CSEF

Centre for Studies in Economics and Finance

WORKING PAPER NO. 476

Subgame Perfect Nash Equilibrium: A Learning Approach Via Costs to Move

Francesco Caruso, Maria Carmela Ceparano and Jacqueline Morgan

June 2017
This version July 2018

University of Naples Federico II | University of Salerno | Bocconi University, Milan

CSEF - Centre for Studies in Economics and Finance
DEPARTMENT OF ECONOMICS – UNIVERSITY OF NAPLES
80126 NAPLES - ITALY
Tel. and fax +39 081 675372 – e-mail: csef@unisa.it

ISSN: 2240-9696

CSEF
Centre for Studies in Economics and Finance

WORKING PAPER NO. 476

Subgame Perfect Nash Equilibrium: A Learning Approach Via Costs to Move[♣]

Francesco Caruso[*], **Maria Carmela Ceparano**[*] **and Jacqueline Morgan**[**]

Abstract
In one-leader one-follower two-stage games, also called Stackelberg games, multiplicity of Subgame Perfect Nash Equilibria (henceforth SPNE) arises when the best reply correspondence of the follower is not a single-valued map. This paper concerns a new method to approach SPNEs which makes use of a sequence of SPNEs of perturbed games where the best reply correspondence of the follower is single-valued. The sequence is generated by a learning method where the payoff functions of both players are modified subtracting a term that represents a physical and behavioral cost to move and which relies on the proximal point methods linked to the Moreau-Yosida regularization. Existence results of SPNEs approached via this method are provided under mild assumptions on the data, together with numerical examples and connections with other methods to construct SPNEs.

Keywords: Non-cooperative game; Stackelberg game; subgame perfect Nash equilibrium; selection; learning method; cost to move; proximal point method.

[♣] This paper was previously circulated with the title "Proximal Approach in Selection of Subgame Perfect Nash Equilibria"

[*] Università di Napoli Federico II. Postal address: Department of Economics, Università di Napoli Federico II, Via Cintia (Monte S. Angelo), 80126 Napoli, Italy. E-mails: francesco.caruso@unina.it and mariacarmela.ceparano@unina.it

[**] Università di Napoli Federico II and CSEF. Postal address: Department of Economics, Università di Napoli Federico II, Via Cintia (Monte S. Angelo), 80126 Napoli, Italy. E-mail: morgan@unina.it

Table of contents

1 Introduction

Throughout the paper, we consider a one-leader one-follower two-stage continuous non-cooperative game Γ, also called Stackelberg game: the players have a continuum of actions and one player acting in the second stage makes a choice after having observed the choice taken by one player acting in the first stage. As usual, we refer to the player moving in the second stage as the *follower* and to the player moving in the first stage as the *leader*.

We denote by X and L the set of actions and the payoff function of the leader, respectively, and by Y and F the set of actions and the payoff function of the follower, respectively, with L and F real-valued functions defined on $X \times Y$. A strategy for the follower is a function from X to Y, so the set of follower's strategies is $Y^X := \{\varphi \mid \varphi\colon X \to Y\}$. The target of each player is to maximize his payoff function.

The solution concept we consider is the *subgame perfect Nash equilibrium* concept (*SPNE* for short), a well-known refinement of the Nash equilibrium widely used in dynamic games ([39]; see also, for example, [20, 28]).

Multiple SPNEs could come up when the optimal reaction of the follower to any choice of the leader is not always unique (i.e. the follower's best reply correspondence is not single-valued). So, we introduce a constructive method in order to select an SPNE by using a learning approach with the following features:

(i) on the one hand, it has the advantage of relieving the leader of learning the follower's best reply correspondence and it allows to overcome the difficulties deriving from the possible non single-valuedness of the best reply correspondence of the follower;

(ii) on the other hand, it has a behavioral interpretation that covers various physical, physiological, psychological, and cognitive aspects of decision making processes.

In fact, we recursively define a sequence $(\Gamma_n)_n$ of Stackelberg games in which the follower's best reply correspondence is single-valued (i.e., a sequence of classical Stackelberg games, see [43, 4]) and a sequence of strategy profiles $(x_n, \varphi_n)_n$ such that $(x_n, \varphi_n) \in X \times Y^X$ is an SPNE of Γ_n for any $n \in \mathbb{N}$: the payoff functions of both players in Γ_n are obtained by subtracting to the payoff functions of Γ a quadratic term depending on the SPNE reached in Γ_{n-1}. Consequently (x_n, φ_n), SPNE of Γ_n, is an update of (x_{n-1}, φ_{n-1}), SPNE of Γ_{n-1}. It will be shown that the limit of such sequence of SPNEs generates an SPNE of Γ.

The quadratic term represents a physical and behavioral *cost to move*, embedding the idea that in real life changing an action or improving the quality of actions has a cost ([3, 2]). The mathematical tools underlying costs to move involve the *proximal point methods*, a class of optimization techniques based on the *Moreau-Yosida regularization* ([29, 27, 36], see also [1] and the references therein). Such methods have already been used to construct Nash equilibria in one-stage games (see, for example, [18, 17, 32, 2])

and to define a new Nash equilibrium refinement for one-stage games when there is uncertainty related to players' strategies (see [6]).
To the best of our knowledge, a learning method based on costs to move has never been used before to construct an SPNE in Stackelberg games, whereas a first attempt to approach an SPNE in a constructive way in Stackelberg games is due to Morgan and Patrone (2006) where Tikhonov regularization ([41]) has been exploited. Nevertheless, although such a regularization allows to generate a sequence of games where the follower's best reply correspondence is single-valued, the method used in [31] does not display a behavioral interpretation.
We emphasize that the idea we propose in order to approach an equilibrium is in the same spirit of the theory of equilibrium refinements for normal form games based on perturbations of the data of the game (see, for example, [44, 19, 38, 33, 34, 21, 14]).

The paper is structured as follows. In Section 2 the method used to approach an SPNE is formulated and further detailed interpretations are provided. Results about the existence of an SPNE achievable via the above mentioned method are presented in Section 3. Connections with the method proposed in [31] and with other solution concepts for Stackelberg games are provided in Section 4. Finally, in Section 5 conclusions and possible directions for future research are discussed. An Appendix contains the main computations of the numerical examples.

2 Constructive procedure and interpretation

Let Γ be a Stackelberg game. We use the notation (X, Y, L, F) to refer to Γ, in order to focus on the relevant features of the game, pointing out that (X, Y, L, F) is not to be understood as the game given in normal form.
Recall that a strategy profile $(\bar{x}, \bar{\varphi}) \in X \times Y^X$ is an SPNE of Γ if the following conditions are satisfied:

(SG1) for each choice x of the leader, the follower reacts maximizing his payoff function, i.e. for any $x \in X$:

$$F(x, \bar{\varphi}(x)) \geq F(x, y), \text{ for any } y \in Y, \tag{1}$$

or, equivalently, $\bar{\varphi}(x) \in \operatorname{Arg\,max}_{y \in Y} F(x, y)$;

(SG2) the leader maximizes his payoff function taking into account his hierarchical advantage, i.e.

$$\bar{x} \in \operatorname*{Arg\,max}_{x \in X} L(x, \bar{\varphi}(x)). \tag{2}$$

Denoting by M the set-valued map that associates with each $x \in X$ the set $M(x)$ of follower's best responses to x, that is

$$M(x) := \operatorname*{Arg\,max}_{y \in Y} F(x, y), \tag{3}$$

condition (SG1) is equivalent to require that $\bar{\varphi}(x) \in M(x)$ for any $x \in X$. The set-valued map M is the so-called *follower's best reply correspondence*. When M is a single-valued map, i.e. $M(x) = \{m(x)\}$ for any $x \in X$, the function m is called *follower's best reply function*.
Before presenting a constructive procedure in order to select an SPNE when the best reply correspondence M is not known to be single-valued, we define a class of games for which such an SPNE is achievable through this procedure.

Definition 2.1 A Stackelberg game $\Gamma = (X, Y, L, F)$ belongs to the family $\mathcal{G}$ if the following assumptions are satisfied:

($\mathcal{A}1$) X is a nonempty compact subset of a Euclidean space $\mathbb{X}$ with norm $\|\cdot\|_{\mathbb{X}}$;

($\mathcal{A}2$) Y is a nonempty convex and compact subset of a Euclidean space $\mathbb{Y}$ with norm $\|\cdot\|_{\mathbb{Y}}$;

($\mathcal{L}1$) L is upper semicontinuous on $X \times Y$;

($\mathcal{L}2$) $L(x, \cdot)$ is lower semicontinuous on Y, for any $x \in X$;

($\mathcal{F}1$) F is upper semicontinuous on $X \times Y$;

($\mathcal{F}2$) for any $(x, y) \in X \times Y$ and for any sequence $(x_k)_k \subseteq X$ converging to x there exists a sequence $(\tilde{y}_k)_k \subseteq Y$ converging to y such that

$$\liminf_{k \to +\infty} F(x_k, \tilde{y}_k) \geq F(x, y);$$

($\mathcal{F}3$) $F(x, \cdot)$ is concave on Y, for any $x \in X$.

Remark 2.1 (on discontinuity) Requiring ($\mathcal{F}1$)-($\mathcal{F}3$) is weaker than requiring the continuity of F. Indeed, the function F defined on $X \times Y$, where $X = [1, 2]$ and $Y = \overline{B}_{((1,0),1)}$ (i.e. Y is the closed ball in $\mathbb{R}^2$ centered in $(1, 0)$ with radius 1), by

$$F(x, (y_1, y_2)) = \begin{cases} -\frac{y_2^2}{2y_1} x, & \text{if } (y_1, y_2) \neq (0, 0) \\ 0, & \text{if } (y_1, y_2) = (0, 0) \end{cases}$$

satisfies ($\mathcal{F}1$)-($\mathcal{F}3$) but $F(x, \cdot)$ is not lower semicontinuous at $(0, 0)$, for any $x \in [1, 2]$, as shown in the Appendix.

Remark 2.2 (on variational convergences) Assumptions ($\mathcal{F}1$)-($\mathcal{F}2$) have implications in term of Γ-convergence or epiconvergence (see, for example, [1, 13]). Indeed, let $x \in X$ and let $(x_k)_k \subseteq X$ be a sequence converging to x and consider the following real-valued functions defined on Y by

$$W_k(y) = F(x_k, y), \text{ for any } k \in \mathbb{N},$$
$$W(y) = F(x, y).$$

Then the sequence of functions $(W_k)_k$ Γ^+-converges to W (that is, $(-W_k)_k$ epiconverges to $-W$).

In the following remark some properties of the family $\mathcal{G}$ are stated. The proofs can be obtained by using Γ-convergence results (see, for example, Proposition 6.16 and Proposition 6.21 in [13]).

Remark 2.3 (on properties of $\mathcal{G}$) Assume $(X, Y, U, V) \in \mathcal{G}$ and $(X, Y, W, Z) \in \mathcal{G}$.

(i) The game $(X, Y, hU, kV) \in \mathcal{G}$ for any $h, k \geq 0$.

(ii) If Ψ and Φ are real-valued functions defined on $\mathbb{R}$ with Ψ continuous and Φ increasing and concave, then the game $(X, Y, (\Psi \circ U), (\Phi \circ V)) \in \mathcal{G}$.

