Centre for Studies in Economics and Finance

WORKING PAPER NO. 471

Uniqueness of Nash Equilibrium in Continuous Weighted Potential Games

Francesco Caruso, Maria Carmela Ceparano and Jacqueline Morgan

April 2017

This version May 2017

University of Naples Federico II

University of Salerno

Bocconi University, Milan

CSEF - Centre for Studies in Economics and Finance
DEPARTMENT OF ECONOMICS – UNIVERSITY OF NAPLES
80126 NAPLES - ITALY
Tel. and fax +39 081 675372 – e-mail: csef@unisa.it
ISSN: 2240-9696

WORKING PAPER NO. 471

Uniqueness of Nash Equilibrium in Continuous Weighted Potential Games

Francesco Caruso*, Maria Carmela Ceparano* and Jacqueline Morgan**

Abstract
The literature results about existence of Nash equilibria in continuous potential games (Monderer and Shapley, 1996) exploits the property that any maximum point of the potential function is a Nash equilibrium of the game (the vice versa being not true) and those about uniqueness use strict concavity of the potential function. Therefore, the following question arises: can we find sufficient conditions on the data of the game which guarantee one and only one Nash equilibrium when existence of a maximum of the potential function is not ensured and the potential function in not strictly concave? The paper positively answers this question for two-player weighted potential games when the strategy sets are not bounded sets of not necessarily finite dimensional spaces. Significative examples infinite dimensional spaces are provided, together with an application in infinite dimensional ones.

Keywords: Non-cooperative game; weighted potential game; uniqueness of Nash equilibrium; fixed point.

* Università di Napoli Federico II. Postal address: Department of Economics, Università di Napoli Federico II, Via Cintia (Monte S. Angelo), 80126 Napoli, Italy.
E-mails: francesco.caruso@unina.it and mariacarmela.ceparano@unina.it

** Università di Napoli Federico II and CSEF. Postal address: Department of Economics, Università di Napoli Federico II, Via Cintia (Monte S. Angelo), 80126 Napoli, Italy. E-mail: morgan@unina.it

Table of contents

1 Introduction

The set of Nash equilibria of a (weighted) potential game with (weighted) potential function P coincides with the set of Nash equilibria of a game in which all the payoff functions of the players are replaced by the (weighted) potential function P ([14]). Any maximum point of P is a Nash equilibrium of the (weighted) potential game but the converse is not true in general: it may exist a Nash equilibrium of the potential game that is not a maximum point of P. Nevertheless, if we assume that the space $\mathbf{X}$ of the strategy profiles is a convex set and P is bounded and concave on $\mathbf{X}$ and continuously differentiable on the interior of $\mathbf{X}$, then any Nash equilibrium of the potential game is also a maximum point of P [16, Corollary of Theorem 1]. If, in addition, P is strictly concave and attains a maximum, then the potential game has one and only one Nash equilibrium which coincides with the maximum point of P. The last result is implied by Theorem 2 in [17]: in fact, if the potential function P is strictly concave then the *diagonal strict concavity* condition holds. However, the strict concavity of P is a very strong assumption and, incidentally, it is not sufficient to ensure by itself the existence of a maximum point of P if the strategy sets are not compact.

In this paper we consider a two-player weighted potential game where the strategy sets are assumed to be real Hilbert spaces and, having in mind to construct algorithms in order to approach Nash equilibria, we present a uniqueness result (existence of one and only one Nash equilibrium) which could be useful when the equilibria cannot be computed explicitly (see Example 3.2) or when P is not strictly concave.

Being the strategy sets real Hilbert spaces, our uniqueness result has not to be compared with the results where the strategy sets are required to be (lower and/or upper) bounded sets, a property which is ruled out by our assumption. See, for example, the results in [17, 12, 10, 5].

The paper is structured as follows.

In Section 2 the concept and some properties of weighted potential games, together with the differentiability of the best reply functions and a majorization of their Lipschitz constant, are recalled. Section 3 contains the main results of the paper illustrated by examples in finite dimensional spaces. In particular, when no one of the two possible compositions of the best reply functions is a contraction, uniqueness of Nash equilibrium is shown to be guaranteed for a non-restrictive class of games (Theorem 3.1). Finally, in Section 4 the class of bilinear weighted potential functions is considered and an application to a differential game is provided.

2 Weighted potential games and best replies properties

For $i \in I := \{1, 2\}$, let X_i be a real Hilbert space. We denote by $(\cdot\,,\cdot)_{X_i}$ the inner product on X_i and by $\|\cdot\|_{X_i}$ the norm on X_i. Let F_i be a real-valued function defined on $\mathbf{X} \coloneqq X_1 \times X_2$.

First, we clarify some notations. For any $i \in I$ we denote by $-i$ the player different from i (i.e., $\{-i\} = I \setminus \{i\}$). So, a strategy profile $\mathbf{x} = (x_1, x_2) \in \mathbf{X}$ could be denoted also by $(x_i, x_{-i}) \in X_i \times X_{-i}$. Furthermore, if $i \in I$ and $x_{-i} \in X_{-i}$, by $F_i(\cdot, x_{-i})$ we denote the real-valued function $x_i \to F_i(x_i, x_{-i})$. Within the paper, let Γ be the two-player game in strategic form where X_i and F_i are respectively the strategy set and the payoff function of player i, that is $\Gamma \coloneqq \{2, X_1, X_2, F_1, F_2\}$.

The game Γ is said to be a *weighted potential game* ([14]) if there exist a vector $\mathbf{w} = (w_1, w_2) \in \mathbb{R}^2_{++} \coloneqq \{(w_1, w_2) \in \mathbb{R}^2 : w_1 > 0, w_2 > 0\}$ and a real-valued function P defined on $\mathbf{X}$ such that

$$F_i(x_i, x_{-i}) - F_i(x_i', x_{-i}) = w_i(P(x_i, x_{-i}) - P(x_i', x_{-i})),$$

for any $x_i, x_i' \in X_i$ and $x_{-i} \in X_{-i}$, for any $i \in I$. P is called *weighted potential* (w-potential in short) of Γ. When $w_1 = w_2 = 1$, Γ becomes a *potential game* and P is a *potential*.

A useful characterization of weighted potential games is given by the following proposition.

