lambda \geq 0$ we have $\omega + \lambda x = x_1 + \lambda\phi_1 + x_2 + \lambda\phi_2 ... + x_l + \lambda\phi_l \in F$, because the preferences are monotone. Hence $p_n(\omega + \lambda x) \geq p_n(w_n)$. Therefore

$$\lambda p_n(x) \geq p_n(w_n - \omega), \text{ for any } \lambda \geq 0.$$

This shows that $p_n(x) \geq 0$, because if we suppose that $p_n(x) < 0$, the above relation cannot hold for any λ. Hence $p_n(x) \geq 0$ for any $x \in X_+$ and p_n is positive.

Since we have assumed that x_0 is an interior point of P, then we can show (see [13, Theorem 3.8.4 and 3.8.5]) that the set

$$B = \{f \in X_+^* \mid f(x_0) = 1\}$$

is a weak-star closed and norm bounded base for the cone X_+^* and therefore weak-star compact. We can also suppose that $p_n \in B$ for any n, because the linear functionals $\dfrac{p_n}{p_n(x_0)}$ separate w_n and F and belong to B. Since the base B is weak star compact, the sequence $\{p_n\}$ has an accumulation point $p \in B$, therefore for any $V \in I$, where I is the set of the neighborhoods of 0 in the weak-star topology, there exists n_V so that $p_{n_V} \in V$ and the net $(p_{n_V})_{V\in I}$ converges to p. So we have $p(x_0) = 1$ and also the subnet $(w_{n_V})_{V\in I}$ converges to ω. Hence, for any $x \in F$ we have

$$p(\omega) = \lim_V p_{n_V}(w_{n_V}) \leq \lim_V p_{n_V}(x) = p(x),$$

therefore p supports F at ω.

To show that p supports the allocation x, suppose that $x \in P_i$ is such that $x \succsim_i x_i$ for some i. Then we have

$$x + \sum_{j=1, j\neq i}^{l} x_j \in F,$$

therefore

$$\left(x + \sum_{j=1, j\neq i}^{l} x_j\right) \geq p(\omega) = p\left(\sum_{j=1}^{l} x_j\right),$$

therefore $p(x) \geq p\left(x_i\right)$ and p supports the allocation x.

If ω is a almost order unit of X then by the Remark 4.4 above we have that $p(\omega) > 0$. ■

Remark 4.6. In [6, Lemma 3.2], it is supposed that the space $X = \mathbf{R}^3$ is ordered by the ice-cream cone $P = \{(x,y,z) \in \mathbf{R}^3 \mid z \geq (x^2+y^2)^{\frac{1}{2}}\}$. In the economy there are two consumers with total endowment $\omega = (0,1,1)$ and utility functions $u_1(x,y,z) = 2z$ and $u_2(x,y,z) = x+2z$. In this example any allocation $x = (x_1, x_2)$ is of the form $x_1 = a\omega$ and $x_2 = (1-a)\omega$ with $a \in [0,1]$ because ω defines an extremal ray of the cone P. It is easy to see that any allocation is Pareto optimal. Moreover, it cannot be supported by a price vector p so that $p(\omega) > 0$. This is not in conflict with our result. Note that P is a cone with nonempty interior but ω is not an interior point of P. Also, all the assumptions of the above theorem are satisfied, therefore by our theorem any allocation must be supported by a nonzero price vector. It is easy to see that the price vector $p = (0,1,1)$ is positive with respect to the cone P, is nonzero on X and that p supports any allocation. But $p(\omega) = 0$.

4.3 Equilibrium

We introduce two more conditions, (A2) and (A3), in order to guarantee existence of equilibria.

(A2) The preferences $\succsim_i$ are defined by the utility functions $u_i : P_i \longrightarrow [0,+\infty)$ with $u_i(0) = 0$ for each $i = 1,2,...,l,$ and each u_i is bounded on the subset

$$\Omega_i = \{x \in P_i \mid 0 \leq x \leq \omega\}$$

of P_i.

Remark 4.7. The condition $u_i(0) = 0$ does not introduce any loss of generality because if we suppose that $\succsim_i$ is defined by the utility function u_i and u_i is monotone, then the utility function $\widetilde{u_i}(x) = u_i(x) - u_i(0)$ is monotone, defines $\succsim_i$ and $\widetilde{u_i}(0) = 0$.

If $\omega \in P_i$ and u_i is monotone, then u_i is bounded on Ω_i. If u_i is the restriction of a monotone utility function on X_+, then it is in any case bounded on the set Ω_i.

Assumption (A3) says what follows.

(A3) There exists $x = (x_1, x_2, ..., x_l) \in \mathcal{K}$ so that $u_i(x_i) > 0$, for any i.

Proposition 4.8. *The condition* $(A1b)$*, and therefore also* $(A1c)$*, implies* $(A3)$.

Proof. Suppose that $(A1b)$ is true. Let $w = (\frac{\omega_1}{2l}, \frac{\omega_2}{2l}, ... \frac{\omega_l}{2l}) \in \mathcal{K}$. Then by $(A1b)$ for any i there exists $y_i \in P_i$ so that $\frac{\omega_i}{2l} + y_i \succ_i \frac{\omega_i}{2l}$ and $0 < y_i < t\omega$ with $t = \frac{1}{2l}$. If $x = (\frac{\omega_1}{2l} + y_1, \frac{\omega_2}{2l} + y_2, ..., \frac{\omega_l}{2l} + y_l)$ then we have that $x \in \mathcal{K}$ with $u_i(x_i) = u_i(\frac{\omega_i}{2l} + y_i) > u_i(\frac{\omega_i}{2l}) \geq 0$ for any i therefore $(A3)$ is true. By definition of $(A1c)$, we have that $(A1c) \Longrightarrow (A3)$. ∎

We recall the notion of radially continuous function which has been defined in [18]. We say that the function u_i is **radially continuous** if for any $x \in P_i$, the restriction of u_i on the half-line $\{tx | t \in \mathbb{R}_+\}$ is continuous. It is easy to show that u_i is radially continuous on P_i if for any $x \in P_i$ and for any sequence $\{t_n\}$ of positive real numbers with $t_n \longrightarrow t$ we have $u_i(t_n x) \longrightarrow u_i(tx)$. A stronger notion we shall also use is that of linear continuity. The function u_i is **linearly continuous**[4] if the restriction of u_i on any line of P_i is continuous. This is equivalent with the property: for any $x, y \in P_i$ with $y \neq 0$ and any sequence of real numbers $t_n \longrightarrow t \in \mathbb{R}$ we have that $u_i(x + t_n y) \longrightarrow u_i(x + ty)$, whenever $x + t_n y,\ x + ty \in P_i$.