(iii) If Z is continuous, then the game $(X, Y, (U + W), (V + Z)) \in \mathcal{G}$.

The Costs to Move Procedure $(\mathcal{CM})$ defined below illustrates the learning method that we use to construct recursively a sequence of games $(\Gamma_n)_n$ and a sequence of strategy profiles $(\bar{x}_n, \varphi_n)_n$.

Procedure $(\mathcal{CM})$

Fix an initial point $(\bar{x}_0, \bar{y}_0) \in X \times Y$ and define for any $n \in \mathbb{N}$

$$(\mathcal{S}_n) \quad \begin{cases} \Gamma_n = (X, Y, L_n, F_n) \\ \{\varphi_n(x)\} = \operatorname{Arg\,max}_{y \in Y} F_n(x, y), \text{ for any } x \in X \\ \bar{x}_n \in \operatorname{Arg\,max}_{x \in X} L_n(x, \varphi_n(x)) \end{cases}$$

where for any $(x, y) \in X \times Y$

$$F_n(x, y) := F(x, y) - \frac{1}{2\gamma_{n-1}} \|y - \varphi_{n-1}(x)\|^2_{\mathbb{Y}}$$

$$L_n(x, y) := L(x, y) - \frac{1}{2\beta_{n-1}} \|x - \bar{x}_{n-1}\|^2_{\mathbb{X}},$$

with $(\gamma_n)_{n \in \mathbb{N} \cup \{0\}} \subseteq]0, +\infty[$ and $\lim_{n \to +\infty} \gamma_n = +\infty$,
$(\beta_n)_{n \in \mathbb{N} \cup \{0\}} \subseteq]0, +\infty[$ and $\lim_{n \to +\infty} \beta_n = +\infty$,
and $\varphi_0(x) := \bar{y}_0$ for any $x \in X$.

Procedure $(\mathcal{CM})$ is well-defined when $F_n(x, \cdot)$ has a unique maximizer on Y, for any $x \in X$ and for any $n \in \mathbb{N}$, and when $L_n(\cdot, \varphi_n(\cdot))$ admits a maximizer on X, for any $n \in \mathbb{N}$. For the class of games introduced in Definition 2.1 such properties are satisfied, as it is proved in the next proposition.

Proposition 2.1. *Assume that $\Gamma \in \mathcal{G}$. Then, Procedure $(\mathcal{CM})$ is well-defined and φ_n is a continuous function on X, for any $n \in \mathbb{N}$.*

Proof. We prove the result by induction on n. Let $n = 1$. By Remark 2.3(iii), $\Gamma_1 \in \mathcal{G}$. Moreover $F_1(x, \cdot)$ is strictly concave for any $x \in X$, therefore $\varphi_1(x)$ is well-defined and the follower's best reply correspondence in Γ_1 is single-valued. Since $\Gamma_1 \in \mathcal{G}$, in particular

(a$_1$) F_1 is upper semicontinuous on $X \times Y$,

(b$_1$) for any $(x, y) \in X \times Y$ and for any sequence $(x_k)_k \subseteq X$ converging to x, there exists a sequence $(\tilde{y}_k)_k \subseteq Y$ converging to y such that

$$\liminf_{k \to +\infty} F_1(x_k, \tilde{y}_k) \geq F_1(x, y).$$

Conditions (a$_1$) and (b$_1$) are sufficient to guarantee that $\lim_{k\to+\infty} \varphi_1(x_k) = \varphi_1(x)$ for any sequence $(x_k)_k$ converging to x, i.e. that φ_1 is continuous (see, for example, Proposition 5.1 in [30]). This fact and the upper semicontinuity of L_1 ensure that $\bar{x}_1$ is well-defined. Hence, the base case is proved.

Assume that the result holds for $n > 1$, so the strategy profile $(\bar{x}_n, \varphi_n)$ is well-defined and φ_n is a continuous function. In light of Remark 2.3(iii), $\Gamma_{n+1} \in \mathcal{G}$ since φ_n is continuous. Furthermore $F_{n+1}(x, \cdot)$ is strictly concave for any $x \in X$, so $\varphi_{n+1}(x)$ is well-defined and φ_{n+1} is the follower's best reply function in Γ_{n+1}. As $\Gamma_{n+1} \in \mathcal{G}$, then

(a$_{n+1}$) F_{n+1} is upper semicontinuous on $X \times Y$,

(b$_{n+1}$) for any $(x, y) \in X \times Y$ and for any sequence $(x_k)_k \subseteq X$ converging to x, there exists a sequence $(\tilde{y}_k)_k \subseteq Y$ converging to y such that

$$\liminf_{k \to +\infty} F_{n+1}(x_k, \tilde{y}_k) \geq F_{n+1}(x, y).$$

By (a$_{n+1}$) and (b$_{n+1}$) it follows that φ_{n+1} is continuous (again in light of, for example, Proposition 5.1 in [30]). Hence $\bar{x}_{n+1}$ is well-defined, since L_{n+1} is upper semicontinuous. So the inductive step is proved and the proof is complete. □

Note that assumption $(\mathcal{L}2)$ in the definition of the family $\mathcal{G}$ is unnecessary in the proof of Proposition 2.1. We assumed $\Gamma \in \mathcal{G}$ in the proposition only for simplicity of exposition.

Interpretation of the procedure At the generic step $(\mathcal{S}_n)$ of the procedure, the follower chooses his strategy φ_n taking into account his previous strategy φ_{n-1}. In making such a choice, he finds an action that compromises between maximizing $F(x, \cdot)$ and being near to $\varphi_{n-1}(x)$, for any $x \in X$. The latter purpose is motivated according to an *anchoring effect*:

> "agents have a (local) vision of their environment which depends on their current actions. Each action is anchored to the preceding one, which means that the perception the agents have of the quality of their subsequent actions depends on the current ones. In economics and management, one may think of actions as routines, ways of doing, while costs to change reflect the difficulty of quitting a routine or entering another one or reacting quickly." *([2, p.1066])*

Such an anchoring effect is formulated by subtracting a quadratic cost to move that reflects the difficulty of changing the previous action. The coefficient γ_{n-1} is linked to the per unit of distance cost to move of the follower and it is related to the trade-off parameter between maximizing $F(x,\cdot)$ and minimizing the distance from $\varphi_{n-1}(x)$. Since the same arguments apply for the preceding steps until going up to step $(\mathcal{S}_1)$, it follows that $\varphi_n(x)$ as well as the limit of $\varphi_n(x)$ embed the willingness of being near to $\bar{y}_0$. Analogous observations hold also for the leader, who chooses an action having in mind to be near to his previous choices.
The use of proximal point methods, underlying costs to move, has also the advantage to regularize even in situations where the functions are possibly non-smooth and extended real-valued (for a more detailed discussion on proximal point methods and their interpretations, see [35]).

In the proof of Proposition 2.1 we showed that the follower's best reply correspondence in Γ_n is single-valued, i.e., Γ_n is a classical Stackelberg game. Moreover, the follower's best reply function φ_n in Γ_n is continuous and the strategy profile $(\bar{x}_n, \varphi_n)$ is an SPNE of Γ_n, for any $n \in \mathbb{N}$. Hence, Procedure $(\mathcal{CM})$ allows to define a perturbation of the game Γ consisting of the sequence of classical Stackelberg games $(\Gamma_n)_n$ and to construct a sequence of SPNEs related to such a perturbation.

In the next proposition, we prove that the limit of the sequence $(\varphi_n)_n$ is a selection of the follower's best reply correspondence. The pointwise convergence of $(\varphi_n)_n$ is obtained by adapting to a parametric optimization context a classical result about the convergence of proximal point methods. Before showing the result, we state the following lemma.

Lemma 2.1 (on parametric proximal point methods)**.** *Let G be a real-valued function defined on $X \times Y$ and $\bar{G}$ be the extended real-valued function defined on $X \times \mathbb{Y}$ by*

$$\bar{G}(x,y) = \begin{cases} G(x,y), & \text{if } y \in Y \\ -\infty, & \text{if } y \notin Y. \end{cases} \tag{4}$$

Let $x \in X$. If the function $G(x,\cdot)$ is upper semicontinuous and concave on Y, then

(i) the function $\bar{G}(x,\cdot)$ is upper semicontinuous and concave on $\mathbb{Y}$;

(ii) $\operatorname{Arg\,max}_{y\in Y} G(x,y) = \operatorname{Arg\,max}_{y\in\mathbb{Y}} \bar{G}(x,y)$*;*

(iii) $\operatorname{Arg\,max}_{y\in Y} G(x,y) - \frac{1}{2\lambda}\|y-v\|^2_{\mathbb{Y}} = \operatorname{Arg\,max}_{y\in\mathbb{Y}} \bar{G}(x,y) - \frac{1}{2\lambda}\|y-v\|^2_{\mathbb{Y}}$*, for any $\lambda > 0$ and $v \in \mathbb{Y}$.*

(iv) $\varphi^*(x) \in \operatorname{Arg\,max}_{y\in Y} G(x,y) \iff \{\varphi^*(x)\} = \mathbf{prox}_{\lambda, G(x,\cdot)}(\varphi^*(x))$*, for any $\lambda > 0$, where* $\mathbf{prox}_{\lambda,G(x,\cdot)}(v) := \operatorname{Arg\,max}_{y\in Y} G(x,y) - \frac{1}{2\lambda}\|y-v\|^2_{\mathbb{Y}}$*, for any $v \in \mathbb{Y}$.*

Proof. Claims *(i)*-*(iii)* are immediate, the proof of claim *(iv)* is analogous to the one, for example, in [35, Section 2.3], taking into account claims *(i)*-*(iii)*. □

Proposition 2.2. *Assume that* $(\mathcal{A}2)$*,* $(\mathcal{F}1)$ *and* $(\mathcal{F}3)$ *hold. Then the sequence* $(\varphi_n)_n$ *pointwise converges to a function* $\varphi \in Y^X$ *and* $\varphi(x) \in M(x)$ *for any* $x \in X$*, where* $M(x) = \operatorname{Arg\,max}_{y\in Y} F(x,y)$*.*

Proof. Let $x \in X$. By assumptions $(\mathcal{F}1)$ and $(\mathcal{F}3)$ and Lemma 2.1*(i)*, the function $-\bar{F}(x,\cdot)$, where $\bar{F}$ is defined on $X \times \mathbb{Y}$ by

$$\bar{F}(x,y) = \begin{cases} F(x,y), & \text{if } y \in Y \\ -\infty, & \text{if } y \notin Y, \end{cases}$$

is lower semicontinuous and convex, is not identically $+\infty$ and does not assume the value $-\infty$ (i.e. $-\bar{F}(x,\cdot)$ is a proper lower semicontinuous convex function). Moreover, in light of Lemma 2.1*(ii)*, the compactness of Y and assumption $(\mathcal{F}1)$,

$$\operatorname*{Arg\,min}_{y\in\mathbb{Y}} -\bar{F}(x,y) = \operatorname*{Arg\,max}_{y\in\mathbb{Y}} \bar{F}(x,y) \neq \emptyset.$$