Proposition 2.1 (Theorem 2.1 [9]). *The following conditions are equivalent:*

(i) Γ *is a weighted potential game with w-potential* P *and weights* $(w_i)_{i \in I}$*;*

(ii) there exists $(h_i)_{i \in I}$*, where* $h_i : X_{-i} \to \mathbb{R}$*, such that*

$$F_i(\mathbf{x}) = w_i P(\mathbf{x}) + h_i(x_{-i}), \quad \textit{for any } \mathbf{x} \in \mathbf{X} \textit{ and } i \in I.$$

Now we recall some notations, following for example [13, 8, 2]. When U and V are Banach spaces, $\mathcal{L}(U, V)$ denotes the (Banach) space of continuous linear operators defined on U and valued on V and $\|\cdot\|_{\mathcal{L}(U,V)}$ denotes the operator norm of $\mathcal{L}(U, V)$, U^* denotes the space of continuous linear operators defined on U and valued on $\mathbb{R}$, i.e. $U^* = \mathcal{L}(U, \mathbb{R})$, and $\langle \cdot, \cdot \rangle_{U^* \times U}$ denotes the duality operation between U^* and U. Furthermore, if f is a twice differentiable function $f : U \to V$, we denote by Df the *Fréchet derivative* of f and by $D^2 f$ the second Fréchet derivative of f. Recall that Df assigns to any $u \in U$ an element of $\mathcal{L}(U, V)$ called the *derivative of* f *at* u and $D^2 f$ assigns to any $u \in U$ an element of $L(U, \mathcal{L}(U, V))$ called the *second derivative of* f *at* u. If g is a twice differentiable function from $\mathbf{X}$ to $\mathbb{R}$ and $j, k \in I$, we denote by $D_{x_j} g$ the *partial derivative of* g *with respect to* x_j and by $D_{x_k}(D_{x_j} g)$ the *second partial derivative of* g *with respect to* x_j *and* x_k; when $k = j$ the standard notation $D^2_{x_j} g \coloneqq D_{x_j}(D_{x_j} g)$ is used. Note that $D_{x_j} g(\mathbf{x}) \in X_j^*$ and $D_{x_k}(D_{x_j} g)(\mathbf{x}) \in \mathcal{L}(X_k, X_j^*)$.

The following result is a direct consequence of Proposition 2.1 and gives a necessary condition for a game to be a weighted potential game when the payoff functions are twice-continuously differentiable.

Corollary 2.1. *If Γ is a weighted potential game and the payoff functions are twice continuously differentiable then there exists $\alpha_i > 0$ such that*

$$D_{x_{-i}}(D_{x_i}F_i) = \alpha_i D_{x_{-i}}(D_{x_i}F_{-i}).$$

Let Γ be a weighted potential game with P as a w-potential and $i \in I$. We denote by B_i the *best reply correspondence of player i*, that is B_i is the set-valued map defined on X_{-i} by

$$B_i(x_{-i}) \coloneqq \operatorname*{Arg\,max}_{x_i \in X_i} F_i(x_i, x_{-i}) \subseteq X_i,$$

i.e., $B_i(x_{-i}) = \{x_i \in X_i \colon F_i(x_i, x_{-i}) \geq F_i(x_i', x_{-i}),$ for any $x_i' \in X_i\}$. By Proposition 2.1, $B_i(x_{-i}) = \operatorname{Arg\,max}_{x_i \in X_i} P(x_i, x_{-i})$. If B_i is (nonempty) single-valued, the function b_i such that $\{b_i(x_{-i})\} \coloneqq B_i(x_{-i})$ is well-defined and called *best reply function of player i*.

From now on we assume that B_i is single-valued, for any $i \in I$.

Remark 2.1 If P is strongly concave in any argument, that is if the function $P(\cdot, x_{-i}) : x_i \in X_i \to P(x_i, x_{-i})$ is strongly concave for any $x_{-i} \in X_{-i}$ and any $i \in I$, then B_i is single-valued, for any $i \in I$ [4, Corollary 11.16]. Clearly a function could be strongly concave in any argument and, at the same time, it could be not concave on $\mathbf{X}$ (take, for example, P defined on $\mathbb{R}^2$ by $P(x_1, x_2) = -x_1^2 - x_2^2 - 5x_1^2 x_2^2$).

Let $i \in I$. In the following we use:

$(\mathcal{A}_i)$ $\left\{ \text{$P$ is a twice continuously differentiable function on $\mathbf{X}$ and the inverse operator $[D^2_{x_i}P(\mathbf{x})]^{-1} : X_i^* \to X_i$ exists and is continuous, for any $\mathbf{x} \in \mathbf{X}$.} \right.$

Proposition 2.2. *Let $i \in I$ and assume $(\mathcal{A}_i)$. Then the best reply function b_i is continuously differentiable on X_{-i}. Moreover, if $\lambda_i \in [0, +\infty[$, where*

$$\lambda_i \coloneqq \sup_{\mathbf{x} \in \mathbf{X}} \|[D^2_{x_i}P(\mathbf{x})]^{-1} \circ D_{x_{-i}}(D_{x_i}P)(\mathbf{x})\|_{\mathcal{L}(X_{-i}, X_i)}$$

then b_i is Lipschitz continuous with Lipschitz constant no greater than λ_i.

Proof. The arguments are similar to those used, for example, in [15], where algorithms were constructed in order to approach a saddle point of two-player zero-sum games. Let $x_{-i} \in X_{-i}$. In light of the differentiability of P on $\mathbf{X}$ and by Proposition 2.1, the pair $(b_i(x_{-i}), x_{-i})$ satisfies the equation:

$$D_{x_i}P(b_i(x_{-i}), x_{-i}) = 0$$

(see, e.g., [2, Théorème 1.2 p. 102]). Hence, by the Implicit Function Theorem (see, e.g., [2, Corollaire 5.2 p. 31]), b_i is continuously differentiable on X_{-i}. Furthermore, $Db_i(x_{-i}) \in \mathcal{L}(X_{-i}, X_i)$ and

$$Db_i(x_{-i}) = [D^2_{x_i}P(b_i(x_{-i}), x_{-i})]^{-1} \circ [D_{x_{-i}}(D_{x_i}P)(b_i(x_{-i}), x_{-i})]. \tag{1}$$

Thus, $\sup_{x_{-i}\in X_{-i}} \|Db_i(x_{-i})\|_{\mathcal{L}(X_{-i},X_i)} \leq \lambda_i$. By the Mean Value Inequality (see, e.g., [2, Corollaire 1.4 p. 19])

$$\begin{aligned}\|b_i(x'_{-i}) - b_i(x''_{-i})\|_{X_i} &\leq \sup_{t\in[0,1]} \|Db_i(tx'_{-i} + (1-t)x''_{-i})\|_{\mathcal{L}(X_{-i},X_i)} \|x'_{-i} - x''_{-i}\|_{X_{-i}} \\ &\leq \lambda_i \|x'_{-i} - x''_{-i}\|_{X_{-i}}\end{aligned}$$

for any x'_{-i}, $x''_{-i} \in X_{-i}$. Therefore, if $\lambda_i \in [0, +\infty[$, then b_i is Lipschitz continuous with Lipschitz constant no greater that λ_i. □

Let $i \in I$ and define the function $\beta_i : X_i \to X_i$ by

$$\beta_i(x_i) \coloneqq (b_i \circ b_{-i})(x_i) = b_i(b_{-i}(x_i)), \quad \text{for any } x_i \in X_i. \tag{2}$$

Remark 2.2 A strategy $\bar{x}_i \in X_i$ is a fixed point of β_i on X_i if and only if $(\bar{x}_i, b_{-i}(\bar{x}_i))$ is a Nash equilibrium of Γ.

