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Abstract

We analyze housing market models *à la* Shapley and Scarf with externalities in consumption; that is, agents care about others and their preferences are defined over allocations rather than over single indivisible goods. After collecting some negative results about the existence of several cooperative solutions, we focus on stable allocations and search for special domains of preferences that can guarantee that they both exist and form a stable set *à la* von Neumann and Morgenstern.

JEL Classification: C70, C78, D51, D62, D64.

Keywords: Indivisible goods; other-regarding preferences; core; stable allocations; stable sets.

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1 Introduction

In this paper we consider the problem of allocating a number of differentiated indivisible objects to individuals in an efficient and stable manner, when there is an externality in consumption. In particular, we consider situations in which traders care not only about the good that they receive, but also about the good that is received by others.

The classical version of the model without externalities is due to Lloyd Shapley and Herbert Scarf (1974) and is known in the literature as the *housing market model*. It considers an exchange economy where n agents trade in indivisible objects, say houses, with no transfers of money. Each agent owns one distinct house when entering the market and each agent desires exactly one house. Agents are allowed to swap their houses among themselves without, however, any money transfers. Traders have complete, reflexive and transitive preference relations over all existing houses and exchange their houses to make a mutually beneficial trade. An outcome in this market is an allocation of houses among individuals such that each individual holds exactly one house.

One can consider several desirable properties of the final allocation of goods. Efficiency or Pareto optimality is one of them: an allocation Pareto improves another if each agent has a greater utility in the first. An allocation is Pareto efficient if there is no allocation which Pareto improves it. Other desirable properties include individual rationality (an allocation must be at least as preferred as the initial endowment) and several forms of stability based on different dominance relations between allocations (e.g., the core or the stability à la von Neumann and Morgenstern).

To determine the outcome, Shapley and Scarf (1974) use the core as solution concept. An allocation is in the core if there is no group of individuals that could make every group member strictly better off by reallocating the houses owned by group members among members of the group itself. In their original model, they prove that the core always exists when no individual is indifferent between any indivisible goods and the good is single-valued [1]. The core allocation can be determined by using a constructive procedure, called the top trading cycles algorithm, which they attribute to David Gale. Few years later, Roth and Postlewaite (1977) pointed out another element for a framework where no trader is indifferent between any indivisible goods: they showed that an allocation a can be in the core of a given market but not in the core of the market in which a itself is the initial endowment. They defined an allocation a to be stable if and only if it is in the core of the market where a itself is the initial endowment. Thus, in contrast with the core notion, they consider a dynamic perspective: an allocation is defined to be stable if no

[1] It is worth, however, noting that the core can be empty when indifferences are allowed in preference relations.

coalition of traders can benefit by further reallocating the items after they have traded. By using the top trading cycles algorithm, they proved that there are always stable points in the core.

Housing market models find applications in highly diverse range of real–world problems, including college admissions and allocation of workers to shifts, landing slots to airlines and houses to people. Most of the literature on housing market models, both theoretical and applied, maintains the assumption that agents are self-interested; that is, they just have a preference relation over the differentiated items to be distributed. However, in some situations, traders may care about both the items they receive and the items that are allocated to the other agents. Consider, for instance, the actual problem of allocating houses to people. It is undeniable that the desirability of a house may depend on the peers in the nearby houses; moreover, social connections among traders that are involved in the market – friendship or, even more, family relationships – influence the assignment process. It can be the case that some or all traders may prefer to live close to their relatives/good friends rather than to people that they do not know at all, and this sort of preference can be even more relevant than the physical characteristics of the house itself such as size, view or floor. This issue concretely emerged in Italy after the earthquake that heavily struck its central regions. For the first time, in fact, the wood-frame houses for the resident population left homeless have been distributed not by drawing, as it was in the past, but keeping into consideration the demands by people, trying to recover the same social and urban structure as in the destroyed villages.

Starting from insights of these real-life situations, in this paper we build upon the original model by Shapley and Scarf where, however, there is also an externality in consumption. In particular, we assume that agents are not self-interested but may have other–regarding preferences that are not independent of what items are allocated to other people in the housing market. Formally, each trader has a preference relation over the set of all allocations rather than over the set of indivisible items. We assume that these preferences are in fact ordinal, that is, linear orders: each trader is able to rank allocations from the best to the worst with no ties allowed.

Recently, a strand of literature on one side matching theory and mechanism design has presented models that manage to focus on preference profiles with externalities. Among them, Sönmez (1999) studies a general class of allocation problems that includes housing markets as a subclass, and proves that there exists an efficient, individually rational and strategy–proof solution only if all allocations in the core are Pareto indifferent and that any such solution selects a core allocation whenever the core is nonempty. Ehlers (2014) obtains the same result by considering a different notion of core which contains the one considered by Sönmez and allows blocking for coalitions with some allocations where the non-blocking agents receive their endowments. Mamcu and Saglam (2007) prove that the core may be empty in housing markets with externalities. In a recent paper, Hong and Park (2017) consider a market model with consumption externalities and analyze two solution concepts based on the core. In particular, they show that the allocation derived from the top trading cycles algorithm is stable and belongs to both these two solution concepts; moreover, under a further preference restriction, it is the unique stable allocation in either of these two cores.

Our contribution to the literature on housing markets with externalities is twofold. In the first part of the paper, we present a set of negative results concerning several cooperative solutions for a housing market with externalites, such as the core, the set of stable allocations and the stable sets. More precisely, we show that: (a) the core may often be empty not just in a general framework as it has been already proved by Mumcu and Saglam (2007), but even when the class of preferences is restricted to special settings; (b) the set of stable allocations can also be empty[2], in contrast with what has been proved for the conventional model with selfish preferences by Roth and Postlewaite (1977); (c) housing markets may not admit stable sets for some dominance relation among allocations. These negative results lead into the second, constructive part of the paper where we consider special classes of other–regarding preferences which guarantee the existence of cooperative solutions different from the core. In particular, we focus on the notion of stable allocation and on two special classes of other-regarding preferences and obtain positive results for the set formed by such allocations. The first class is formed by other–regarding preferences that are compatible with a linear order over the indivisible goods; that is, each trader initially has a primitive linear order over the indivisible goods and ranks allocations according to this order [3]. As an example, if trader i prefers item 1 to item 2, then all the allocations that assign item 1 to him will be ranked before those ones that give him the second item. We show that this restriction on the class of other–regarding preferences is not enough to guarantee the nonemptiness of the core; on the contrary, we prove that in this setting the set of stable allocations is nonempty and forms a stable set à la von Neumann and Morgenstern. The latter two results rest on the corresponding ones provided by Roth and Postlewaite (1977) and Kawasaki (2015), respectively, for a standard housing market with selfish preferences. The second class that we consider is formed by other–regarding preferences that meet some form of coalitional altruism. We

[2] We remark that the set of stable allocations can be interpreted as the core of the housing market based on a different blocking mechanism, where the status–quo allocation in place of the initial endowment is considered.

[3] This class is the same considered in Hong and Park (2017) where it is called *egocentric*.

provide an example to show that this class is not included in the previous one. Then, we prove that, also for this class of preferences, the set of stable allocations is nonempty and forms a stable set. The technique to get the latter result is adapted from Kawasaki (2015).

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 contains some basic facts along with negative results about the existence of several cooperative solutions (core allocations, stable allocations, stable sets). Finally, in Section 4 we introduce two particular classes of preferences and prove that in both cases the set of stable allocations is nonempty and forms a stable set à la von Neumann and Morgenstern.

2 Model

We consider the Shapley-Scarf housing market (1974) where, however, traders care about what houses are allocated to the other individuals in the society. The model is the following.

There is a set $N = \{1, \ldots, n\}$ of n traders and a set $H = \{h_1, \ldots, h_n\}$ of n indivisible objects, say houses, that may be called also goods or items. Each trader $i \in N$ is initially endowed with one indivisible good e_i; let $e = (e_i)_{i \in N}$ denote the initial endowment distribution [4].

An allocation is an assignment of items to agents, that is, a bijection map $a : N \to H$. By slightly abusing notation, a_i will be used in place of $a(i)$ to denote the indivisible good allocated to trader i by allocation a. Moreover, an allocation a will be often denoted as a vector $a = (a_1, \ldots, a_n)$ where $a_i \in H$ is the item allocated to trader i. $\mathcal{A}$ will denote the set of all the allocations in the housing market.

A nonempty subset S of N is called a coalition. Whenever necessary, the notation $S \subset N$ will be used, in place of $S \subseteq N$, to state that S is strictly included in N. For any coalition $S \subseteq N$ and any allocation $a \in \mathcal{A}$, let $a(S)$ be the set of houses assigned to the members of S; that is, $a(S) = \{a_i \in H : i \in S\}$.

Given a coalition S, the notation $a = (a^S, b^{N \setminus S})$ will be used for the vector whose components are equal to a_i if $i \in S$ and b_i if $i \in N \setminus S$.