Given the above, and since $\lim_{n\to+\infty} \gamma_n = +\infty$ with $(\gamma_n)_n \subseteq]0,+\infty[$, the function $-\bar{F}(x,\cdot)$ satisfies the hypotheses for the convergence of proximal point methods stated in [5, Theorem 27.1]. Then, the sequence $(z_n)_n$ defined by

$$\{z_n\} \coloneqq \operatorname*{Arg\,min}_{y\in\mathbb{Y}} -\bar{F}(x,y) + \frac{1}{2\gamma_{n-1}} \|y - z_{n-1}\|^2_{\mathbb{Y}} \quad \text{for any } n \in \mathbb{N},$$

where $z_0 \coloneqq \bar{y}_0$, converges to a point in $\operatorname{Arg\,min}_{y\in\mathbb{Y}} -\bar{F}(x,y)$ by Theorem 27.1 in [5]. So, equivalently,

$$\{z_n\} = \operatorname*{Arg\,max}_{y\in\mathbb{Y}} \bar{F}(x,y) - \frac{1}{2\gamma_{n-1}} \|y - z_{n-1}\|^2_{\mathbb{Y}} \quad \text{for any } n \in \mathbb{N},$$

and $(z_n)_n$ converges to a point in $\operatorname{Arg\,max}_{y\in\mathbb{Y}} \bar{F}(x,y)$. Since the unique maximizer of $\bar{F}(x,\cdot) - \frac{1}{2\gamma_{n-1}} \|\cdot - \varphi_{n-1}(x)\|^2_{\mathbb{Y}}$ over $\mathbb{Y}$ coincides with the unique maximizer of $F(x,\cdot) - \frac{1}{2\gamma_{n-1}} \|\cdot - \varphi_{n-1}(x)\|^2_{\mathbb{Y}}$ over Y in light of Lemma 2.1*(iii)*, then $z_n = \varphi_n(x)$ for any $n \in \mathbb{N}$. Furthermore, since the set of maximizers of $\bar{F}(x,\cdot)$ over $\mathbb{Y}$ coincides with the set of maximizers of $\bar{F}(x,\cdot)$ over Y in light of Lemma 2.1*(ii)*, sequence $(\varphi_n(x))_n$ converges to a maximizer of $F(x,\cdot)$ over Y. Hence, the function φ that associates with each $x \in X$ the point $\varphi(x) \coloneqq \lim_{n\to+\infty} \varphi_n(x) \in Y$ is well-defined and $\varphi(x) \in M(x)$ for any $x \in X$. □

3 SPNE existence result

The next theorem provides an existence result of an SPNE achievable via Procedure $(\mathcal{CM})$ for $\Gamma = (X, Y, L, F) \in \mathcal{G}$. Recall that $(\bar{x}_n, \varphi_n)_n$ is the sequence of strategy profiles generated by Procedure $(\mathcal{CM})$, which is well-defined in light of Proposition 2.1.

Theorem 3.1. *Assume that* $\Gamma \in \mathcal{G}$ *and that the sequence of action profiles* $(\bar{x}_n, \varphi_n(\bar{x}_n))_n \subseteq X \times Y$ *converges to* $(\bar{x}, \bar{y}) \in X \times Y$*. Then the strategy profile* $(\bar{x}, \bar{\varphi}) \in X \times Y^X$*, where*

$$\bar{\varphi}(x) \coloneqq \begin{cases} \bar{y}, & \textit{if } x = \bar{x} \\ \lim_{n\to+\infty} \varphi_n(x), & \textit{if } x \neq \bar{x}, \end{cases}$$

is a subgame perfect Nash equilibrium of Γ.

Proof. We start to prove (SG1). Let $x \in X$ and $\varphi(x) = \lim_{n\to+\infty} \varphi_n(x)$, as defined in Proposition 2.2. If $x \neq \bar{x}$, Proposition 2.2 ensures that $\bar{\varphi}(x) = \varphi(x) \in M(x)$. If $x = \bar{x}$, pick $y \in Y$. By assumption $(\mathcal{F}2)$, there exists a sequence $(\tilde{y}_n)_n$ converging to y such that

$$\liminf_{n\to+\infty} F(\bar{x}_n, \tilde{y}_n) \geq F(\bar{x}, y). \tag{5}$$

By $(\mathcal{F}1)$ we have:

$$\begin{aligned} F(\bar{x}, \bar{y}) &\geq \limsup_{n\to+\infty} F(\bar{x}_n, \varphi_n(\bar{x}_n)) \\ &= \limsup_{n\to+\infty} \left[F(\bar{x}_n, \varphi_n(\bar{x}_n)) - \frac{1}{2\gamma_{n-1}} \|\varphi_n(\bar{x}_n) - \varphi_{n-1}(\bar{x}_n)\|^2_{\mathbb{Y}} \right] \\ &= \limsup_{n\to+\infty} F_n(\bar{x}_n, \varphi_n(\bar{x}_n)), \end{aligned} \tag{6}$$

where the first equality holds since the second addend in the $\limsup$ converges to 0 being $(\gamma_n)_n$ a divergent sequence of positive real numbers and Y a compact set, and the second equality comes from the definition of F_n in Procedure $(\mathcal{CM})$. By the definition of $\varphi_n(\bar{x}_n)$ we get

$$\begin{aligned} \limsup_{n\to+\infty} F_n(\bar{x}_n, \varphi_n(\bar{x}_n)) &\geq \limsup_{n\to+\infty} F_n(\bar{x}_n, \tilde{y}_n) \\ &= \limsup_{n\to+\infty} \left[F(\bar{x}_n, \tilde{y}_n) - \frac{1}{2\gamma_{n-1}} \|\tilde{y}_n - \varphi_{n-1}(\bar{x}_n)\|^2_{\mathbb{Y}} \right]. \end{aligned} \tag{7}$$

Recalling the properties of $(\gamma_n)_n$ and the compactness of Y, by (5)-(7) we have

$$\begin{aligned} F(\bar{x}, \bar{y}) &\geq \limsup_{n\to+\infty} \left[F(\bar{x}_n, \tilde{y}_n) - \frac{1}{2\gamma_{n-1}} \|\tilde{y}_n - \varphi_{n-1}(\bar{x}_n)\|^2_{\mathbb{Y}} \right] \\ &= \limsup_{n\to+\infty} F(\bar{x}_n, \tilde{y}_n) \geq \liminf_{n\to+\infty} F(\bar{x}_n, \tilde{y}_n) \geq F(\bar{x}, y). \end{aligned}$$

Hence, $\bar{y} \in M(\bar{x})$ and (SG1) is satisfied.
In order to prove condition (SG2), we have to show that $L(\bar{x}, \bar{y}) \geq L(x, \bar{\varphi}(x))$ for any $x \in X$. So, let $x \in X \setminus \{\bar{x}\}$. In light of $(\mathcal{L}1)$ we get

$$\begin{aligned} L(\bar{x}, \bar{y}) &\geq \limsup_{n\to+\infty} L(\bar{x}_n, \varphi_n(\bar{x}_n)) \\ &= \limsup_{n\to+\infty} \left[L(\bar{x}_n, \varphi_n(\bar{x}_n)) - \frac{1}{2\beta_{n-1}} \|\bar{x}_n - \bar{x}_{n-1}\|^2_{\mathbb{X}} \right] \\ &\geq \limsup_{n\to+\infty} \left[L(x, \varphi_n(x)) - \frac{1}{2\beta_{n-1}} \|x - \bar{x}_{n-1}\|^2_{\mathbb{X}} \right] \\ &\geq \liminf_{n\to+\infty} \left[L(x, \varphi_n(x)) - \frac{1}{2\beta_{n-1}} \|x - \bar{x}_{n-1}\|^2_{\mathbb{X}} \right] \\ &= \liminf_{n\to+\infty} L(x, \varphi_n(x)) \geq L(x, \varphi(x)) \end{aligned}$$

where the first (resp. second) equality holds since the second addend in the $\limsup$ (resp. $\liminf$) converges to 0 being $(\beta_n)_n$ a divergent sequence of positive real numbers and X

a compact set, the second inequality comes from the definition of $\bar{x}_n$ in Procedure $(\mathcal{CM})$, and the last inequality follows by $(\mathcal{L}2)$. As $x \in X \setminus \{\bar{x}\}$, then $L(x, \varphi(x)) = L(x, \bar{\varphi}(x))$ and, therefore, $L(\bar{x}, \bar{y}) \geq L(x, \bar{\varphi}(x))$. Hence (SG2) holds, and the proof is complete. □

Remark 3.1 (on the dependence on $(\bar{x}_0, \bar{y}_0)$) The SPNE selected according to Theorem 3.1 is affected, in general, by the choice of the initial point $(\bar{x}_0, \bar{y}_0)$ in Procedure $(\mathcal{CM})$: in fact, such an SPNE reflects both the leader's willingness of being near to $\bar{x}_0$ and the follower's willingness of being near to $\bar{y}_0$, as discussed in the interpretation of the procedure in Section 2.
The next trivial example, whose main computations are provided in the Appendix, emphasizes this dependence especially from the follower's perspective, whereas in Example 3.2 and Example 3.3 these insights are more evident also from the leader's point of view.

Example 3.1 Let $\Gamma = (X, Y, L, F)$ where $X = Y = [-1, 1]$ and

$$L(x, y) = x, \qquad F(x, y) = -xy.$$

The follower's best reply correspondence M is defined on $[-1, 1]$ by

$$M(x) = \begin{cases} \{1\}, & \text{if } x \in [-1, 0[\\ [-1, 1], & \text{if } x = 0 \\ \{-1\}, & \text{if } x \in]0, 1]. \end{cases} \tag{8}$$

Let $(\bar{x}_0, \bar{y}_0) \in [-1, 1] \times [-1, 1]$ be the initial point of the procedure and let $\beta_n = \gamma_n = 2^n$ for any $n \in \mathbb{N} \cup \{0\}$. Then Procedure $(\mathcal{CM})$ generates the following sequence $(\bar{x}_n, \varphi_n)_n$ of strategy profiles:

$$\bar{x}_n = \begin{cases} \min\{1 + \bar{x}_0, 1\}, & \text{if } n = 1 \\ 1, & \text{if } n \geq 2, \end{cases} \qquad \varphi_n(x) = \begin{cases} 1, & \text{if } x \in \left[-1, \frac{\bar{y}_0 - 1}{a_n}\right[\\ \bar{y}_0 - a_n x, & \text{if } x \in \left[\frac{\bar{y}_0 - 1}{a_n}, \frac{\bar{y}_0 + 1}{a_n}\right] \\ -1, & \text{if } x \in \left]\frac{\bar{y}_0 + 1}{a_n}, 1\right], \end{cases} \tag{9}$$

where the sequence $(a_n)_n$ is recursively defined by

$$\begin{cases} a_1 = 1 \\ a_{n+1} = a_n + 2^n \quad \text{for any } n \geq 1. \end{cases}$$

The SPNE of Γ selected according to Theorem 3.1 is $(\bar{x}, \bar{\varphi})$, where

$$\bar{x} = 1, \qquad \bar{\varphi}(x) = \begin{cases} 1, & \text{if } x \in [-1, 0[\\ \bar{y}_0, & \text{if } x = 0 \\ -1, & \text{if } x \in]0, 1]. \end{cases} \tag{10}$$

Let us note that all the SPNEs of Γ are obtained when varying $\bar{y}_0 \in [-1, 1]$ in (10). Hence $\bar{\varphi}$ is, among all the follower's strategies being part of an SPNE, the follower's strategy such that $\bar{\varphi}(x)$ minimizes the distance from the follower's initial point $\bar{y}_0$, for any $x \in [-1, 1]$. Therefore the SPNE constructed by our method is the nearest SPNE to the initial point $(\bar{x}_0, \bar{y}_0)$ in the sense illustrated in Section 2, Interpretation of the procedure.