We conclude this section with the following result on the derivative of β_i.

Proposition 2.3. *Assume $(\mathcal{A}_i)$ for any $i \in I$. Then, for any $i \in I$, the function $\beta_i = b_i \circ b_{-i}$ is continuously differentiable on X_i and, for any $x_i \in X_i$, $D\beta_i(x_i) \in \mathcal{L}(X_i, X_i)$ is defined by*

$$\begin{aligned}D\beta_i(x_i) =& [D^2_{x_i} P(\beta_i(x_i), b_{-i}(x_i))]^{-1} \circ [D_{x_{-i}}(D_{x_i}P)(\beta_i(x_i), b_{-i}(x_i))] \\ &\circ [D^2_{x_{-i}} P(x_i, b_{-i}(x_i))]^{-1} \circ [D_{x_i}(D_{x_{-i}}P)(x_i, b_{-i}(x_i))].\end{aligned} \tag{3}$$

Moreover, if λ_1, $\lambda_2 \in [0, +\infty[$, then β_i is Lipschitz continuous with Lipschitz constant no greater than λ, where

$$\lambda \coloneqq \lambda_1 \cdot \lambda_2. \tag{4}$$

Proof. By the chain rule (see, e.g., [2, p. 14]), β_i is continuously differentiable on X_i, $D\beta_i(x_i) = Db_i(b_{-i}(x_i)) \circ Db_{-i}(x_i)$ for any $x_i \in X_i$, and equality in (3) follows by (1). Furthermore, in light of Parapraph 2.4 in [2, p. 122]

$$\sup_{x_i\in X_i} \|D\beta_i(x_i)\|_{\mathcal{L}(X_i,X_i)} \leq \lambda_1 \cdot \lambda_2 = \lambda,$$

which implies, as a consequence of the Mean Value Inequality, that

$$\|\beta_i(x'_i) - \beta_i(x''_i)\|_{X_i} \leq \lambda \|x'_i - x''_i\|_{X_i}, \text{ for any } x'_i,\ x''_i \in X_i. \tag{5}$$

Hence, if λ_1, $\lambda_2 \in [0, +\infty[$, then β_i is Lipschitz continuous with Lipschitz constant no greater that λ. □

3 Uniqueness result

In this section assume:

$$(\mathcal{H}) \begin{cases} \Gamma \text{ is a weighted potential game where } X_1 \text{ and } X_2 \text{ are real Hilbert spaces,} \\ P \text{ (w-potential) satisfies assumption } (\mathcal{A}_i) \text{ for any } i \in I, \text{ and the best reply} \\ B_i \text{ is single-valued, for any } i \in I. \end{cases}$$

Let λ be defined as in (4) and β_i be defined as in (2), for any $i \in I$. When $\lambda < 1$, any β_i, for $i \in I$, is a contraction. Then, a well-known Nash equilibrium uniqueness result is obtained by using the Contraction Mapping Theorem [1, Theorem 7 p. 244].

Proposition 3.1. *Assume* $(\mathcal{H})$ *and* $\lambda \in [0, 1[$. *Then* Γ *has one and only one Nash equilibrium.*

Remark 3.1 When $X_1 = X_2 = \mathbb{R}$, the *strict diagonal dominance* condition (used in [10] in the case where the strategy spaces are $X_1 = X_2 = [0, +\infty[$) is equivalent to require[1] $|D_{x_i}(D_{x_j}P)(\mathbf{x})/D^2_{x_i}P(\mathbf{x})| < 1$ for any $\mathbf{x} \in \mathbb{R}^2$ and $i \in I$, which implies $\lambda \leq 1$.

Remark 3.2 If $\lambda = 1$ nothing can be said about existence or about uniqueness of Nash equilibrium. Indeed, consider the two-player games with $X_1 = X_2 = \mathbb{R}$ and:

- $F_1(x_1, x_2) = -x_1^2/2 + x_1x_2$ and $F_2(x_1, x_2) = -x_2^2/2 + x_2 + x_1x_2$. This game is a potential game with $P(x_1, x_2) = -x_1^2/2 - x_2^2/2 + x_1x_2 + x_2$ and its best replies are single valued by Remark 2.1. Furthermore, $\lambda = 1$ and the game has no equilibria;
- $F_1(x_1, x_2) = -x_1^2/2 + x_1x_2$ and $F_2(x_1, x_2) = -x_2^2/2 + x_1x_2$. This game is a potential game with $P(x_1, x_2) = -x_1^2/2 - x_2^2/2 + x_1x_2$. Furthermore, $\lambda = 1$ and each strategy profile (a, a), with $a \in \mathbb{R}$, is a Nash equilibrium;
- $F_1(x_1, x_2) = -e^{x_1^2} + x_1x_2$ and $F_2(x_1, x_2) = -x_2^2/2 + 2x_1x_2$. This game is a weighted potential game with $P(x_1, x_2) = -e^{x_1^2} - x_2^2/4 + x_1x_2$ and weights $(w_1, w_2) = (1, 2)$. Furthermore, $\lambda = 1$ and the unique Nash equilibrium of the game is the pair $(x_1^*, x_2^*) = (0, 0)$.

When $\lambda > 1$, β_i could be not a contraction but the existence of one and only one Nash equilibrium of the weighted potential game Γ will be guaranteed adding the following hypothesis:

$$(\mathcal{G}) \begin{cases} \text{There exist } i_0 \in I \text{ and } \gamma_{i_0} \in]1, +\infty[\text{ such that, for any } \varphi \in X_{i_0},\ x'_{i_0},\ x''_{i_0} \in \\ X_{i_0} \text{ and } x_{-i_0} \in X_{-i_0}, \text{ we have:} \\ \qquad (G_{i_0}(x'_{i_0}, x''_{i_0}, x_{-i_0})\varphi, \varphi)_{X_{i_0}} \geq \gamma_{i_0} \|\varphi\|^2_{X_{i_0}}; \end{cases}$$

where $G_i(x'_i, x''_i, x_{-i}) : X_i \to X_i$ is the operator defined as:

$$\begin{aligned} G_i(x'_i, x''_i, x_{-i}) :=& [D^2_{x_i}P(x'_i, x_{-i})]^{-1} \circ D_{x_{-i}}(D_{x_i}P)(x'_i, x_{-i}) \\ & \circ [D^2_{x_{-i}}P(x''_i, x_{-i})]^{-1} \circ D_{x_i}(D_{x_{-i}}P)(x''_i, x_{-i}), \end{aligned}$$

for any x'_i, $x''_i \in X_i$, $x_{-i} \in X_{-i}$ and $i \in I$.