Let us introduce the **total utility function**

$$u(x) = (u_1(x_1), u_2(x_2), \dots, u_l(x_l)), \quad x = (x_1, x_2, ..., x_l) \in P_1 \times P_2 \times ... \times P_l,$$

and the **utility space**

$$\mathcal{U} = \{u(x) \mid x \in \mathcal{K}\}.$$

Proposition 4.9. *If (A2) is satisfied, the utility space $\mathcal{U}$ has the properties:*

(i) $\mathcal{U}$ is bounded.

(ii) If the preferences are radially continuous, then $\mathcal{U}$ is solid, in the sense that for any $\alpha \in \mathcal{U}$ the order interval $[0, \alpha]$ of $\mathbb{R}^l$ is contained in $\mathcal{U}$, i.e. $[0, \alpha] \subseteq \mathcal{U}$.

(iii) If the preferences are radially continuous and $(A3)$ is true, then the set $\mathcal{U}$ contains the positive part of the ball of $\mathbb{R}^l$ with center zero and radius r.

Proof. (i) By $(A2)$ each u_i is bounded on the set Ω_i. So for any $x = (x_1, x_2, ..., x_l) \in \mathcal{K}$ we have $0 \leq x_i \leq \omega$, therefore $0 \leq u_i(x_i) \leq a_i$, where a_i is an upper bound of u_i on Ω_i. Therefore the set $\mathcal{U}$ is bounded.

(ii) Let $\alpha \in \mathcal{U}$. Then there exists $x \in \mathcal{K}$ with $\alpha = u(x)$. Suppose that $\beta \in \mathbb{R}^l_+$ with $0 \leq \beta \leq \alpha$. We claim that $\beta \in \mathcal{U}$. We have

$$0 \leq \beta_i \leq u_i(x_i) \text{ for each } \quad i.$$

Since the restriction of u_i on the line segment

$$\{tx_i \mid t \in [0, 1]\},$$

is continuous, by the mean value theorem, there exists $t_i \in [0, 1]$ so that $u_i(t_i x_i) = \beta_i$. For these t_i we have

$$u(t_1 x_1, t_2 x_2, \dots, t_l x_l) = \beta,$$

[4] This notion of linear continuity is very similar to a notion of weak continuity of preferences, used by G. Debreu in [11], see Remark 4.15 below.

therefore $\beta \in \mathcal{U}$.
(iii) By $(A3)$, there exists $x = (x_1, x_2, ...x_l) \in \mathcal{K}$ with $u_i(x_i) > 0$ for any i. Then $u(x) \in \mathcal{U}$. We put $r = min_i\{u_i(x_i)\}$. Then $r > 0$ because $u_i(x_i) > 0$ for any i. If we suppose that $z \in \mathbb{R}^l_+$ with $||z|| \le r$, we have that $0 \le z_i \le r$ for each i, therefore $0 \le z \le u(x)$ and by (ii) we have that $z \in \mathcal{U}$. Hence (iii) is true.

■

Denote by Δ the unit simplex of $\mathbb{R}^l_+$.

Under the assumption $(A2)$ and the hypothesis that the utility functions u_i are monotone, by Proposition 4.9, the utility space is bounded. We introduce the function ρ given by

$$\rho(s) = \sup\{\alpha \in \mathbb{R}_+ \mid \alpha s \in \mathcal{U}\},\ s \in \Delta$$

which is well defined (see [2, p. 156]).

We say that the utility space $\mathcal{U}$ is **radially closed** if for any $\alpha \in \mathcal{U}$ the intersection of $\mathcal{U}$ with the half-line $\{t\alpha \mid t \ge 0\}$ is closed.

Proposition 4.10. *If $(A1), (A2), (A3)$ are satisfied, the utility functions are monotone and radially continuous and the utility space $\mathcal{U}$ is radially closed, then for each $s \in \Delta$, we have:*

(i) *$\rho(s) > 0$ and $\rho(s)s \in \mathcal{U}$;*

(ii) *the intersection of $\mathcal{U}$ with the half-line $\{t\rho(s)s \mid t \ge 0\}$ is the line segment defined by 0 and $\rho(s)s$;*

(iii) *if $x \in \mathcal{K}$ with $u(x) \ge \rho(s)s$, then we have that $u(x_i) = \rho(s)s_i$ for at least one i, where s_i is the i-coordinate of s;*

(iv) *if $x \in \mathcal{A}$ with $u(x) \ge \rho(s)s$, then x is weakly Pareto optimal;*

(v) *there exists a weakly Pareto optimal allocation y so that $u(y) \ge \rho(s)s$;*

(vi) *if $P_i = X_+$ for any i, and the utility functions u_i are linearly continuous, there exists a weakly Pareto optimal allocation y so that $u(y) = \rho(s)s$.*

Proof. (i) By Proposition 4.9, (iii), $\mathcal{U}$ contains the positive part D_r of a ball of $\mathbb{R}^l$ with center zero and radius $r > 0$. For any $s \in \Delta$ we have that $\frac{r}{||s||}s \in D_r$, therefore $\frac{r}{||s||}s \in \mathcal{U}$. Hence, by the definition of ρ, we have that $\rho(s) \ge \frac{r}{||s||} \ge \frac{r}{d}$ where $d = sup_{s\in\Delta}||s||$.

Also, by the definition of $\rho(s)$, there exists an increasing sequence of real numbers $\{\alpha_n\}$ which converges to $\rho(s)$ so that $\alpha_n s \in \mathcal{U}$. Then $\alpha_n s \longrightarrow \rho(s)s$, therefore $\rho(s)s \in \mathcal{U}$ because $\mathcal{U}$ is radially closed.

(ii) Let us show now that the intersection of $\mathcal{U}$ with the half-line $\{t\rho(s)s\}$ is the line segment defined by 0 and $\rho(s)s$.

Since the order interval $[0, u(x)]$ of $\mathbb{R}^l$ is contained in $\mathcal{U}$ for any $x \in \mathcal{K}$, we have that for each $0 \le t \le 1$, $t\rho(s)s \in \mathcal{U}$. So the line segment defined by 0 and $\rho(s)s$ is contained in $\mathcal{U}$. By the definition of $\rho(s)$ we have that for each $t > 1$, $t\rho(s)s \notin \mathcal{U}$, therefore the intersection of $\mathcal{U}$ with the semiline $\{ts|t \in \mathbb{R}_+\}$, is the line segment defined by 0 and $\rho(s)s$.