Remark 3.2 (on the pointwise limit of $(\varphi_n)_n$) The follower's strategy $\bar{\varphi}$ in the SPNE defined according to Theorem 3.1 differs from the pointwise limit φ of sequence $(\varphi_n)_n$ at most in one point. In fact if the two limits

$$\lim_{n \to +\infty} \varphi_n(\bar{x}_n) \quad \text{and} \quad \lim_{n \to +\infty} \varphi_n(\bar{x}), \tag{11}$$

where $\bar{x} = \lim_{n \to +\infty} \bar{x}_n$, coincide, then $\bar{\varphi}(x) = \varphi(x)$ for any $x \in X$ and the strategy profile $(\bar{x}, \varphi)$ is an SPNE of Γ in light of Theorem 3.1. Instead, if the two limits in (11) do not coincide, then $\bar{\varphi}(\bar{x}) \neq \varphi(\bar{x})$ and the strategy profile $(\bar{x}, \varphi)$ could be not an SPNE of Γ, hence we need the follower's strategy $\bar{\varphi}$ as in statement of Theorem 3.1 in order to get an SPNE. The following two examples illustrate the two cases described above: in the first one the two limits in (11) are equal, whereas, in the second one the two limits in (11) are different. The main computations of both examples are provided in the Appendix.

Example 3.2 Let $\Gamma = (X, Y, L, F)$ where $X = Y = [-1, 1]$ and

$$L(x, y) = y, \qquad F(x, y) = -xy.$$

The follower's best reply correspondence M is defined on $[-1, 1]$ by

$$M(x) = \begin{cases} \{1\}, & \text{if } x \in [-1, 0[\\ [-1, 1], & \text{if } x = 0 \\ \{-1\}, & \text{if } x \in]0, 1]. \end{cases} \tag{12}$$

Let $(\bar{x}_0, \bar{y}_0) = (1, 1)$ be the initial point of the procedure and let $\beta_n = \gamma_n = 2^n$ for any $n \in \mathbb{N} \cup \{0\}$. Then Procedure $(\mathcal{CM})$ generates the following sequence $(\bar{x}_n, \varphi_n)_n$ of strategy profiles:

$$\bar{x}_n = 0, \qquad \varphi_n(x) = \begin{cases} 1, & \text{if } x \in [-1, 0[\\ 1 - a_n x, & \text{if } x \in [0, 2/a_n] \\ -1, & \text{if } x \in]2/a_n, 1]\,, \end{cases} \tag{13}$$

where the sequence $(a_n)_n$ is recursively defined by

$$\begin{cases} a_1 = 1 \\ a_{n+1} = a_n + 2^n \quad \text{for any } n \geq 1. \end{cases}$$

Hence, the SPNE of Γ selected according to Theorem 3.1 is $(\bar{x}, \bar{\varphi})$, where

$$\bar{x} = 0, \qquad \bar{\varphi}(x) = \begin{cases} 1, & \text{if } x \in [-1, 0] \\ -1, & \text{if } x \in]0, 1]. \end{cases}$$

In this case, $\bar{\varphi}$ coincides with the pointwise limit of $(\varphi_n)_n$ since $\lim_n \varphi_n(\bar{x}_n) = 1 = \lim_n \varphi_n(\lim_n \bar{x}_n)$.

Let us note that Γ has infinitely many SPNEs. In fact, denoted with $\hat{\varphi}^\alpha$ the function defined on $[-1, 1]$ by

$$\hat{\varphi}^\alpha(x) := \begin{cases} 1, & \text{if } x \in [-1, 0[\\ \alpha, & \text{if } x = 0 \\ -1, & \text{if } x \in]0, 1], \end{cases}$$

the set of SPNEs of Γ is $\{(\hat{x}, \hat{\varphi}^\alpha) \mid \hat{x} \in [-1, 0[, \ \alpha \in [-1, 1]\} \cup \{(0, \hat{\varphi}^1)\}$, only one of which is obtained via our method.

Hence, the selection method defined by means of Procedure $(\mathcal{CM})$ is effective.

Example 3.3 Let $\Gamma = (X, Y, L, F)$ where $X = [1/2, 2]$, $Y = [-1, 1]$ and

$$L(x, y) = -x - y, \qquad F(x, y) = \begin{cases} 0, & \text{if } x \in [1/2, 1] \\ (1 - x)y, & \text{if } x \in]1, 2]. \end{cases}$$

The follower's best reply correspondence M is given by

$$M(x) = \operatorname*{Arg\,max}_{y \in [-1,1]} F(x, y) = \begin{cases} [-1, 1], & \text{if } x \in [1/2, 1] \\ \{-1\}, & \text{if } x \in]1, 2]. \end{cases} \tag{14}$$

Let $(\bar{x}_0, \bar{y}_0) = (1, 1)$ and $\beta_n = \gamma_n = n + 1$ for any $n \in \mathbb{N} \cup \{0\}$. Then Procedure $(\mathcal{CM})$ generates the following sequence $(\bar{x}_n, \varphi_n)_n$ of strategy profiles:

$$\bar{x}_n = \begin{cases} 1/2, & \text{if } n = 1 \\ 1 + 2/a_n, & \text{if } n \geq 2, \end{cases} \qquad \varphi_n(x) = \begin{cases} 1, & \text{if } x \in [1/2, 1] \\ a_n + 1 - a_n x, & \text{if } x \in]1, 1 + 2/a_n] \\ -1, & \text{if } x \in]1 + 2/a_n, 2], \end{cases} \tag{15}$$

where the sequence $(a_n)_n$ is recursively defined by

$$\begin{cases} a_1 = 1 \\ a_{n+1} = a_n + n + 1 \quad \text{for any } n \geq 1. \end{cases}$$

Hence, the SPNE of Γ selected according to Theorem 3.1 is $(\bar{x}, \bar{\varphi})$, where

$$\bar{x} = 1, \qquad \bar{\varphi}(x) = \begin{cases} 1, & \text{if } x \in [1/2, 1[\\ -1, & \text{if } x \in [1, 2]. \end{cases} \tag{16}$$

As mentioned in Remark 3.2, in this case

$$\lim_n \varphi_n(\bar{x}_n) = -1 \neq 1 = \lim_n \varphi_n(\lim_n \bar{x}_n)$$

and, furthermore, the strategy profile $(1, \varphi)$, where φ is the pointwise limit of $(\varphi_n)_n$, is not an SPNE of Γ since $\operatorname{Arg\,max}_{x \in [1/2,2]} L(x, \varphi(x)) = \emptyset$.

Remark 3.3 (on the implementation of the method) The method based on Procedure $(\mathcal{CM})$ could be clearly implemented in any finite game in mixed strategies and for any game where the players have a continuum of actions and the functions φ_n can be analytically determined for any $n \in \mathbb{N}$.

Remark 3.4 (on lower semicontinuity of the correspondence M) If the sequence $(\bar{x}_n, \varphi_n(\bar{x}_n))_n$ in the statement of Theorem 3.1 does not converge, the thesis of Theorem 3.1 still holds replacing $(\bar{x}, \bar{y})$ with the limit of a convergent subsequence $(\bar{x}_{n_k}, \varphi_{n_k}(\bar{x}_{n_k}))_k \subseteq (\bar{x}_n, \varphi_n(\bar{x}_n))_n$, whose existence is guaranteed by the compactness of X and Y. Therefore, assumption $\Gamma \in \mathcal{G}$ ensures the existence of SPNEs regardless of the lower semicontinuity of the follower's best reply correspondence. Indeed, in the examples above, the follower's best reply correspondences in (12) and (14) are not lower semicontinuous set-valued maps.

Remark 3.5 (on leader's costs to move) An existence result for SPNEs analogous to Theorem 3.1 can be obtained if the leader's payoff function is not modified in Procedure $(\mathcal{CM})$, that is if the learning approach via costs to move only concerns the follower stage (i.e., $L_n = L$, for any $n \in \mathbb{N}$).

The definition of $(\varphi_n)_n$ in Procedure $(\mathcal{CM})$ is based on a parametric proximal point method. Since proximal point methods require that an initial point has to be fixed, we have taken in Procedure $(\mathcal{CM})$ the constant function $\varphi_0 \in Y^X$ defined by $\varphi_0(x) = \bar{y}_0$ as the follower's initial point. However, Procedure $(\mathcal{CM})$ could be also defined choosing any continuous function $\varphi_0 \in Y^X$ as follower's initial point and all the results of Sections 2 and 3 would be still valid (in particular, Proposition 2.1, Proposition 2.2 and Theorem 3.1).
The next two propositions state some further properties of our constructive method when in Procedure $(\mathcal{CM})$ the initial constant function defined by $\bar{y}_0$ is replaced with a continuous function $\varphi_0 \in Y^X$. For the sake of simplicity, we continue to refer to $(\varphi_n)_n$ as the sequence generated by this modified procedure.

Proposition 3.1. *Let $\Gamma \in \mathcal{G}$ and let the follower's initial point $\varphi_0 \in Y^X$ be a continuous function. Assume that $\varphi_0(x) \in M(x)$ for any $x \in X$. Then $\varphi_n = \varphi_0$ for any $n \in \mathbb{N}$. Moreover, φ_0 is the strategy chosen by the follower in the SPNE selected according to Theorem 3.1.*

Proof. We prove the first part of the result by induction. Firstly, note that the function F satisfies the assumptions of Lemma 2.1 as $\Gamma \in \mathcal{G}$.
Let $n = 1$. Since $\varphi_0(x) \in M(x)$ for any $x \in X$, in light of Lemma 2.1*(iv)* and the definition of φ_1, we have

$$\{\varphi_0(x)\} = \operatorname*{Arg\,max}_{y \in Y} F(x, y) - \frac{1}{2\gamma_0}\|y - \varphi_0(x)\|_{\mathbb{Y}}^2 = \{\varphi_1(x)\}, \text{ for any } x \in X,$$

so, the base case is satisfied. Let $n > 1$ and suppose that $\varphi_n = \varphi_0$. Then $\varphi_n(x) \in M(x)$ for any $x \in X$ and, by Lemma 2.1*(iv)* and definition of φ_{n+1}, we get

$$\{\varphi_n(x)\} = \operatorname*{Arg\,max}_{y \in Y} F(x, y) - \frac{1}{2\gamma_n}\|y - \varphi_n(x)\|_{\mathbb{Y}}^2 = \{\varphi_{n+1}(x)\}, \text{ for any } x \in X,$$

thus, the inductive step is proved. Hence, $\varphi_n = \varphi_0$ for any $n \in \mathbb{N}$ and the first part of the proof is complete.
Since $\varphi_n = \varphi_0$ for any $n \in \mathbb{N}$ and φ_0 is continuous, then, for any sequence $(x_n)_n \subseteq X$ converging to $x \in X$, the sequence $\varphi_n(x_n)$ converges to $\varphi_0(x)$. So, φ_0 is the follower's strategy in the SPNE selected according to Theorem 3.1. □

