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Modeling Stochastic Innovation Races

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Abstract

We consider a firm moving towards a stochastic final destination, to be chosen from a discrete set after a decision period. The decision period itself may be deterministic or stochastic. We assume the firm can move at variable innovation (R&D) speed associated with a monotone nondecreasing variable cost, and it can also stop and move anywhere. There is a fixed cost per time unit "carried" by the firm as well, associated with keeping at the knowledge (technology) frontier. We investigate various types of the firm's optimal trajectory in the R&D race during the decision period.

Keywords. Innovation, Race, Competition, Strategy, Industrial Economics

Table of contents

1. Introduction

2. Characateristics of Innovation Races

3. Stochastic Race in a Deterministic Decision Period

The Speed Race Problem

Choosing t Optimally

The Stopping Line and the Waiting Region

4. The Stochastic Decision Period Case

5. Matching an Innovation Race

6. Multi-Stage Races

7. Conclusions

References

Appendix

‘The process of innovative competition is like a race in which it is necessary to run and run well to endure and to prosper, but the race is more like a marathon than a sprint.’

Fisher, McGowan and Greenwood in Folded, Spindled and Mutilated and US vs IBM (1983)

1. Introduction

Firms’ strategic decisions may sometimes change from stage to stage. At any given stage the next stage may be known to be tentative in nature even when the firm reveals its strategy in private or in public. We may say then that the firm is actually headed towards a stochastic destination. We follow a decision-theoretic rather a game-theoretic line of reasoning , in the sense that we look at the individual firm’s options against any or all of their rivals. This relates to Kamien and Schwartz’ (1982) exploration where they showed that the intensity of rivalry between market participants leads to an increased speed of R&D which is the main characteristic of a frontier race. To motivate the problem we give a characterization of racing behaviour in Section 2. Since the firm faces extensive uncertainty in its strategic positioning, we assume the decision period is stochastic. However, to formulate the problem we first let the decision period be deterministic (in finite time) but the destination be stochastic, in Section 3. At a further step we also assume the decision period to be stochastic (Section 4). If the decision period is stochastic, the problem of identifying the optimal trajectory in terms of strategic positioning,i.e., in the plane based innovation race is a dynamic optimization problem which can be solved by dynamic programming , or other calculus of variations or numerical methods. In Section 5 and 6 we deal with special issues of racing behaviour among firms. Section 7 draws conclusions.

2. Characateristics of Innovation Races

The concept of a race is intrinsic to sports events, crossing the finishing line first is everything to a racer, the rewards may be immense by reaching for the gold. In general, if such a race evolves, the race looks like a sequential machine (finite automaton) acting under resource and time constraints, until a winner clearly emerges. A winner may establish himself at the first trials or runs, leaving very little chance for those left behind to catchup. The situation of competitive rivalry among firms or businesses in high technology industries may resemble more complex paradigms of a race that appear more difficult to describe than a sports event. First of all, the finishing line may not be sharply defined. It could be a greater market share than any of the rivals attain, it may be a higher profitability given the share, or a higher growth potential. In terms of process, it could be even a slow race at the beginning which might accelerate to whatever the finishing line constitutes of. It may be a race that is open to new entrants along the way, in a dormant, low innovation - driven industry that brings changes to this industry. It may allow moves among rivals, unheard of in conventional races, such as "leapfrogging", "take a breath and recharge" or redefining a new race through mergers and acquisitions, in the course of the given one. Races may be endogeneous, induced by changes in innovation patterns, market structures and productivity cycles. All these issues of complexity may justify to set up a racing model that captures many of the essential features. This would be a model of a stochastic race which is proposed. Let us describe the characteristics of such a race on achieving technological and market supremacy (Gottinger, 1996). A finishing line would be ahead of a present technological frontier which would be the common ground for starting the race.

Let *TF*(C) be each racing company's technological knowledge frontier while *TF*(*I*) would be the respective industry's frontier represented by the most advanced company as a benchmark. All firms engage in pushing their frontier forward which determines the extent to which movements in the individual *TF*(*C*) of the racing firms translate into movements of the *TF*(*I*). While a variety of situations may emerge, the extremal cases involve: either one firm may push the frontier at all times, with the others following closely behind or all firms share more or less equally in the task of advancing the *TF*(*I*). The first situation corresponds to the existence of a unique technogical leader for a particular race, and a number of quick followers. The other situation corresponds to the existence of multiple technological leaders. In some industries firms share the task for pushing the frontier forward more equally than in other industries. This is usually the case the more high paced and dynamic is the race in an industry. In any race of the sort "closeness" is an important but relative attribute. The races are all close by construction, however, some might be closer than others. As a closeness measure of the race at any particular time one could define $c(t) = \sum_0^N \left[TF(C_i) - TF(I)\right]^2 / N(t)$ where *N*(*t*) is the number of active firms in that industry at time t. The measure thus constructed has a lowest value of 0, which corresponds to a 'perfectly close' race. Higher values of the measure correspond to races that are less close. Unlike other characteristics such as the domination period length during a race, innovation when ahead versus when behind, leapfrogging versus frontier sticking, which describe the behaviour of a particular feature of the race and of a particular firm in relation to the frontier, the closeness measure is more of an aggregate statistic of how close the various racing parties are at a point in time. The closeness measure is simply an indication of the distance to approach a benchmark, and it does not say anything about the evolution of the technological frontier. To see this, note that if none of the frontiers were evolving, the closeness measure would be 0, as it would if all the frontiers were advancing in perfect lock-step with one another.