(iii) Assume that $x \in \mathcal{K}$ and $u(x) \geq \rho(s)$s.

If we suppose that $u_i(x_i) > \rho(s)s_i$ for any i, and if we put

$$t' = min\left\{\frac{u_i(x_i)}{\rho(s)s_i} \mid \text{for any } i \text{ with } \rho(s)s_i > 0\right\},$$

then we have that $t' > 1$ and it is easy to show that $u(x) \geq t'\rho(s)s$, therefore $t'\rho(s)s \in \mathcal{U}$. Consequently we have $t'\rho(s)s \in \mathcal{U}$ with $t' > 1$, a contradiction. So, we conclude that $u_i(x_i) = \rho(s)s_i$ for at least one i.

(iv) Suppose that $x \in \mathcal{A}$ with $u(x) \geq \rho(s)s$. If we suppose that an allocation z exists so that $z_i \succ_i x_i$ for any i, we have that $u_i(z_i) > \rho(s)s_i$, for any i, a contradiction by (iii). Therefore x is weakly Pareto optimal.

(v) Suppose that $x \in \mathcal{K}$ with $u(x) = \rho(s)s$. Then by Proposition 4.3, there exists $y \in \mathcal{A}$ so that $y \succeq x$, therefore y is weakly Pareto optimal by (iv).

(vi) Suppose that $x \in \mathcal{K}$ with $u(x) = \rho(s)s$. If we suppose that $x \in \mathcal{A}$, then x is weakly Pareto optimal by (iv) and (vi) is true. So we suppose that

$$z = \omega - \sum_{i=1}^{l} x_i > 0$$

and we take the allocation w with $w_i = x_i + \frac{z}{l}$ for any i. Since the utility functions are monotone we have that $u(w) \geq u(x) = \rho(s)s$, therefore by (iii) we have that there exists j so that $u_j(x_j + \frac{z}{l}) = u_j(x_j) = \rho(s)s_j$ and we fix such a j.

We define the vector $y^1 \in \mathcal{K}$ so that

$$y_i^1 = x_i, \quad \text{for any } i \neq j \text{ and } y_j^1 = x_j + \frac{z}{l},$$

where j is the above fixed index. Then we have

$$u(y^1) = u(x) \text{ and } \omega - \sum_{i=1}^{l} y_i^1 = z\frac{l-1}{l}.$$

As before, starting by y^1 we consider the allocation w so that $w_i = y_i^1 + z\frac{l-1}{l}\frac{1}{l}$ for any i. Then as we have noted before, we have that $u_j(w_j) = u_j(y_j^1) = u_j(x_j)$ for at least one j and we fix such a j. We define the vector $y^2 \in \mathcal{K}$ so that

$$y_i^2 = y_i^1, \quad \text{for any } i \neq j \text{ and } y_j^2 = y_j^1 + z\left(\frac{l-1}{l}\right)\frac{1}{l}, \quad \text{for the fixed } j.$$

Then we have

$$u(y^2) = u(y^1) = u(x) \text{ and } \omega - \sum_{i=1}^{l} y_i^2 = z\left(\frac{l-1}{l}\right)^2.$$

By continuing this process we get a sequence $\{y^n\}$ of $\mathcal{K}$ so that

$$u(y^n) = u(x) \text{ for any } n \text{ and } \omega - \sum_{i=1}^{l} y_i^n = z\left(\frac{l-1}{l}\right)^n.$$

By this process we have that for any fixed i, the sequence $\{y_i^n\}$ is of the form

$$y_i^n = x_i + \frac{z}{l}\left(a_1 + a_2\left(\frac{l-1}{l}\right) + a_3\left(\frac{l-1}{l}\right)^2 + ... + a_n\left(\frac{l-1}{l}\right)^n\right),$$

where $a_k = 1$ if during the k step $i = j$ and $a_k = 0$ if $i \neq j$. So we have that $\{y_i^n\}$ is an increasing sequence of the line segment defined by the vectors x_i and az where

$$a = \frac{1}{l}\sum_{i=0}^{\infty}\left(\frac{l-1}{l}\right)^i$$

and this sequence is Cauchy because

$$||y_i^{r+m} - y_i^r|| \leq \frac{||z||}{l}\sum_{s=r+1}^{r+m}\left(\frac{l-1}{l}\right)^s.$$

Therefore the sequence is convergent to a vector y_i of the line segment and $y_i \in X_+$ because $x_i,\ az \in X_+$ and X_+ is convex. Then, by taking limits we have that $\sum_{i=1}^{l} y_i = \omega$ therefore we have that $y = (y_1, y_2, ..., y_l)$ is an allocation. Hence $u(y) \geq u(x)$, therefore the allocation y is weakly Pareto optimal by (iv). Also we have $u_i(y_i) = \lim_n u_i(y_i^n) = u_i(x_i)$ for any i, because we have assumed that the functions u_i are linearly continuous. Hence $y = (y_1, y_2, ..., y_l)$ is a weakly Pareto optimal allocation with $u(y) = u(x)$ and the proof of (vi) has been completed.

■

In the Proposition 4.10 above we have proved that for any $s \in \Delta$, there exists a weakly Pareto optimal allocation y so that $u(y) \geq \rho(s)s$. Therefore, if $x \in \mathcal{K}$ and $u(x) = \rho(s)s$, we have $u_i(x_i) \leq u_i(y_i)$ for any i. The next condition (A4) strengthens this property by adding the extra condition that $u_i(y_i) \leq u_i(\omega_i)$, for any i with $\rho(s)s_i = 0$.

(A4) If $x \in \mathcal{K}$ with $u(x) = \rho(s)s$, there exists a weakly Pareto optimal allocation y so that

(i) $u_i(x_i) \leq u_i(y_i)$ for any i and

(ii) for any i with $u_i(x_i) = 0$ we have $u_i(y_i) \leq u_i(\omega_i)$.

Note that in the case where $P_i = X_+$ for any i and the utility functions are linearly continuous, we have by (vi) of Proposition 4.10, that a weakly Pareto optimal allocation y exists so that $u(y) = \rho(s)s$. Therefore for any i with $\rho(s)s_i = 0$ we have $u_i(y_i) = \rho(s)s_i = 0 \leq u_i(\omega_i)$ hence $(A4)$ is satisfied.