Proposition 3.2. *Let $\Gamma \in \mathcal{G}$ and let the follower's initial point $\varphi_0 \in Y^X$ be a continuous function. Assume that there exists $\nu \in \mathbb{N}$ such that $\varphi_\nu = \varphi_{\nu-1}$. Then $\varphi_\nu(x) \in M(x)$ for any $x \in X$ and $\varphi_n = \varphi_\nu$ for any $n > \nu$. Moreover, φ_ν is the strategy chosen by the follower in the SPNE selected according to Theorem 3.1.*

Proof. By the definition of φ_ν and since $\varphi_\nu = \varphi_{\nu-1}$, we have

$$\begin{aligned}\{\varphi_\nu(x)\} &= \operatorname*{Arg\,max}_{y \in Y} F(x, y) - \frac{1}{2\gamma_{\nu-1}}\|y - \varphi_{\nu-1}(x)\|_{\mathbb{Y}}^2 \\ &= \operatorname*{Arg\,max}_{y \in Y} F(x, y) - \frac{1}{2\gamma_{\nu-1}}\|y - \varphi_\nu(x)\|_{\mathbb{Y}}^2, \text{ for any } x \in X.\end{aligned}$$

Then, in light of Lemma 2.1*(iv)* we get $\varphi_\nu(x) \in M(x)$ for any $x \in X$.
Consider the new constructive procedure whose follower's initial point is the continuous function φ_ν and with $(\gamma_{\nu+n})_{n\in\mathbb{N}\cup\{0\}}$ instead of $(\gamma_n)_{n\in\mathbb{N}\cup\{0\}}$ (such a procedure is nothing but the original procedure taken away the first $\nu - 1$ steps). Applying Proposition 3.1 we have $\varphi_n = \varphi_\nu$ for any $n > \nu$. Given the above and by the continuity of φ_ν, arguing as in the last part of the proof of Proposition 3.1, it follows that φ_ν is the strategy chosen by the follower in the SPNE selected according to Theorem 3.1. □

4 Connections with another constructive method and other solution concepts

In this section, firstly we analyze the relation between our learning method based on costs to move and the method proposed in Morgan and Patrone (2006), then we compare the SPNE achievable via Theorem 3.1 with the SPNEs obtainable through the weak

Stackelberg equilibrium and the strong Stackelberg equilibrium. We just investigate the connections with the above mentioned three methods since, to our knowledge, only these ones provide the construction of an SPNE in games of perfect information where the players have a continuum of actions and, hence, also in Stackelberg games.

4.1 Connections with Morgan and Patrone (2006)

In [31] a constructive method based on Tikhonov regularization is used in order to approach an SPNE in Stackelberg games. More precisely, the authors consider the following regularized second-level problem

$$P_{\alpha_n}(x): \quad \min_{y\in Y} F(x,y) + \alpha_n \|y\|^2,$$

where $x \in X$ and $(\alpha_n)_n$ is a decreasing sequence of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = 0$. Denoted by $\bar{\rho}_n(x)$ the unique solution to $P_{\alpha_n}(x)$ and by $\hat{\rho}(x)$ the unique minimum norm solution to the problem

$$P(x): \quad \min_{y\in Y} F(x,y),$$

classical results on Tikhonov regularization ([41]) ensure that the sequence $(\bar{\rho}_n(x))_n$ converges to $\hat{\rho}(x)$. Let $\bar{x}_n$ be a solution to the regularized problem

$$S_{\alpha_n}: \quad \min_{x\in X} L(x, \bar{\rho}_n(x))$$

and assume that the sequence $(\bar{x}_n, \bar{\rho}_n(\bar{x}_n))_n$ converges to $(\bar{x}, \bar{y})$, then, under suitable assumptions, the strategy profile $(\bar{x}, \tilde{\rho}) \in X \times Y^X$ where

$$\tilde{\rho}(x) = \begin{cases} \bar{y}, & \text{if } x = \bar{x} \\ \hat{\rho}(x), & \text{if } x \neq \bar{x}, \end{cases}$$

is an SPNE of the initial game (see [31, Theorem 3.1]).

We note that the way in which the SPNE is constructed via the method described above does not involve any task of learning step by step. Indeed, $P_{\alpha_n}(x)$ is not recursively defined and therefore, at a given step n, neither the follower's strategy $\bar{\rho}_n$ is an updating of his previous strategy $\bar{\rho}_{n-1}$ nor $\bar{x}_n$ is an updating of $\bar{x}_{n-1}$. Hence, the anchoring effects arising in Procedure $(\mathcal{CM})$ do not appear in this framework, as well as other kinds of behavioral motivation. As a matter of fact, in general, Procedure $(\mathcal{CM})$ and procedure in [31] (adapted to maximization frameworks) do not generate the same SPNE, as shown in the next example.

Example 4.1 Let Γ be the game defined in Example 3.3. The SPNE constructed by using the approach in [31] is $(1, \tilde{\rho})$, where

$$\tilde{\rho}(x) = \begin{cases} 0, & \text{if } x \in [1/2, 1[\\ -1, & \text{if } x \in [1, 2]; \end{cases}$$

that does not coincide with the SPNE found out in (16).

4.2 Connections with Weak and Strong Stackelberg equilibria

In Stackelberg games where the follower's best reply correspondence is not always single-valued, two extreme behaviors of the leader could arise regarding his beliefs about how the follower chooses inside his own set of optimal actions in response to each action chosen by the leader. In the first case, the leader is optimistic and believes that the follower chooses the best action for the leader; whereas in the second one, the leader is pessimistic and believes that the follower could choose the worst action for the leader. These behaviors lead to two widely investigated problems (originally named *generalized Stackelberg* problems, see [22]): the *strong Stackelberg*, also called *optimistic Stackelberg* (see, for example, [8, 42, 24, 15, 12], and references therein), and the *weak Stackelberg*, also called *pessimistic Stackelberg* (see, for example, [30, 25, 26, 45, 16, 23], and references therein) problems, respectively, described below.

$$\text{(s-S)} \begin{cases} \max_{x\in X} \max_{y\in M(x)} L(x,y) \\ \text{where } M(x) \text{ is defined in (3),} \end{cases} \qquad \text{(w-S)} \begin{cases} \max_{x\in X} \min_{y\in M(x)} L(x,y) \\ \text{where } M(x) \text{ is defined in (3).} \end{cases}$$

An action profile $(x^*, y^*) \in X \times Y$ is said to be

(i) *strong Stackelberg equilibrium* (or *optimistic equilibrium*) if

$$x^* \in \operatorname*{Arg\,max}_{x\in X} \max_{y\in M(x)} L(x,y) \text{ and } y^* \in \operatorname*{Arg\,max}_{y\in M(x^*)} L(x^*,y),$$

(ii) *weak Stackelberg equilibrium* (or *pessimistic equilibrium*) if

$$x^* \in \operatorname*{Arg\,max}_{x\in X} \min_{y\in M(x)} L(x,y) \text{ and } y^* \in M(x^*).$$

Starting from a strong or a weak Stackelberg equilibrium one could derive an SPNE according to the two different behaviors of the leader. In fact:

(i) if the action profile (x^*, y^*) is a strong Stackelberg equilibrium, then the strategy profile (x^*, φ^*) is an SPNE when $\varphi^*(x) \in \operatorname{Arg\,max}_{y\in M(x)} L(x,y)$ for any $x \in X$;

(ii) if the action profile (x^*, y^*) is a weak Stackelberg equilibrium, then the strategy profile (x^*, φ^*) is an SPNE when $\varphi^*(x) \in \operatorname{Arg\,min}_{y\in M(x)} L(x,y)$ for any $x \in X$.

Nevertheless, in the optimistic (resp. pessimistic) situation, the computation of strong (resp. weak) Stackelberg equilibria, and related SPNEs, would require the leader to know the best reply correspondence of the follower. Instead, an SPNE obtainable through the learning approach with costs to move described in Procedure $(\mathcal{CM})$ relieves the leader of knowing the follower's best reply correspondence. Moreover, let us note that the SPNE obtained via Procedure $(\mathcal{CM})$ does not coincide, in general, with the SPNEs associated with optimistic or pessimistic equilibria. To show this fact, it is sufficient to check if the limit $(\bar{x}, \bar{y})$ of the sequence of actions $(\bar{x}_n, \varphi_n(\bar{x}_n))_n$ obtained through Procedure $(\mathcal{CM})$ is a strong or a weak Stackelberg equilibrium. This lack of connection is exhibited in the following example.

Example 4.2 Let Γ be the game defined in Example 3.3. The follower's best reply correspondence M is given in (14). Since for any $x \in [1/2, 2]$

$$\max_{y\in M(x)} L(x,y) = -x+1, \qquad \min_{y\in M(x)} L(x,y) = \begin{cases} -x-1, & \text{if } x \in [1/2,1] \\ -x+1, & \text{if } x \in]1,2], \end{cases}$$

then

$$\operatorname*{Arg\,max}_{x\in[1/2,2]} \max_{y\in M(x)} L(x,y) = \{1/2\}, \qquad \operatorname*{Arg\,max}_{x\in[1/2,2]} \min_{y\in M(x)} L(x,y) = \emptyset.$$

Hence, the strong Stackelberg equilibrium is the action profile $(1/2,-1)$ as $\{-1\} = \operatorname{Arg\,max}_{y\in M(1/2)} L(1/2,y)$. Instead, the weak Stackelberg equilibrium does not exist.
Procedure $(\mathcal{CM})$ generates the sequence $(\bar{x}_n, \varphi_n)_n$ defined in (15). The sequence of actions $(\bar{x}_n, \varphi_n(\bar{x}_n))_{n\geq 2} = (1+2/a_n, 1)_{n\geq 2}$ converges to $(1,-1)$, which is neither a strong nor a weak Stackelberg equilibrium.

5 Conclusion

In this paper we presented a theoretical method to construct a Subgame Perfect Nash Equilibrium of a one-leader one-follower two-stage game by using a learning approach via costs to move. The method is based on a procedure that allows to overcome the difficulties occurring when the follower's best reply correspondence is not single-valued. In fact, we constructed recursively a sequence of SPNEs of classical Stackelberg games whose payoff functions are obtained by subtracting to the payoff functions of the initial game a cost to move term depending on the SPNE reached at the previous step. Hence, we showed the existence of an SPNE achievable via this learning method under mild assumptions on the data of the game.

The analysis for one-leader two-follower two-stage games is presently in progress. In this case, the nonuniqueness of the parametric Nash equilibria obtained as the optimal reaction of the followers, will be possibly overcome by applying a learning method based on costs to move and known results about uniqueness of Nash equilibria as [37], [10], or [9].

Another direction for future research is the extension of our learning method to Stackelberg differential games. In fact, starting from Chen and Cruz ([11]) and Simaan and Cruz ([40]), the literature on Stackelberg differential games has dealt essentially with situations where, for any control path chosen by the leader, the follower's optimal control path is unique. Using a generalization of the proposed constructive procedure with costs to move, we aim to approach an SPNE even in Stackelberg differential games whose follower's optimal control path is not uniquely determined.

Furthermore, we purpose to adapt the method presented in this paper to *semivectorial bilevel optimal control problems* ([7]), that are differential games with hierarchical play where one leader in the first stage faces a scalar optimal control problem and more followers in the second stage solve a cooperative differential game. In fact, our learning approach via costs to move could be useful to construct SPNEs when the followers' Pareto control paths is not unique requiring only convexity assumptions, whereas in [7] the non single-valuedness of the followers' best reply correspondence is overcome in the optimistic and the pessimistic situations associated with the problem by means of some strict convexity assumptions.