[1] If the strict diagonal dominance holds then $D^2_{x_i}P(\mathbf{x}) \neq 0$ for any $\mathbf{x} \in \mathbb{R}^2$.

Remark 3.3 When $X_1 = X_2 = \mathbb{R}$ the derivatives $D^2_{x_i}P$ and $D_{x_{-i}}(D_{x_i}P)$ are real-valued functions defined on $\mathbb{R}^2$, $\left[D^2_{x_i}P(\mathbf{x})\right]^{-1}$ exists provided that $D^2_{x_i}P(\mathbf{x}) \neq 0$ and $\left[D^2_{x_i}P(\mathbf{x})\right]^{-1} = 1/D^2_{x_i}P(\mathbf{x})$. Then $(\mathcal{G})$ holds if there exist $i \in I$ and $\gamma_i > 1$ such that

$$G_i(x_i', x_i'', x_{-i}) = \frac{D_{x_{-i}}(D_{x_i}P)(x_i', x_{-i})\, D_{x_i}(D_{x_{-i}}P)(x_i'', x_{-i})}{D^2_{x_i}P(x_i', x_{-i})\, D^2_{x_{-i}}P(x_i'', x_{-i})} \geq \gamma_i,$$

for any x_i', $x_i'' \in X_i$ and $x_{-i} \in X_{-i}$.

Now, we introduce the function $g_i^\delta : X_i \to X_i$ defined by

$$g_i^\delta(x_i) := \delta x_i - (\delta - 1)\beta_i(x_i), \tag{6}$$

where $\delta \in \mathbb{R}$ and $i \in I$. When $\delta > 1$ we call such a function *δ-inverse convex combinator* since in this case x_i is a convex combination of $g_i^\delta(x_i)$ and $\beta_i(x_i)$, for any $x_i \in X_i$: this justifies the use of term "inverse".

Lemma 3.1. *Let $\delta \neq 1$. A point $\bar{x}_i$ is a fixed point of g_i^δ on X_i if and only if $\bar{x}_i$ is a fixed point of β_i on X_i.*

Proof. By definition, $\bar{x}_i \in X_i$ is a fixed point of g_i^δ on X_i if and only if $g_i^\delta(\bar{x}_i) = \bar{x}_i$, i.e., $\delta\bar{x}_i - (\delta - 1)\beta_i(\bar{x}_i) = \bar{x}_i$ which is equivalent to $\beta_i(\bar{x}_i) = \bar{x}_i$ being $\delta \neq 1$. □

Theorem 3.1. *Assume $(\mathcal{H})$, $(\mathcal{G})$ and $\lambda \in]1, +\infty[$. Then Γ has one and only one Nash equilibrium.*

Proof. Let $\lambda \in]1, +\infty[$ and let $i_0 \in I$ and $\gamma_{i_0} > 1$ be such that $(\mathcal{G})$ holds. Let $g_{i_0}^\delta$ be the δ-inverse convex combinator where

$$\delta = \begin{cases} 2, & \text{if } 1 < \lambda \leq \sqrt{2\gamma_{i_0} - 1} \\ \frac{\lambda^2 - \gamma_{i_0}}{\lambda^2 - 2\gamma_{i_0} + 1}, & \text{if } \lambda > \sqrt{2\gamma_{i_0} - 1}. \end{cases} \tag{7}$$

Note that $\delta > 1$ since both λ and γ_{i_0} are greater than 1. By Remark 2.2 and Lemma 3.1, Γ has a unique Nash equilibrium if and only if $g_{i_0}^\delta$ has a unique fixed point on X_{i_0}. Let x_{i_0}', $x_{i_0}'' \in X_{i_0}$. Then

$$\begin{aligned} \|g_{i_0}^\delta(x_{i_0}') - g_{i_0}^\delta(x_{i_0}'')\|^2_{X_{i_0}} &= \|\delta[x_{i_0}' - x_{i_0}''] - (\delta - 1)[\beta_{i_0}(x_{i_0}') - \beta_{i_0}(x_{i_0}'')]\|^2_{X_{i_0}} \\ &= \delta^2\|x_{i_0}' - x_{i_0}''\|^2_{X_{i_0}} + (\delta - 1)^2\|\beta_{i_0}(x_{i_0}') - \beta_{i_0}(x_{i_0}'')\|^2_{X_{i_0}} \\ &\quad - 2\delta(\delta - 1)(\beta_{i_0}(x_{i_0}') - \beta_{i_0}(x_{i_0}''), x_{i_0}' - x_{i_0}'')_{X_{i_0}}. \end{aligned} \tag{8}$$

By applying the Mean Value Theorem for real-valued functions to the function φ defined by

$$\varphi(\theta) := (\beta_{i_0}(\theta x_{i_0}' + (1 - \theta)x_{i_0}''), x_{i_0}' - x_{i_0}'')_{X_{i_0}}, \quad \text{for any } \theta \in [0, 1],$$

there exists $t \in]0, 1[$ such that

$$\begin{aligned} (\beta_{i_0}(x_{i_0}') - &\beta_{i_0}(x_{i_0}''), x_{i_0}' - x_{i_0}'')_{X_{i_0}} \\ &= (D\beta_{i_0}(t x_{i_0}' + (1 - t)x_{i_0}'')(x_{i_0}' - x_{i_0}''), x_{i_0}' - x_{i_0}'')_{X_{i_0}}. \end{aligned} \tag{9}$$

Note that $D\beta_{i_0}(x_{i_0}) = G(\beta_{i_0}(x_{i_0}), x_{i_0}, b_{-i_0}(x_{i_0}))$ by (3). Hence, hypothesis ($\mathcal{G}$) and condition (9) imply that

$$(\beta_{i_0}(x'_{i_0}) - \beta_{i_0}(x''_{i_0}), x'_{i_0} - x''_{i_0})_{X_{i_0}} \geq \gamma_{i_0} \|x'_{i_0} - x''_{i_0}\|^2_{X_{i_0}}, \tag{10}$$

that is β_{i_0} is strongly monotone with constant γ_{i_0}. Thus, in light of (8), (10) and (5) we have

$$\|g^\delta_{i_0}(x'_{i_0}) - g^\delta_{i_0}(x''_{i_0})\|^2_{X_{i_0}} \leq \left[\delta^2 + (\delta-1)^2\lambda^2 - 2\delta(\delta-1)\gamma_{i_0}\right] \|x'_{i_0} - x''_{i_0}\|^2_{X_{i_0}}.$$