A sufficient condition to ensure that $(A4)$ holds true is given in the following

Proposition 4.11. *If the preferences are monotone and satisfy the next conditions*

(A4a) any preference relation u_i has the property:

$$x_i, y_i \in P_i,\ u_i(x_i) = 0 \Longrightarrow u_i(y_i) = u_i(x_i + y_i),$$

(A4b) for each $x \in X,\ 0 \leq x \leq \omega$ there exist vectors $x_i \in P_i$ so that $x = \sum_{i=1}^{l} x_i$ with $0 \preceq_i x_i \preceq_i \omega_i$, for any i,

then $(A4)$ is true.

Proof. Suppose that $x \in \mathcal{K}$ with $u(x) = \rho(s)s$, for some $s \in \Delta$. Let $z = \omega - \sum_{i=1}^{l} x_i$. Then by $(A4b)$, $z = \sum_{i=1}^{l} z_i$ with $z_i \in P_i$ and $0 \preceq_i z_i \preceq_i \omega_i$ for any i. The allocation y with $y_i = x_i + z_i$ for any i is weakly Pareto optimal, by Proposition 4.10 (iv), with $y \succeq x$. Also for any i with $u_i(x_i) = 0$, by $(A4a)$, we have that $u_i(x_i + z_i) = u_i(z_i) \leq u_i(\omega_i)$. Hence (A4) is satisfied. ■

It is worthwhile to observe that

Proposition 4.12. *If any preference relation is strictly monotone, or if any preference relation is monotone and strictly convex, then condition $(A4a)$ in Proposition 4.11 is satisfied.*

Proof. Suppose first that any preference relation is strictly monotone. Suppose that $x_i, y_i \in P_i$ with $u_i(x_i) = 0$. Since $x_i \geq 0$, $u_i(0) = 0$ and u_i is strictly monotone we have $x_i = 0$, therefore $u_i(y_i) = u_i(x_i + y_i)$ and $(A4a)$ is true.

If any preference relation $\succeq_i$ is monotone and strictly convex, we have that any preference relation $\succeq_i$ is strictly monotone, as follows: For any $x, y \in P_i$ with $x > y$ we have $x = y + (x - y)$, and by the strict convexity of the preference $\succeq_i$, for any $t \in (0, 1)$ we have $t(y + (x - y)) + (1 - t)y \succ_i y$, therefore $t(x - y) + y \succ_i y$. But $\succeq_i$ is monotone, therefore by the relation $x > t(x - y) + y$, we have $x \succeq_i t(x - y) + y \succ_i y$ and $x \succ_i y$. Hence by the first part of proof, $(A4a)$ is satisfied. ■

Proposition 4.13. *If $P_i = X_+$ for any i, the cone X_+ has the Riesz decomposition property and the preferences $\succeq_i$ are monotone, then condition $(A4b)$ in Proposition 4.11 is satisfied.*

Proof. Let $x \in X$ with $0 \leq x \leq \omega$. Since $\omega = \sum_{i=1}^{l} \omega_i$, by the Riesz decomposition property, there exist vectors $x_i \in X$, $i = 1, 2, ..., l$ so that $0 \leq x_i \leq \omega_i$, for any i and $x = \sum_{i=1}^{l} x_i$. Since the preferences are monotone we have $0 \preceq_i x_i \preceq_i \omega_i$, for any i. ■

For each price vector p we denote by G_p the function

$$G_p(x) = (p(x_1), p(x_2), \dots, p(x_l)),$$

where $x = (x_1, x_2, \dots, x_l) \in (X)^l$. Also we shall denote by

$$w = (\omega_1, \omega_2, ..., \omega_l),$$

the initial allocation.

As it is standard, we say that a NFD-allocation $x \in \mathcal{A}$ is:

- a **quasi equilibrium**, supported by the linear functional p, if for any i and any $z \in P_i$ we have $z \succeq_i x_i \Longrightarrow p(z) \geq p(\omega_i)$;
- a **quasi-valuation equilibrium**, if moreover $p(\omega) > 0$

In the literature, in the above definitions the continuity of the supporting functional p is also required. In our terminology, we do not require the continuity of p in order to specify in our results the topology of the continuity of p.

For the next theorem we recall (Proposition 4.8) that $(A1c)$ implies $(A3)$.

Theorem 4.14. *Suppose that in the economy $\mathcal{E}$ the conditions $(A1)$, $(A1c)$, $(A2)$, are satisfied, the preferences are monotone and convex, the utility functions are radially continuous and the utility space $\mathcal{U}$ is radially closed. If the interior of X_+ is non-empty and $(A4)$ is satisfied, then a quasi equilibrium allocation x exists supported by a continuous, positive linear functional p of X.*
If moreover ω is an almost order unit of X, then we have $p(\omega) > 0$ and x is a quasi-valuation equilibrium.

Proof. Suppose that x_0 is an interior point of P. Then we can show, see [13, Theorem 3.8.4 and 3.8.5], that the set

$$B = \{f \in X_+^* \mid f(x_0) = 1\},$$

is a weak-star closed and norm bounded base for the dual cone $X_+^* = \{f \in X^* \mid f(x) \geq 0,\ \text{for any } x \in X_+\}$ of X_+, therefore the base B is weak-star compact.

For each $s \in \Delta$, we select an element $x_s = (x_{s1}, x_{s2}, ..., x_{sl})$ of $\mathcal{K}$ so that $u(x_s) = \rho(s)s$. Then by using Proposition 4.10, we can select a weakly Pareto optimal allocation $x^s = ({x_1}^s, {x_2}^s, ..., {x_l}^s)$ so that $u({x_i}^s) \geq \rho(s)s$. Also by $(A4)$ we may suppose that for any i with $u_i({x_i}^s) = 0$ we have $u_i({x_i}^s) \leq u_i(\omega_i)$.

By Theorem 4.5, there exists a price vector $p \in B$ which supports x^s and we denote the set of these supporting prices by $L(s)$, i.e.

$$L(s) = \{p \in B \mid p \text{ supports } x^s\}.$$

We define the sets

$$\Phi(s) = \{G_p(w - x^s) = (p(\omega_1 - x_1^s), p(\omega_2 - x_2^s), ..., p(\omega_l - x_l^s)) \mid p \in L(s)\}$$

and

$$\Psi(s) = s + \Phi(s).$$

If we suppose that $s \in \Delta$ is a fixed point of Ψ, i.e. $s \in \Psi(s)$, then we get a quasi equilibrium allocation as follows:

$$s = s + h, \text{ where } h \in \Phi(s), \text{ therefore } h = 0 \in \Phi(s).$$

So there exists $p \in L(s)$ so that $G_p(w - x^s) = 0$, hence $p \cdot \omega_i = p \cdot x_i^s$ for each i.
Since p supports the allocation x^s we have

$$x \in P_i, x \succsim_i x_i \Rightarrow p \cdot x \geq p \cdot x_i^s = p \cdot \omega_i,$$

hence $(x_1^s, x_2^s, \ldots, x^s{}_l)$ is a quasi equilibrium allocation supported by p.