On the basis of theoretical works on these issues, e.g. Fudenberg, (1983), Harris and Vickers (1985,1987), Reinganum (1989) etc. there have been attempts to categorize similarities and differences among various races due to a range of behaviour rules in races though very little empirical work has been done to substantiate these claims (Lerner, 1997). The very robust feature that appears to be common to all races is that there is a pronounced tendency for a firm to innovate more when it falls behind in the race. In less dynamic industries the race seems most prone to catchup behaviour rather than frontier pushing behaviour. Further, even the catchup behaviour evidenced by firms in this race is less aggressive in that it seldom tries to leapfrog the frontier. Rather, the firms tend to exhibit more frontier sticking behaviour than the firms in high technology, fast pace industries. Overall, these facts seem to suggest that the incremental returns to a firm that occupies the race leader position seem to be lower in the first than in the second category. The first category also tends to be occupied by firms with the most unequal frontier pushing activity.

By taking the (large mainframe) computer industry as an example, an interesting point to note is that each frontier advance is embodied in a machine that is frequently the product of years of planning. Frequently, the performance target of a computer mainframe is set when the project begins, though there is some uncertainty in the time taken to achieve this target performance. This, in conjunction with the racing behaviour, implies that firms must constantly anticipate their rivals' actions when deciding on their technology strategy (Fisher, 1983). This is marvellously described in the novel by Tracy Kidder (1981). Empirical studies confirm that such anticipation does, in fact, crucially impact the targeting decision. This validates the emphasis of strategic interaction placed by the 'racing behaviour' perspective.

3. Stochastic Race in a Deterministic Decision Period

The Problem: On an Euclidean plane let *N* be a set of *n* points (x_i, y_i); *i* = 1,... *n* ; let *n* probabilities $p_i; i = 1, \dots, n$ be given such that $\sum p_i = 1$. We use the Euclidean distance on a plane because innovation characteristics are at least two-dimensional, that is, it would apply to so-called system products that consist of at least two components. The probabilities will most likely be subjective probabilities determined by the individual firm's chances to position itself , endogeneously determined by its distance to the finishing line or its proximity to the next rival in the race. They may be formed by considering the firm's own position in the race as well as depending on the stochasticity of the rivals' efforts. As a first approximation we may let the firm's R&D investment x_i, in relation to the total investment of its rivals $\sum x_j$, determine the probability $p_i = \frac{x_i}{\sum x_j}$. Let a starting point, point (x_0, y_0) or (point 0) also be given; let *f*(*S*); *S* ≥ 0 be a function such that

(1) $$f(0) = 0,$$

(2) $$f(S) > 0\,;\forall S > 0,$$

(3) $$f(S + \varepsilon) \geq f(S)\,;\forall S, \varepsilon > 0,$$

and such that except for *S* = 0, *f*(*S*) is (not necessarily strictly) convex and represents the cost of racing at speed *S*; let *F* > 0 be given (the fixed time value); and finally let *T* > 0 be given (the decision period). It is required to minimize the following function by choosing $t \equiv (x_t, y_t)$ and *S* (i.e., choose a point t, to be at *T* time units from now, and a speed *S* with which to proceed afterwards, so that the expected total cost to cross the 'finishing line' will be minimized:

(4) $$Z(t, S) = FT + f\left(\frac{d(0,t)}{T}\right) d(0,t) + \left(f(S) + \frac{F}{S}\right) \sum_{i=1}^{n} p_i d(t,i)$$

where *d*(*i*, *j*) is the Euclidean distance between points *i* and *j*. We denote the optimal *S* by *S**, and similarly we have *t** and *Z** = *Z*(*t**, *S**). Note that *FT* is a constant, so we can actually neglect it; the second term is the cost of getting to *t* during *T* time units, i.e., at a speed of *d*(0, *t*)/*T*. Now, clearly the problem of finding *S** can be solved separately, and indeed we start by solving it.

If we look at the list of stipulations for $f(S)$, (1) just means that we can stop and wait at zero marginal cost (which we first keep as a strict assumption to justify the flexibility of the race). (2) is evident, and (3) is redundant, given (1), since if f is not monotone for $S > 0$, then it has a global minimum for that region at some S, say $S_{\min}$, where the function assumes the value $f_{\min} \leq f(S)$; $\forall\ S > 0$. Now suppose we wish to move at a speed of λS_{min}; $\lambda \varepsilon$ (0, 1], during T time units, thus covering a distance of λTS_{min}; then who is to prevent us from waiting $(1 - \lambda)T$ time units, and then go at S_{min} during the remaining λT time units, at a variable cost of f_{min} per distance unit? As for the convexity requirement, which we actually need from S_{min} and up only, this is not a restriction at all! Not only do all the firms we mentioned behave this way in practice generally, but even if they did not, we could use the convex support function of f as our 'real' f, by a policy, similar to the one discussed above, of moving part time at a low speed and part time at a higher one at a cost which is a linear convex combination of the respective f's. Figure 1 'sums' our treatment of an ill-behaved function (in dotted lines where never used). We will also assume that f is continuously differentiable, see Zang (1981)as to why this is not restrictive in practice. Hence, our only real assumption is that we can stop and wait at zero cost, i.e., (1).