Observe that $\left[\delta^2 + (\delta-1)^2\lambda^2 - 2\delta(\delta-1)\gamma_{i_0}\right] < 1$ or, equivalently, that

$$\delta + 1 + (\delta - 1)\lambda^2 - 2\delta\gamma_{i_0} < 0. \tag{11}$$

Indeed, when $1 < \lambda \leq \sqrt{2\gamma_{i_0} - 1}$ we have $\delta = 2$ and then inequality (11) becomes $\lambda^2 - 4\gamma_{i_0} + 3 < 0$ that is satisfied since $\sqrt{4\gamma_{i_0} - 3} > \sqrt{2\gamma_{i_0} - 1}$, being $\gamma_{i_0} > 1$.
Instead, when $\lambda > \sqrt{2\gamma_{i_0} - 1}$, factoring out δ in inequality (11), we get

$$\delta(\lambda^2 - 2\gamma_{i_0} + 1) < \lambda^2 - 1,$$

that is satisfied since $\gamma_{i_0} > 1$.
Thus, $g^\delta_{i_0}$ defined in (6) is a contraction when δ is given by (7) and therefore Γ has one and only one Nash equilibrium. □

Let us note that condition $\lambda \in]1, +\infty] :=]0, +\infty[\cup \{+\infty\}$ and hypothesis ($\mathcal{G}$) are related. Indeed:

Proposition 3.2. *If ($\mathcal{G}$) holds, then $\lambda \in]1, +\infty]$.*

Proof. Without loss of generality, suppose that ($\mathcal{G}$) holds with $i_0 = 1$. Let $\varphi \in X_1 \setminus \{0\}$, x'_1, $x''_1 \in X_1$ and $x_2 \in X_2$. Hypothesis ($\mathcal{G}$) ensures that

$$(G_1(x'_1, x''_1, x_2)\varphi, \varphi)_{X_1} > \|\varphi\|^2_{X_1}. \tag{12}$$

In light of the Cauchy-Schwarz inequality and the definition of operator norm

$$(G_1(x'_1, x''_1, x_2)\varphi, \varphi)_{X_1} \leq \|G_1(x'_1, x''_1, x_2)\|_{\mathcal{L}(X_1,X_1)} \|\varphi\|^2_{X_1}. \tag{13}$$

Define the operators

$$A(x'_1, x_2) := [D^2_{x_1} P(x'_1, x_2)]^{-1} \circ D_{x_2}(D_{x_1} P)(x'_1, x_2) \in \mathcal{L}(X_2, X_1)$$
$$B(x''_1, x_2) := [D^2_{x_2} P(x''_1, x_2)]^{-1} \circ D_{x_1}(D_{x_2} P)(x''_1, x_2) \in \mathcal{L}(X_1, X_2).$$

By Parapraph 2.4 in [2, p. 122], Proposition 2.2 and (4)

$$\begin{aligned} \|G_1(x'_1, x''_1, x_2)\|_{\mathcal{L}(X_1,X_1)} &\leq \|A(x'_1, x_2)\|_{\mathcal{L}(X_2,X_1)} \|B(x''_1, x_2)\|_{\mathcal{L}(X_1,X_2)} \\ &\leq \lambda_1\lambda_2 = \lambda. \end{aligned} \tag{14}$$

Hence, In light of (12)-(14), we have

$$\lambda\|\varphi\|^2_{X_1} \geq \|G_1(x'_1, x''_1, x_2)\|_{\mathcal{L}(X_1,X_1)} \|\varphi\|^2_{X_1} > \|\varphi\|^2_{X_1},$$

that is $\lambda \in]1, +\infty]$. □

Remark 3.4 In Theorem 3.1 hypothesis $(\mathcal{G})$ cannot be dropped, as shown in the following example.

Example 3.1 Let $X_1 = X_2 = \mathbb{R}$, $F_1(x_1, x_2) = -e^{x_1^2} + 3x_1x_2$ and $F_2(x_1, x_2) = -x_2^2/2 + 3x_1x_2$. This game is a potential game with potential $P(x_1, x_2) = -e^{x_1^2} - x_2^2/2 + 3x_1x_2$. Moreover, $D^2_{x_1}P(\mathbf{x}) = -2e^{x_1^2}(1 + 2x_1^2)$, $D_{x_{-i}}(D_{x_i}P)(\mathbf{x}) = D_{x_1}(D_{x_2}P)(\mathbf{x}) = 3$ and $D^2_{x_2}P(\mathbf{x}) = -1$, for any $\mathbf{x} \in \mathbb{R}^2$. So,

$$\lambda = 3 \cdot \sup_{x_1 \in \mathbb{R}} \frac{3}{|-2e^{x_1^2}(1 + 2x_1^2)|} = \frac{9}{2}$$

and $G_1(1, x_1'', x_2) = G_2(x_2', x_2'', 1) = 9/(6e) < 1$ for any x_1'', x_2', $x_2'' \in \mathbb{R}$. Hence, for any $i \in I$ there does not exist $\gamma_i > 1$ such that $(\mathcal{G})$ holds. Such a game has the following three Nash equilibria: $(-a, -3a)$, $(0, 0)$, $(a, 3a)$, with $a = \sqrt{\ln 9 - \ln 2}$ (the first and the third are maximum points of P, the second is not).

Remark 3.5 Theorem 3.1 crucially depends on the smoothness of the potential but note that it is also the case of the results in [16]: see *Remarks* in [16, p. 226] for examples where the uniqueness of Nash equilibrium is not guaranteed when P is not smooth even if strictly concave.

Assume now that $X_1 = X_2 = \mathbb{R}$. The next Proposition explores how the hypotheses of Theorem 3.1 are related to the existence of maximum points of the w-potential.