To show that Ψ has a fixed point, we prove first that the set $\Psi(s)$ is convex for each s and that the graph of Ψ is closed. It is enough to show that Φ has these properties. The first is obvious. To show that the graph of Φ is closed, we suppose that

$$s_n \longrightarrow s, \ \gamma_n \in \Phi(s_n) \text{ and } \gamma_n \longrightarrow \gamma$$

and we have to show that $\gamma \in \Phi(s)$. Let

$$\gamma_n = G_{p_n}(w - x^{s_n}), \text{ where } p_n \in L(s_n).$$

Since the base B is weak star compact, the sequence $\{p_m\}$ has an accumulation point $p \in B$, therefore for any $V \in I$, where I is the set of the neighborhoods of 0 in the weak-star topology, there exists m_V so that $p_{m_V} \in V$ and the net $(p_{m_V})_{V \in I}$ converges to p. we shall show that p supports the allocation x^s. Consider $x \in P_i$ such that $x \succsim_i x_i^s$. Then by $(A1c)$, for any $t > 0$ there exists $y_{i,x,t} \in P_i$ so that $y_{i,x,t} \leq t\omega$ with

$$lim_{t \longrightarrow 0} y_{i,x,t} = 0 \text{ and } x + y_{i,x,t} \succ_i x \succeq_i x_i^s,$$

therefore $u_i(x + y_{i,x,t}) > u_i(x_i^s)$ for each $t > 0$. Since p_n supports the allocation x^{s_n} we have

$$p_n(x + y_{i,x,t}) = p_n(x) + p_n(y_{i,x,t}) \geq p_n(x_i^{s_n}) \text{ for each } t > 0,$$

and by taking limits as $t \longrightarrow 0$ we have that $p_n(x) \geq p_n(x_i^{s_n})$, for each n. Therefore we have

$$p_n(x) \geq p_n(x_i^{s_n}) = p_n(\omega_i) - p_n(\omega_i - x_i^{s_n}) = p_n(\omega_i) - \gamma_n^i,$$

where γ_n^i is the i coordinate of γ_n.
Since $p_{n_V} \longrightarrow p$ in the weak star topology of X^*, by the previous relation and the hypothesis that $\gamma_n \longrightarrow \gamma$ we have that $\gamma_{n_V} \longrightarrow \gamma$, and also that

$$p(x) \geq p(\omega_i) - \gamma^i \text{ for each } i,$$

where γ^i is the i coordinate of γ. But

$$p(\omega_i) - \gamma^i = p(\omega_i) - p(\omega_i - x_i^s) = \ p(x_i^s),$$

therefore $p(x) \geq p(x_i^s)$ and p supports the allocation x^s. Also by the above relation we have that $\gamma^i = p \cdot (\omega_i - x_i^s)$, hence

$$\gamma = G_p(w - x^s),$$

therefore $\gamma \in \Phi(s)$ and the function Φ has closed graph.

Let us show that the set Φ is bounded valued. For any i we have $\omega_i - x_i^s \leq \omega_i \leq \omega$ and $\omega_i - x_i^s \geq \omega_i - \omega \geq -\omega$. Therefore for any $p \in L(s)$ and any i we have

$$0 \leq |p(\omega_i - x_i^s)| \leq p(\omega) \leq M = max\{p(\omega) \mid p \in B\}.$$

Note that the linear functional $\widehat{\omega} \in X^*$ defined by ω, i.e. the linear functional $\widehat{\omega}(f) = f(\omega)$, for any $f \in X^*$, is weak-star continuous. Therefore $\widehat{\omega}$ takes maximum on the weak-star compact base B of X_+^* and the maximum in the above relation exists.

Therefore our function Φ takes values in a closed ball of $\mathbf{R}^l$. Hence Φ and also Ψ take value in a compact metric space. So , by the closed graph theorem for multivalued functions, see [1, Theorem 17.11], we have that Ψ is upper hemicontinuous.

Let us prove now that Ψ is inward pointing, i.e. for each $s \in \Delta$ there exists $y \in \Psi(s)$ and $\lambda > 0$ such that

$$s + \lambda(y - s) \in \Delta.$$

Precisely, we shall show that for any $s \in \Delta$ and any $z \in \Phi(s)$ i.e. for any $z = G_p(w - x^s)$, where $p \in L(s)$, the vector

$$y = s + z$$

of $\Psi(s)$ satisfies this condition. We have to show that the vector

$$b = s + \lambda(y - s) = s + \lambda z$$

belongs to Δ, where λ is a suitable real number which will be defined below.

Let $s \in \Delta$ be fixed. For this s we have selected an element x^s of $\mathcal{K}$ so that $u(x^s) = \rho(s)s$. To simplify our notations below, we shall denote x^s by x.

For any i with $s_i > 0$, there exists a real number $\lambda_i > 0$ such that $s_i + \lambda_i z_i > 0$. Therefore we can select a real number $\lambda = \lambda_0 > 0$ so that $b_i = s_i + \lambda_0 z_i > 0$, for each i with $s_i > 0$.

For any i with $s_i = 0$ we have $u_i(x_i^s) \geq \rho(s)s_i = 0$ and according to our remarks in the beginning of the proof, we have $\omega_i \succeq_i x_i^s$. Hence $p(\omega_i) \geq p(x_i^s)$, because p supports the allocation x^s.

For any i with $s_i = 0$ we have

$$z_i = p(\omega_i - x_i^s) \geq 0 \text{ and } b_i = s_i + \lambda_0 z_i \geq 0.$$

So, for the real number λ_0, we have $b_i = s_i + \lambda_0 z_i \geq 0$, for any i, hence $b \geq 0$.