Lemma 1: let $\bar{f}(S)$; $S > 0$ be any positive cost function associated with moving at speed S continuously and let (1) hold, then by allowing mixed speed strategies, we can obtain a function $f(S)$; $S > 0$ such that f is positive, monotone nondecreasing and convex, and reflects the real variable costs.

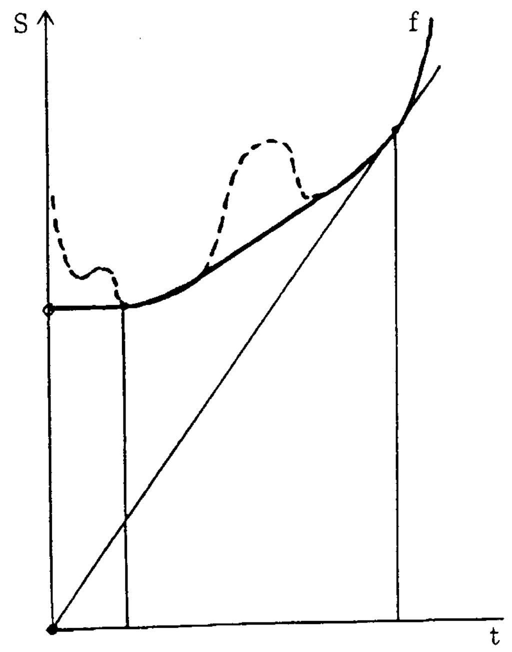

FIG. 1

A WELL BEHAVED (FAT) AND ILL BEHAVED (DOTTED) SPEED FUNCTION

Now, since each time unit cost is F, and we can go S distance units during it, each distance unit's 'fair share' is F/S. To this add $f(S)$, to obtain the cost of a distance unit at a speed of S when the firm knows where it is going, and requires their fixed costs to be covered. (On the other hand, not knowing what it wants to do means that the firm has to lose the F money, or part of it.) Denote the total cost as above by $TC(S)$, or

(5) $$TC(S) = f(S) + F/S.$$

But, F/S is strictly convex in S, and $f(S)$ is convex too, so $TC(S)$ is strictly convex. Further, $\lim_{\varepsilon \to 0} + TC(\varepsilon) = \infty$, so $TC(S)$ has a unique minimum, S^*. Since we practically assume differentiability, then

(6) $$S^* = \arg\left\{f'(S) = F/S^2\right\}$$

and we can obtain it numerically. (S^* is also depicted in Figure 1, where a ray from the origin supports f.)

Choosing t Optimally

Our problem is to find the point t, or the 'decision point', where we elect to be at the end of the decision period. Then, we will know with certainty what we have to do, so we will proceed at S^* to whichever point i chosen, at a cost of $TC(S^*)\, d(t, i)$. Denoting $TC(S^*) = TC^*$, we may rewrite (4) as follows:

$$Z(t) = FT + f\left(\frac{d(0,t)}{T}\right)d(0,t) + TC^* \sum_{i=1}^{n} p_i d(t,i)$$

Theorem 1: $Z(t)$ is strictly convex in t.

Proof: Clearly FT is a constant so it is convex. Let $h(w) = f(w/T)w$, hence our second term, $f(d(0,t)/T)d(0,t)$ is $h(d(0,t))$. By differentiation we can show that $h(w)$ is strictly convex, monotone increasing and nonnegative. $d(0, t)$ is convex (being a norm), and it follows that $h(d(0,t))$ is strictly convex as well (see Theorem 5.1 in Rockafellar (1970), for instance), As for the third term it is clearly convex (since $\{p_i\}_{i = 1, \dots n}$ are nonnegative probabilities), and our result follows for the sum.

It follows that a unique minimum, Z^* exists for Z, within the convex hull of the n + 1 points 0, 1, ..., n. In order to find this minimum we look for a point such that the gradient ∇Z is zero. We now examine the two components of ∇Z, by x_t and y_t.

(8) $$\frac{\delta Z}{\delta x_t} = \frac{x_t - x_0}{d(0,t)} = \left(f(S) + Sf'(S)\right) + TC^* \sum_{i=1}^{n} p_i \frac{x_t - x_i}{d(t,i)} ,$$

(9) $$\frac{\delta Z}{\delta y_t} = \frac{y_t - y_0}{d(0,t)} = \left(f(S) + Sf'(S)\right) + TC^* \sum_{i=1}^{n} p_i \frac{y_t - y_i}{d(t,i)}$$

where

(10) $$S = d(0, t)/T.$$

The 'length' of the gradient L_t is

(11) $$L_t = \left[\left(\frac{\delta Z}{dx_t}\right)^2 + \left(\frac{\delta Z}{dy_t}\right)^2\right]^{1/2}.$$

We can gain some more insight into the problem if we consider two limiting cases : (i) $T \to \infty$; and (ii) $T \to 0$.

(i) <u>The $T \to \infty$ Case:</u> Here we assume $S_{min} > 0$. Under this assumption, at a cost of f_{min} per distance unit, the firm can arrive anywhere during the decision time. Hence we have $p_0 = f_{min}/TC^*$, and our problem is solved. As usual, denote the solution point by t^* , and clearly for T large enough we are not going to move during the whole decision period, but rather only during T^* time units of it, where

(12) $$T^* = d(0, t^*)/S_{min}.$$

Hence the same solution is obtained for any $T > T^*$.