Proposition 3.3. *Under the assumptions of Theorem 3.1 with $X_1 = X_2 = \mathbb{R}$, P does not admit a maximum point on $\mathbb{R}^2$ (and, therefore, the unique Nash equilibrium of Γ is not a maximum point of the w-potential P). Moreover P is not strictly concave.*

Proof. First, let $i_0 \in I$ and $\gamma_{i_0} > 1$ be such that $(\mathcal{G})$ holds and let $\mathbf{x} = (x_{i_0}, x_{-i_0}) \in \mathbb{R}^2$. Choosing $x_{i_0}' = x_{i_0}'' = x_{i_0}$, by Remark 3.3:

$$G_{i_0}(x_{i_0}, x_{i_0}, x_{-i_0}) = \frac{[D_{x_{-i_0}}(D_{x_{i_0}}P)(x_{i_0}, x_{-i_0})]^2}{D^2_{x_{i_0}}P(x_{i_0}, x_{-i_0})D^2_{x_{-i_0}}P(x_{i_0}, x_{-i_0})} \geq \gamma_{i_0} > 1,$$

that is

$$D^2_{x_{i_0}}P(x_{i_0}, x_{-i_0})D^2_{x_{-i_0}}P(x_{i_0}, x_{-i_0}) - [D_{x_{-i_0}}(D_{x_{i_0}}P)(x_{i_0}, x_{-i_0})]^2 < 0,$$

i.e., the Hessian matrix of P is indefinite at $\mathbf{x}$. As $\mathbf{x}$ is arbitrary, we have that P does not attain a maximum in $\mathbb{R}^2$ (see, e.g., [6, Problem 1.5 p. 279]) and P is not strictly concave on $\mathbb{R}^2$. (see, e.g.,[6, Theorem 2.18 p. 260]). □

Remark 3.6 The following example illustrates a class of games which satisfies the assumptions of Theorem 3.1 and where best replies and the Nash equilibria could not be computed explicitly. However, by applying Theorem 3.1, one can conclude that the game has one and only one Nash equilibrium.

Example 3.2 Let Γ be a weighted potential game with $X_1 = X_2 = \mathbb{R}$ and P defined for any $\mathbf{x} \in \mathbb{R}^2$ by:

$$P(\mathbf{x}) = f_1(x_1) + f_2(x_2) + bx_1x_2, \tag{15}$$

where $b \in \mathbb{R} \setminus \{0\}$ and f_i is twice continuously differentiable for any $i \in I$. Then $D^2_{x_i} P(\mathbf{x}) = D^2 f_i(x_i)$ and $D_{x_{-i}}(D_{x_i} P)(\mathbf{x}) = b$.
For any $i \in I$, define $M_i := -\inf_{x_i \in \mathbb{R}} D^2 f_i(x_i)$, $m_i := -\sup_{x_i \in \mathbb{R}} D^2 f_i(x_i)$ and assume:

(i) $m_i > 0$;
(ii) $\frac{b^2}{M_1 M_2} > 1$.

By (i), $D^2_{x_i} P(\mathbf{x}) \le \sup_{x_i \in \mathbb{R}} D^2 f_i(x_i) = -m_i < 0$; hence P is strongly concave in any argument and the best replies are single-valued, so $(\mathcal{H})$ is satisfied. Since

$$\lambda_i = \sup_{\mathbf{x} \in \mathbb{R}^2} \left| \frac{D_{x_{-i}}(D_{x_i} P)(x_i, x_{-i})}{D^2_{x_i} P(x_i, x_{-i})} \right| = \frac{|b|}{\inf_{\mathbf{x} \in \mathbb{R}^2} |D^2_{x_i} P(x_i, x_{-i})|} = \frac{|b|}{m_i},$$

then $\lambda = \lambda_1 \lambda_2 = \frac{b^2}{m_1 m_2} > \frac{b^2}{M_1 M_2} > 1$ in light of (ii). Hence $\lambda \in]1, +\infty[$. Moreover, by (ii), for any x_1', x_1'', x_2', $x_2'' \in \mathbb{R}$:

$$G_1(x_1', x_1'', x_2') = G_2(x_2', x_2'', x_1') = \frac{b^2}{D^2 f_1(x_1') D^2 f_2(x_2')} \ge \frac{b^2}{M_1 M_2} > 1.$$

Hence, $(\mathcal{G})$ holds. Then, in light of Theorem 3.1, Γ has one and only one Nash equilibrium.

In particular, conditions (i)-(ii) are satisfied when we consider (15) with $b = -12$ and $f_i(x_i) = 1/(1 + x_i^2) - 4x_i^2 + x_i$ for any $x_i \in \mathbb{R}$ and $i \in I$, that is

$$P(x_1, x_2) = \frac{1}{1 + x_1^2} + \frac{1}{1 + x_2^2} - 4x_1^2 + x_1 - 4x_2^2 + x_2 - 12x_1x_2.$$

Indeed, (i)-(ii) hold since $0 < 15/2 = m_i < M_i = 10$ for any $i \in I$ and $b^2/(M_1 M_2) = 36/25$.

4 Bilinear weighted potential games and applications

Now, we present an application of Theorem 3.1 when Γ is a *bilinear weighted potential game*, i.e. when P is defined on $X_1 \times X_2$ by

$$P(x_1, x_2) = -a_1(x_1, x_1) - a_2(x_2, x_2) + b(x_2, x_1) + L_1(x_1) + L_2(x_2) + c, \tag{16}$$

where X_1 and X_2 are real Hilbert spaces, $a_1 : X_1 \times X_1 \to \mathbb{R}$, $a_2 : X_2 \times X_2 \to \mathbb{R}$ and $b : X_2 \times X_1 \to \mathbb{R}$ are bilinear continuous operators, $L_1 : X_1 \to \mathbb{R}$ and $L_2 : X_2 \to \mathbb{R}$ are linear continuous operators and $c \in \mathbb{R}$.
Furthermore, for any $i \in I$, assume that a_i is symmetric and that there exists $\alpha_i \in \mathbb{R}_{++}$ such that

$$a_i(x_i, x_i) \ge \alpha_i \|x_i\|^2_{X_i}, \quad \text{for any } x_i \in X_i. \tag{17}$$

Operators a_1, a_2 and b define the linear continuous operators $A_1 \in \mathcal{L}(X_1, X_1^*)$, $A_2 \in \mathcal{L}(X_2, X_2^*)$ and $B \in \mathcal{L}(X_2, X_1^*)$, respectively, such that

$$a_i(x_i', x_i'') = \langle A_i x_i', x_i'' \rangle_{X_i^* \times X_i}, \quad \text{for any } x_i',\, x_i'' \in X_i \text{ and } i \in I, \tag{18}$$

$$b(x_2, x_1) = \langle B x_2, x_1 \rangle_{X_1^* \times X_1}, \quad \text{for any } x_1 \in X_1 \text{ and } x_2 \in X_2. \tag{19}$$