Appendix

Main computations of Remark 2.1 Firstly, we show that the function F defined in Remark 2.1 satisfies $(\mathcal{F}_1)$-$(\mathcal{F}_3)$.

(i) Proof of $(\mathcal{F}_1)$:
We need to show the upper semicontinuity of F only at $(x,(0,0))$, as F is continuous for any $(x,(y_1,y_2)) \in X \times (Y \setminus \{(0,0)\})$. Let $x \in X$ and let $(x_k,(y_{1,k},y_{2,k}))_k \subseteq X \times Y$ be a sequence converging to $(x,(0,0))$. Since $F(x,(y_1,y_2)) \leq 0$ for any $(x,(y_1,y_2)) \in X \times Y$ and $F(x,(0,0)) = 0$, then

$$\limsup_{k\to+\infty} F(x_k,(y_{1,k},y_{2,k})) \leq F(x,(0,0)).$$

Therefore $(\mathcal{F}_1)$ holds.

(ii) Proof of $(\mathcal{F}_2)$:
We need to show $(\mathcal{F}_2)$ only at $(x,(0,0))$, as F is continuous for any $(x,(y_1,y_2)) \in X \times (Y \setminus \{(0,0)\})$. Let $x \in X$ and let $(x_k)_k \subseteq X$ be a sequence converging to x. Define $(\tilde{y}_{1,k},\tilde{y}_{2,k}) := (1/k,0) \in Y$ for any $k \in \mathbb{N}$. Since $F(x_k,(\tilde{y}_{1,k},\tilde{y}_{2,k})) = 0$ for any $k \in \mathbb{N}$ and $F(x,(0,0)) = 0$, then

$$\liminf_{k\to+\infty} F(x_k,(\tilde{y}_{1,k},\tilde{y}_{2,k})) = F(x,(0,0)).$$

Therefore $(\mathcal{F}_2)$ holds.

(iii) Proof of $(\mathcal{F}_3)$:
Let $x \in X$. In order to prove the concavity of $F(x,(\cdot,\cdot))$ on $Y \setminus \{(0,0)\}$, we consider the twice-continuously differentiable function $g\colon]0,+\infty[\times\mathbb{R} \to \mathbb{R}$ defined by

$$g(y_1,y_2) := -\frac{y_2^2}{2y_1}x.$$

The Hessian matrix of g at (y_1,y_2) is

$$Hg(y_1,y_2) = \begin{pmatrix} -\frac{y_2^2}{y_1^3}x & \frac{y_2}{y_1^2}x \\ \frac{y_2}{y_1^2}x & -\frac{1}{y_1}x \end{pmatrix}.$$

Since $Hg(y_1,y_2)$ is negative semi-definite for any $(y_1,y_2) \in]0,+\infty[\times\mathbb{R}$ (being $x \in [1,2]$), then g is concave on $]0,+\infty[\times\mathbb{R}$. Therefore, $F(x,(\cdot,\cdot))$ is concave on $Y \setminus$

$\{(0,0)\}$, as $F(x,(y_1,y_2)) = g(y_1,y_2)$ for any $(y_1,y_2) \in Y \setminus \{(0,0)\}$. The concavity of $F(x,(\cdot,\cdot))$ on Y follows by the equality

$$F(x, t(0,0) + (1-t)(y_1,y_2)) = tF(x,(0,0)) + (1-t)F(x,(y_1,y_2)),$$

that holds for any $t \in [0,1]$ and $(y_1,y_2) \in Y$. Hence $(\mathcal{F}_3)$ is satisfied.

Let $x \in X$. We show that the function $F(x,(\cdot,\cdot))$ is not lower semicontinuous at $(0,0)$. In fact, let $(\bar{y}_{1,k}, \bar{y}_{2,k}) := (1/k, 1/\sqrt{k}) \in Y$ for any $k \in \mathbb{N}$. Since $F(x,(\bar{y}_{1,k},\bar{y}_{2,k})) = -x/2 \in [-1/2,-1]$ and $F(x,(0,0)) = 0$, then

$$\liminf_{k\to+\infty} F(x,(\bar{y}_{1,k},\bar{y}_{2,k})) \not\geq F(x,(0,0)).$$

Main computations of Example 3.1 Firstly, note that $\Gamma \in \mathcal{G}$. We prove (9) by induction on n. Let $n = 1$, then

$$\{\varphi_1(x)\} = \operatorname*{Arg\,max}_{y\in Y} F_1(x,y) = \operatorname*{Arg\,max}_{y\in[-1,1]} -xy - \frac{(y-\bar{y}_0)^2}{2} = \begin{cases} 1, & \text{if } x \in [-1, \bar{y}_0 - 1[\\ \bar{y}_0 - x, & \text{if } x \in [\bar{y}_0 - 1, \bar{y}_0 + 1] \\ -1, & \text{if } x \in]\bar{y}_0 + 1, 1]. \end{cases}$$

and

$$\{\bar{x}_1\} = \operatorname*{Arg\,max}_{x\in X} L_1(x,\varphi_1(x)) = \operatorname*{Arg\,max}_{x\in[-1,1]} x - \frac{(x-\bar{x}_0)^2}{2} = \begin{cases} 1 + \bar{x}_0, & \text{if } \bar{x}_0 \in [-1,0] \\ 1, & \text{if } \bar{x}_0 \in]0,1]. \end{cases}$$

As $a_1 = 1$, the base case is fulfilled. Assume that (9) holds for $n > 1$. So

$$F_{n+1}(x,y) = \begin{cases} P_1(x,y) = -\frac{y^2}{2^{n+1}} - \left(x - \frac{1}{2^n}\right)y - \frac{1}{2^{n+1}}, & \text{if } x \in \left[-1, \frac{\bar{y}_0-1}{a_n}\right[\\ P_2(x,y) = -\frac{y^2}{2^{n+1}} - \left(x + \frac{a_n x - \bar{y}_0}{2^n}\right)y - \frac{(\bar{y}_0 - a_n x)^2}{2^{n+1}}, & \text{if } x \in \left[\frac{\bar{y}_0-1}{a_n}, \frac{\bar{y}_0+1}{a_n}\right] \\ P_3(x,y) = -\frac{y^2}{2^{n+1}} - \left(x + \frac{1}{2^n}\right)y - \frac{1}{2^{n+1}}, & \text{if } x \in \left]\frac{\bar{y}_0+1}{a_n}, 1\right], \end{cases}$$

and

$$L_{n+1}(x,y) = -\frac{x^2}{2^{n+1}} + \left(1 + \frac{1}{2^n}\right)x - \frac{1}{2^{n+1}}.$$

If $x \in [-1, (\bar{y}_0 - 1)/a_n[$, then the unique maximizer of $P_1(x,\cdot)$ on $Y = [-1,1]$ is 1 since the abscissa of the vertex of the parabola $\mathcal{P}_1 := \{(y,z) \in R^2 \mid z = P_1(x,y)\}$ is $1 - 2^n x > 1$. If $x \in [(\bar{y}_0 - 1)/a_n, (\bar{y}_0 - 1)/(2^n + a_n)[$, then the unique maximizer of $P_2(x,\cdot)$ on $Y = [-1,1]$ is 1 since the abscissa of the vertex of the parabola $\mathcal{P}_2 := \{(y,z) \in R^2 \mid z = P_2(x,y)\}$ is $\bar{y}_0 - (2^n + a_n)x > 1$. If $x \in [(\bar{y}_0 - 1)/(2^n + a_n), (\bar{y}_0 + 1)/(2^n + a_n)]$, then the unique maximizer of $P_2(x,\cdot)$ on $Y = [-1,1]$ is $\bar{y}_0 - (2^n + a_n)x$ since the abscissa of the vertex of the parabola $\mathcal{P}_2$ is $\bar{y}_0 - (2^n + a_n)x \in [-1,1]$. If $x \in](\bar{y}_0 + 1)/(2^n + a_n), (\bar{y}_0 + 1)/a_n]$, then the unique maximizer of $P_2(x,\cdot)$ on $Y = [-1,1]$ is -1 since the abscissa of the vertex of the parabola $\mathcal{P}_2$ is $\bar{y}_0 - (2^n + a_n)x < -1$. If $x \in [(\bar{y}_0 + 1)/a_n, 1[$, then the unique maximizer of $P_3(x,\cdot)$ on $Y = [-1,1]$ is -1 since the abscissa of the vertex of the

parabola $\mathcal{P}_3 := \{(y,z) \in R^2 \mid z = P_3(x,y)\}$ is $-(2^n x + 1) < -1$.
Given the above, since $2^n + a_n = a_{n+1}$,

$$\{\varphi_{n+1}(x)\} = \operatorname*{Arg\,max}_{y \in Y} F_{n+1}(x,y) = \begin{cases} 1, & \text{if } x \in \left[-1, \frac{\bar{y}_0 - 1}{a_{n+1}}\right[\\ \bar{y}_0 - a_{n+1}x, & \text{if } x \in \left[\frac{\bar{y}_0-1}{a_{n+1}}, \frac{\bar{y}_0+1}{a_{n+1}}\right] \\ -1, & \text{if } x \in \left]\frac{\bar{y}_0+1}{a_{n+1}}, 1\right], \end{cases} \tag{17}$$

for any $x \in [-1,1]$. Since $L_{n+1}(x, \varphi_{n+1}(x)) = L_{n+1}(x,y)$ for any $(x,y) \in X \times Y$ and the abscissa of the vertex of the parabola $\mathcal{T} = \{(x,z) \in \mathbb{R}^2 \mid z = L_{n+1}(x, \varphi_{n+1}(x))\}$ is $2^n + 1 > 1$, then

$$\{\bar{x}_{n+1}\} = \operatorname*{Arg\,max}_{x \in [-1,1]} L_{n+1}(x, \varphi_{n+1}(x)) = \{1\}. \tag{18}$$

Equalities (17)-(18) prove the inductive step, so (9) holds.
As $\lim_{n\to+\infty} a_n = +\infty$, we get

$$\bar{x} = \lim_{n\to+\infty} \bar{x}_n = 1, \qquad \varphi(x) = \lim_{n\to+\infty} \varphi_n(x) = \begin{cases} 1, & \text{if } x \in [-1, 0[\\ \bar{y}_0, & \text{if } x = 0 \\ -1, & \text{if } x \in]0,1]. \end{cases}$$

Since $\lim_{n\to+\infty} \varphi_n(\bar{x}_n) = 1$, then the SPNE constructed according to Theorem 3.1 is $(1, \bar{\varphi}) = (1, \varphi)$.