Also we have that $b \in \Delta$ because $s \in \Delta$, and $\sum_{i=1}^{l} z_i = 0$. The last equality is true because $z = G_p(w - x^s) = (p(\omega_1 - x_1^s), p(\omega_2 - x_2^s), ..., p(\omega_l - x_l^s))$ and w, x^s are NFD-allocations. Therefore Ψ is inward pointing. By the theorem of Halpern-Bergman, see [1, Theorem 17.54,], we have that Ψ has a fixed point and the proof has been completed. If ω is a almost order unit of X_+, by Remark 4.4 we have $p(\omega) > 0$. ■

Remark 4.15. We briefly recall some main equilibrium existence results given in frameworks close to that of our paper. Among the papers present in the literature dealing with commodity space of infinite dimension, [11] is pioneering. In the latter paper, the hypothesis of free disposal is assumed and also a weak axiom of continuity for preferences (the assumption III of page 590) is used. This assumption is the following:
For any i, any $x_i, x_i', x_i'' \in P_i$ and any real sequence $t_n \in [0, 1]$ with $t_n \longrightarrow t$, we have:

$$t_n x_i' + (1 - t_n)x_i'' \succeq_i x_i \textit{ for each } n \Longrightarrow t x_i' + (1 - t)x_i'' \succeq_i x_i,$$

and

$$t_n x_i' + (1 - t_n)x_i'' \preceq_i x_i \textit{ for each } n \Longrightarrow t x_i' + (1 - t)x_i'' \preceq_i x_i.$$

It is easy to see that our assumption of linear continuity of the utility functions implies the assumption III of Debreu and if in our economy $\mathcal{E}$ we add the extra assumption $(*)$ below, then assumption III implies the linear continuity of the utility functions[5] and these two conditions are equivalent. For this equivalence, the closedness of the cones P_i, the monotonicity of the utility functions and property $(*)$ are only needed. The condition is a kind of completeness of the preferences and is the following: *We say that assumption*

[5]The proof is the following: Suppose that III is satisfied , $x_i, y_i \in P_i$, $z_n = t_n x_i + (1 - t_n)y_i$, $t_n \longrightarrow t$ and $z = t x_i + (1 - t)y_i$. If there exists a subsequence of $u_i(z_n)$ that we denote again by $u_i(z_n)$ and a real number $a > u_i(z)$ with $u_i(z_n) \geq a$, for any n, then there exists $w \in P_i$ so that $u_i(z_1) \geq a > u_i(w) > u_i(z)$. Then $u_i(z_n) \geq a > u_i(w) > u_i(z)$ for each n, hence $z_n \succeq w$ for any n, therefore $z \succeq w$ by III. So we have $z \succeq w \succ z$, a contradiction. Similarly, $u_i(z_n) \leq a < u_i(z)$ for a subsequence of $u_i(z_n)$, there exists $w \in P_i$ with $u_i(z_1) \leq a < u_i(w) < u_i(z)$ and we have the contradiction $z \preceq w \prec z$, by III. Therefore $u(z_n) \longrightarrow u(z)$ and u_i is linearly continuous.

$(*)$ *is satisfied if for any* i, *any* $x_i, y_i \in P_i$ *and any real number* $a_i \in (u_i(x_i), u_i(y_i))$ *there exist* $z_i, w_i \in P_i$ *with* $u_i(x_i) < u_i(z_i) < a_i < u_i(w_i) < u_i(y_i)$.

As it is made clear by [12], and also evident in [9], two important points matter in the case of equilibrium existence with general consumption sets: one is the possibility to have relative compactness of bounded sets with respect to a weak-star topology on the commodity space and the other one concerns the free disposal assumption. The approach of [15] and [4], which has the advantage to admit the possibility of an empty interior for the positive cone, strongly relies on the lattice structure of the commodity space and on the fact that consumption sets coincide with the positive cone. In the same context, the paper by [8] considers general consumption sets but, in order to work with them, assumes a form of comprehensiveness with respect to vectors of the positive cone. An approach slightly different, based on a separating argument in the space of allocations, is presented in [14], again under the assumption that the consumption sets coincide with the positive cone. Finally, the asymmetric information model considered in [17], produces an existence theorem which generalizes results by [9] and [12] at several instances. Notice that in this paper, the non-free disposal assumption entails that the linear supporting price is not necessarily positive.

4.3.1 The case where $P_i = X_+$ for any i.

In order to show that our assumptions are natural extensions or variations of the usual ones, in this subsection we consider the case where all the consumers have as consumption set the positive cone X_+ of X, i.e. $P_i = X_+$ for any i. In this case $(A1)$ is trivially true and $(A1b)$ and $(A1c)$ are equivalent to the hypothesis that ω is an extremely desirable bundle for any consumer. In particular, if the preferences are strictly monotone, then ω is an extremely desirable bundle for any consumer. Moreover, if the preferences are monotone and strictly convex, then they are strictly monotone (for this see the proof of Proposition 4.12) hence ω is an extremely desirable bundle for any consumer.

If the preferences $\succeq_i$ are defined by monotone utility functions u_i, we have that $(A2)$ is true as follows: As we have observed after the definition of $(A2)$ we may suppose that $u_i(0) = 0$ for any i. Also, by the definition of Ω_i and our assumption that $P_i = X_+$ for any i, we have that $\Omega_i = [0,\ \omega]$ for any i, therefore, for any $x \in \Omega_i$, we have that $0 \leq u_i(x) \leq u_i(\omega)$ hence u_i is bounded on Ω_i and $(A2)$ is true.

By statement (vi) of Proposition 4.10 and the above remarks we have: If the preferences $\succeq_i$ are defined by monotone utility functions, ω is an extremely desirable for any consumer and the utility functions are linearly continuous, then $(A4)$ is true.

Recall that u_i is linearly continuous if the restriction of u_i on any line of P_i is continuous. After these remarks, by Theorem 4.14 we have:

Theorem 4.16. *Suppose that in the economy* $\mathcal{E}$ *we have* $P_i = X_+$ *for any* i. *Suppose also that the preferences* $\succeq_i$ *are convex and defined by monotone utility functions. If* ω *is an extremely desirable bundle for any consumer, the utility space* $\mathcal{U}$ *is radially closed,* X_+

has nonempty interior and $(A4)$ is satisfied (in particular if the utility functions are linearly continuous), then a quasi equilibrium allocation x exists supported by a continuous, positive linear functional p of X.
If moreover ω is a almost order unit of X, then we have $p(\omega) > 0$, therefore x is a quasi-valuation equilibrium.

5 Cones with nonempty semi-interior and equilibrium

In this section we still study the exchange economy as defined in subsection 4.1. However, we shall add to conditions $(A1)-(A4)$ the non emptiness of the set of semi-interior, rather than interior, points of X_+.

