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# *Blocking Coalitions and Fairness in Asset Markets and Asymmetric Information Economies*

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**Abstract**

This paper analyses two properties of the core in a two-period exchange economy under uncertainty: the veto power of arbitrary sized coalitions; and coalitional fairness of core allocations. We study these properties in relation to classical (static) and sequential (dynamic) core notions and apply our results to asset markets and asymmetric information models. We develop a formal setting where consumption sets have no lower bound and impose a series of general restrictions on the first period trades of each agent. All our results are applications of the same lemma about improvements to an allocation that is either non-core or non-coalitionally fair. Roughly speaking, the lemma states that if all the members of a coalition achieve a better allocation in some way (for instance, by blocking the status quo allocation or because they envy the net trade of other coalitions) then an alternative improvement can be obtained through a perturbation of the initial improvement.

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# 1 Introduction

This paper analyses the core of a two-period exchange economy under uncertainty. The study focuses on two properties: the robustness of core allocations with respect to the restrictions imposed on the size of the blocking coalitions; and the coalitional fairness of the core allocations. Both properties have been studied in the context of asymmetric information economy models, but not for the core of asset markets. Our analysis includes both scenarios, in a general frame that allows for consumption sets with no lower bound and for restrictions on the first period trades of agents. We study the above mentioned properties in the context of classical (static) and sequential (dynamic) core notions and all the main results are applications of the same lemma on allocation improvements. The resulting characterizations of the core are new for asset market models and, also, provide improvements to the results for asymmetric information economies.

In a pure exchange economy with uncertainty, agents subscribe contracts at time $\tau = 0$ (ex-ante) that are contingent on the realization of an uncertain state of nature at time $\tau = 1$ (ex-post), using a decision process based on maximizing the expected utility. Two notions of core have been investigated for such an economy: ex-ante and sequential. According to the "classical" ex-ante notion, an allocation in the core is assigned to each agent in each state of nature which can be realized tomorrow, in the form of a bundle of goods with the property that no coalition can improve it ex-ante. The ex-ante core includes special features that facilitate analysis of general equilibrium models with uncertainty. For instance, the core may enable analysis of competitive price equilibrium concepts and, also, may constitute a useful alternative to them. Sequential core notions modify the ex-ante core and take account of forms of incompleteness that can arise from a contracting process over two states of nature. In defining the classical core, it is generally assumed that agents in a coalition coordinate over a set of actions which are enforceable, and that all commitments are binding. The latter assumption may be questionable in frameworks characterized by uncertainty and which are implicitly dynamic in nature. In fact, agents can reconsider their distribution of resources after the uncertainty is resolved and coalitions might be able to improve the allocation at the ex-post stage, even if this allocation is ex-ante stable with respect to coalition improvements. Consequently, in a context of sequential trades, the classical core notion can be modified in several directions in order to guarantee that an allocation is stable not only at the ex-ante stage, but also in the face of coalition improvements when the state of nature is realized[1].

The notions of ex-ante and sequential core satisfy interesting properties in two specific models with uncertainty which are relevant to our study. The first is the asset markets case. Here, a competitive equilibrium allocation exists in a generic sense (Duffie and Shafer, 1985), but there are exceptional cases when the competitive equilibrium fails to exist. This is the case in Hart (1975), who proposes a situation where the core can be proved to be non-empty. The proof of core existence is given by Koutsougeras (1998), which captures the case of arbitrary short sales through consumption sets with no lower bounds. The second model is economies with asymmetric information. In this case, the core allows analysis of the cooperative behaviour of agents, coordinating their contingent plans at time $\tau = 0$, under the assumption that they will receive partial information about the the true state of nature at time $\tau = 1$. Typically, private information is introduced into the Arrow-Debreu state contingent model with the further assumption that the agent's allocation is compatible with private information[2]. The corresponding ex-ante and sequential private core outcomes are shown to exist and to be incentive compatible (see Yannelis, 1991, Koutsougeras, 1998). Notice, also, that both these examples share the feature that an agent's trades at time $\tau = 0$ are restricted. This is due either to the incomplete market or to informational constraints. In asset markets, it is possible that not every consumption bundle can be implemented at the ex-ante stage via existing assets; in models with asymmetric information, agents can exchange only consumption bundles which they are able to verify according to their private information[3].

In this paper, we consider a general model to study core properties under the forms of market and/or contract incompleteness described above. Specifically, we consider a two-period exchange economy, in which the set of agents includes some large agents and a continuum of small agents (oligopolistic market). This representation of the

[1]This is the perspective adopted by several authors including Gale (1978), Repullo (1988), Koutsougeras (1998), Predtetchinski et al. (2002), Predtetchinski et al. (2006), who introduce sequential core notions by requiring that the allocation is not blocked at time $\tau = 0$ and in any possible state at time $\tau = 1$. In all of these papers, the different types of blocking mechanisms proposed at $\tau = 0$ lead to the notions such as a two-stage core, a strong sequential core and a weak sequential core, among the others.

[2]i.e. measurable with respect to the agent's information partition.

[3]Daher et al. (2007) studied the existence of an equilibrium in a complete asset market framework assuming that agents may have different information. To this end, they considered an economy with finitely many agents and an arbitrary space of states of nature, and introduced the assumption of coalitional independence property.

agents' space allows simoultaneous consideration of the case of a finite economy, an atomless economy, and an economy with atoms[4] (mixed market models). In such an economy, each agent is characterized by a state-dependent utility function, a random initial endowment and a prior belief. We follow the approach in Koutsougeras (1998), where consumption sets may have no lower bound and a series of general restrictions is introduced and applied in the first period trades of each agent. The corresponding notions of core are defined for both the static and the sequential cases and the following questions are investigated: 1. Given an atomless economy, is it possible to consider only coalitions of a fixed size in order to find all the core allocations? 2. In a mixed economy, is a core allocation always coalitionally fair?