Preferences represent the departing point from the standard model by Shapley and Scarf; indeed, we assume that each agent $i \in N$ has a linear order over the set $\mathcal{A}$ of allocations, denoted by $\rhd_i$ [5]; that is, he is able to rank all allocations from the best to the worst with

[4] We remark that models with no initial endowment can be also analyzed within the same framework we consider in this paper with slightly modifications; for instance, in settings related to inheritance and/or divorce settlements, there is not an initial ownership of the items to be allocated.

[5] The formal definitions concerning binary relations are collected in the Appendix.

no ties allowed. Given $a, b \in \mathcal{A}$, the notation $a \rhd_i b$ means that trader i prefers allocation a to allocation b.

Hence, a housing market with an externality is a collection $\mathcal{H} = (N, H, e, (\rhd_i)_{i \in N})$.

An allocation $a = (a_1, \ldots, a_n) \in \mathcal{A}$ is individually rational if one of the two conditions are true: $a = e$ or, for all $i \in N$ $a \rhd_i e$.

We introduce now the dominance relations that will be used throughout the paper and represent the basis for both the cooperative solution notions of core and stable sets.

Generally speaking, given two allocations a and b, we say that the alternative allocation a dominates the status quo b if there exists a coalition $S \subseteq N$ such that a is a redistribution of the status–quo over S and it ensures a better outcome to each one of its members. It is clear that the externality in consumption much influences the last point and leads to several distinct definitions. In fact, since preferences depend on the items of all traders, a potentially blocking coalition S has to take into account the reaction that the complementary coalition $N \setminus S$ might oppose to a deviation a from the status quo b. Here, two settings are considered, both characterized by an absence of reaction by the traders in $N \setminus S$. In the first case, traders in $N \setminus S$ just stick to their initial endowment $e_{N \setminus S}$; in the second one, they stick to the status–quo allocation $b_{N \setminus S}$.

The two dominance relations, denoted by α_1 and α_2 respectively, are formally defined as follows.

Definition 2.1 (α_1-Dominance) Let a and b be two allocations of the market $\mathcal{H}$. We say that a α_1–dominates b, denoted by $a \succ_{\alpha_1} b$, if there exists a non–empty coalition $S \subseteq N$ such that:

a. $a(S) = e(S)$;

b. $(a_S, e_{N \setminus S}) \rhd_i b\,,$ for all $i \in S\,.$

Definition 2.2 (α_2-Dominance) Let a and b be two allocations of the market $\mathcal{H}$. We say that a α_2–dominates b, denoted by $a \succ_{\alpha_2} b$, if there exists a non–empty coalition $S \subseteq N$ such that:

a. $a(S) = b(S)$;

b. $(a_S, b_{N \setminus S}) \rhd_i b\,,$ for all $i \in S\,.$

The dominance relations between allocations can be equivalently expressed by using the notion of blocking mechanism: we will say that coalition S blocks allocation b through a if a dominates b, that is, if its members are able to redistribute the commodities they own (the initial endowment in the α_1 dominance relation or the goods allocated by the allocation b in the α_2 dominance relation) among themselves so that they are all better off.

Based on these dominance relations and the associated blocking mechanisms, the notions of Pareto optimality, core and stable allocations can be defined.

Definition 2.3 (Pareto optimal allocation) An **allocation** $a \in \mathcal{A}$ is said to be Pareto optimal (or, efficient) for the market $\mathcal{H}$ if it cannot be blocked by the grand coalition N through another allocation b. That is, there does not exist another allocation b such that:

$$b \rhd_{i \in N} a, \quad \text{for every } i \in N\,.$$

Definition 2.4 (Core allocation) An allocation $a \in \mathcal{A}$ is said to be a **core allocation** for the market $\mathcal{H}$ if it cannot be α_1–blocked by any coalition.

The set of the core allocations for the housing market $\mathcal{H}$ will be denoted by $C(\mathcal{H})$.

Definition 2.5 (Stable allocation) An allocation $a \in \mathcal{A}$ is stable à la Roth and Postlewaite (stable allocation, henceforth) if it cannot be α_2–blocked by any coalition.

The set of all stable allocations for the housing market $\mathcal{H}$ will be denoted by $\mathcal{S}(\mathcal{H})$.

Hence, an allocation $a \in \mathcal{A}$ is stable if it belongs to the core of the housing market $\bar{\mathcal{H}} = (N, H, a, (\rhd_i)_{i \in N})$ where the initial endowment is given by a.

Two remarks are worth noting. First, given an allocation $a \in \mathcal{A}$, singletons cannot block any allocation in the housing market $\bar{\mathcal{H}} = (N, H, a, (\rhd_i)_{i \in N})$; that is, in contrast with the core blocking mechanism where even singletons have the power to block an allocation, in the stable allocation notion only coalitions formed by more than one trader can be considered. Second, the notion of stable allocation is, as a matter of fact, independent of the initial endowment e.

We end this section by introducing the notion of stable set.

Let $\succ$ denote a dominance relation among allocations, that is, a binary relation on the set $\mathcal{A}$.

A nonempty set V of allocations is said to be:

- internally stable with respect to $\succ$ if the following condition holds:

$$\text{if } a \in V, \text{ then there is no } b \in V \text{ such that } b \succ a;$$

- externally stable with respect to $\succ$ if the following condition holds:

$$\text{if } a \in \mathcal{A} \setminus V, \text{ then there is } b \in V \text{ such that } b \succ a;$$

- a (Von Neumann–Morgenstern) stable set with respect to $\succ$ if it is both internally and externally stable.

It is clear by the previous definition that different stable sets can be conceived according to what dominance relation is considered on the set of allocations.

3 Motivating examples and preparatory results

We list now some positive facts [P], whose proofs are trivial consequence of the previous definitions, and some negative facts [N], illustrated through examples. All of them contribute to make the cooperative solutions framework clearer; the negative ones, in particular, have to be interpreted as the main motivation for the next section. Some of them are even more relevant when they are contrasted with the conventional model populated by self-interested traders. Connections and comparisons between the two models are concisely outlined, separately for each fact, whenever important.

Fact 1. [**P**] Every core allocation in the housing market $\mathcal{H}$ is efficient.
Fact 2. [**P**] Every stable allocation in the housing market $\mathcal{H}$ is efficient.
Fact 3. [**N**] The core of a housing market with an externality in consumption may be empty.
An example that shows this fact has been provided by Mumcu and Saglam (2007) and is the following.
Consider a housing market with three agents, $N = \{1, 2, 3\}$, and three goods, $H = \{h_1, h_2, h_3\}$. The possible allocations for this market are the following:

$$a_1 = (h_1, h_2, h_3) \quad a_2 = (h_1, h_3, h_2) \quad a_3 = (h_2, h_1, h_3)$$

$$a_4 = (h_2, h_3, h_1) \quad a_5 = (h_3, h_1, h_2) \quad a_6 = (h_3, h_2, h_1)$$

Suppose that $e = a_1$. The preference relations for each trader are displayed below:

Agent 1:	$a_6 \rhd_1 a_3 \rhd_1 a_2 \rhd_1 a_1 \rhd_1 a_4 \rhd_1 a_5$
Agent 2:	$a_3 \rhd_2 a_2 \rhd_2 a_6 \rhd_2 a_1 \rhd_2 a_4 \rhd_2 a_5$
Agent 3:	$a_2 \rhd_3 a_1 \rhd_3 a_6 \rhd_3 a_3 \rhd_3 a_4 \rhd_3 a_5$

It is easy to show that the core of this housing market is empty (see, Mamcu and Saglam, 2007).
This negative result, that has to be considered jointly with a companion result for a special class of preference relations (see Example 2, Section 4), becomes fruitful when contrasted

with the self-interested preference case where the core has been proved to be nonempty for strict preferences. In fact, it pushes to consider alternative cooperative solution concepts for models with indivisible goods and externalities. One possible candidate is the set of stable allocations for which we can firstly state the following two facts.

Fact 4. [**N**] Not all core allocations of a housing market with an externality in consumption are stable.

For the standard model of Shapley and Scarf, an example concerning this point has been provided by Roth and Postlewaite (1977) that also state: "It is also worth noting that the existence of unstable allocations in the core is a phenomenon that results directly from the indivisibility of goods in the market. In a market with divisible goods (and with continuous and insatiable preferences), every allocation (commodity bundle) in the core is stable."