Hence, P is twice continuously differentiable on $\mathbf{X}$ and

$$\begin{aligned} &D_{x_1} P(\mathbf{x}) = -2A_1 x_1 + B x_2 + L_1, \qquad D^2_{x_1} P(\mathbf{x}) = -2A_1, \qquad D_{x_2}(D_{x_1} P)(\mathbf{x}) = B \\ &D_{x_2} P(\mathbf{x}) = -2A_2 x_2 + B^t x_1 + L_2, \qquad D^2_{x_2} P(\mathbf{x}) = -2A_2, \qquad D_{x_1}(D_{x_2} P)(\mathbf{x}) = B^t, \end{aligned}$$

where $B^t := B^* J$ and B^* is the adjoint of B and J is the natural embedding of H into H^{**} (see, e.g., VI.2.1 and II.3.18 in [8]). Therefore, the best replies are single-valued since P is strongly concave. In light of Lax-Milgram Theorem (see, e.g., [4, Example 26.9]) the operators A_1 and A_2 are invertible, so $(\mathcal{H})$ is satisfied. Moreover, $\lambda_1 = \frac{1}{2}\|A_1^{-1} \circ B\|_{\mathcal{L}(X_2, X_1)} < +\infty$, $\lambda_2 = \frac{1}{2}\|A_2^{-1} \circ B^t\|_{\mathcal{L}(X_1, X_2)} < +\infty$ and

$$\begin{aligned} G_1(x_1', x_1'', x_2) &= \frac{1}{4}[A_1^{-1} \circ B \circ A_2^{-1} \circ B^t], \quad \text{for any } x_1',\, x_1'' \in X_1 \text{ and } x_2 \in X_2; \\ G_2(x_2', x_2'', x_1) &= \frac{1}{4}[A_2^{-1} \circ B^t \circ A_1^{-1} \circ B], \quad \text{for any } x_1 \in X_1 \text{ and } x_2',\, x_2'' \in X_2. \end{aligned}$$

Therefore, hypothesis $(\mathcal{G})$ holds when there exists $\gamma > 1$ such that

$$([A_1^{-1} \circ B \circ A_2^{-1} \circ B^t]\varphi, \varphi)_{X_1} \geq 4\gamma \|\varphi\|^2_{X_1}, \quad \text{for any } \varphi \in X_1;$$

or

$$([A_2^{-1} \circ B^t \circ A_1^{-1} \circ B]\varphi, \varphi)_{X_2} \geq 4\gamma \|\varphi\|^2_{X_2}, \quad \text{for any } \varphi \in X_2.$$

When the strategy sets of both players coincide and the operators a_1, a_2 and b are linear functions of the inner product of the Hilbert space, $(\mathcal{G})$ becomes easy to prove, as illustrated in the following proposition.

Proposition 4.1. *Let $X_1 = X_2 = H$ be a real Hilbert space and P defined as in (16) with*

$$a_i(x_i', x_i'') = \alpha_i \cdot (x_i', x_i'')_H, \quad \textit{for any } x_i',\, x_i'' \in H \textit{ and } i \in I, \tag{20}$$

$$b(x_2, x_1) = \rho \cdot (x_2, x_1)_H, \quad \textit{for any } x_1,\, x_2 \in H, \tag{21}$$

where α_1, $\alpha_2 \in \mathbb{R}_{++}$ and $\rho \in \mathbb{R}$.
Assume that $\frac{\rho^2}{\alpha_1 \alpha_2} > 4$. Then, Γ has one and only one Nash equilibrium.

Proof. Let $\varphi \in H$. Then $B^t \varphi \in H^*$; moreover in light of the definitions of B^* and J, and by (19) and (21):

$$\begin{aligned} \langle B^t \varphi, x_2 \rangle_{H^* \times H} &= \langle B^* J \varphi, x_2 \rangle_{H^* \times H} = \langle J\varphi, B x_2 \rangle_{H^{**} \times H^*} \\ &= \langle B x_2, \varphi \rangle_{H^* \times H} = \rho \cdot (x_2, \varphi)_H, \quad \text{for any } x_2 \in H. \end{aligned} \tag{22}$$

Consider the operator $A_2^{-1} \in \mathcal{L}(H^*, H)$. Then, $A_2^{-1}(B^t\varphi)$ is the unique $x_2 \in H$ such that $A_2 x_2 = B^t\varphi$, that is $\langle A_2 x_2, k\rangle_{H^*\times H} = \langle B^t\varphi, k\rangle_{H^*\times H}$ for any $k \in H$. In light of (18), (20) and (22)

$$\alpha_2 \cdot (x_2, k)_H = \rho \cdot (k, \varphi)_H, \quad \text{for any } k \in H;$$

so $A_2^{-1}(B^t\varphi) = x_2 = \frac{\rho}{\alpha_2}\varphi$.
The operator $B(A_2^{-1}(B^t\varphi)) \in H^*$ is defined on H by:

$$\langle B(A_2^{-1}(B^t\varphi)), h\rangle_{H^*\times H} = \frac{\rho^2}{\alpha_2} \cdot (\varphi, h)_H, \quad \text{for any } h \in H. \tag{23}$$

Finally, consider the operator $A_1^{-1} \in \mathcal{L}(H^*, H)$. Then, $A_1^{-1}(B(A_2^{-1}(B^t\varphi)))$ is the unique $x_1 \in H$ such that $A_1 x_1 = B(A_2^{-1}(B^t\varphi))$, i.e. $\langle A_1 x_1, h\rangle_{H^*\times H} = \langle B(A_2^{-1}(B^t\varphi)), h\rangle_{H^*\times H}$ for any $h \in H$. Therefore, by (23)

$$\alpha_1 \cdot (x_1, h)_H = \left(\frac{\rho^2}{\alpha_2}\varphi, h\right)_H, \quad \text{for any } h \in H;$$

so $A_1^{-1}(B(A_2^{-1}(B^t\varphi))) = x_1 = \frac{\rho^2}{\alpha_1\alpha_2}\varphi$. Hence, $(\mathcal{G})$ holds since

$$([A_1^{-1} \circ B \circ A_2^{-1} \circ B^t]\varphi, \varphi)_H = \frac{\rho^2}{\alpha_1\alpha_2}\|\varphi\|_H^2, \quad \text{for any } \varphi \in H,$$

and $\frac{\rho^2}{\alpha_1\alpha_2} > 4$. Then, Γ has one and only one equilibrium by Theorem 3.1. □

Proposition 4.1 allows to prove the existence of a unique open-loop Nash equilibrium (see, e.g., [3, 7, 11]) of the following differential game.