In relation to the first question, a positive response for atomless economies without uncertainty is provided in the literature in the so-called Vind's theorem (see Vind, 1972, Schmeidler, 1972). The Vind's theorem states that whenever a pure exchange economy allocation does not belong to the core of the economy, then for any measure $\varepsilon$ less than the measure of the whole set of agents, there is a coalition $S$ whose measure is exactly equal to $\varepsilon$ such that the allocation is improved by $S$. Applications of this result cover several different problems of interest to a study of the core. The immediate implications include the fact that a core allocation can be implemented only through the formation of small coalitions. Further implications are normative: since, as consequence of Vind's theorem, arbitrary large sized coalitions are entitled to block each allocation outside the core, the core can be seen as a solution supported by an arbitrary large majority of agents[5]. In the case of the ex-ante core and the *sequential core* (Definition 4.2), we formulate versions of Vind's theorem adapted to our framework (Theorem 3.4 and Theorem 4.4) and show that any allocation not in the core can be blocked by a coalition of any given measure, but smaller than the size of the grand coalition. Applications are provided for asset markets and asymmetric information economies, while a counter-example shows that the result does not hold for the *two-stage core* notion in Koutsougeras (1998) (Definition 4.3).

The second question refers to the coalitional fairness of allocations, a property of

[4]Atoms are usually identified with individual agents with an initially large (compared to the total market endowment) ownership of some commodities. Also, in the case where the initial endowment is spread over a continuum of small agents, atoms can arise if agents combine in non negligible large coalitions (see Gabszewicz and Shitovitz, 1992).

[5]Compare Evren and Hüsseinov (2008); related papers include Hervés-Beloso et al. (2000), Hervés-Beloso et al. (2005), Pesce (2010), Bhowmik and Cao (2012), Bhowmik (2015), Bhowmik and Graziano (2015).

equity in which bundle comparisons are allowed between groups of agents according to the concept of coalitional envy[6]. An allocation is coalitionally fair (briefly c-fair), if there is no coalition envy related to the net trade of another coalition, under the given distribution of resources. We show that any ex-ante core allocation is coalitionally fair in the sense that no coalition in the atomless sector envies the net trade of any other coalition comprising atoms, and vice versa (Theorem 3.15). So, large agents, despite their privileged initial position, can not enforce a core allocation because this would render the allocation unfair towards some atomless coalitions, and vice versa. Again our results refer to both models: an asset market with the possibility of arbitrary short-selling, and an asymmetric information economy where agents contract over contingenties in the first period. Note that, the coalitional fairness of core allocations follows easily from the core-Walras equivalence theorem in a standard model of an atomless economy. However, this does not apply to our framework since the choice of an appropriate Walrasian equilibrium concept is problematic for both models considered[7]. Our main result for the coalitional fairness of ex-ante core allocations can be extended to the sequential core (Theorem 4.11) while a counter-example shows that it does not hold for the two-stage core.

It is worth noting that the results described so far are new and have no counterpart in models where the consumption sets are not lower bounded. Moreover, in the context of applications, there are no counterparts in the incomplete asset markets literature, which, instead, mostly addresses issues related to competitive equilibria. Our analysis goes further to provide a deeper understanding of the cooperative solutions in such a framework and shows that the core not only exists as proved in Koutsougeras (1998), but also fulfills additional relevant properties. Finally, our results provide a first contribution to equilibrium analysis in incomplete markets with small and large agents. Similar remarks apply to sequential core notions where questions related to measure of blocking coalitions and fairness have yet to be investigated.

The remainder of the paper is organized as follows: section 2 presents the model and its specifications in the case of asset markets economies and asymmetric information economies. Section 3 focuses on the ex-ante core: we prove Vind's theorem and the

[6]This idea was proposed by Varian (1974), (see also Schmeidler and Vind, 1972) and, although on a slightly different basis, by Gabszewicz (1975) and was developed in Bhowmik (2015).

[7]An attempt to obtain two-stage core equivalence results with a measure space of agents and an incomplete asset markets, is proposed in Koutsougeras (1996). The Core equivalence is proved for the notion of a sequential core that is adapted to account for externalities deriving from the interdependency between trade at the ex-ante and ex-post stage.

coalitional fairness of ex-ante core allocations, assuming consumption sets with no lower bound and imposing general restrictions on the first period trades of each agent. Section 4 discusses these properties in relation to the sequential core. Applications are presented in section 5 and the main proofs are provided in appendix A. Finally, appendix B specifies some additional elements required for applications to asset markets which are not necessarily complete. The paper's key result is a technical lemma (Lemma 6.1), which states that if all the members of a coalition prefer what they obtain from some allocation $h$ rather than what they obtain from a given allocation $f$, then there is an alternative allocation $y$, which is a perturbation of $h$, such that all members of the coalition prefer what they obtain from $y$ to what they obtain from $f$.

# 2 Description of the model

We consider a standard pure exchange economy extending over two time periods, $\tau = 0$ and $\tau = 1$, with uncertainty in the second period. The space of *economic agents* is described by a probability measure space $(T, \mathscr{T}, \mu)$ where $\mu$ is a complete, finite and positive measure. Since $\mu(T) < \infty$, the set $T$ of agents can be decomposed in the disjoint union of two parts: the set $T_0$ of small or negligible agents which coincides with the atomless part of $T$; the set $T_1$ of large or non-negligible agents, which coincides with the countable union of atoms of $\mu$. This general representation permits to cover simultaneously the case of an economy with a finite set of agents (when $T_0$ is empty and $T_1$ is finite), the case of an atomless economy (when $T_1$ is empty), the case of mixed markets in which an ocean of small agents coexists with few influential agents (when both $T_0$ and $T_1$ have positive measure). Moving from this representation, in the $\sigma$-algebra of coalitions of agents $\mathscr{T}$ we can identify two relevant subfamilies by defining

$$\mathscr{T}_0 := \{S \in \mathscr{T} : S \subseteq T_0\} \quad \text{and} \quad \mathscr{T}_1 := \{S \in \mathscr{T} : T_1 \subseteq S\}.$$

Thus, $\mathscr{T}_0$ is the subfamily of $\mathscr{T}$ formed by all coalitions of negligible agents, while $\mathscr{T}_1$ is formed by coalitions containing all large agents. Finally, we denote by

$$\mathscr{T}_2 := \mathscr{T}_0 \cup \mathscr{T}_1 = \{S \in \mathscr{T} : S \in \mathscr{T}_0 \text{ or } S \in \mathscr{T}_1\}$$

the subfamily of $\mathscr{T}$ formed by coalitions containing either no large agents or all of them.