It may be advisable to try solving under the assumption that $T > T^*$, and then check the assumption. This way, even if $f'(s)$ jumps at S_{min}, we will not have any problems with it. If $T > T^*$, we are through, and else we know that $S > S_{min}$.

(ii) <u>The $T \to 0^+$ Case:</u> recall (5), with S^* as per (6) we have $TC^* = f(S^*) + F/S^*$, and from (6) we easily obtain

(13) $$F/S^* = S^* f'(s^*)$$

Now substitute (13) to TC^*, and we have

(14) $$TC^* = f(S^*) + S^* f'(S^*).$$

Let $W(S)$ as defined below

(15) $$W(S) \equiv f(S) + S f'(S),$$

be the relative weight of the starting point 0 in (8) and (9). We observe that $W(S)$ is a monotone increasing function (since f, f' and $f'' > 0$), and that $W(S^*) = TC^*$. But at t^* (8) and (9) are zero, hence

(16) $$\frac{x_0 - x_t^*}{d(0,t^*)} W(S) = TC^* \sum_{i=1}^{n} p_i \frac{x_r^* - x_i}{d(t^*,i)}$$

(17) $$\frac{y_0 - y_t^*}{d(0,t^*)} W(S) = TC^* \sum_{i=1}^{n} p_i \frac{y_r^* - y_i}{d(t^*,i)}$$

Squaring (16) and (17), adding them and taking the square root again, we obtain

(18) $$W(S) = TC^* \left[\left(\sum_{i=1}^{n} p_i \frac{x_r^* - x_i}{d(t^*,i)} \right)^2 + \left(\sum_{i=1}^{n} p_i \frac{y_t^* - y_i}{d(t^*,i)} \right)^2 \right]^{1/2} .$$

Clearly $W(S) \leq TC^*$ (the magnitude of a vector sum is less than the sum of the magnitudes), with equality only in the special case where all the points, including the starting point are colinear, and both 0 and t are to the same side of all the rest (in which case the firm can behave as if it knows where it is going, since it has to reach the first point at least, and it knows the decision will be made by the time it gets there). But W(S) is monotone, hence if $W(S) < TC^*$ then

$$S < S^*,$$

and

(19) $$d(0, t^*) \leq TS^*$$

Following (18) we define $G(t)$ for any $t \, \varepsilon \, \{ E^2 - N \}$ (i.e., any point on the plane and 0, but not $i \, \varepsilon \, N$)

(20) $$G(t) = TC^* \left[\left(\sum_{i=1}^{n} p_i \frac{x_r^* - x_i}{d(t^*,i)} \right)^2 + \left(\sum_{i=1}^{n} p_i \frac{y_t^* - y_i}{d(t^*,i)} \right)^2 \right]^{1/2} .$$

For $t = t^*$, and S chosen optimally (18) plus (20) yield

(21) $$G(t^*) = W(S).$$

Now (for the first time) we use the data $T \to 0$, and by (18) we have

(22) $$\lim_{T \to 0^*} d(0, t^*) = 0 .$$

I.e., we only have to determine in which direction and at what speed to proceed, but we will not get very far. The direction we choose is that of $-\nabla Z(t^*)$, as we always have to; but now we can take 0 instead of t^*, using (22) so we do not have to search for this value. As for the speed, we choose S^∇ (the 'gradient' speed), such that

(23) $$S^{\nabla} \stackrel{\Delta}{=} \arg\{W(S) = G(0)\},$$

since by (21) this is the value for $t^* = 0$.

Since the speed is one of the parameters we are interested in, we present a theorem which will also hold for the stochastic decision period case.

Theorem 2: The gradient speed S^{∇} as defined at any point, is an upper bound for the optimal speed at that point, and S^* is an upper bound for S^{∇}.

Proof: By Theorem 1, $Z(t)$ is strictly convex, hence along the segment $\overline{0,t^*}$ it is also strictly convex, and since $Z(t^*) \leq Z(t)$; $\forall t$, it is monotone decreasing along the segment. Let $g(t)$ be the absolute value of the directed derivative along $\overline{0,t^*}$. Clearly $g(t)$ is monotone decreasing for c when $t = \lambda 0 + (1-\lambda)t^*$ (i.e., $x_t = \lambda x_0 + (1-\lambda)x_t^*$, $y_t = \lambda y_0 + (1-\lambda)y_t^*$, 0 being an index and not a number here). For $\lambda = 0$ the slope $g(0)$ is bounded from above by $G(0)$, since $G(0)$ reflects the steepest descent (in the direction of $-\nabla Z$). At t^*, the direction of $\overline{0,t^*}$ is the steepest descent direction itself, by (18). Summing these assertions we have

(24) $$G(0) \geq g(0) > g(t^*) = G(t);\ 0 \neq t^*$$

It follows that the gradient speed S^{∇} is an upper bound on the speed for any movement from 0, and we can designate any point as 0. I.e.

(25) $$S \leq S^{\nabla} \leq S^*.$$

So S^{∇}, which is rather easy to compute, is an upper bound on our speed anywhere, and it would be easy to extend the proof to the stochastic decision period case, using the basic attributes of the expectation.