Example 4.1 Consider a two-player differential game with state equation given by

$$\dot{x}(t) = u_1(t) + u_2(t) - m x(t), \qquad x(0) = x_0, \tag{24}$$

where $t \in [0, T]$, $T \in]0, +\infty[$, x is continuously differentiable on $[0, T]$, u_1, $u_2 \in U := L^2([0, T])$, $m \in \mathbb{R}_{++}$ and $x_0 \in \mathbb{R}_{++}$.
Player i, $i \in I$, has an instantaneous profit at time t equal to

$$\pi_i(x(t), u_1(t), u_2(t)) = x(t) - \alpha_i[u_i(t)]^2 + \rho u_1(t) u_2(t),$$

where $\alpha_1 > 0$, $\alpha_2 > 0$ and $\rho \in \mathbb{R}$. So, player i's objective functional is

$$J_i(x, u_1, u_2) = \int_0^T e^{-rt}\pi_i(x(t), u_1(t), u_2(t))\, dt, \tag{25}$$

where $r \geq 0$ is the common discount rate. The differential game (24)-(25) is similar to Example 7.1 in [7] which describes a situation where two individuals invest in a public stock of knowledge (see also Section 9.5 in [7]).
The solution to the first-order differential equation (24) is

$$x(t) = x_0 e^{-mt} + e^{-mt}\int_0^t [u_1(s) + u_2(s)]e^{ms}\, ds. \tag{26}$$

Denote by F_i the real-valued function defined on $U \times U$ obtained by substituting (26) in (25), that is

$$F_i(u_1, u_2) := \int_0^T e^{-rt}\left[x_0 e^{-mt} + e^{-mt}\int_0^t [u_1(s) + u_2(s)]e^{ms}\,ds\right] dt$$
$$- \int_0^T e^{-rt}\{\alpha_i[u_i(t)]^2 - \rho u_1(t)u_2(t)\}\,dt.$$

The game $\Gamma = \{2, U, U, F_1, F_2\}$ is a potential game with potential

$$P(u_1, u_2) = F_1(u_1, u_2) - \int_0^T e^{-rt}\alpha_2[u_2(t)]^2\,dt.$$

Such a potential belongs to the class of functions considered in (16), where:

$$a_i(u_i', u_i'') = \alpha_i \int_0^T e^{-rt}u_i'(t)u_i''(t)\,dt \quad \text{for any } u_i',\, u_i'' \in U \text{ and } i \in I; \tag{27}$$

$$b(u_2, u_1) = \rho \int_0^T e^{-rt}u_1(t)u_2(t)\,dt \quad \text{for any } u_1,\, u_2 \in U; \tag{28}$$

$$L_i(u_i) = \int_0^T e^{-(r+m)t}\left[\int_0^t e^{ms}u_i(s)\,ds\right] dt \quad \text{for any } u_i \in U;$$

$$c = \int_0^T x_0 e^{-(r+m)t}\,dt.$$

Note that the operators a_i and b in (27)-(28) are of the same type of (20)-(21) where $H = U$ and U is endowed with the inner product[2] defined by

$$(u_1, u_2)_U := \int_0^T e^{-rt}u_1(t)u_2(t)\,dt, \quad \text{for any } u_1,\, u_2 \in U. \tag{29}$$

Hence, arguing as in Proposition 4.1 we can conclude that the differential game defined by (24)-(25) has one and only one open-loop Nash equilibrium if $\frac{\rho^2}{\alpha_1\alpha_2} > 4$.

References

[1] J.-P. Aubin and I. Ekeland. *Applied nonlinear analysis*. New York: Dover Publications, 2006. ISBN: 9780486453248.

[2] A. Avez. *Calcul Différentiel*. Paris: Masson, 1983. ISBN: 2-225-79079-5.

[3] T. Başar and G.J. Olsder. *Dynamic noncooperative game theory*. SIAM, 1999. ISBN: 978-0-89871-429-6.

[4] H.H. Bauschke and P.L. Combettes. *Convex analysis and monotone operator theory in Hilbert spaces*. New York: Springer Science & Business Media, 2011. ISBN: 978-1-4419-9466-0.

[2] One can show that $L^2([0, T])$ with the inner product defined in (29) is a Hilbert space.

[5] M.C. Ceparano and F. Quartieri. "Nash equilibrium uniqueness in nice games with isotone best replies". In: *J Math Econom* (2017), pp. 154–165. DOI: 10.1016/j.jmateco.2017.02.011.

[6] A. de la Fuente. *Mathematical methods and models for economists*. Cambridge: Cambridge University Press, 2000. ISBN: 0-521-58529-5.

[7] E. Dockner, S. Jørgensen, N. Van Long, and G. Sorger. *Differential games in economics and management science*. New York: Cambridge University Press, 2000. ISBN: 978-0-521-63732-9.

[8] N. Dunford and J.T. Schwartz. *Linear operators*. New York: Wiley-interscience pubilshers, 1971. ISBN: 0 470 22605 6.

[9] G. Facchini, F. van Megen, P. Borm, and S. Tijs. "Congestion models and weighted Bayesian potential games". In: *Theory Decis* 42 (1997), pp. 193–206. DOI: 10.1023/A:1004991825894.

[10] D. Gabay and H. Moulin. "On the uniqueness and stability of Nash equilibrium in noncooperative games". In: *Applied stochastic control in econometrics and management science*. Ed. by A Bensoussan, P.R. Kleindorfer, and C.S. Tapiero. Amsterdam: North-Holland, 1980, pp. 271–293.

[11] A. Haurie, J.B. Krawczyk, and G. Zaccour. *Games and dynamic games*. World Scientific, 2012. ISBN: 978-981-4401-26-5.

[12] S. Karamardian. "The nonlinear complementarity problem with applications, Part 2". In: *J Optim Theory Appl* 4 (1969), pp. 167–181.

[13] L.A. Lusternik and V.J. Sobolev. *Elements of Functional Analysis*. Delhi, India: Hindustan Publishing Corp., 1974.

[14] D. Monderer and L.S. Shapley. "Potential games". In: *Game Econ Behav* 14 (1996), pp. 124–143. DOI: 10.1006/game.1996.0044.

[15] J. Morgan. "Méthode directe de recherche du point de selle d'une fonctionnelle convexe-concave et application aux problèmes variationnels elliptiques avec deux contrôles antagonistes". (French). In: *Int J Comput Math* 4 (1974), pp. 143–175. DOI: 10.1080/00207167408803086.

[16] A. Neyman. "Correlated equilibrium and potential games". In: *Int J Game Theory* 26 (1997), pp. 223–227. DOI: 10.1007/BF01295851.

[17] J.B. Rosen. "Existence and uniqueness of equilibrium points for concave n-person games". In: *Econometrica* 33 (1965), pp. 520–534. DOI: 10.2307/1911749.