The exogenous uncertainty is described by a measurable space $(\Omega, \mathscr{F})$, where $\Omega$ is a finite set denoting all possible states of nature at time $\tau = 1$ and the $\sigma$-algebra $\mathscr{F}$ denotes all events. The *commodity space* is the $\ell$-dimensional Euclidean space $\mathbb{R}^\ell$. The order on $\mathbb{R}^\ell$ is denoted by $\leq$, and $\mathbb{R}^\ell_+ := \{x \in \mathbb{R}^\ell : x \geq 0\}$ denotes the positive cone of $\mathbb{R}^\ell$. The symbol $x \gg 0$ is employed to mean that $x$ is an interior point of $\mathbb{R}^\ell_+$, and the notation $\mathbb{R}^\ell_{++} := \{x \in \mathbb{R}^\ell_+ : x \gg 0\}$ is used for the interior of the positive cone. Suppose that $(\mathbb{R}^\ell)^\Omega$ is endowed with the point-wise algebraic operations, the point-wise order and the product norm. An element $y \in (\mathbb{R}^\ell)^\Omega$ can be identified with the function $y : \Omega \to \mathbb{R}^\ell$ and vice versa.

We assume that agents engage in a sequential trade where renegotiation is allowed ex-post, i.e. after the resolution of uncertainty. At time $\tau = 0$ (ex-ante stage) there is uncertainty about the state of nature that will be realized at time $\tau = 1$ (ex-post stage). At the ex-ante stage, agents arrange future delivery of commodities that will be carried out at time $\tau = 1$. When the state of nature is realized, agents carry out their trades and with this new initial position they can (possibly) exchange again[8].

The *economy* is defined as the pair $\{\mathscr{E}, \mathscr{G}\}$ with $\mathscr{E} := \{(X_t, u_t, e(t,\cdot), \mathbb{P}_t) : t \in T\}$ and the following specifications:

(i) $X_t : \Omega \rightrightarrows \mathbb{R}^\ell$ denotes the (state-contingent) consumption set of agent $t \in T$ and $X_t(\omega)$ its projection on the coordinate $\omega$[9];

(ii) $u_t : \Omega \times \mathbb{R}^\ell \to \mathbb{R}$ is the state-dependent utility function of agent $t$;

(iii) $e(t,\cdot) : \Omega \to \mathbb{R}^\ell$ is the random initial endowment of agent $t$;

(iv) $\mathbb{P}_t : \Omega \to [0,1]$ is the prior of agent $t$;

the set

$$\mathscr{G} := \{\mathscr{G}_i \subseteq (\mathbb{R}^\ell)^\Omega : i \in \mathbb{K}\},$$

where $\mathbb{K}$ is either a finite set or a countably infinite set, denotes the ex-ante choice space of agents, i.e. a collection of restrictions that may apply in the trades of each agent at the ex-ante stage. Define the set $I_i := \{t \in T : \mathscr{G}_t = \mathscr{G}_i\}$, where $\mathscr{G}_t$ is a Borel measurable subset of $(\mathbb{R}^\ell)^\Omega$ denoting the restriction imposed to agent $t$'s trades at $\tau = 0$. We assume that $I_i$ is a member of $\mathscr{T}$ and that $T = \bigcup\{I_i : i \in \mathbb{K}\}$. So $\{I_i : i \in \mathbb{K}\}$ represents a measurable partition of the set of agents into types, where each type is defined by a possible trade restriction at the ex-ante stage. The ex-ante

[8]For simplicity, we assume that there are no endowments and thus no consumption at $\tau = 0$. Hence, agents are only concerned with allocating their second period ($\tau = 1$) endowments.

[9]Notice that we do not impose non-negative constraints on consumption sets. Thus, short sales are allowed.

consumption bundles available for $t \in T$ under the restrictions defined by $\mathscr{G}$, are given by

$$\mathscr{A}_t = \left\{x \in (\mathbb{R}^\ell)^\Omega : x(\omega) \in X_t(\omega) \text{ for all } \omega \in \Omega \text{ and } x - e(t,\cdot) \in \mathscr{G}_t\right\},$$

where $x - e(t,\cdot)$ represents the state contingent net trade of $t$. For any $x : \Omega \to \mathbb{R}^\ell$, define the ex ante expected utility of agent $t$ by the usual formula

$$V_t(x) = \sum_{\omega \in \Omega} u_t(\omega, x(\omega))\mathbb{P}_t(\omega).$$

The next step is the statement of the main assumptions to be used throughout the paper. We assume that:

($\mathbf{A}_1$) For all $(t,\omega) \in T \times \Omega$, $X_t(\omega)$ is a closed convex cone.

($\mathbf{A}_2$) The correspondence $Y : T \times \Omega \rightrightarrows \mathbb{R}^\ell$, defined by $Y(t,\omega) := X_t(\omega)$, is such that $Y(\cdot,\omega)$ is $\mathscr{T}$-measurable for all $\omega \in \Omega$.

($\mathbf{A}_3$) The mapping $e(\cdot,\omega) : T \to \mathbb{R}^\ell$ is $\mathscr{T}$-measurable for all $\omega \in \Omega$ and $e(t,\omega)$ is an interior point of $X_t(\omega)$ for all $\omega \in \Omega$.

($\mathbf{A}_4$) The mapping $\varphi : T \to [0,1]^\Omega$, defined by $\varphi(t) = \mathbb{P}_t$, is $\mathscr{T}$-measurable.

($\mathbf{A}_5$) For all $(t,\omega) \in T_1 \times \Omega$, $u_t(\omega,\cdot)$ is concave.

($\mathbf{A}_6$) For all $(t,\omega) \in T \times \Omega$, $u_t(\omega,\cdot)$ is continuous and for all $x \in \mathbb{R}^\ell$, $t \mapsto u_t(\omega,x)$ is $\mathscr{T}$-measurable.

($\mathbf{A}_7$) For all $(t,\omega) \in T \times \Omega$, $u_t(\omega,y) > u_t(\omega,x)$ for all $x,y \in X_t(\omega)$ with $y \geq x$ and $x \neq y$.