The Stopping Line and the Waiting Region

For $T \geq T^*$, we obtain $S = S_{min}$, and by (15) it follows that $W(S) = f_{min}$. Using (21) we have

(26) $G(t^*) = f_{min}$.

Now, starting at different points, but such that $G(0) > f_{min}$ and $T > T^*$ as defined for them we should stop at different decision points respectively (unless we start from colinear points, on $\overline{0,t^*}$) each satisfying (25). Actually there is a locus of points satisfying (26), which we call D as follows

(27) $$D = \{\, t \,\varepsilon\, E^2 \,|G(t) = f_{min}\}\ .$$

We call D the stopping line (although it may happen to be a point). Now denote the area within D, inclusive, as C, or

(28) $$C = \{\ t\ \varepsilon\ E^2\ |G(t) = f_{min}\}.$$

C is also called the waiting area, since being there during the decision period would imply waiting. Clearly $C \subseteq D$, with $C = D$ for the special case where one of the points $N \cup 0$ is the only solution for a large T. In case $C \neq D$, however, we have a nonempty set E as follows

(29) $$E = C - D \text{ (or } C/D).$$

Specifically, there is a point in C, and in E if $E \neq \emptyset$, for which $G = 0$ (if $C = D$, we have to define G as 0 there, since it includes 0/0 terms); we denote this point by t_{min}, i.e.,

(30) $$G(t_{min}) = 0.$$

Clearly, in order to identify t_{min}, we do not need any information about the starting point or any of the costs we carry, but just the information on N and $\{p_i\}$

We are now ready to discuss the stochastic decision period case. In that connection, note that some of our results so far, such as Theorem 1, the stopping line, etc., are not dependent upon T, hence they can serve us for the stochastic decision period case as well.

4. The Stochastic Decision Period Case

Our problem is exactly as before, except that T is a random variable (RV) now. Conceivably the p_i values could be influenced by information such as "the decision has not yet been made", but we do not consider this case in detail (i.e., we assume statistical independence between T and the choice). Our RV may be discrete (contact with management is at predetermined times), continuous or mixed. We discuss the discrete case in detail, and show how to accommodate the continuous case by the discrete one. We assume that the distribution of T is given. (Bayesians will find no fault with that assumption, hopefully. Other will have to take it at face value). Let

(31) $$P(T = h_j) = q_j;\ j\ \varepsilon\ J = \{1, 2, ...\},$$

where $q_j \geq 0;\ \forall j$ and they sum to one, of course; J may be finite or not, the index 0 is maintained for the start as before, and we may assume $h_0 = q_0 = 0$ for it. Our problem is to find the best set of decision points t_j (or t_j^* when optimality is assumed), such that as long as the decision is not yet made by $T = h_j$ we proceed from t_j^* to t_{j+1}^* (starting at $t_0 = t_0^*$). Let v_j be the conditional probability of decision at h_j, given it has not been made yet, i.e.

(32) $$v_j = P(T - h_j \mid T > h_{j-1}) = \frac{q_i}{1 - \sum_{k=1}^{j-1} q_k};$$

and, following (7) we define $Z(t_{j-1}, t_j)$;

(33) $$Z\left(t_{j-1},t_j\right)=f(h_j-h_{j-1})+f\left(\frac{d(t_{j-1},t_j)}{(h_j-h_{j-1})}\right)d(t_{j-1},t_j)+$$

$$+v_j TC*\sum_{i=1}^{n} p_i\ d(t_j,i)+(1-v_j)Z_{j+1}\left(t_j,t^*_{j+1}\right)$$

This formulation lends itself to dynamic programming very naturally, and assuming optimality we define $Z^*_j(t_{j-1})$:

(34) $$Z^*_j(t_{j-1})=\min_{t_j}\left\{Z_j(t_{j-1},t_j)\right\}=Z_j(t_{j-1},t^*_j).$$

Before proceeding further with the general solution, two limiting cases will help us to confine our search to a manageable area. These are analogs of cases we discussed above, and here is the payoff for the effort there.

The $P(T < T^*) \rightarrow 0$ Case: This is the analog of the $T \geq T^*$ case, so we proceed at S_{min} to the correct spot along the stopping line. We refer to the solution as the "slow" trajectory.

The $P(T < \varepsilon) \rightarrow 1;\ \forall\ \varepsilon > 0$ Case: This case to which we refer as the "gradient" case, is analog to the $T \rightarrow 0^+$ case, since it stipulates that with probability approaching 1 this is indeed anticipated. Therefore we move at a speed of S^∇ in the $-\nabla Z(t)$ direction. Now, under the stipulation , the probability that we will go far before the decision is negligible, but this does not deter us from defining the steepest descent, or (minus) gradient trajectory all the way until the stopping line. The "gradient" speed we use is a function of t, which may be obtained by

(35) $$S^\nabla(t) = \arg\{f(S^\nabla(t)) + S^\nabla(t)\, f'(S^\nabla(t)) - G(t) = 0\},$$

which is a direct extension of (23).

Since by Theorem 2(which extends almost directly to the stochastic decision period case) $S^\nabla(t)$ is an upper bound on S. We also refer to this as the fast trajectory. It is interesting (although intuitively clear) to note that S^∇ is decreasing along the fast trajectory.