For a model with an externality in consumption, the following example can be considered. The market is the same described in Fact 3 except that preferences are now as follows:

Agent 1:	$a_5 \rhd_1 a_6 \rhd_1 a_3 \rhd_1 a_4 \rhd_1 a_1 \rhd_1 a_2$
Agent 2:	$a_3 \rhd_2 a_5 \rhd_2 a_6 \rhd_2 a_1 \rhd_2 a_2 \rhd_2 a_4$
Agent 3:	$a_2 \rhd_3 a_5 \rhd_3 a_1 \rhd_3 a_3 \rhd_3 a_4 \rhd_3 a_6$

The core is formed by the allocations a_2, a_3 and a_5. That is, $C(\mathcal{H}) = \{a_2, a_3, a_5\}$. However, neither a_2 nor a_3 are stable. Indeed, coalition $S_1 = \{1, 2\}$ can α_2–block a_2 through the redistribution (h_3, h_1) as well as $S_2 = \{1, 3\}$ can α_2–block a_3 through the redistribution (h_3, h_2). That is, the following relations hold true:

$$a_5 \succ_{\alpha_2}^{S_1} a_2$$
$$a_5 \succ_{\alpha_2}^{S_2} a_3$$

In this example, $S(\mathcal{H}) = \{a_5\}$.

Fact 4 represents the main reason why stable allocations have been considered a solution concept in its own right as well as a proper refinement to the core. For traditional preferences, relevant positive results have been provided for such solution notion: firstly, the set of stable allocations is nonempty when no trader is indifferent between any indivisible goods (Roth and Postlewaite, 1977); moreover, when a suitable dominance relation among allocations is considered, stable allocations form a stable set à la von Neumann–Morgenstern (Kawasaki, 2015). Unfortunately, the former property does not hold in general for our model, as shown by the next fact.

Fact 5. [**N**] The set of stable allocations in a housing market with an externality in

consumption may be empty.
An example is provided by the same housing market considered in the previous two facts where preferences are modified again, as shown in the next table.

Agent 1:	$a_5 \rhd_1 a_3 \rhd_1 a_2 \rhd_1 a_1 \rhd_1 a_4 \rhd_1 a_6$
Agent 2:	$a_3 \rhd_2 a_1 \rhd_2 a_6 \rhd_2 a_2 \rhd_2 a_4 \rhd_2 a_5$
Agent 3:	$a_1 \rhd_3 a_2 \rhd_3 a_6 \rhd_3 a_5 \rhd_3 a_4 \rhd_3 a_3$

The following blocking mechanisms are easy to check: the grand coalition $N = \{1, 2, 3\}$ blocks both a_6 and a_4 through a_1; coalition $S = \{2, 3\}$ α_2–blocks a_5 and a_2 through a_6 and a_1, respectively; coalition $S = \{1, 3\}$ α_2–blocks a_3 through a_5; coalition $S = \{1, 2\}$ α_2–blocks a_1 through a_3. Hence, for this housing market stable allocations do not exist.
The next fact is even more serious; it states that stable sets may not exist when the dominance relation $\succ_{\alpha_1}$ is taken into account. The same problem holds for a house barter market with no externalities, as shown by the Example 4 in Wako and Muto (2012).
Fact 6. [**N**] An housing market with an externality in consumption may not admit stable sets when the dominance relation $\succ_{\alpha_1}$ is considered.
Indeed, consider a housing market $\mathcal{H}$ with three individuals and three items to be allocated among them. The allocations are the same as listed in Fact 3, that is:

$$a_1 = (h_2, h_3, h_1) \quad a_2 = (h_2, h_1, h_3) \quad a_3 = (h_1, h_3, h_2)$$

$$a_4 = (h_3, h_2, h_1) \quad a_5 = (h_3, h_1, h_2) \quad a_6 = (h_1, h_2, h_3)$$

Suppose that the initial endowment is a_6 and that each individual has the following strict preference relation over the allocations:

Agent 1:	$a_2 \rhd_1 a_1 \rhd_1 a_4 \rhd_1 a_5 \rhd_1 a_3 \rhd_1 a_6$
Agent 2:	$a_3 \rhd_2 a_1 \rhd_2 a_2 \rhd_2 a_5 \rhd_2 a_4 \rhd_2 a_6$
Agent 3:	$a_4 \rhd_3 a_1 \rhd_3 a_5 \rhd_3 a_3 \rhd_3 a_6 \rhd_3 a_2$

The allocations a_5 and a_6 are dominated by a_1 through the grand coalition $N = \{1, 2, 3\}$. Moreover, it is easy to check that:

$$a_1 \not\succ_{\alpha_1} a_k, \quad k = 2, 3, 4\,. \tag{1}$$

Suppose that K is a stable set for the housing market $\mathcal{H}$.
Suppose that $a_1 \notin K$. Then, by the external stability of the stable set, there exists $a \in K$ such that $a \succ_{\alpha_1} a_1$ which is impossible. Hence, $a_1 \in K$ and, as a consequence of the internal stability, $a_5, a_6 \notin K$.

Moreover, $K \cap \{a_2, a_3, a_4\} \neq \emptyset$ because, if $K \cap \{a_2, a_3, a_4\} = \emptyset$, then $a_2 \notin K$ and $a_1 \succ_{\alpha_1} a_2$. It also holds that:

$$\begin{aligned} a_2 &\succ_{\alpha_1}^{\{1,2\}} a_4; \\ a_4 &\succ_{\alpha_1}^{\{1,3\}} a_3; \\ a_3 &\succ_{\alpha_1}^{\{2,3\}} a_2. \end{aligned} \tag{2}$$

Hence, by the internal stability of K, it cannot be the case that $\{a_i, a_j\} \subset K$ for $i, j = 1, 2, 3$ and $i \neq j$. That is, there exists exactly one allocation $a_k \in \{a_2, a_3, a_4\}$ such that $a_k \in K$.
Consider now the two allocations in $\{a_2, a_3, a_4\}$ different from a_k. Since they do not belong to K, by the external stability of K, they have to be dominated by an allocation in K. But this cannot be the case as a consequence of (1) and (2).
Hence, K cannot be externally stable. We can thus conclude that the housing market $\mathcal{H}$ considered in this example does not admit stable sets with respect to the dominance relation α_1.

The following conclusions can be drawn from the previous facts. On one hand, Fact 3 and Fact 5, in particular, are the main rationale for developing and investigating particular classes of other–regarding preferences that allow to recover positive results for cooperative solution concepts, both in terms of existence and stability. On the other hand, Fact 6 pushes to test whether the dominance relation $\succ_{\alpha_2}$ is more suitable than $\succ_{\alpha_1}$ to provide such positive results.
The purpose of the next section is to answer the following question: can we find suitable profiles of preferences such that stable allocations exist and form a stable set à la von Neumann and Morgenstern?

4 New classes of other–regarding preferences

In this Section we consider two particular classes of other–regarding preferences. In the first one, we assume that traders have a primitive linear order over the indivisible goods traded in the market and we consider other–regarding preferences that are "compatible" with this order; that is, each trader ranks allocations according to this order over commodities. For instance, if trader i prefers commodity h_1 to commodity h_2, then all the allocations which assign him h_1 have to precede in the linear order (or, are strictly preferred to) the allocations that give him the item h_2. In the second case, on the contrary, we consider linear orders on the set $\mathcal{A}$ of allocations that meet some form of "coalitional altruism".

4.1 Other–regarding linear orders compatible with selfish ones

We suppose that there is another element among the primitives of the model described in Section 2: each trader $i \in N$ is equipped with a linear order $>_i$ over the set of houses H that represents his selfish strict preference over indivisible commodities. The relation between the selfish linear order and the other–regarding linear order is expressed in the next definition.

Definition 4.1 *For the trader $i \in N$, the linear order $\rhd_i$ over the set $\mathcal{A}$ of allocations is compatible with the linear order $>_i$ over the set H of commodities if for each pair of allocations $a = (a_i, a_{N\setminus\{i\}})$ and $b = (b_i, b_{N\setminus\{i\}})$ that assign to trader i distinct goods, that is $a_i \neq b_i$, it holds that:*

$$a \rhd_i b \Leftrightarrow a_i >_i b_i \ .$$

The preference profile $(\rhd_i)_{i\in N}$ is said to be compatible with the preference relations $(>_i)_{i\in N}$ if for all $i \in N$ the linear order $\rhd_i$ is compatible with the preference $>_i$.

Let us consider the following example.