($\mathbf{A}_8$) For all $(t,\omega) \in T \times \Omega$, $x \in \mathscr{A}_t$ and $\varepsilon > 0$, there is an $y \in \bigcap\{\varepsilon\mathscr{G}_i : i \in \mathbb{K}\} \cap \mathbb{B}(0,\varepsilon)^\Omega$ such that $x + y \in X_t$ and $u_t(\omega, x(\omega) + y(\omega)) > u_t(\omega, x(\omega))$[10].

($\mathbf{A}_9$) $\mathscr{G}_i$ is a convex cone containing 0 and $-\mathscr{G}_i \subseteq \mathscr{G}_i$ for all $i \in \mathbb{K}$.

In the next two subsections, following Koutsougeras (1998), we present two natural environments for the applications of our results. We do so by specifying restrictions to ex-ante trades assigned by means of the general collection

$$\mathscr{G} := \{\mathscr{G}_i \subseteq (\mathbb{R}^\ell)^\Omega : i \in \mathbb{K}\}.$$

Applications refer to the asset market and the asymmetric information frameworks. Notice that, compared to Koutsougeras (1998), our model allows the simultaneous presence of small and large agents in both contexts.

[10] $\mathbb{B}(0,\varepsilon)$ denotes the ball centered at 0 and radius $\varepsilon$ in $\mathbb{R}^\ell$.

## 2.1 Asset markets economies

In asset market economies, ex-ante contracts take the form of asset trades. At time $\tau = 0$, agents trade assets and trading is described by a choice of portfolio of assets. At time $\tau = 1$, the state of nature is realized and observable to all the agents. Asset contracts are signed and result in allocations of commodity bundles. To allow the possibility that agents can go arbitrarily short in asset trading, the consumption set is assumed to be lower unbounded. This is because some agents may promise to deliver large quantities of commodities in some states and then plan to retrieve some quantity if one of these states occurs. In particular, we assume that $X_t = (\mathbb{R}^\ell)^\Omega$, for all $t \in T$.

Given the structure of trades, the ex-ante choice space $\mathscr{G}$ of individuals is specified in terms of portfolio of assets.

We view a (real) asset as a title to receive a commodity in amounts that may depend on which state of nature occurs. Assume that there are $J$ assets, denoted by $S^1, \cdots, S^J$, with $S^j \in (\mathbb{R}^\ell)^\Omega$, for each $j = 1, \ldots, J$ and $S^j(\omega)$ interpreted as the amount of each of the $l$-commodities given by the asset $S^j$ if the state $\omega$ occurs. At the beginning of the first period, agent $t$'s ex-ante contract is determined by a portfolio of assets $\theta(t) \in \mathbb{R}^J$, where $\theta^j(t)$ specifies the number of the $j^{\text{th}}$ asset that consumer $t$ holds. Here, $\theta^j(t) > 0$ denotes the consumer $t$ demands of the $j^{\text{th}}$ asset and $\theta^j(t) < 0$ denotes the consumer $t$ supplies of the $j^{\text{th}}$ asset. Moreover, $\sum_{j=1}^J \theta^j(t) \cdot S^j(\omega)$ is the result of the ex-ante trade of agent $t$ if he chooses the portfolio $\theta(t)$. The restriction on ex-ante contracts is thus given, for each agent $t \in T$, by

$$\mathscr{G}_t = \left\{ \sum_{j=1}^J \theta^j(t) \cdot S^j : \theta(t) \in \mathbb{R}^J \right\}.$$

Upon the realization of the state of nature $\omega$, agent $t$ holding a portfolio $\theta_t$ commands the commodity bundle given by

$$e(t, \omega) + \sum_{j=1}^J \theta^j(t) \cdot S^j(\omega)$$

for all $\omega \in \Omega$. The utility $W_t(\theta(t))$ of agent $t$ for a portfolio $\theta(t)$ is given by

$$W_t(\theta(t)) = V_t \left( e(t, \cdot) + \sum_{j=1}^J \theta(t)^j \cdot S^j(\cdot) \right).$$

## 2.2 Asymmetric information economies

An alternative situation that our general model captures is the model of economies with asymmetric information. In such an economy, each economic agent trades state-contingent commodity vectors in the first period, where, for each physical commodity $l = 1, \cdots, \ell$ and state $\omega \in \Omega$, a unit of state-contingent commodity $l\omega$ is a title to receive a unit of the physical good $l$ if and only if $\omega$ occurs. Moreover, ex-ante (net) trades of each agent are compatible with the private information $\mathscr{F}_t$[11] of the corresponding agent about the state of nature which will be realized at time $\tau = 1$. It is assumed that at time $\tau = 0$ agents have imperfect information about the true state of the nature. Let $\mathscr{F}_t(\omega_0)$ be the event in the partition generating $\mathscr{F}_t$ containing the state $\omega_0$. If $\omega_0 \in \Omega$ occurs at $\tau = 1$, the agent $t$ only knows that the realized state belongs to $\mathscr{F}_t(\omega_0)$, that is, the agent knows that one of the states in $\mathscr{F}_t(\omega_0)$ is the realized state, but does not know the true state unless $\mathscr{F}_t(\omega_0)$ contains only one element. Since $\Omega$ is a finite set, only finitely many different private information sets are possible, which are denoted by the collection $\{\mathscr{F}_i : 1 \leq i \leq m\}$. Define

$$J_i = \{t \in T : \mathscr{F}_t = \mathscr{F}_i\}$$

and

$$\mathscr{G}_i = \{x : \Omega \to \mathbb{R}^\ell_+ : x \text{ is } \mathscr{F}_i\text{-measurable}\}.$$

Note that, $\mathscr{G}_i$ is a Borel measurability subset of $(\mathbb{R}^\ell)^\Omega$, for every $1 \leq i \leq m$. We assume that $J_i$ is a member of $\mathscr{T}$ and $T = \bigcup\{J_i : 1 \leq i \leq m\}$. Restrictions in trades at the ex-ante stage are asymmetric and defined by the requirement that each agent only subscribes those contracts that he is able to distinguish according to his private information. Thus, for each agent $t \in T$, we set $\mathscr{G}_t = \mathscr{G}_i$ if $t \in J_i$.