Suppose that X is a normed space ordered by the positive cone X_+. we shall denote by $||.||$ the initial norm of X and by $|||.|||$ the norm of X defined by the positive cone X_+ of X. Recall that the $||.||$-topology of X is coarser than the $|||.|||$-topology of X. Recall also that $x \in X_+$ is a semi-interior of X_+ if $x - \rho U_+ \subseteq X_+$ for some real number $\rho > 0$, where U is the unit ball of the initial norm $||.||$ of X.

Theorem 5.1. *Suppose that in the economy $\mathcal{E}$ the conditions $(A1)$ and $(A1b)$ are satisfied and that the preferences are monotone and convex. If x_0 is a semi-interior point of X_+, then any weakly Pareto optimal allocation is supported by a positive, $|||.|||$-continuous linear functional p of X with $p(x_0) = 1$. If moreover ω is a $|||.|||$-almost order unit of X, we have that $p(\omega) > 0$.*

Proof. Consider the space X equipped with the $|||.|||$-topology defined by the positive cone X_+. Since the cone X_+ and the subcones P_i of X_+ are $||.||$-closed, we have that the cone X_+ and the subcones P_i are $|||.|||$-closed. Since x_0 is a semi-interior point of X_+, by Proposition 3.2, x_0 is an $|||.|||$-interior point of X_+. So we consider as commodity space the ordered normed space $(X, |||.|||)$. Then all the assumptions of Theorem 4.5 are satisfied, hence any weakly Pareto optimal allocation is supported by a positive, linear functional p of X, which is continuous with respect to the $|||.|||$-topology of X and such that $p(x_0) = 1$.
If ω is a $|||.|||$-almost order unit of X, by Remark 4.4 we have that $p(\omega) > 0$. ■

Theorem 5.2. *Suppose that in the economy $\mathcal{E}$ the conditions $(A1)$, $(A1c)$, $(A2)$, are satisfied, the preferences are monotone and convex and the utility functions are radially continuous. If the utility space $\mathcal{U}$ is radially closed, X_+ has at least one semi-interior point and $(A4)$ is satisfied, then a quasi equilibrium allocation x exists which is supported by a positive, $|||.|||$-continuous linear functional p of X, where $|||.|||$ is the norm of X defined by the cone X_+. If moreover ω is a $|||.|||$-almost order unit of X, we have that $p(\omega) > 0$ and x is a $|||.|||$-quasi-valuation equilibrium.*

Proof. Consider, as commodity space, the ordered normed space $(X, |||.|||)$. Then, by Proposition 3.2, x_0 is a $|||.|||$- interior point of X_+. The cone X_+ and the subcones P_i of X_+ are $|||.|||$-closed. Also the utility functions are radially continuous with respect to the $|||.|||$-topology of X because the $|||.|||$-topology of X is finer than the $||.||$ -topology. So for the commodity space $(X, |||.|||)$, all the assumptions of Theorem 4.14 are satisfied, hence a quasi equilibrium allocation x exists which is supported by a positive $|||.|||$-continuous linear functional p of X. If moreover ω is a $|||.|||$-almost order unit of X, we have that $p(\omega) > 0$ by Remark 4.4 and x is a $|||.|||$-quasi-valuation equilibrium. ■

In the case where $P_i = X_+$ for any i, Theorem 4.16 takes the following form and can be applied, for example, with reference to the space of Example 2.5

Theorem 5.3. *Suppose that in economy $\mathcal{E}$ we have $P_i = X_+$ for any i. Suppose also that the preferences $\succeq_i$ are convex and defined by monotone utility functions. If ω is an extremely desirable bundle for any consumer, the utility space $\mathcal{U}$ is radially closed, X_+ has at least one semi-interior point and $(A4)$ is satisfied (in particular if the utility functions are linearly continuous), then a quasi equilibrium allocation x exists supported by a $|||.|||$-continuous, positive linear functional p of X.*
If moreover ω is an $|||.|||$-almost order unit of X, we have $p(\omega) > 0$ and x is a $|||.|||$-quasi-valuation equilibrium.

6 Strongly reflexive cones

In this section we apply our results in the case where X is a normed space ordered by a strongly reflexive and normal cone P, i.e $X_+ = P$. Recall that the cone P is strongly reflexive if the positive part $U_+ = U \cap P$ of the unit ball U of X is compact. Strongly reflexive cones have been studied in [10], where it is shown that this class of cones is a rich one. Also our examples 2.6 and 2.7 are examples of strongly reflexive cones with semi-interior points.

In the next theorem we avoid the closedness condition. In particular we avoid the assumption of Theorem 5.2 that the utility space is radially closed. However, the radial continuity is replaced by the stronger condition of the continuity of the utility functions.

Theorem 6.1. *Suppose that in the economy $\mathcal{E}$ the commodity space X is a normed space ordered by the normal and strongly reflexive cone X_+. Suppose that the conditions $(A1), (A1c), (A2)$, are satisfied, the preferences are monotone and convex and the utility functions are continuous. If X_+ has at least one semi-interior point and $(A4)$ is satisfied, then a quasi equilibrium allocation x exists supported by a positive linear functional p of X and p is continuous with respect to the $|||.|||$ norm of X which is defined by the cone X_+. If moreover ω is a $|||.|||$-almost order unit of X, then $p(\omega) > 0$.*

Proof. According to Theorem 5.2 it is enough to show that the utility space $\mathcal{U}$ is radially closed. Since the cone X_+ is normal, there exists a real constant $c > 0$ so that for any $x, y \in X$ we have: $0 \leq x \leq y \Longrightarrow ||x|| \leq c||y||$.

For any $x = (x_1, x_2, ..., x_l) \in \mathcal{K}$ we have $x_i \in [0, \omega]$ for any i, then we have $||x_i|| \leq c||\omega||$ and $x_i \in c||\omega||U_+$.

Then the set
$$W = c||\omega||U_+,$$
is a compact subset of X because U_+ is compact and it is easy to show that the set $\mathcal{K}$ is a closed subset of the compact subset W^l of X^l. Therefore $\mathcal{K}$ is compact. Since the total utility function u is continuous, we have that the utility space $\mathcal{U}$, as the image of $\mathcal{K}$ via the total utility, is compact. Therefore $\mathcal{U}$ is closed and hence radially closed and the theorem is true.