Lemma 2: When moving along the gradient trajectory, which we denote by X(t), in the $-\nabla Z(t)$ direction, $S^\nabla(t)$ is monotone nonincreasing.

Proof: Let z(t) be the expected distance to the final destination from t given a decision (recall, a decision is due immediately), i.e.,

(36) $$z\left(X(t)\right)=TC*\sum_{i=1}^{n} p_i d(t,i)$$

Then $G(t)$ as per (20), is z's directional derivative along $X(t)$, i.e.,

(37) $$G(t) = |z'(X(t))|$$

We want to show that $G(t)$ is monotone nonincreasing (which will imply our lemma by (35) and the montonicity of $W(S)$). We know that $G(t)$ decreases from $G(0)$ to f_{min} without changing signs along $X(t)$, so it will suffice to show that $z''(X(t)) \geq 0$. But $X(t)$ is a trajectory in the steepest descent direction, hence if we differentiate it by t, twice, we get

$$\dot{X}(t) = -\nabla z(X(t))(\mathrm{t})\,, \tag{38}$$

$$\ddot{X}(t) = -\nabla^2 z(X(t))\dot{X}(t) = \nabla^2 z(X(t))\nabla z(X(t)) \tag{39}$$

We continue and differentiate $z(X(t))$, twice again, to obtain

$$z'(X(t)) = \nabla z(X(t))^T \dot{X}(t)\,, \tag{40}$$

$$z''(X(t)) = \dot{X}(t)^T \nabla^2 z(X(t))\dot{X}(t) + \nabla z(X(t))^T \ddot{X}(t)\,. \tag{41}$$

Finally, by substituting (39) and (41) we have

$$z''(X(t)) = \left(\dot{X}(t) + \nabla z(X(t))\right)^T \nabla^2 z(X(t))\left(\dot{X}(t) + \nabla z(X(t))\right), \tag{42}$$

which is a bilinear form ($Y^T\nabla^2 zY$). Now, z is clearly a convex function, hence $\nabla^2 z$ is positive semidefinite (at least), and our result follows.

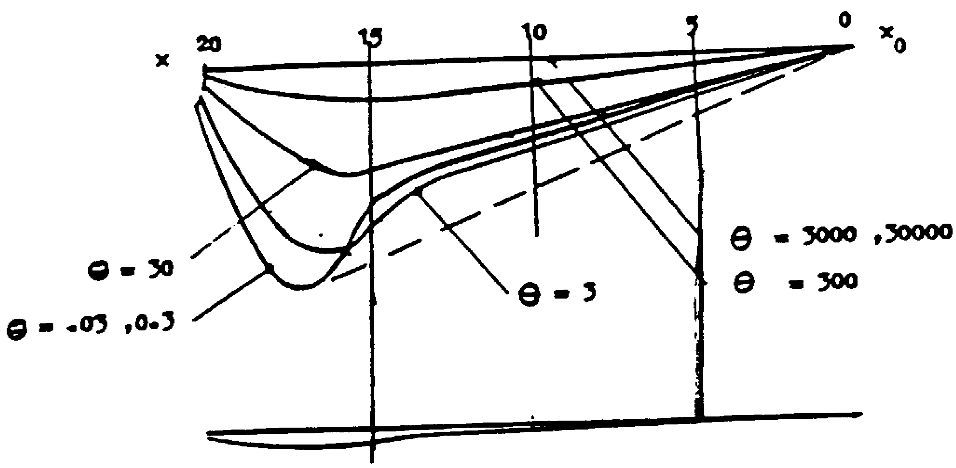

FIG. 2

EXPONENTIALLY DISTRIBUTED DECISION PERIOD, FOR EXPECTATIONS AND FOR RANDOMLY CHOSEN POINTS

Figure 2 depicts the results of a program run for an exponentially distributed decision period, for seven expectations θ , and for seven randomly chosen points of randomly chosen weights (probabilities). For large θ's, the trajectories were virtually the same as the slow trajectory. For small θ's, a similar behaviour was observed relative to the fast trajectory. Interestingly, though all the trajectories, were within the convex hull of the area between these two trajectories, one of them actually intersected the fast trajectory. The speed race obeyed Theorem 2. In general one might say that the faster is the decision due, the faster we should move, and the longer our total trajectory may be – since we do not expect to stick to it for a long while: the slower is the decision due, the more we tend to go slowly and along a "mildly curving" trajectory (if not exactly straight). In any case, as far as the physical location of the trajectory is concerned, we should not find ourselves out of the area defined by the convex hull of the extreme case trajectories, minus *E* (which we should never enter even if it is in that convex hull). If we do, we can always find a better trajectory for immediate or delayed advantage, within this area. We denote this result in Theorem 3.

Theorem 3: One should never leave the convex hull of the slow and fast trajectories, minus *E*.

Proof: By negation, as described above.

Incidentally, the results in Figure 2 even show that the convex hull of a "medium" speed trajectory and the slow one contains all the "slower" speed trajectories. However, this may not be extendable to more general distributions, where the relative "speeds" may not be uniquely implied. Note, however, that if we do not use exaggerated vertical scale, all trajectories seem rather straight! Practically there is no doubt that location wise we should pick some straight trajectory, even the slow one, and just optimize the speed race. This would yield most of the potential gain, with the additional benefit of a less complex race problem.