Example 1. [Preference profiles $(\rhd_i)_{i\in N}$ that are compatible and not compatible with preference relations $(>_i)_{i\in N}$ over houses]

Consider a housing market with three individuals and three items to be allocated among them. The allocations are the following:

$$a_1 = (h_1, h_2, h_3) \quad a_2 = (h_1, h_3, h_2) \quad a_3 = (h_2, h_1, h_3)$$

$$a_4 = (h_2, h_3, h_1) \quad a_5 = (h_3, h_1, h_2) \quad a_6 = (h_3, h_2, h_1)$$

Consider the following linear orders over allocations:

Agent 1:	$a_1 \rhd_1 a_2 \rhd_1 a_3 \rhd_1 a_4 \rhd_1 a_5 \rhd_1 a_6$
Agent 2:	$a_1 \rhd_2 a_6 \rhd_2 a_3 \rhd_2 a_5 \rhd_2 a_2 \rhd_2 a_4$
Agent 3:	$a_4 \rhd_3 a_6 \rhd_3 a_2 \rhd_3 a_5 \rhd_3 a_1 \rhd_3 a_3$

They are compatible with the following linear orders over the three items:

Agent 1:	$h_1 >_1 h_2 >_1 h_3$
Agent 2:	$h_2 >_2 h_1 >_2 h_3$
Agent 3:	$h_1 >_3 h_2 >_3 h_3$

On the contrary, the preference profile $(\rhd_i)_{i\in N}$ represented below is not compatible with any selfish preference profile over the set of houses H:

Agent 1:	$a_3 \rhd_1 a_6 \rhd_1 a_1 \rhd_1 a_4 \rhd_1 a_2 \rhd_1 a_6$
Agent 2:	$a_6 \rhd_2 a_5 \rhd_2 a_4 \rhd_2 a_3 \rhd_2 a_2 \rhd_2 a_1$
Agent 3:	$a_1 \rhd_3 a_6 \rhd_3 a_2 \rhd_3 a_5 \rhd_3 a_4 \rhd_3 a_3$

Denote now by $\mathcal{H}^{self}$ a house market which is identical to the one defined in Section 2 apart that each trader has a selfish preferences $>_i$ over the indivisible goods, as it is in conventional housing market models. As regards the core notion, it is easy to check that the following inclusion holds when the linear orders $(\rhd_i)_{i\in N}$ over allocations are compatible with $(>_i)_{i\in N}$:

$$C(\mathcal{H}) \subseteq C(\mathcal{H}^{self})\,.$$

However, as shown by the next example, the problem of the nonexistence of core allocations is not solved either by considering other-regarding preferences which are compatible with selfish linear orders over commodities.

Example 2. [Empty core for linear orders over allocations that are compatible with a preference relations over houses]

Let us consider a housing market with three agents and three indivisible objects. The initial endowment is given by $e = (h_3, h_2, h_1)$ while the possible allocations for this market are the same as described in Section 2, that is:

$$a_1 = (h_1, h_2, h_3) \quad a_2 = (h_1, h_3, h_2) \quad a_3 = (h_2, h_1, h_3)$$

$$a_4 = (h_2, h_3, h_1) \quad a_5 = (h_3, h_1, h_2) \quad a_6 = (h_3, h_2, h_1)$$

Suppose that traders have other–regarding preferences given as follows:

Agent 1:	$a_1 \rhd_1 a_2 \rhd_1 a_4 \rhd_1 a_3 \rhd_1 a_6 \rhd_1 a_5$
Agent 2:	$a_6 \rhd_2 a_1 \rhd_2 a_4 \rhd_2 a_2 \rhd_2 a_5 \rhd_2 a_3$
Agent 3:	$a_1 \rhd_3 a_3 \rhd_3 a_5 \rhd_3 a_2 \rhd_3 a_4 \rhd_3 a_6$

The preference profile $(\rhd_i)_{i\in N}$ is compatible with the linear orders $(>_i)_{i\in N}$ over commodities shown in the next table:

Agent 1:	$h_1 >_1 h_2 >_1 h_3$
Agent 2:	$h_2 >_2 h_3 >_2 h_1$
Agent 3:	$h_3 >_3 h_2 >_3 h_1$

The following facts are easy to check. Coalition $S = \{2\}$ α_1–blocks the allocation a_1 through $a_6 = e$. The grand coalition N α_1–blocks: a_2 through a_1; a_3 through a_1; a_4 through a_1; a_5 through a_1. Finally, coalition $S = \{1, 3\}$ α_1–blocks the allocation a_6 through a_1 (and, also, a_5 through a_1). Hence, we can conclude that the core of the housing market described in this example is empty.

The fact that the core can frequently be empty when externalities are considered, even in very simple settings as shown by the previous example, calls for alternative cooperative solution concepts. The rest of the present section tries to answer this call by considering the set of stable allocations as a possible suitable candidate. To corroborate such a choice, we will prove that stable allocations not only exist but form a stable set à la von Neumann and Morgenstern when the dominance relation $\succ_{\alpha_2}$ is considered. About the dominance relation to be considered, we remark that stable sets with respect to the dominance relation $\succ_{\alpha_1}$ may not exist, even when the other–regarding preferences are compatible with a linear order over houses. Indeed, this is clearly shown by the example contained in Fact 6 where the traders' preference orders are compatible with the following orders over the set of houses:

Agent 1:	$h_2 >_1 h_3 >_1 h_1$
Agent 2:	$h_3 >_2 h_1 >_2 h_2$
Agent 3:	$h_1 >_3 h_2 >_3 h_3$

We are now interested in comparing the sets of stable allocations in the two housing markets $\mathcal{H}$ and $\mathcal{H}^{self}$ in the case when, for each trader $i \in N$, his linear order $\rhd_i$ over allocations is compatible with a selfish strict preference relation $>_i$ over the indivisible goods H.

Next result first shows that, whenever other–regarding preferences $(\rhd_i)_{i\in N}$ are considered that are compatible with a system of selfish linear orders $(>_i)_{i\in N}$, there is no distinction between the stable allocations for the two housing markets, $\mathcal{H}$ and $\mathcal{H}^{self}$.

Proposition 4.1 *Let the preference profile $(\rhd_i)_{i\in N}$ be compatible with the preference relations $(>_i)_{i\in N}$. Then, the following equality holds true:*

$$\mathcal{S}(\mathcal{H}^{self}) = \mathcal{S}(\mathcal{H}).$$

Proof. Let $a \in \mathcal{S}(\mathcal{H}^{self})$. By way of contradiction, suppose that $a \notin \mathcal{S}(\mathcal{H})$.
Then, there exist a coalition $S \subseteq N$ and a redistribution b over S such that:

a. $b(S) = a(S)$;

b. $(b_S, a_{N\setminus S}) \rhd_i a$, for all $i \in S$.

Let S' be the sub-coalition of S formed by all traders that receive the same good either in a and b. That is,

$$S' = \{i \in S : a_i = b_i\}.$$

Notice that S' and S cannot be equal due to condition b.; moreover, $S \setminus S'$ cannot be a singleton, while S' can be empty.
Since $\rhd_i$ is compatible with $>_i$ for each $i \in S'$, by condition b. we get that:

$$b_i >_i a_i, \text{ for all } i \in S \setminus S' .$$

Moreover,

$$b(S \setminus S') = b(S) \setminus b(S') = a(S) \setminus a(S') = a(S \setminus S') .$$

Hence, we can conclude that $a \notin \mathcal{S}(\mathcal{H}^{self})$ because it is α_2–blocked by the coalition $S \setminus S'$ through b.
To prove the converse inclusion, let now $a \in \mathcal{S}(\mathcal{H})$. By way of contradiction, suppose that $a \notin \mathcal{S}(\mathcal{H}^{self})$.
Then, there exist a coalition $S \subseteq N$ and a redistribution b over S such that:

a. $b(S) = a(S)$;

b. $b_i >_i a_i$, for all $i \in S$.

By condition b., along with the assumption that $\rhd_i$ is compatible with $>_i$ for all $i \in N$, it follows that:

$$(b_S, a_{N \setminus S}) \rhd_i a, \text{ for all } i \in S .$$

Hence, $a \notin \mathcal{S}(\mathcal{H})$ and the proof ends. □

As a consequence of the previous result, next corollary can be stated whose proof is just based on the corresponding result for the housing market $\mathcal{H}^{self}$ provided by Roth and Postlewaite (1977) and is omitted.

Corollary 4.1 *If the preference profile $(\rhd_i)_{i \in N}$ is compatible with the preference relations $(>_i)_{i \in N}$, then $\mathcal{S}(\mathcal{H}) \neq \emptyset$.*

As said before, another important feature characterizes the set of all stable allocations; indeed, it can be easily shown that it forms a stable set à la von Neumann and Morgenstern with respect to the dominance relation α_2.
The proof grounds on the corresponding result provided by Kawasaki (2015) for the standard housing market model.