■

In the case where $P_i = X_+$ for any i, and X_+ is strongly reflexive, Theorem 4.16 takes the form:

Theorem 6.2. *Suppose that in the economy $\mathcal{E}$ we have $P_i = X_+$ for any i and that the cone X_+ is strongly reflexive and normal. Suppose also that the preferences $\succeq_i$ are convex and defined by monotone utility functions. If the utility functions are continuous and X_+ has at least one semi-interior point, then a weak equilibrium allocation x exists supported by a $|||.|||$-continuous, positive linear functional p of X.*
If moreover ω is an almost order unit of X, then we have $p(\omega) > 0$, therefore x is a $|||.|||$-quasi-valuation equilibrium.

In the next examples we continue Example 2.6 and Example 2.7 by proving the existence of equilibrium. In these examples E is a Banach lattice, P is a subcone of E_+, and $X = P - P$ is the subspace of E generated by the cone P. In both cases the cone P is strongly reflexive with semi-interior points. Note also that P as a subcone of E_+ is normal. We consider X ordered by the cone P and our economic model is considered with respect to the new ordering. Therefore, by the previous theorem, a quasi valuation equilibrium allocation exists. Note that in both examples the closedness (compactness) of utility space $\mathcal{U}$ is not needed. Also the space X is dense in the initial space E.

Example 2.6 continued. Let E be an infinite dimensional Banach lattice with a normalized, positive Schauder basis $\{e_i\}$. For example, we may suppose that E is one of the spaces c_0 or ℓ_p for $1 \leq p < +\infty$. Suppose that P is the cone of Example 2.6. Suppose that in our economy of subsection 4.1 the commodity space is the subspace $X = P - P$ of E and suppose that X is ordered by the cone P, i.e. $X_+ = P$.

As we have noted in this example X is dense in E and also P has semi-interior points. Suppose that the total endowment ω is such a semi-interior point of X_+. Suppose also that all the consumers have the same consumption set i.e. $P_i = X_+$ for any i and that

the preferences are continuous, monotone and strictly convex. Then the preferences are strictly monotone hence ω is extremely desirable for any consumer. Then by Theorem 6.2 a quasi equilibrium allocation x exists supported by a $|||.|||$-continuous, positive linear functional p of X, where $|||.|||$ is the norm of X defined by the cone X_+.
Also ω, as a semi-interior point of X, is an order unit of X therefore $p(\omega) > 0$ and x is a $|||.|||$-quasi-valuation equilibrium.

Example 2.7 continued. Let E be the space $L_1\,[0,1]$ and suppose that P is the subcone of E_+ of Example 2.7. Suppose that in our economy the commodity space is the subspace $X = P - P$ of E ordered by the cone P, i.e. $X_+ = P$. As we have noted in this example X is dense in $L_1\,[0,1]$ and P has semi-interior points. Suppose that the total endowment ω is such a semi-interior point. Suppose also that all the consumers have the same consumption set i.e. $P_i = X_+$ and that the preferences are monotone and strictly convex. Then the preferences are strictly monotone and ω is extremely desirable for any consumer. Then a quasi equilibrium allocation x exists supported by a $|||.|||$-continuous, positive linear functional p of X, where $|||.|||$ is the norm of X defined by the cone X_+. As above x is a $|||.|||$-quasi-valuation equilibrium.

7 Appendix: Ordered linear spaces

In this paragraph, we give some essential notions and results from the theory of (partially) ordered linear spaces which are needed in this paper. For more information see in [3], [13] and [1].

An **ordered vector space** is a linear space X equipped with a reflexive, antisymmetric and transitive relation $\geq$ with the property: $x \geq y \Longrightarrow x + z \geq y + z$ and $\lambda x \geq \lambda y$, for any $x, y, z \in X$ and any real number $\lambda \geq 0$. Then we say that $\geq$ is an order relation of X and $X_+ = \{x \in X \mid x \geq 0\}$ is the positive cone of X. Note that the order relation of X is not necessarily complete.

Let X be a vector space and let P be a **cone** of X (i.e. P is a nonempty, convex subset of X so that $\lambda P \subseteq P$ for every real number $\lambda \geq 0$ and $P \cap (-P) = \{0\}$). The cone $P \subseteq X$ induces the order relation $\geq$ in X so that $x \geq y$ if and only if $x - y \in P$, for any $x, y \in X$. Then X is an ordered vector space with positive cone the cone P, i.e. $X_+ = P$.

If $P - P = X$ the cone P is **generating** or **reproducing**. For any $x, y \in X$ with $x \leq y$, the set $[x,\ y] = \{z \in X \mid x \leq z \leq y\}$ is the **order interval** defined by x, y. A linear functional f of X is **positive** if $f(x) \geq 0$ for each $x \in P$ and **strictly positive** if $f(x) > 0$ for each $x \in P, x \neq 0$. A subset B of P is a **base for the cone** P if a strictly positive linear functional f of X exists so that, $B = \{x \in P \mid f(x) = 1\}$. Then we say that the base B is defined by the functional f. A vector $x \in P$ is **strictly positive** if for any non-zero, positive and continuous linear functional f of X we have $f(x) > 0$.

If for any $x, y \in E$ the supremum $x \vee y$ and the infimum $x \wedge y$ of the set $\{x, y\}$ exist in X, then X is a **vector lattice** and we denote by $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ and $|x| = x \vee (-x)$ the positive part, the negative part and the absolute value of x. A vector $e \in X_+$ is an **order unit** of X if $X = \cup_{n=1}^{\infty}[-ne, ne]$.

If X is a normed space, then every interior point of P is an order unit of X, [3, Lemma 2.5], but the converse is not always true. For the converse the completeness of X is needed. We have: If X is a Banach space and P is closed, then every order unit of P is an interior point of P [3, Theorem 2.8]. If X is a normed space and $int(P) \neq \emptyset$ we have: a vector $x \in P$ is strictly positive if and only if x is an interior point of P, [3, Lemma 2.17].

Recall that for any $A \subseteq X$ we denote by $\overline{A}$, the closure of A, by $int(A)$ the set of interior points of A by $\mathrm{co}(A)$ the convex hull of A and by $\overline{\mathrm{co}}(A)$ the closed convex hull of A.

If X is a Banach space, the sequence $\{x_n\}$ is a **positive basis** of X if it is a Schauder basis of X and the positive cone P of X and the positive cone of the basis $\{x_n\}$ (i.e. the set of elements of X with positive coordinates) coincide. Note that the usual bases of the spaces ℓ_p , $1 \leq p < \infty$ and the space c_0 are the simplest examples of positive bases.

Finally note that a **Banach lattice** is an ordered Banach space X which is a lattice, with the property: $|x| \geq |y| \Longrightarrow ||x|| \geq ||y||$, for any $x, y \in X$.

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