If we return now to (33) or (34), we can see that the stopping line and the search area are what we need practically to obtain a working dynamic programming model. We may start by assuming the slow trajectory, which would make our decision variable univariate, and we can fold back satisfactorily by assuming that the, say, k^{th} step will bring us to the stopping line. It may happen, that we overshoot the starting point, but it should not be difficult to adjust. However, we do not suggest using this method here, since it does not seem to justify the programming effort, and we can simply use a multivariable library search method instead. A problem might be if local minima exist besides the global minimum; the next theorem removes this obstacle.

Theorem 4 : The problem of locating the decision points t_j^* so as to minimize $Z_1^*(t_0)$ is convex.

Proof: By iterative application of Theorem 1.

5. Matching an Innovation Race

In tracking the evolution of part of the Japanese telecommunication industry over several years it shows that strategic interactions between the firms play a substantial role in determining firm level and industry level of technological evolution. In particular, we identify several ‘races’, each of which is the result of a subset of firms jockeying for a position either as a race leader or for a position not too far behind the race leader. The identification and interpretation of the races relies on the fact that different firms take very different technological paths to reach a common ‘cycle time’ level. In view of the previous description of a stochastic race it is pertinent to distinguish between two kinds of racing behaviour. A lagging firm might simply try to close the gap between itself and the technological leader at any point in time (‘frontier-sticking’ behaviour or catch-up race), or it might try to actually usurp the position of the leader by “leapfrogging” it (frontier race). When there are disproportionately large payoffs to being in the technical lead (relative of the payoffs that a firm can realize if it is simply close enough the technical frontier), then one would expect that leapfrogging behaviour would occur more frequently than frontier-sticking behaviour. All attempts to leapfrog the current technological leader might not be successful since many lagging firms might be attempting to leapfrog the leader simultaneously, and the leader might be trying to get further ahead simultaneously. In this regard, one should both report the attempted leapfroggings and the realized leapfroggings. Thus, we may distinguish between two-layer races, in the first one leapfroggings may stochastically occur, in the second one, followers or imitators may just try to catch up with the frontier, by frontier-sticking behaviour.

6. Multi-Stage Races

All the existing work on multistage races is in the patent race framework. Harris and Vickers (1987) show that the leader invests more than the follower in a multi-stage patent race scenario. Their result generalizes a similar result due to Grossman and Shapiro (1987) for two-stage games. In contrast, instead of analyzing aggregate resource allocation, we discuss how given resources are allocated. In line with the stochastic race model suggested in Sections 3 and 4, there could be an important strategic advantage of being aggressive or fast in each stage of a multi-stage race. Our focus will be on characterizing the differences in the expected payoff functions of firms as they get ahead of their rivals (or fall behind) and closer to the finishing line. Our explanation here will be intuitive, and analytical treatment is given in the appendix. We can speak of the monopoly (or duopoly) term becoming more important in a payoff expression as the ratio of its coefficient to that of the duopoly (or monopoly) term rises. It can be established that the monopoly term in the expected payoff expression of the leading firm in a two-firm multi-stage race becomes progressively more important as it gets further ahead of its rival, provided the lead meets a mimimum threshold. The threshold lead is smaller the closer is the lead firm to the finishing line. Conversely, the duopoly term in the expected payoff expression of the lagging firm becomes more important as it falls further behind, subject to the same threshold lead considerations as the leading firm. We assume the leading firm’s payoff, when it has finished all stages and is reaping monopoly profits, as a function of the lead it has over its rival. Let the lead firm receive a payoff when it has finished all n stages, and the rival is in the first stage. Then the coefficient on the monopoly term rises faster than that on the duopoly term as the lead increases. Then, using this property of the coefficients, we consider the leading firm's payoff as a function of its lead, when it is in the last stage of the n stage race. Once again, we show that as long as the lead exceeds a threshold lead (which may be 0), the coefficient on the monopoly term rises faster than that on the duopoly term as the lead increases. A method of recursion can be applied, where the relationship of the coefficients when the lead firm is at stage s of the race is, is used to derive similar relationships when the lead firm is at stage $s – 1$. The procedure is similar for the lagging firm. In this case we can show that the duopoly coefficients rise faster than the monopoly ones as the lead increases, subject to the threshold lead considerations. The same is shown to be true recursively when the lead position is at position $n – 1$, $n – 2$, etc. This characterization highlights two forces that influence a firm's choices in various stages: proximity to the finishing line and distance between the firms. The probability of reaping monopoly profits is higher the farther ahead a firm is of its rival, and even more so the closer the firm is to the finishing line. If the lead firm is far from the finishing line, even a sizeable lead may not translate into the dominance of the monopoly profit term, since there is plenty of time for the lead situation to be reversed and failure to finish first remains a probable outcome. In contrast, the probability that the lagging firm will get to be a monopolist becomes smaller as it falls behind the lead firm. This raises the following question. What kinds of actions cause a firm to get ahead? Intuitively, one would expect that a firm that is ahead of its rival at any time *t*, in the sense of having completed more stages by time t, is likely to have chosen the faster, less aggressive strategy more often. The monopoly term is increasingly important to a firm that falls behind. Further simple calculations suggest that the firm that is ahead is likely to have made less aggressive choices than the firm that is behind in the race.