# 3 The ex-ante core

In this section, we introduce the classical *ex-ante core* for a two period exchange economy with uncertainty, and study its properties. In line with the classical core concept, it is assumed implicitly that trades take place at time $\tau = 0$ and that contracts are binding: they are carried out after the resolution of uncertainty and there is no possibility of their renegotiation. When compared to the literature in this area, our notion does not assume that consumption sets are lower bounded. Also, there may be additional restrictions imposed on ex-ante trades. Therefore, general definitions of resources assignments and allocations can be formulated as follows.

An *assignment* is a function $f : T \times \Omega \to \mathbb{R}^\ell$ such that $f(t, \omega) \in X_t(\omega)$ $\mu$-a.e. on $T$ and for all $\omega \in \Omega$. Define a sub-restriction $\mathscr{H}$ of the restriction $\mathscr{G}$ as a set

$$\mathscr{H} := \{\mathscr{H}_i \subseteq \mathscr{G}_i : i \in \mathbb{K}\}.$$

[11] $\mathscr{F}_t$ is a $\sigma$-algebra of sets generated by a partition of the set $\Omega$.

Let $\mathscr{H}_t = \mathscr{H}_i$ if $t \in I_i$ and $i \in \mathbb{K}$. As assignment $f$ is called *$\mathscr{H}$-allocation* if $f(t,\cdot) - e(t,\cdot) \in \mathscr{H}_t$ $\mu$-a.e. on $T$. Every $\mathscr{G}$-allocation is simply termed as an *allocation*. We denote by $\mathscr{A}^{\mathscr{H}}$ the set of all $\mathscr{H}$-allocations and by $\mathscr{A}$ the set of all allocations. An assignment $f$ is *feasible* if $\int_T f(\cdot,\omega)d\mu = \int_T e(\cdot,\omega)d\mu$, for all $\omega \in \Omega$. A *coalition* is a member of $\mathscr{T}$ whose measure is non-zero. Furthermore, a *sub-coalition* $S'$ of a coalition $S$ is a coalition $S'$ such that $S' \subseteq S$. For any coalition $S$, define $S_i = S \cap I_i$ for $i \in \mathbb{K}$ and let

$$\Lambda^S := \{i \in \mathbb{K} : \mu(S_i) > 0\} \text{ and } a^S := \inf\{\mu(S_i) : i \in \Lambda^S\}.$$

It follows that the set $\Lambda^S$ is finite if $a^S > 0$ and infinite if $a^S = 0$.

**Definition 3.1.** An assignment $f$ is *ex ante blocked by a coalition $S$ via an assignment $h$* if $\mu$-a.e. on $S$ and for all $\omega \in \Omega$:

(i) $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$;

(ii) $\int_S h(\cdot,\omega)d\mu = \int_S e(\cdot,\omega)\, d\mu.$

The *ex ante core* is the set of feasible allocations that are not ex ante blocked by any coalition via some allocation and it is denoted by $\mathscr{C}(\mathscr{E})$.

**Remark 3.2.** The existence of ex-ante core allocations with arbitrary short sales is proved in Koutsougeras (1998). The proof is for a finite economy (i.e. when $T_0$ is empty and $T_1$ is finite) with unbounded consumption sets. Assumptions are imposed on preferences, which are close to the generally adopted non-arbitrage conditions to show the existence of competitive equilibria (see e.g. Page, 1987). In the case of an atomless economy (i.e., the case where $T_1$ is empty), sufficient (non-arbitrage) conditions for the existence of competitive equilibria in line with Page (1987) are formulated in Le Van and Magnien (2005). Hence, adopting an approach similar to that in Koutsougeras (1998), it can be shown that the ex-ante core in our model is non-empty when the measure space of agents is atomless. The proceed towards the existence of ex-ante core allocations in general mixed markets, we build on the atomless case. Following Bhowmik and Graziano (2015), the mixed economy $\mathscr{E}$ can be associated to an atomless economy $\mathscr{E}^*$. Broadly speaking, this new economy arises by "splitting" each large agent into a continuum of small agents, whose characteristics are the same as those of the large agent. Therefore, a natural correspondence can be constructed between allocations in the two markets (compare Bhowmik and Graziano, 2015, Lemma 4.3, Proposition 4.4).

## 3.1 The ex-ante core and the size of blocking coalitions

The main result of this section is the extension of Vind's theorem (Vind, 1972) to the ex-ante core. The Vind's theorem states that whenever there exists a coalition that improves a given allocation $f$, then, for any measure $\varepsilon \in (0, \mu(T))$, there is a coalition $R$ whose measure is exactly equal to $\varepsilon$ and is such that $f$ is blocked by $R$. The proof requires the assumption that the space is atomless and depends on the validity of Lyapunov's convexity theorem for the range of a finite dimensional vector measure. We now make two assumptions related to a sub-restriction $\mathscr{H}$ on the ex-ante choice set which is helpful to obtain our main results.

($\mathbf{A}_8'$) For any $(t, \omega) \in T \times \Omega$, $x \in X_t$ with $x - e(t, \cdot) \in \mathscr{H}_t$ and $\varepsilon > 0$, there is an $y \in \bigcap\{\varepsilon\mathscr{H}_i : i \in \mathbb{K}\} \cap \mathbb{B}(0, \varepsilon)^\Omega$ such that $x + y \in X_t$ and $u_t(\omega, x(\omega) + y(\omega)) > u_t(\omega, x(\omega))$.

($\mathbf{A}_9'$) $\mathscr{H}_i$ is a convex set with $0 \in \mathscr{H}_i$ and $-\mathscr{H}_i \subseteq \mathscr{H}_i$ for all $i \in \mathbb{K}$.

**Remark 3.3.** If $\mathscr{H}_i = \mathscr{G}_i$ for all $i \in \mathbb{K}$, then ($\mathbf{A}_8'$) and ($\mathbf{A}_9'$) are satisfied under ($\mathbf{A}_8$) and ($\mathbf{A}_9$), respectively.

We proceed with a general formulation of the Vind's result. The first theorem takes a form which allows us the aforementioned application to asset market models.