Proposition 4.2 *If the preference profile $(\rhd_i)_{i \in N}$ is compatible with the preference relations $(>_i)_{i \in N}$, then $\mathcal{S}(\mathcal{H})$ is a stable set with respect to the dominance relation α_2.*

Proof. For the internal stability, suppose that there exist two allocations a and b such that a dominates b in $\mathcal{S}(\mathcal{H})$. Since $a, b \in \mathcal{S}(\mathcal{H}^{self})$, by the same line of reasoning as in Proposition 4.1, we can conclude that $S \setminus S'$ blocks b through a in $\mathcal{S}(\mathcal{H}^{self})$. That is, $b \notin \mathcal{S}(\mathcal{H}^{self})$, which is equivalent to $b \notin \mathcal{S}(\mathcal{H})$.
As to the external stability, let $a \notin \mathcal{S}(\mathcal{H})$. As a consequence of Proposition 4.1, this is equivalent to $a \notin \mathcal{S}(\mathcal{H}^{self})$. By the external stability of $\mathcal{S}(\mathcal{H}^{self})$, it follows that there exists an allocation $b \in \mathcal{S}(\mathcal{H}^{self})$ that dominates a through a coalition S in the housing market $\mathcal{H}^{self}$. But b also belongs to $\mathcal{S}(\mathcal{H})$. Moreover, since $\rhd_i$ is compatible with $>_i$ for all $i \in N$, it easily follows that b dominates a also in the market $\mathcal{H}$ and this concludes the proof. □

4.2 Other–regarding linear orders that meet some form of coalitional altruism

Here we consider a different class of other–regarding preferences that meet some form of coalitional altruism. The main rationale to introduce this second class of profiles is that the setting described in the previous section leaves few room for altruism, being each individual primarily interested in the house that is allotted to himself. In particular, we formulate two distinct assumptions and compare them both among themselves and with the anonymity assumption used, among others, by Velez (2016) in the case of indivisible commodities and by Borglin (1973) and Dufwenberg and al. (2011) for divisible goods.
By restricting the preference domain, we are able prove that the set of all stable allocations is stable à la von Neumannn and Morgenstern and, moreover, it is the unique stable set.
Assumption A1
Let $S_1 \subset N$ and $S_2 \subset N$ be two disjoint coalitions and suppose that:

$$a = (a_{S_1}, a_{S_2}, a_{N\setminus(S_1\cup S_2)}) \rhd_i b\,, \text{ for all } i \in S_1$$

and

$$a' = (a_{S_1}, a'_{S_2}, a_{N\setminus(S_1\cup S_2)}) \neq b \rhd_j a\,, \text{ for all } j \in S_2.$$

Then, $a' \rhd_i b$, for all $i \in S_1$.

The interpretation for this assumption is the following. Suppose that all traders in a given coalition S_1 rank the allocation a first than b and that an allocation where they all receive the same as in a is preferred by all traders in a disjoint coalition S_2. Then, all traders in S_1 prefer this new allocation to b. That is to say, the preference ranking of traders (in S_1) exhibits some form of altruism towards the other individuals in the market. Note that in the previous assumption, coalition S_2 cannot be a singleton. Moreover, Assumption (A1) trivially holds whenever all traders have the same linear order over the set of allocations $\mathcal{A}$.

Assumption A2

Let $S \subseteq N$ be a coalition.
If $a = (a_S, b_{N \setminus S}) \rhd_i b = (b_S, b_{N \setminus S})$ for all $i \in S$, then:

$$a = (a_S, b_{N \setminus S}) \rhd_j b = (b_S, b_{N \setminus S}) \text{ for all } j \in N \setminus S.$$

First of all, let us compare the two assumptions outlined before.

Proposition 4.3 (altruism comparison) *If the profile of preferences $(\rhd_i)_{i \in N}$ meets Assumption (A.2), then it also meets Assumption (A.1).*

Proof. Let us consider two disjoint coalitions S_1 and S_2 and suppose that:

$$a \quad = (a_{S_1}, a_{S_2}, a_{N \setminus (S_1 \cup S_2)}) \rhd_i b\,, \text{ for all } i \in S_1$$

and

$$a' \quad = (a_{S_1}, a'_{S_2}, a_{N \setminus (S_1 \cup S_2)}) \neq b \rhd_j a\,, \text{ for all } j \in S_2.$$

The last relation can be rewritten as:

$$a' = (a_{N \setminus S_2}, a'_{S_2}) \rhd_j (a_{N \setminus S_2}, a_{S_2})\,, \text{ for all } j \in S_2\,,$$

and, by Assumption (A2), it implies that:

$$a' = (a_{N \setminus S_2}, a'_{S_2}) \rhd_j (a_{N \setminus S_2}, a_{S_2})\,, \text{ for all } i \in N.$$

That is:

$$(a_{S_1}, a'_{S_2}, a_{N \setminus (S_1 \cup S_2)}) \rhd_i (a_{S_1}, a_{S_2}, a_{N \setminus (S_1 \cup S_2)})\,, \text{ for all } i \in S_1.$$

Thus, by transitivity, we conclude that:

$$(a_{S_1}, a'_{S_2}, a_{N \setminus (S_1 \cup S_2)}) \rhd_i b\,, \text{ for all } i \in S_1\,. \qquad \square$$

The converse does not hold; that is, Assumption (A1) does not imply Assumption (A2). The following example focuses on this point; more importantly, it also shows that the class of preferences that meets Assumption (A1) is not included in the class of other–regarding linear orders compatible with selfish preferences analyzed in the previous section.

Example 4. [A preference profile $(\rhd_i)_{i \in N}$ in a housing market with three items that meets Assumption (A1) but not Assumption (A2)]

Consider a housing market $\mathcal{H}$ with three traders and three items: $N = \{1, 2, 3\}$ and $H = \{h_1, h_2, h_3\}$. The possible allocations are:

$$a_1 = (h_1, h_2, h_3) \quad a_2 = (h_3, h_1, h_2) \quad a_3 = (h_2, h_3, h_1)$$

$$a_4 = (h_2, h_1, h_3) \quad a_5 = (h_3, h_2, h_1) \quad a_6 = (h_1, h_3, h_2)$$

Suppose that each trader has other–regarding preferences over the set of possible allocations expressed as follows:

Agent 1:	$a_3 \rhd_1 a_6 \rhd_1 a_1 \rhd_1 a_5 \rhd_1 a_2 \rhd_1 a_4$
Agent 2:	$a_3 \rhd_2 a_6 \rhd_2 a_1 \rhd_2 a_5 \rhd_2 a_2 \rhd_2 a_4$
Agent 3:	$a_3 \rhd_3 a_5 \rhd_3 a_6 \rhd_3 a_2 \rhd_3 a_1 \rhd_3 a_4$

Two points are worth remarking in this preference profile. First, note that the linear orders of Trader 2 and Trader 3 are compatible with selfish orders given by, respectively:

Agent 2:	$h_3 >_2 h_2 >_2 h_1$
Agent 3:	$h_1 >_3 h_2 >_3 h_3$

On the contrary, the linear order of trader 1 over allocations is not compatible with any strict preference relation over the three indivisible items. Trader 1's linear order over the set of allocations $\mathcal{A}$ can be interpreted as a totally benevolent behavior towards Trader 2: that is, Trader 1 always prefers what is better for Trader 2. Hence, the preferences profile $(\rhd_i)_{i \in N}$ over the set $\mathcal{A}$ is not compatible with any set of linear orders $(>_i)_{i \in N}$ over the set H.

The second important point is that the linear order of Trader 3 takes into some account the preference list of Trader 2; indeed, among all allocations which assign him the same house, Trader 3 always prefers the one which gives Trader 2 his preferred item. On the contrary, the order of Trader 2 is independent of how Trader 3 ranks the items. For instance, Trader 2 prefers allocation a_1 to a_5, ignoring that $h_1 >_3 h_3$.

Let us prove now that Assumption (A1) holds for the profile $(\rhd_i)_{i \in N}$. Since the coalition S_2 cannot be a singleton, we have just to check the three cases where S_1 is a singleton and S_2 is formed by the remaining two agents. As an example, consider $S_1 = \{1\}$ and $S_2 = \{2, 3\}$. Starting with a_1, the coalition S_2 can just move to a_6 which is preferred to a_1 by both traders. And, in fact, also Trader 1 prefers a_6 to a_1. As to a_2, the coalition S_2 can just move to a_5 which is preferred to a_2 by both traders. And, also in this case, Trader 1 prefers a_5 to a_2. For a_3, the coalition S_2 can just move to a_5 which, however, is not preferred to a_3 by Trader 2. For the allocation a_4, the coalition S_2 can just move to a_3 which is preferred to a_4 by both Traders 2 and 3. And, in fact, also Trader 1 prefers a_3 to a_4. For a_5, the coalition S_2 can just move to a_1 which, however, is not preferred to a_5 by Trader 3. Finally, for the allocation a_6, the coalition S_2 can just move to a_1 which, however, is not preferred to a_6 by Trader 2. We can conclude that Assumption (A1) holds for the profile of other–regarding preferences $(\rhd_i)_{i \in N}$ described in this example. On the contrary, Assumption (A2) does not hold; for instance, it holds that Trader 1 and Trader 2 both prefer allocation a_6 to allocation a_5 while Trader 3 does not.

Let us consider another example of housing market where the Assumption (A1) hold.