A further interesting question is whether a lead results in greater likelihood of increasing lead and then in an increased chance of leapfrogging (as in a frontier race) or in more catchup behaviour (as in a catchup). The existing literature (Grossman and Shapiro, 1987; Harris and Vickers, 1987) has suggested that a firm that surges ahead of its rival increases its investment in R&D and speeds up while a lagging firm reduces ist investment in R&D and slows down. Consequently , these papers suggest that the lead continues to increase. However, when duopoly competition and dichotomy of the race (in frontier pushing and frontier sticking behaviour) are accounted for, the speeding up of a leading firm occurs only under special circumstances. In high tech industries, such as computers and telecommunications, it could be expected that monopoly profits do not change substantially with increased aggressiveness, but duopoly profits do change substantially with increased aggressiveness. Then a firm getting far enough ahead such that the monopoly term dominates its payoff expression will always choose the fast strategy, while a firm that gets far enough behind will always choose the slow and aggressive approach. Then the lead is likely to continue to increase. If, on the other hand, both monopoly and duopoly profits increase substantially with increased aggressiveness then even large leads can vanish with significant probabilities.

7. Conclusions

Our stochastic model of a race embraces several features that resemble moving objects towards a stochastic final destination. In contrast to game-theoretic treatments of racing behaviour we look into racing patterns of individual firms in view of their strategic responses to their racing environment. Among those features we identified the speed race problem, the selection of an optimal decision point t^*, to optimize a gradient trajectory (of technological evolution) and to determine the 'stopping line and the waiting region'. Such model would be compatible with observations on innovation races in high technology industries, in particular, with race-type behaviours such as leapfrogging and catchup, striking a balance between moving ahead and waiting. The model can be improved by incorporating constraints into it. For example, constraints on an innovation path could be given by road blocks such as a bankruptcy constraint or an R&D uncertain payoff constraint. Some of these constraints may be conceptually easy to introduce, others may be tougher such as an investment constraint if the total innovation effort en route to t* plus the worst case would violate it. In such a case one may want to weight the distant finishing line unproportionally.

It is interesting to speculate on the implications of the way the firms in major hi-tech markets, such as telecommunications, split clearly into the two technology races, with one set of firms clearly lagging the other technologically. The trajectories of technological evolution certainly seem to suggest that firms from one frontier cannot simply jump to another trajectory. Witness, in this regard, the gradual process necessary for the firm in the catchup race to approach those in the frontier race. There appears to be a frontier 'lock-in' in that once a firm is part of a race, the group of rivals within that same race are the ones whose actions influence the firm's strategy the most. Advancing technological capability is a cumulative process. The ability to advance to a given level of technical capability appears to be a function of existing technical capability. Given this path dependence, the question remains: why do some firms apparently choose a path of technological evolution that is less rapid than others. Two sets of possible explanations could be derived from our case analysis, which need not be mutually exclusive. The first explanation lingers primarily on the expensive nature of R&D in industries like telecommunications and computers which rely on novel discovery for their advancement. Firms choosing the catchup race will gain access to a particular technical level later than those choosing the frontier, but will do so at a lower cost.

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APPENDIX

On the Expected Gain by the Model

Our model is based on the fact that the decision period is going to be completely wasted, unless we utilize it. This places an obvious upper bound on our expected gain, V, namely

(A1) $$V \leq FT$$

In the stochastic case, similarly

(A2) $$V \leq FE(T)$$

Clearly, the only way we can approach this upper bound is if $G(t)$ approaches TC^* along the trajectory, throughout the decision period; for instance, if the points are close to each other and far from the start. In this case we behave as if the destination is known. However, in both the deterministic and the stochastic decision period case, if T is large, we cannot do anything at least part of the time. It is obvious that in the deterministic case the gain cannot exceed FT^*, which makes us rewrite $G(t)$, and similarly (A2).

(A3) $$V \leq F \min \{T, T^*\}$$

(A4) $$V \leq F \min \{E(T), T^*\}.$$

But, suppose now that $T \geq T^*$, can we really expect to gain even FT^*? The answer of course is no. In this case $G(t)$ is rather low, at least towards the stopping line where it reaches f_{min}. We may compute V for this case by the following formula

(A5) $$V = TC^*\left(\sum_{i=1}^{n} p_i\left(d(0,i) - d(t^*,i)\right)\right) - f_{\min} d(0,t^*),$$

where the gross gain is the improvement in the expected „future“ total costs to reach the final destination, but we have to subtract the “present” variable costs, in this case $f_{\min}\, d(0, t^*)$. By substituting $f\left(d(0, t^*)/T\right)d(0, t^*)$ for these costs, we obtain the expected gain in the deterministic case.

(A6) $$V = TC^*\left(\sum_{i=1}^{n} p_i\left(d(0,i) - d(t^*,i)\right)\right) - f\left(d(0,t^*)/T\right) d(0,t^*).$$

In the stochastic case, similarly, we have the following result.

(A7) $$V = FE(T) + TC^*\left(\sum_{i=0}^{n} p_i\left(d(0,i)\right)\right) - z_1^*(0).$$

A similar result can be obtained at any stage, given that we reached it without decision, but we omit it. Note however that this expected gain is monotone nonincreasing. For instance, once we reach the stopping line it drops to zero, since there is nothing useful we can do any more. Note that if we start anywhere in C, all the formulas above, including the bounds (A3) and (A4) yield $V = 0$ (e.g., $T^* = 0$ in this case).