**Theorem 3.4.** *Assume that the economy $\mathscr{E}$ is atomless and consider a sub-restriction $\mathscr{H}$ that satisfies ($\mathbf{A}_8'$) and ($\mathbf{A}_9'$). Let $f$ be a feasible $\mathscr{H}$-allocation ex ante blocked by a coalition $S$ via some $\mathscr{H}$-allocation $h$. Under ($\mathbf{A}_1$)-($\mathbf{A}_7$), for any given $0 < \varepsilon < \mu(T)$, there is a coalition $R$ and an assignment $y$ such that $\mu(R) = \varepsilon$ and $f$ is ex ante blocked by $R$ via $y$ and $y(t, \cdot) - e(t, \cdot) \in 4\mathscr{H}_t$ $\mu$-a.e. on $T$.*

The second formulation follows from Theorem 3.4 and Remark 3.3

**Corollary 3.5.** *Assume that the economy $\mathscr{E}$ is atomless. Let $f$ be a feasible allocation not in the ex-ante core. Under ($\mathbf{A}_1$)-($\mathbf{A}_9$), for any given $0 < \varepsilon < \mu(T)$, there is a coalition $R$ such that $\mu(R) = \varepsilon$ and $f$ is ex ante blocked by $R$ via some allocation.*

As consequence, the ex-ante core is equal to the set of allocations that are neither blocked by arbitrary small coalitions nor by arbitrary large coalitions. In particular, whenever the formation of coalitions implies costs proportional to their size $\mu$, natural interpretations arise.

**Remark 3.6.** We note that, when the commodity space is infinite dimensional, the conclusion of Vind's theorem is not confirmed in the absence of some additional assumptions. The case of infinitely many commodities is of interest since it permits, for example, treatment of the allocation problems over an infinite time horizon and study of economies with commodity differentiation. In these cases, Lyapunov's convexity theorem does not hold or holds only in "weak forms". Moreover, in markets with infinitely many commodities, it may be true, also, that the positive cone of the commodity space lacks interior points. Their absence has been shown to cause further technical problems in reproducing the classical proof of Vind's result (see the discussion in Bhowmik and Graziano, 2015). Recent extensions of Vind's theorem to infinite dimensional commodity spaces include Hervés-Beloso et al. (2000), Evren and Hüsseinov (2008), Bhowmik and Cao (2012), Bhowmik and Cao (2013), Bhowmik and Graziano (2015). In particular, results provided in Evren and Hüsseinov (2008) take account of consumption sets that are not lower bounded, in an ordered Banach space whose positive cone has interior points. It is worth mentioning that, compared to our Theorem 3.4 and Corollary 3.5, their extension of Vind's theorem cannot be applied to asset markets case.

**Remark 3.7.** Lyapunov's theorem also does not hold for the case of an agent space which may include atoms. In Bhowmik and Graziano (2015), the classical theorem on the size of blocking coalitions is extended to the case of atoms by considering the general class of Aubin coalitions. Results for the size of ordinary blocking coalitions in mixed market are provided in Evren and Hüsseinov (2008), while the support of Aubin blocking coalitions is studied by Pesce (2014).

## 3.2 Coalitional fairness of ex-ante core allocations

In this section we study the fairness properties of ex-ante core allocations, that is, the extent to which net trades which are stable with respect to ex-ante coalition improvements are also equitable. The model described in the above sections, rules out consideration of classical individual notions of envy-freeness and fairness such as those proposed by Foley (1967) and Varian (1974). Indeed, with individual fairness, each individual evaluates a given allocation by making utility comparisons between the commodity bundle he receives and the commodity bundles received by the other agents. So, it might be that agents are expected to compare bundles that are not compatible with restrictions imposed ex-ante on their trades through $\mathscr{G}$. Consequently, we address the equity properties of ex-ante core allocations by adopting the notion of coalitional fairness.

Starting from the idea of fair net trades for coalitions, proposed for mixed economies by Gabszewicz (1975), in this section we define the notion of *coalitional fairness* for allocations in a two-period exchange economy. An allocation is considered coalitionally fair (c-fair) if no coalition can redistribute among its members the net trade of any other coalition, in such a way that each of them is better off. For an atomless economy without uncertainty, competitive equilibria are c-fair, and c-fair allocations are in the core. Hence, as a consequence of the core equivalence result, coalitional fairness turns out to be equivalent to the core property. In our model with consumption sets not lower bounded and restrictions imposed on ex-ante trade, there are no simple results that hold true. Indeed, as mentioned before, it is unclear both whether the core equivalence theorem holds and which competitive equilibrium notion should be adopted. Additional difficulties arise due to the presence of atoms, the case where the set of c-fair allocations is strictly smaller than the core (see Gabszewicz, 1975).

The following theorems ensure weaker versions of the result. Under a suitable set of assumptions, each ex-ante core allocation of a mixed economy is such that no coalition of small agents can benefit from achieving the net trade of any other coalition comprised of all the large agents, letting each member exchange according to restrictions on trades, and vice versa.

The first notion of fairness requires that no coalition of small agents envies the net trade of a disjoint coalition comprised of all large agents.

**Definition 3.8.** A feasible allocation $f$ is called *ex-ante* $\mathscr{C}_{(\mathscr{T}_0,\mathscr{T}_1)}(\mathscr{E})$*-fair* if there do not exist two disjoint elements $S_1 \in \mathscr{T}_0$, $S_2 \in \mathscr{T}_1$ and an allocation $h$ such that $\mu$-a.e. on $S_1$ and for each $\omega \in \Omega$:

(i) $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$;

(ii) $\displaystyle\int_{S_1} (h(\cdot,\omega) - e(\cdot,\omega))d\mu = \int_{S_2} (f(\cdot,\omega) - e(\cdot,\omega))d\mu.$

Ex-ante core allocations satisfy coalitional fairness with respect to $\mathscr{T}_0$ and $\mathscr{T}_1$. Like in the previous section, we provide two formulations of our main results.