Main computations of Example 3.2 Firstly, note that $\Gamma \in \mathcal{G}$. We prove (13) by induction on n. Let $n = 1$, then

$$\{\varphi_1(x)\} = \operatorname*{Arg\,max}_{y \in Y} F_1(x,y) = \operatorname*{Arg\,max}_{y \in [-1,1]} -xy - \frac{(y-1)^2}{2} = \begin{cases} 1, & \text{if } x \in [-1,0[\\ 1-x, & \text{if } x \in [0,1]. \end{cases}$$

and

$$\{\bar{x}_1\} = \operatorname*{Arg\,max}_{x \in X} L_1(x, \varphi_1(x)) \quad \text{where } L_1(x, \varphi_1(x)) = \begin{cases} -\frac{x^2-2x-1}{2} & \text{if } x \in [-1,0[\\ -\frac{x^2-1}{2}, & \text{if } x \in [0,1], \end{cases}$$

that is $\bar{x}_1 = 0$. As $a_1 = 1$, the base case is fulfilled. Assume that (13) holds for $n > 1$. So

$$F_{n+1}(x,y) = \begin{cases} P_1(x,y) = -\frac{y^2}{2^{n+1}} - \left(x - \frac{1}{2^n}\right) y - \frac{1}{2^{n+1}}, & \text{if } x \in [-1,0[\\ P_2(x,y) = -\frac{y^2}{2^{n+1}} - \left(x + \frac{a_n x - 1}{2^n}\right) y - \frac{(1-a_n x)^2}{2^{n+1}}, & \text{if } x \in \left[0, \frac{2}{a_n}\right] \\ P_3(x,y) = -\frac{y^2}{2^{n+1}} - \left(x + \frac{1}{2^n}\right) y - \frac{1}{2^{n+1}}, & \text{if } x \in \left]\frac{2}{a_n}, 1\right], \end{cases}$$

and

$$L_{n+1}(x,y) = -\frac{x^2}{2^{n+1}} + y. \tag{19}$$

If $x \in [-1, 0[$, then the unique maximizer of $P_1(x, \cdot)$ on $Y = [-1, 1]$ is 1 since the abscissa of the vertex of the parabola $\mathcal{P}_1 \coloneqq \{(y, z) \in \mathbb{R}^2 \mid z = P_1(x, y)\}$ is $1 - 2^n x > 1$. If $x \in [0, 2/(2^n + a_n)]$, then the unique maximizer of $P_2(x, \cdot)$ on $Y = [-1, 1]$ is $1 - (2^n + a_n)x$ since the abscissa of the vertex of the parabola $\mathcal{P}_2 \coloneqq \{(y, z) \in \mathbb{R}^2 \mid z = P_2(x, y)\}$ is $1 - (2^n + a_n)x \in [-1, 1]$. If $x \in]2/(2^n + a_n), 2/a_n]$, then the unique maximizer of $P_2(x, \cdot)$ on $Y = [-1, 1]$ is -1 since the abscissa of the vertex of the parabola $\mathcal{P}_2$ is $1 - (2^n + a_n)x < -1$. If $x \in]2/a_n, 1]$, then the unique maximizer of $P_3(x, \cdot)$ on $Y = [-1, 1]$ is -1 since the abscissa of the vertex of the parabola $\mathcal{P}_3 \coloneqq \{(y, z) \in \mathbb{R}^2 \mid z = P_3(x, y)\}$ is $-(2^n x + 1) < -1$.
Given the above, since $2^n + a_n = a_{n+1}$,

$$\{\varphi_{n+1}(x)\} = \operatorname*{Arg\,max}_{y \in Y} F_{n+1}(x, y) = \begin{cases} 1, & \text{if } x \in [-1, 0[\\ 1 - a_{n+1}x, & \text{if } x \in \left[0, \frac{2}{a_{n+1}}\right] \\ -1, & \text{if } x \in \left]\frac{2}{a_{n+1}}, 1\right], \end{cases} \tag{20}$$

for any $x \in [-1, 1]$. Evaluating the function L_{n+1} given in (19) at $(x, \varphi_{n+1}(x))$, we get

$$L_{n+1}(x, \varphi_{n+1}(x)) = \begin{cases} T_1(x) = -\frac{x^2}{2^{n+1}} + 1, & \text{if } x \in [-1, 0[\\ T_2(x) = -\frac{x^2}{2^{n+1}} - a_{n+1}x + 1, & \text{if } x \in \left[0, \frac{2}{a_{n+1}}\right] \\ T_3(x) = -\frac{x^2}{2^{n+1}} - 1, & \text{if } x \in \left]\frac{2}{a_{n+1}}, 1\right], \end{cases}$$

Since

(i) the abscissa of the vertexes of the parabolas $\mathcal{T}_1 = \{(x, z) \in \mathbb{R}^2 \mid z = T_1(x)\}$ and $\mathcal{T}_3 = \{(x, z) \in \mathbb{R}^2 \mid z = T_3(x)\}$ is 0;

(ii) the abscissa of the vertex of the parabola $\mathcal{T}_2 = \{(x, z) \in \mathbb{R}^2 \mid z = T_2(x)\}$ is $-2^n a_{n+1} < 0$;

(iii) $L_{n+1}(\cdot, \varphi_{n+1}(\cdot))$ is continuous on $[-1, 1]$,

then

$$\{\bar{x}_{n+1}\} = \operatorname*{Arg\,max}_{x \in [-1,1]} L_{n+1}(x, \varphi_{n+1}(x)) = \{0\}. \tag{21}$$

Equalities (20)-(21) prove the inductive step, so (13) holds.
As $\lim_{n \to +\infty} a_n = +\infty$, we get

$$\bar{x} = \lim_{n \to +\infty} \bar{x}_n = 0, \qquad \varphi(x) = \lim_{n \to +\infty} \varphi_n(x) = \begin{cases} 1, & \text{if } x \in [-1, 0] \\ -1, & \text{if } x \in]0, 1]. \end{cases}$$

Since $\lim_{n \to +\infty} \varphi_n(\bar{x}_n) = 1$, then the SPNE constructed according to Theorem 3.1 is $(0, \bar{\varphi}) = (0, \varphi)$.

Main computations of Example 3.3 Firstly, note that $\Gamma \in \mathcal{G}$. We prove (15) by induction on n. Let $n = 1$, then

$$\{\varphi_1(x)\} = \operatorname*{Arg\,max}_{y\in Y} F_1(x,y) \quad \text{where } F_1(x,y) = \begin{cases} -\frac{(y-1)^2}{2} & \text{if } x \in \left[\frac{1}{2}, 1\right] \\ -\frac{y^2}{2} + (2-x)y - \frac{1}{2}, & \text{if } x \in]1,2], \end{cases}$$

that is

$$\varphi_1(x) = \begin{cases} 1 & \text{if } x \in \left[\frac{1}{2}, 1\right] \\ 2-x, & \text{if } x \in]1,2] \end{cases}$$

Moreover

$$\{\bar{x}_1\} = \operatorname*{Arg\,max}_{x\in X} L_1(x,\varphi_1(x)) \quad \text{where } L_1(x,\varphi_1(x)) = \begin{cases} -\frac{x^2+3}{2} & \text{if } x \in \left[\frac{1}{2}, 1\right] \\ -\frac{x^2-2x+5}{2}, & \text{if } x \in]1,2], \end{cases}$$

that is $\bar{x}_1 = \frac{1}{2}$. As $a_1 = 1$, the base case is fulfilled. Assume that (15) holds for $n > 1$. So

$$F_{n+1}(x,y) = \begin{cases} P_1(x,y), & \text{if } x \in \left[\frac{1}{2}, 1\right] \\ P_2(x,y), & \text{if } x \in \left]1, 1+\frac{2}{a_n}\right] \\ P_3(x,y), & \text{if } x \in \left]1+\frac{2}{a_n}, 2\right], \end{cases}$$

where

$$\begin{aligned} P_1(x,y) &= -\frac{(y-1)^2}{2(n+1)}, \\ P_2(x,y) &= -\frac{y^2}{2(n+1)} + \left(1 - x + \frac{a_n+1-a_n x}{n+1}\right) y - \frac{(a_n+1-a_n x)^2}{2(n+1)}, \\ P_3(x,y) &= -\frac{y^2}{2(n+1)} + \left(1 - x - \frac{1}{n+1}\right) y - \frac{1}{2(n+1)}, \end{aligned}$$

and

$$L_{n+1}(x,y) = -\frac{x^2}{2(n+1)} - \left(1 - \frac{a_n+2}{(n+1)a_n}\right) x - \frac{(a_n+2)^2}{2(n+1)a_n^2} - y. \tag{22}$$

If $x \in \left[\frac{1}{2}, 1\right]$, then the unique maximizer of $P_1(x,\cdot)$ on $Y = [-1,1]$ is 1 since the abscissa of the vertex of the parabola $\mathcal{P}_1 := \{(y,z) \in \mathbb{R}^2 \mid z = P_1(x,y)\}$ is 1. If $x \in \left]1, 1+\frac{2}{a_n+n+1}\right]$, then the unique maximizer of $P_2(x,\cdot)$ on $Y = [-1,1]$ is $a_n + n + 2 - (n+1+a_n)x$ since the abscissa of the vertex of the parabola $\mathcal{P}_2 := \{(y,z) \in \mathbb{R}^2 \mid z = P_2(x,y)\}$ is $a_n + n + 2 - (n+1+a_n)x \in [-1,1]$. If $x \in \left]1+\frac{2}{a_n+n+1}, 1+\frac{2}{a_n}\right]$, then the unique maximizer of $P_2(x,\cdot)$ on $Y = [-1,1]$ is -1 since the abscissa of the vertex of the parabola $\mathcal{P}_2$ is $a_n + n + 2 - (n+1+a_n)x < -1$. If $x \in \left]1+\frac{2}{a_n}, 2\right]$, then the unique maximizer of $P_3(x,\cdot)$ on $Y = [-1,1]$ is -1 since the abscissa of the vertex of the parabola $\mathcal{P}_3 := \{(y,z) \in \mathbb{R}^2 \mid z = P_3(x,y)\}$ is $n - (n+1)x < -1$.

Given the above, since $n+1+a_n = a_{n+1}$,

$$\{\varphi_{n+1}(x)\} = \operatorname*{Arg\,max}_{y\in Y} F_{n+1}(x,y) = \begin{cases} 1, & \text{if } x\in \left[\frac{1}{2},1\right] \\ a_{n+1}+1-(a_{n+1})x, & \text{if } x\in \left]1,1+\frac{2}{a_{n+1}}\right] \\ -1, & \text{if } x\in \left]1+\frac{2}{a_{n+1}},2\right], \end{cases} \tag{23}$$

for any $x\in\left[\frac{1}{2},2\right]$. Evaluating the function L_{n+1} given in (22) at $(x,\varphi_{n+1}(x))$, we get

$$L_{n+1}(x,\varphi_{n+1}(x)) = \begin{cases} T_1(x), & \text{if } x\in \left[\frac{1}{2},1\right] \\ T_2(x), & \text{if } x\in \left]1,1+\frac{2}{a_{n+1}}\right] \\ T_3(x), & \text{if } x\in \left]1+\frac{2}{a_{n+1}},2\right], \end{cases}$$

where

$$\begin{aligned} T_1(x) &= -\frac{x^2}{2(n+1)} - \left(1-\frac{a_n+2}{(n+1)a_n}\right)x - \frac{(a_n+2)^2}{2(n+1)a_n^2} - 1, \\ T_2(x) &= -\frac{x^2}{2(n+1)} - \left(1-a_{n+1}-\frac{a_n+2}{(n+1)a_n}\right)x - \frac{(a_n+2)^2}{2(n+1)a_n^2} - a_{n+1} - 1, \\ T_3(x) &= -\frac{x^2}{2(n+1)} - \left(1-\frac{a_n+2}{(n+1)a_n}\right)x - \frac{(a_n+2)^2}{2(n+1)a_n^2} + 1. \end{aligned}$$