Example 5. [A linear order in a housing market with four commodities that meets Assumption (A1)]

Consider a housing market $\mathcal{H}$ with four traders and four items: $N = \{1, 2, 3, 4\}$ and $H = \{h_1, h_2, h_3, h_4\}$. The possible allocations are:

$$
\begin{array}{llll}
a_1 = (h_1, h_2, h_3, h_4) & a_2 = (h_4, h_1, h_2, h_3) & a_3 = (h_3, h_4, h_1, h_2) & a_4 = (h_2, h_3, h_4, h_1) \\
a_5 = (h_2, h_1, h_3, h_4) & a_6 = (h_4, h_2, h_1, h_3) & a_7 = (h_3, h_4, h_2, h_1) & a_8 = (h_1, h_3, h_4, h_2) \\
a_9 = (h_1, h_2, h_4, h_3) & a_{10} = (h_3, h_1, h_2, h_4) & a_{11} = (h_4, h_3, h_1, h_2) & a_{12} = (h_2, h_4, h_3, h_1) \\
a_{13} = (h_1, h_3, h_2, h_4) & a_{14} = (h_4, h_1, h_3, h_2) & a_{15} = (h_2, h_4, h_1, h_3) & a_{16} = (h_3, h_2, h_4, h_1) \\
a_{17} = (h_4, h_3, h_3, h_1) & a_{18} = (h_1, h_4, h_2, h_3) & a_{19} = (h_3, h_1, h_4, h_2) & a_{20} = (h_2, h_3, h_1, h_4) \\
a_{21} = (h_3, h_2, h_1, h_4) & a_{22} = (h_4, h_3, h_2, h_1) & a_{23} = (h_1, h_4, h_3, h_2) & a_{24} = (h_2, h_1, h_4, h_3)
\end{array}
$$

Suppose that traders 1, 2 and 3 have the same other–regarding preferences over the set of possible allocations that are expressed as follows:

Agent 1:	$a_1 \rhd_1 a_8 \rhd_1 a_9 \rhd_1 a_{13} \rhd_1 a_{18} \rhd_1 a_{23} \rhd_1 a_4 \rhd_1 a_5 \rhd_1 a_{12} \rhd_1 a_{15} \rhd_1 a_{20} \rhd_1 a_{24} \rhd_1$ $a_3 \rhd_1 a_7 \rhd_1 a_{10} \rhd_1 a_{16} \rhd_1 a_{19} \rhd_1 a_{21} \rhd_1 a_{12} \rhd_1 a_6 \rhd_1 a_{11} \rhd_1 a_{14} \rhd_1 a_{17} \rhd_1 a_{22}$
Agent 4:	$a_1 \rhd_4 a_{13} \rhd_4 a_5 \rhd_4 a_{20} \rhd_4 a_{10} \rhd_4 a_{21} \rhd_4 a_4 \rhd_4 a_{12} \rhd_4 a_7 \rhd_4 a_{16} \rhd_4 a_{17} \rhd_4 a_{22} \rhd_4$ $a_8 \rhd_4 a_{23} \rhd_4 a_3 \rhd_4 a_{19} \rhd_4 a_{11} \rhd_4 a_{14} \rhd_4 a_9 \rhd_4 a_{18} \rhd_4 a_{15} \rhd_4 a_{24} \rhd_4 a_2 \rhd_4 a_6$

The preference profile $(\rhd_i)_{i \in N}$ meets Assumption (A1) (the proof is omitted).

In the Appendix a different setting with four traders is presented where Assumption (A1) holds as well.

It is easy to see how the preference order of Trader 4 in the previous example can be modified so that Assumption (A1) does not hold anymore. This is illustrated in the next example.

Example 6. [A preference profile $(\rhd_i)_{i \in N}$ in a housing market with four items that does not meet Assumption (A1)]

Consider the same housing market $\mathcal{H}$ with four traders and four items as before where, however, the preference order for Trader 4 is modified as follows:

Agent 4:	$a_{10} \rhd_4 a_{21} \rhd_4 a_5 \rhd_4 a_{20} \rhd_4 a_1 \rhd_4 a_{13} \rhd_4 a_{11} \rhd_4 a_{14} \rhd_4 a_3 \rhd_4 a_{19} \rhd_4 a_8 \rhd_4 a_{23} \rhd_4$ $a_{17} \rhd_4 a_{22} \rhd_4 a_{16} \rhd_4 a_7 \rhd_4 a_4 \rhd_4 a_{12} \rhd_4 a_2 \rhd_4 a_6 \rhd_4 a_{15} \rhd_4 a_{24} \rhd_4 a_9 \rhd_4 a_{18}$

To show that Assumption (A1) does not hold, consider the following two disjoint coalitions: $S_1 = \{4\}$ and $S_2 = \{1, 2\}$. Trader 4 prefers allocation a_{10} to allocation a_5 and Trader 1 and Trader 2 both prefer a_{13} to a_{10}. However, $a_{13} \not\triangleright_4 a_5$.
Despite Assumption (A1) does not hold in this example, it is worth noting that the order rank of Trader 4 is still compatible with a preference relation over houses which is the following:

$$h_4 >_4 h_2 >_4 h_1 >_4 h_3 \,.$$

The reason why the assumption does not hold is that Trader 4' s linear order over allocations exhibits a conflicting interest towards the rest of traders. As an example, consider the block of allocations that allot to Trader 4 his most preferred house h_4, that is: $a_1, a_5, a_{10}, a_{13}, a_{21}, a_{20}$. These allocations are ranked by Trader 4 in a way which contrasts with Trader 1' preferences over houses: indeed, the allocations that assign h_3 to Trader 1 (a_{10} and a_{21}) appear first than the allocations that assign him the item h_2 (a_5 and a_{20}).
A final remark is in order. Assumption (A1) also holds when each trader has two primitive linear orders, one over items and the other one over traders, and he perfectly knows the orders of all other traders in the market. Each individual uses these orders to rank allocations. For instance if Trader 1 prefers item 1 to item 2 and Trader 2 to Trader 3, he would rank allocations in such a way that all those giving him item 1 are ranked highly than those that give him item 2; and, among all the allocations that give him item 1, the allocations that give Trader 2 his most preferred item are ranked highly than those that give Trader 3 his most preferred item. Note that the example we illustrated before is not included in this class of preferences.
We prove that if the preference profile $(\triangleright_i)_{i \in N}$ satisfies Assumption (A1), then the set of stable allocations for the housing market $\mathcal{H}$ is nonempty and is the unique stable set à la von Neumann and Morgenstern. The following lemma is needed.

Lemma 4.1 *Let the linear order profile $(\triangleright_i)_{i \in N}$ meet Assumption (A1). If $a \in \mathcal{A}$ is not stable, then there exists $b \in \mathcal{A}$ such that:*

a. *b is stable;*

b. *b α_2–dominates a.*

Proof.
The proof consists in constructing an allocation $b \in \mathcal{A}$ with the desired properties. We proceed by steps.
Step 1.
Since a is not stable, there exists a coalition $T_1 \subseteq N$ and a redistribution b_1 such that

I. $b_1(T_1) = a(T_1)$;

II. $(b_1^{T_1}, a^{N \setminus T_1}) \rhd_i a, \forall\, i \in T_1$.

Moreover, T_1 and the associated redistribution b_1 can be chosen so that T_1 is the maximal coalition that blocks a[6]. Consider the following allocation in $\mathcal{A}$:

$$b_1 = (b_1^{T_1}, a^{N \setminus T_1})\,.$$

If b_1 is stable, then the proof ends.
Suppose that b_1 is not stable; then, there exist a coalition $T_2 \subseteq N$ and a redistribution b_2 such that:

III. $b_2(T_2) = b_1(T_2)$;

IV. $(b_2^{T_2}, b_1^{N \setminus T_2}) \rhd_i b_1, \forall\, i \in T_2$.

We want to prove that the inclusion $T_2 \subseteq T_1$ has then to hold.
Suppose that $T_2 \cap T_1 = \emptyset$, that is, $T_2 \subset N \setminus T_1$.
Let us consider the following allocation in $\mathcal{A}$:

$$b_2 = \left(b_2^{T_2}, b_1^{T_1}, a^{N \setminus (T_2 \cup T_1)}\right)$$

and show that the coalition $T_2 \cup T_1$ blocks a via b_2. It holds that:

$$b_2(T_2 \cup T_1) = b_2(T_2) \cup b_2(T_1) = b_2(T_2) \cup b_1(T_1) = b_1(T_2) \cup a(T_1) = a(T_2) \cup a(T_1) = a(T_2 \cup T_1).$$

Moreover, by (II), (IV) and Assumption (A1), it follows that:

$$(b_2^{T_2 \cup T_1}, a^{N \setminus (T_2 \cup T_1)}) \rhd_i a, \forall\, i \in T_1.$$

By way of contradiction, suppose now that there exists $\underline{i} \in T_2$ such that:

$$a \rhd_{\underline{i}} (b_2^{T_2 \cup T_1}, a^{N \setminus (T_2 \cup T_1)})\,,$$

which is equivalent to:

$$(a^{T_2}, a^{T_1}, a^{N \setminus (T_2 \cup T_1)}) \rhd_{\underline{i}} (b_2^{T_2}, b_1^{T_1}, a^{N \setminus (T_2 \cup T_1)})\,.$$

By the previous relation, along with Assumption (A1) and (II), it follows that:

$$(a^{T_2}, b_1^{T_1}, a^{N \setminus (T_2 \cup T_1)}) \rhd_{\underline{i}} (b_2^{T_2}, b_1^{T_1}, a^{N \setminus (T_2 \cup T_1)})\,,$$