**Theorem 3.9.** *Consider a sub-restriction $\mathscr{H}$ that satisfies* ($\mathbf{A}_8'$) *and* ($\mathbf{A}_9'$). *Let $f$ and $h$ be two $\mathscr{H}$-allocations. Assume further that there exist two disjoint elements $S_1 \in \mathscr{T}_0$ and $S_2 \in \mathscr{T}_1$ such that $\mu$-a.e. on $S_1$ and for each $\omega \in \Omega$:*

(i) $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$*;*

(ii) $\displaystyle\int_{S_1} (h(\cdot,\omega) - e(\cdot,\omega))d\mu = \int_{S_2} (f(\cdot,\omega) - e(\cdot,\omega))d\mu.$

*Under* $(\mathbf{A}_1)$-$(\mathbf{A}_7)$, *there is a coalition* $R$ *and an assignment* $y$ *such that* $f$ *is ex ante blocked by* $R$ *via* $y$*; and* $y(t,\cdot) - e(t,\cdot) \in 4\mathscr{H}_t$ $\mu$*-a.e. on* $T$.

**Corollary 3.10.** *Let* $(\mathbf{A}_1)$-$(\mathbf{A}_9)$ *be satisfied. Then any ex-ante core allocation is ex-ante* $\mathscr{C}_{(\mathscr{T}_0,\mathscr{T}_1)}(\mathscr{E})$*-fair.*

In the next notion of fairness the role of coalitions is exchanged. The fair criterion requires that no coalition containing all large agents envies a disjoint coalition of small agents because its net trade would make its members better off.

**Definition 3.11.** A feasible allocation $f$ is called *ex-ante* $\mathscr{C}_{(\mathscr{T}_1,\mathscr{T}_0)}(\mathscr{E})$*-fair* if there do not exist two disjoint elements $S_1 \in \mathscr{T}_1$, $S_2 \in \mathscr{T}_0$ and an allocation $h$ such that $\mu$-a.e. on $S_1$ and for each $\omega \in \Omega$:

(i) $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$;

(ii) $\displaystyle\int_{S_1} (h(\cdot,\omega) - e(\cdot,\omega))d\mu = \int_{S_2} (f(\cdot,\omega) - e(\cdot,\omega))d\mu.$

The following fairness property of ex-ante core allocations holds true.

**Theorem 3.12.** *Consider a sub-restriction* $\mathscr{H}$ *that satisfies* $(\mathbf{A}'_8)$ *and* $(\mathbf{A}'_9)$*. Let* $f$ *and* $h$ *be two* $\mathscr{H}$*-allocations. Assume further that there exist two disjoint elements* $S_1 \in \mathscr{T}_1$ *and* $S_2 \in \mathscr{T}_0$ *such that* $\mu$*-a.e. on* $S_1$ *and for each* $\omega \in \Omega$*:*

(i) $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$;

(ii) $\displaystyle\int_{S_1} (h(\cdot,\omega) - e(\cdot,\omega))d\mu = \int_{S_2} (f(\cdot,\omega) - e(\cdot,\omega))d\mu.$

*Under* $(\mathbf{A}_1)$-$(\mathbf{A}_7)$, *there is a coalition* $R$ *and an assignment* $y$ *such that* $f$ *is ex ante blocked by* $R$ *via* $y$*; and* $y(t,\cdot) - e(t,\cdot) \in 4\mathscr{H}_t$ $\mu$*-a.e. on* $T$*, provided any one of the following two conditions is satisfied:*

(a) $a^{S_1} > 0$.

(b) $a^{S_1} = 0$ *and there is an* $\varepsilon > 0$ *such that* $e(t,\omega) + \mathbb{B}(0,\varepsilon) \subseteq X_t(\omega)$ *for all* $(t,\omega) \in T \times \Omega$*, there are some* $i_0 \in \Lambda^{S_1}$ *and* $\delta > 0$ *such that for any measurable subset* $D$ *of* $S_1$ *with* $\mu(D) < \delta$,
$$\frac{2}{\mu(S_{i_0})}\int_D (h-e)d\mu \in \mathscr{H}_{i_0}.$$

**Corollary 3.13.** *Let* $(\mathbf{A}_1)$-$(\mathbf{A}_9)$ *be satisfied and* $a^S > 0$*, for any coalition* $S$*. Then any ex-ante core allocation is ex-ante* $\mathscr{C}_{(\mathscr{T}_1,\mathscr{T}_0)}(\mathscr{E})$*-fair.*

Our final fairness requirement states that no coalition of small agents envies the net trade of a disjoint coalition containing all large agents and vice versa.

**Definition 3.14.** A feasible allocation $f$ is called *ex-ante* $\mathscr{C}_{\{\mathscr{T}_0,\mathscr{T}_1\}}(\mathscr{E})$*-fair* if it is ex-ante $\mathscr{C}_{(\mathscr{T}_0,\mathscr{T}_1)}(\mathscr{E})$-fair and ex-ante $\mathscr{C}_{(\mathscr{T}_1,\mathscr{T}_0)}(\mathscr{E})$-fair.

As consequences of Corollaries 3.10 and 3.13, we obtain the following property of ex-ante core allocations.

**Theorem 3.15.** *Let* $(\mathbf{A}_1)$*-*$(\mathbf{A}_9)$ *be satisfied and* $a^S > 0$*, for any coalition* $S$*. Then any ex-ante core allocation is ex-ante* $\mathscr{C}_{\{\mathscr{T}_0,\mathscr{T}_1\}}(\mathscr{E})$*-fair.*

**Remark 3.16.** The study of c-fair allocations in mixed markets with asymmetric information is not new. For example, positive results in the presence of atoms are provided in Donnini et al. (2014) and Bhowmik (2015), among others. To our knowledge, the fairness of ex-ante core allocations has not been studied in relation to asset market economies. The consequences for this specification of our model are discussed in section 5. Notice, also, that if the economy is atomless, choosing the empty coalition as coalition $S_2$ in Definition 3.11, shows that each ex-ante $\mathscr{C}_{\{\mathscr{T}_0,\mathscr{T}_1\}}(\mathscr{E})$-fair allocation is a core allocation. Consequently, Theorem 3.15 implies that the ex-ante core of an atomless economy coincides with the set of ex-ante $\mathscr{C}_{\{\mathscr{T}_0,\mathscr{T}_1\}}(\mathscr{E})$-fair allocations.

# 4 The sequential core

This section considers sequential core notions to analyze both the Vind's theorem and coalitional fairness.