Since

(i) the abscissa of the vertexes of the parabolas $\mathcal{T}_1 = \{(x,z)\in\mathbb{R}^2 \mid z = T_1(x)\}$ and $\mathcal{T}_3 = \{(x,z)\in\mathbb{R}^2 \mid z = T_3(x)\}$ is $\frac{2}{a_n} - n < \frac{1}{2}$;

(ii) the abscissa of the vertex of the parabola $\mathcal{T}_2 = \{(x,z)\in\mathbb{R}^2 \mid z = T_2(x)\}$ is $(n+1)a_{n+1} - n + \frac{2}{a_n} > 1+\frac{2}{a_n} > 1+\frac{2}{a_{n+1}}$;

(iii) $T_1\left(\frac{1}{2}\right) < T_3(2)$;

(iv) $L_{n+1}(\cdot,\varphi_{n+1}(\cdot))$ is continuous on $\left[\frac{1}{2},2\right]$,

then

$$\{\bar{x}_{n+1}\} = \operatorname*{Arg\,max}_{x\in[-1,1]} L_{n+1}(x,\varphi_{n+1}(x)) = \left\{1+\frac{2}{a_{n+1}}\right\}. \tag{24}$$

Equalities (23)-(24) prove the inductive step, so (15) holds.
As $\lim_{n\to+\infty} a_n = +\infty$, we get

$$\bar{x} = \lim_{n\to+\infty} \bar{x}_n = 1, \qquad \varphi(x) = \lim_{n\to+\infty} \varphi_n(x) = \begin{cases} 1, & \text{if } x\in\left[\frac{1}{2},1\right] \\ -1, & \text{if } x\in]1,2]. \end{cases}$$

Since $\lim_{n\to+\infty} \varphi_n(\bar{x}_n) = -1$, then the SPNE constructed according to Theorem 3.1 is $(0,\bar{\varphi})$, where

$$\bar{\varphi}(x) = \begin{cases} 1, & \text{if } x\in\left[\frac{1}{2},1\right[\\ -1, & \text{if } x\in[1,2]. \end{cases}$$

References

[1] H. Attouch. *Variational convergence for functions and operators*. Boston: Pitman Advanced Publishing Program, 1984.

[2] H. Attouch, P. Redont, and A. Soubeyran. "A new class of alternating proximal minimization algorithms with costs-to-move". In: *SIAM J Optim* 18 (2007), pp. 1061–1081. DOI: https://doi.org/10.1137/060657248.

[3] H. Attouch and A. Soubeyran. "Inertia and reactivity in decision making as cognitive variational inequalities". In: *J Convex Anal* 13 (2006), p. 207.

[4] T. Başar and G. Olsder. *Dynamic Noncooperative Game Theory*. SIAM, 1999.

[5] H. Bauschke and P. Combettes. *Convex analysis and monotone operator theory in Hilbert spaces*. Springer Science & Business Media, 2011.

[6] P. Bich. "Prudent equilibria and strategic uncertainty in discontinuous games". Preprint. 2016. URL: https://halshs.archives-ouvertes.fr/halshs-01337293/file/prudent-equilibria-preprint-24-juin.pdf.

[7] H. Bonnel and J. Morgan. "Semivectorial Bilevel Convex Optimal Control Problems: Existence Results". In: *SIAM J Control Optim* 50 (2012), pp. 3224–3241. DOI: 10.1137/100795450.

[8] M. Breton, A. Alj, and A. Haurie. "Sequential Stackelberg equilibria in two-person games". In: *J Optim Theory Appl* 59 (1988), pp. 71–97. DOI: 10.1007/BF00939867.

[9] F Caruso, MC Ceparano, and J Morgan. "Uniqueness of Nash equilibrium in continuous two-player weighted potential games". In: *J Math Anal Appl* 459 (2018), pp. 1208–1221. DOI: https://doi.org/10.1016/j.jmaa.2017.11.031.

[10] MC Ceparano and F Quartieri. "Nash equilibrium uniqueness in nice games with isotone best replies". In: *J Math Econ* 70 (2017), pp. 154–165. DOI: https://doi.org/10.1016/j.jmateco.2017.02.011.

[11] C. Chen and J. Cruz. "Stackelberg solution for two-person games with biased information patterns". In: *IEEE Trans Autom Control* 17 (1972), pp. 791–798. DOI: 10.1109/TAC.1972.1100179.

[12] B. Colson, P. Marcotte, and G. Savard. "An overview of bilevel optimization". In: *Ann Oper Res* 153 (2007), pp. 235–256. DOI: https://doi.org/10.1007/s10479-007-0176-2.

[13] G. Dal Maso. *An introduction to Γ-convergence*. Basel: BirkhÃďuser, 1993.

[14] G. De Marco and J. Morgan. "Slightly Altruistic Equilibria". In: *J Optim Theory and Appl* 137 (2008), pp. 347–362. DOI: 10.1007/s10957-008-9353-y.

[15] S. Dempe. *Foundations of bilevel programming*. US: Springer, 2002.

[16] S. Dempe, B.S. Mordukhovich, and A.B. Zemkoho. "Necessary optimality conditions in pessimistic bilevel programming". In: *Optim* 63.4 (2014), pp. 505–533. DOI: https://doi.org/10.1080/02331934.2012.696641.

[17] S.D. Flåm and A.S. Antipin. "Equilibrium programming using proximal-like algorithms". In: *Math Program* 78 (1996), pp. 29–41. DOI: https://doi.org/10.1007/BF02614504.

[18] S.D. Flåm and G.H. Greco. "Non-cooperative games; methods of subgradient projection and proximal point". In: *Advances in Optimization: Proceedings of the 6th French-German Colloquium on Optimization Held at Lambrecht, FRG, June 2–8, 1991*. Ed. by W. Oettli and D. Pallaschke. Berlin, Heidelberg: Springer, 1992, pp. 406–419.

[19] J. Harsanyi. "Oddness of the number of equilibrium points: a new proof". In: *Int J Game Theory* 2 (1973), 235âĂŞ–250.

[20] A. Haurie, J.B. Krawczyk, and G. Zaccour. *Games and dynamic games*. World Scientific Publishing Company, 2012.

[21] M. Kojima, A. Okada, and S. Shindoh. "Strongly stable equilibrium points of n-person noncooperative games". In: *Math Oper Res* 10 (1985), pp. 650–663.

[22] G. Leitmann. "On generalized Stackelberg strategies". In: *J Optim Theory Appl* 26 (1978), pp. 637–643. DOI: 10.1007/BF00933155.

[23] M.B. Lignola and J. Morgan. "Inner regularizations and viscosity solutions for pessimistic bilevel optimization problems". In: *J Optim Theory Appl* 173 (2017), pp. 183–202. DOI: 10.1007/s10957-017-1085-4.

[24] M.B. Lignola and J. Morgan. "Stability of regularized bilevel programming problems". In: *J Optim Theory Appl* 93 (1997), pp. 575–596. DOI: 10.1023/A:1022695113803.

[25] M.B. Lignola and J. Morgan. "Topological existence and stability for stackelberg problems". In: *J Optim Theory Appl* 84 (1995), pp. 145–169. DOI: https://doi.org/10.1007/BF02191740.

[26] P. Loridan and J. Morgan. "Weak via strong Stackelberg problem: new results". In: *J Global Optim* (1996), pp. 263–287. DOI: https://doi.org/10.1007/BF00121269.

[27] B. Martinet. "Brève communication. Régularisation d'inéquations variationnelles par approximations successives". In: *ESAIM: Mathematical Modelling and Numerical Analysis - Modélisation Mathématique et Analyse Numérique* 4 (1970), pp. 154–158. URL: http://eudml.org/doc/193153.

[28] M. Maschler, E. Solan, and S. Zamir. *Game Theory*. Cambridge: Cambridge University Press, 2013. DOI: 10.1017/CBO9780511794216.

[29] J.J. Moreau. “Proximité et dualité dans un espace hilbertien”. In: *Bulletin de la Société mathématique de France* 93 (1965), pp. 273–299.

[30] J. Morgan. “Constrained Well-Posed Two-Level Optimization Problems”. In: *Nonsmooth Optimization and Related Topics*. Ed. by F.H. Clarke, V.F. Dem’yanov, and F. Giannessi. Boston, MA: Springer US, 1989, pp. 307–325.

[31] J. Morgan and F. Patrone. “Stackelberg Problems: Subgame Perfect Equilibria via Tikhonov Regularization”. In: *Advances in Dynamic Games: Applications to Economics, Management Science, Engineering, and Environmental Management*. Ed. by A. Haurie, S. Muto, L.A. Petrosjan, and T.E.S. Raghavan. Boston, MA: Birkhäuser Boston, 2006, pp. 209–221.

[32] A. Moudafi. “Proximal point algorithm extended to equilibrium problems”. In: *J Nat Geom* 15 (1999), pp. 91–100.

[33] R. Myerson. “Refinements of Nash equilibrium concept”. In: *Int J Game Theory* 7 (1978), pp. 73–80.

[34] A. Okada. “On stability of perfect equilibrium points”. In: *Int J Game Theory* 10 (1981), pp. 67–73.

[35] N. Parikh and S. Boyd. “Proximal Algorithms”. In: *Found Trends Optim* 1 (2014), pp. 127–239. DOI: http://dx.doi.org/10.1561/2400000003.

[36] R.T. Rockafellar. “Monotone operators and the proximal point algorithm”. In: *SIAM J Control Optim* 14 (1976), pp. 877–898. DOI: https://doi.org/10.1137/0314056.

[37] JB Rosen. “Existence and Uniqueness of Equilibrium Points for Concave N-Person Games”. In: *Econometrica* 33 (1965), pp. 520–534.

[38] R. Selten. “Reexamination of the perfectness concept for equilibrium points in extensive games”. In: *Int J Game Theory* 4 (1975), pp. 25–55.

[39] R. Selten. “Spieltheoretische behandlung eines oligopolmodells mit nachfrageträgheit”. In: *Zeitschrift für die gesamte Staatswissenschaft* (1965), pp. 301–324, 667–689.

[40] M. Simaan and J. B. Cruz. “On the Stackelberg strategy in nonzero-sum games”. In: *J Optim Theory Appl* 11 (1973), pp. 533–555. DOI: 10.1007/BF00935665.

[41] A.N. Tikhonov. “Solution of incorrectly formulated problems and the regularization method”. In: *Soviet Math Dokl* 4 (1963), pp. 1035–1038.

[42] L.N. Vicente and P.H. Calamai. “Bilevel and multilevel programming: A bibliography review”. In: *J Global Optim* 5 (1994), pp. 291–306. DOI: https://doi.org/10.1007/BF01096458.

[43] H. von Stackelberg. *The theory of the market economy*. Oxford: Oxford University Press, 1952.

[44] W. Wen-Tsun and J. Jia-He. "Essential equilibrium points of n-person non-cooperative games". In: *Scientia Sinica* 11 (1962), pp. 1307–1322.

[45] W. Wiesemann, A. Tsoukalas, P.-M. Kleniati, and B. Rustem. "Pessimistic bilevel optimization". In: *SIAM J Optim* 23 (2013), pp. 353–380. DOI: https://doi.org/10.1137/120864015.