[6] Notice that, by the definition of blocking, T_1 can be equal to the grand coalition N but it cannot be equal to singletons.

that is,

$$b_1 \rhd_{\underline{i}} (b_2^{T_2}, b_1^{N\setminus T_2}),$$

which contradicts (IV). Hence, we can conclude that $T_1 \cup T_2$ blocks the allocation a through b_2. Since this contradicts the maximality of T_1, it follows that:

$$T_2 \cap T_1 \neq \emptyset$$

Suppose now that $T_2 \cap T_1 \neq \emptyset$ and $T_2 \cap (N \setminus T_1) \neq \emptyset$.
Consider the allocation $c \in \mathcal{A}$ defined as follows:

$$c = \left(b_1^{T_1\setminus T_2}, b_2^{T_2}, a^{N\setminus(T_2\cup T_1)} \right)$$

Let us show that $T_2 \cup T_1$ blocks the allocation a via c. In fact, it is easily shown that c is a redistribution of a over $T_2 \cup T_1$:

$$\begin{aligned} c(T_2 \cup T_1) &= c(T_1 \setminus T_2) \cup c(T_2) = b_1(T_1 \setminus T_2) \cup b_2(T_2) = b_1(T_1 \setminus T_2) \cup b_1(T_2) = \\ &= b_1(T_1 \cup T_2) = b_1(T_1) \cup b_1(T_2 \setminus T_1) = a(T_1) \cup a(T_2 \setminus T_1) = a(T_2 \cup T_1). \end{aligned}$$

Moreover, (II) and (IV) can be rewritten, respectively, as:

$$\left(b_1^{T_1\setminus T_2}, b_1^{T_1\cap T_2}, a^{T_2\setminus T_1}, a^{N\setminus(T_2\cup T_1)} \right) \rhd_i a, \forall\, i \in T_1;$$
$$\left(b_2^{T_2\setminus T_1}, b_2^{T_1\cap T_2}, b_1^{T_1\setminus T_2}, a^{N\setminus(T_2\cup T_1)} \right) \rhd_j b_1, \forall\, j \in T_2.$$

By the previous two equations and by Assumption (A1), it follows that for all $i \in T_1 \setminus T_2$:

$$c = \left(b_1^{T_1\setminus T_2}, b_2^{T_2}, a^{N\setminus(T_2\cup T_1)} \right) \rhd_i a.$$

Moreover, by transitivity, for all $i \in T_1 \cap T_2$:

$$\left(b_2^{T_2\setminus T_1}, b_2^{T_1\cap T_2}, b_1^{T_1\setminus T_2}, a^{N\setminus(T_2\cup T_1)} \right) \rhd_i b_1 \rhd_i a.$$

Let now $i \in T_2 \setminus T_1$. By way of contradiction, suppose that there exists $\underline{i} \in T_2 \setminus T_1$ such that:

$$a \rhd_{\underline{i}} \left(b_1^{T_1\setminus T_2}, b_2^{T_2}, a^{N\setminus(T_2\cup T_1)} \right).$$

By Assumption (A1) and by (II), it follows that:

$$\left(a_{T\setminus T_1}, b_1^{T_2\cup T_1}, b_1^{T_1\setminus T_2}, a^{N\setminus(T_2\cup T_1)} \right) \rhd_{\underline{i}} \left(b_1^{T_1\setminus T_2}, b_2^{T_2}, a^{N\setminus(T_2\cup T_1)} \right),$$

that is:

$$b_1 \rhd_{\underline{i}} (b_2^{T_2}, b_1^{N\setminus T_2}),$$

that contradicts the (IV).
Hence, we can conclude that $T_1 \cup T_2$ blocks the allocation a through the allocation c which contradicts the maximality of T_1.
Therefore, $T_2 \subseteq T_1$.
Take the coalition T_2 such that it is maximal among all coalitions that block b_1, consider the associated redistribution b_2 and the allocation $b_2 = (b_2^{T_2}, b_1^{N\setminus T_2}) \in \mathcal{A}$. If $T_2 = T_1$, then relabel the allocation b_2 as b_1 and repeat Step 1 [7]. If $T_2 \subset T_1$, move to Step 2.
Step 2 .
Consider the allocation $b_2 = \left(b_2^{T_2}, b_1^{T_1\setminus T_2}, a^{N\setminus(T_2\cup T_1)}\right) \in \mathcal{A}$.
If b_2 is stable, then the proof ends. Indeed, it can be easily shown that the allocation b_2 α_2–blocks a via T_1. First of all, it holds that:

$$b_2(T_1) = b_2(T_2) \cup b_2(T_1 \setminus T_2) = b_1(T_2) \cup b_1(T_1 \setminus T_2) = b_1(T_1) = a(T_1)$$

Moreover, by the following two relations:

$$(b_1^{T_1}, a^{N\setminus T_1}) = (b_1^{T_2}, b_1^{T_1\setminus T_2}, a^{N\setminus T_1}) \rhd_i a, \forall\, i\, \in T_1\,,$$
$$(b_2^{T_2}, b_1^{N\setminus T_2}) = (b_2^{T_2}, b_1^{T_1\setminus T_2}, a^{N\setminus T_1}) \rhd_i b_1, \forall\, i\, \in T_2\,,$$

along with the equality $b_2(T_2) = b_1(T_2)$ and the Assumption (A1) applied to the disjoint coalitions $T_1 \setminus T_2$ and T_2 , we get that:

$$(b_2^{T_2}, b_1^{T_1\setminus T_2}, a^{N\setminus T_1}) \rhd_i a, \forall\, i\, \in T_1 \setminus T_2\,.$$

To complete the proof, take $i \in T_2$. Then, by the transitivity of the preference relation $\rhd_i$, we get:

$$(b_2^{T_1}, a^{N\setminus T_1}) = (b_2^{T_2}, b_1^{T_1\setminus T_2}, a^{N\setminus T_1}) \rhd_i b_1 = (b_1^{T_1}, a^{N\setminus T_1}) \rhd_i a\,,$$

where the last step derives from the fact that $i \in T_1$. This concludes the proof that b_2 α_2–blocks a via T_1.
If b_2 is not stable, there exists a coalition $T_3 \subseteq N$ and a redistribution b_3 such that:

V. $b_3(T_3) = b_2(T_3)$;

VI. $(b_3^{T_3}, b_2^{N\setminus T_3}) \rhd_i b_2, \forall\, i\, \in T_3$.

As in the previous step, choose T_3 maximal among all coalitions that block b_2. By following the same line of reasoning as in the previous step, it can be proved that $T_3 \subseteq T_2$.
If $T_3 = T_2$, we repeat Step 2 with $b_3 = b_2$. If $T_3 \subset T_2$, we move to the next step.

[7] Since preferences are strict and allocations are finite, Step 1 cannot be repeated an infinite number of times otherwise we would get $(a^{T_1}, a^{N\setminus T_1}) \rhd_i a$, for all $i \in T_1$, which is a contradiction.

This process must end in a finite number of steps.
Suppose that it ends at Step q. At this last step, we get the following allocation in $\mathcal{A}$:

$$b_q = (b_q^{T_q}, b_{q-1}^{T_{q-1}\setminus T_q}, \dots, b_1^{T_1\setminus T_2}, a^{N\setminus T_1}) \,.$$

We want to show that this allocation has the two properties stated in the lemma.
First, it is stable since the process ends at Step q. Let us prove that the coalition T_1 α_2–blocks a through the allocation b_q; that is, we need to show that:

VII. $b_q(T_1) = a(T_1)$;

VIII. $(b_q^{T_1}, a^{N\setminus T_1}) \rhd_i a, \forall\, i\, \in T_1$.

As to the first point, we have:

$$\begin{aligned} b_q(T_1) &= b_q(T_q) \cup b_q(T_1 \setminus T_q) = b_{q-1}(T_q) \cup b_{q-1}(T_1 \setminus T_q) = \\ &= b_{q-1}(T_1) = b_{q-1}(T_{q-1}) \cup b_{q-1}(T_1 \setminus T_{q-1}) = b_{q-2}(T_{q-1}) \cup b_{q-2}(T_1 \setminus T_{q-1}) = \\ &= \dots = \\ &= b_1(T_1) = a(T_1) \,. \end{aligned}$$

As to point (VI), it can be easily derived by the following claim.

Claim 4.1 *Let $k = 2, 3, \dots, q \in \mathbb{N}$ and consider the following allocation in $\mathcal{A}$:*

$$b_k = (b_k^{T_k}, b_{k-1}^{T_{k-1}\setminus T_k}, \dots, b_1^{T_1\setminus T_2}, a^{N\setminus T_1}) \,.$$

For any $k = 2, 3, \dots, q$, if the allocation b_{k-1} α_2–blocks a via T_1 and the allocation b_k α_2–blocks b_{k-1} via T_k, then b_k α_2–blocks a via T_1.

Proof.
We want to show that:

$$(b_k^{T_1}, a^{N\setminus T_1}) \rhd_i a, \forall\, i\, \in T_1 \,.$$

By assumption, we know that:

$$\begin{aligned} (b_{k-1}^{T_1}, a^{N\setminus T_1}) = (b_{k-1}^{T_{k-1}}, b_{k-2}^{T_{k-2}\setminus T_{k-1}}, \dots, b_1^{T_1\setminus T_2}, a^{N\setminus T_1}) = \\ = \quad (b_{k-1}^{T_k}, b_{k-1}^{T_{k-1}\setminus T_k}, \dots, b_1^{T_1\setminus T_2}, a^{N\setminus T_1}) \rhd_i a, \forall\, i\, \in T_1 \,, \end{aligned} \tag{3}$$

and

$$(b_k^{T_k}, b_{k-1}^{N\setminus T_k}) \rhd_i b_{k-1}, \forall\, i\, \in T_k \,. \tag{4}$$

Since $b_{k-1}(T_k) = b_k(T_k)$, we can apply Assumption (A1) to the disjoint sets $T_1 \setminus T_k$ and T_k and we get:

$$(b_k^{T_k}, b_{k-1}^{N\setminus T_k}) \rhd_i a, \forall\, i\, \in T_1 \setminus T_k \,.$$

Moreover, for $i \in T_k$ we get the conclusion by the transitivity of the preference relation $\rhd_i$ and by the relation (4). This completes the proof of the Claim. $\square$
By the Claim, we can conclude that the allocation b_q α_2–blocks a via the coalition T_1 and the proof of the Lemma ends. $\square$

Theorem 4.1 *If the linear order profile $(\rhd_i)_{i\in N}$ meets Assumption (A1), then the set $\mathcal{S}(\mathcal{H})$ formed by all stable allocations is the unique stable set à la von Neumann–Morgenstern with respect to the dominance relation α_2.*

Proof.
The internal stability is trivial while the external stability of $\mathcal{S}(\mathcal{H})$ is an easy consequence of Lemma 4.1.
Let us prove now that $\mathcal{S}(\mathcal{H})$ is the unique stable set for the house market $\mathcal{H}$ with respect to the dominance relation α_2.
By way of contradiction, suppose that there exists another stable set $V \subseteq \mathcal{A}$. It holds that $\mathcal{S}(\mathcal{H}) \subseteq V$. Indeed, if $a \in \mathcal{S}(\mathcal{H})$ and $a \notin V$, by the external stability of V, it follows that there is $b \in V$ such that $b \succ_{\alpha_2} a$. But then, by definition of stable allocation, $a \notin \mathcal{S}(\mathcal{H})$ and we get a contradiction.
If $\mathcal{S}(\mathcal{H}) = V$, the proof ends.
Then, suppose that there is $a \in V \setminus \mathcal{S}(\mathcal{H})$. By the external stability of $\mathcal{S}(\mathcal{H})$, we get $z \in \mathcal{S}(\mathcal{H})$ such that $z \succ_{\alpha_2} a$. We get a contradiction with the internal stability of V and the proof ends. $\square$

Remark 4.1 Velez (2016) assumes that each agents has complete, transitive, and continuous preferences on the set $\mathcal{A}$ of allocations. Thanks to the continuity assumption, he can introduce a continuous utility function u_i defined on $\mathcal{A}$ that represents trader i's preferences. Given this description, an assumption is done on the utility functions: in fact, it is assumed that externalities are anonymous, i.e., an agent's welfare is not affected by reshuffling the consumption bundles of the other agents. Formally, for $i \in N$ and $a \in \mathcal{A}$, if $\pi : N \to N$ is a permutation such that $\pi(i) = i$, then $u_i(a) = u_i(a_\pi)$.
Despite his model cannot be compared with that introduced in the present paper since we consider neither money nor ties in preferences, it can be stated that the framework analyzed in Velez (2016) leaves few room for altruism or envy issues.

5 Appendix

5.1 Binary relations

A binary relation $\succ$ on a set X is a subset of the cartesian product $X \times X$. If $(a, b) \in \succ$, we write equivalently $a \succ b$.
Given a binary relation $\succ$ on X and $a, b \in X$, define:

$$\begin{aligned} a \sim b \quad &\Leftrightarrow (\text{ not } a \succ b \text{ and } \text{ not } b \succ a)\,; \\ a \succeq b \quad &\Leftrightarrow (a \sim b \text{ or } a \succ b)\,. \end{aligned}$$

A binary relation $\succ$ on X is said to be: **irreflexive** if, for all $a \in X$, not $a \succ a$; **asymmetric** if, for all $a, b \in X$, $a \succ b$ implies that not $b \succ a$; **transitive** if, for all $a, b, c \in X$, $a \succ b$ and $b \succ c$ imply that $a \succ c$; **negatively transitive** if, for all $a, b, c \in X$, $a \succ b$ imply $a \succ c$ or $c \succ b$; **a weak order** if it is asymmetric and negatively transitive; **a linear order** if it is a weak order, and for all $a, b \in X$ with $a \neq b$, either $a \succ b$ or $b \succ a$. It is worth noting that asymmetry and negative transitivity together imply transitivity. As a consequence, all weak and linear orders are transitive.

5.2 A setting with four traders where Assumption (A1) holds

We show here that Assumption (A1) holds more generally for a housing market $\mathcal{H}$ with four traders and four items characterized as follows. The set of agents is divided into two symmetric blocks: the first one, denoted by G_1, is formed by Trader 1 and Trader 2 that have the same linear order over the set of allocations $\mathcal{A}$; this linear order is one compatible with a preference relation $>_1$ over the set of houses, H. The second group, denoted by G_2, is formed by the remaining two individuals, Trader 3 and Trader 4, with the same linear order on the set $\mathcal{A}$, possibly different from that in G_1 [8]. Like in the previous group, this linear order is compatible with a preference relation $>_3$ over the set of houses. Moreover, we assume that Trader 1 is "partially altruistic" towards Trader 3 and, vice versa, Trader 3 is "partially altruistic" towards Trader 1. Roughly speaking, this means that Trader 1 ranks the allocations that assign him the same item in a way which accounts for the Trader 3's preference relation over the set H. Formally speaking, given two allocations $a, b \in \mathcal{A}$, if $a(1) = b(1)$ and $a(3) >_3 b(3)$, then $a \rhd_1 b$. The same is true, *mutatis mutandis*, for Trader 3.
This setting models a framework where Trader 2 has a totally benevolent behavior towards Trader 1 (that is, he always prefers what is preferred by Trader 1) while Trader 4 is totally

[8] The case where all four traders have the same linear order over the set of allocations $\mathcal{A}$ is trivial.

benevolent towards Trader 3; moreover, there is some form of partial altruism between the two groups of individuals.

We want to show that Assumption (A1) holds in such a setting.

Consider two disjoint coalitions S_1 and S_2 and two allocations a and a' in $\mathcal{A}$ such that:

$$a \quad = (a_{S_1}, a_{S_2}, a_{N\setminus(S_1\cup S_2)}) \rhd_i b\,, \text{ for } i \in S_1 \tag{5}$$

and

$$a' \quad = (a_{S_1}, a'_{S_2}, a_{N\setminus(S_1\cup S_2)}) \neq b \rhd_j a\,, \text{ for all } j \in S_2. \tag{6}$$

We have to prove that $a' \rhd_i a\,,$ for $i \in S_1$.

As already stated in Section 4.2, S_2 cannot be a singleton. Moreover, notice that all cases when S_2 intersects both $G_1 = \{1, 2\}$ and $G_2 = \{3, 4\}$ are trivial.

Now, consider $S_2 = \{1, 2\}$ and $S_1 = \{4\}$. Trader 4, as a member of coalition S_1, gets the same item by the allocations a and a'; the same is true for Trader 3, as a member of $N \setminus (S_1 \cup S_2)$. That is,

$$\begin{cases} a(3) = a'(3) \\ a(4) = a'(4) \end{cases}$$

Hence, Trader 1 and Trader 2 have to swap their houses, otherwise $a = a'$. That is, $a(1) \neq a'(1)$.

It cannot be the case that $a(1) >_1 a'(1)$; indeed, if $a(1) >_1 a'(1)$, then $a \rhd_1 a'$, which contradicts (6).

Then, $a'(1) >_1 a(1)$; since Trader 3 is partially altruistic towards Trader 1, we get:

$$a' \rhd_3 a\,,$$

and hence:

$$a' \rhd_4 a\,,$$

that is what we had to conclude.

The case $S_2 = \{1, 2\}$ and $S_1 = \{3\}$ is equivalent to the previous one as well as the case $S_2 = \{1, 2\}$ and $S_1 = \{3, 4\}$.

All other remaining cases are symmetric.

Hence, we can conclude that Assumption (A1) holds in general for the housing market $\mathcal{H}$ with four traders and four items described at the beginning of this section.

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