In our definition of core allocation, we discard the usual premise that agents in a coalition coordinate over a set of actions, which are enforceable, and that all commitments are binding. Hence, agents are allowed to reconsider their positions after the resolution of uncertainty and coalitions may be able, at the ex-post stage, to improve upon the allocation received. Based on these assumptions about the nature of sequential trades, the classical core notion can be modified to guarantee that an allocation is both stable at the ex-ante stage and stable to coalition improvements when the state of nature is realized. In other words, in this perspective, a core allocation is unblocked not only at time $\tau = 0$, but also in any possible state at time $\tau = 1$.

Analysis of the literature shows that differences in the blocking mechanism proposed at $\tau = 0$, lead to different existing sequential core notions. This applies, for example, to the cases of a two-stage core, a strong sequential core and a weak sequential core, among others.

In what follows, a sequential core allocation is typically formed by a pair $(f, g)$: $g$ is the assignment of the bundles to be consumed after re-contracting at $\tau = 1$; $f$ is the allocation of resources chosen by agents at $\tau = 0$ according to their constrained consumption sets and which makes $g$ an admissible choice for the agents at $\tau = 1$. To make $g$ admissible, it is enough that, with the initial resources at $\tau = 1$ given by $f$, agents have no incentive to deviate from $g$. At the same time, $g$ must be robust to deviations at $\tau = 0$.

Here, we consider two notions of core: the *sequential core* in which the component $f$ of sequential trade is required to belong to the ex-ante core; and the weaker two stage core (see Koutsougeras, 1998). We extend Vind's theorem and study coalitional fairness in relation to the sequential core. We compare our results for the sequential core notion with those for the two-stage core. A similar comparison can be conducted for the other sequential core notions. For completeness, an overview of these notions and their relationships is provided at the end of the section.

Given an assignment $f$, define $\mathscr{E}(f)$ to be the economy which is the same as $\mathscr{E}$ except for the initial endowment assignment which is replaced by $f$. A *two-period trading plan* $(f, g)$ is a combination of an allocation $f$ and an assignment $g$ in $\mathscr{E}(f)$. For the formal introduction of sequential core notions, we need to consider the family of economies $\{\mathscr{E}(f; \omega)\}_{\omega \in \Omega}$, one for each possible state of nature $\omega \in \Omega$ following the resolution of uncertainty. The economy $\mathscr{E}(f; \omega)$ has the following specification

$$\mathscr{E}(f; \omega) := \{(X_t(\omega), u_t(\omega, \cdot), f(t, \omega)) : t \in T\}.$$

**Definition 4.1.** Given an assignment $f$, let $\mathscr{C}(\mathscr{E}; f)$ denote the set of assignments $g$ in $\mathscr{E}(f)$ such that

(i) $\displaystyle\int_T g(\cdot, \omega) d\mu = \int_T f(\cdot, \omega) d\mu$, for all $\omega \in \Omega$;

(ii) there are no state $\omega_0$, coalition $S$ and assignment $h$ in $\mathscr{E}(\omega_0)$ such that

1. $u_t(\omega_0, h(t)) > u_t(\omega_0, g(t, \omega_0))$ $\mu$-a.e. on $S$, and
2. $\displaystyle\int_S h d\mu = \int_S f(\cdot, \omega_0) d\mu.$

Notice that the set $\mathscr{C}(\mathscr{E}; f)$ introduced with Definition 4.1, can be identified with the ex-post core of the economy $\mathscr{E}(f)$ and the relative blocking mechanism with the ex-post blocking mechanism. This implies, in particular, that each $g \in \mathscr{C}(\mathscr{E}; f)$ is ex-post individually rational.

The following definition introduces formally the sequential core.

**Definition 4.2.** A two-period trading plan $(f, g)$ is in the *sequential core* if $f \in \mathscr{C}(\mathscr{E})$ and $g \in \mathscr{C}(\mathscr{E}; f)$.

The notion of two-stage core is due to Koutsougeras (1998).

**Definition 4.3.** A two-period trading plan $(f, g)$ is in the *two-stage core* if the following are satisfied:

(i) $\int_T f(\cdot, \omega) d\mu = \int_T e(\cdot, \omega) d\mu$ for all $\omega \in \Omega$ and $g \in \mathscr{C}(\mathscr{E}; f)$;

(ii) there are no coalition $S$ and assignment $h$ such that

1. $h(t, \cdot) \in \mathscr{A}_t$ and $V_t(h(t, \cdot)) > V_t(g(t, \cdot))$, $\mu$-a.e. on $S$;
2. $\int_S h(\cdot, \omega) d\mu = \int_S e(\cdot, \omega) d\mu$, for all $\omega \in \Omega$.

Let us denote by $SC(\mathscr{E})$ and $TSC(\mathscr{E})$, respectively, the sequential core and the two-stage core of the economy $\mathscr{E}$. Notice that the blocking mechanism introduced with point (ii) of Definition 4.3 makes no use of allocation $f$ at all. The only role played by $f$ is to make $g$ an admissible choice at $\tau = 1$. Improvements at time $\tau = 0$ must be compatible with restrictions on trades for the members of a blocking coalition and no exchange is expected within the coalition after the resolution of uncertainty. Therefore, Definition 4.3 requires a comparison of ex-ante utilities in assignments $h$ and $g$, where $h$ is compatible with ex-ante restrictions on trade while $g$ is not. This technical asymmetry between properties of $h$ and $g$ makes the results valid in the standard case non immediately applicable.

Consider now a two-period trading plan $(f, g)$ in the sequential core $SC(\mathscr{E})$. Since the allocation $g$ in the trading plan $(f, g)$ of $SC(\mathscr{E})$ is ex-post individually rational in the economy $\mathscr{E}(f)$, it is true that $u_t(\omega, g(t, \omega)) \geq u_t(\omega, f(t, \omega))$, $\mu$-a.e. on $T$ and for each $\omega \in \Omega$. So, condition 1. in (ii) of Definition 4.3, when formulated for $(f, g)$, would imply also that $V_t(h(t, \cdot)) > V_t(f(t, \cdot))$, $\mu$-a.e. on $S$, contradicting the fact that $f$ is an ex-ante core allocation. Consequently, one has the inclusion $SC(\mathscr{E}) \subseteq TSC(\mathscr{E})$. In fact, the sequential core $SC(\mathscr{E})$ has been used to show that the two-stage core of a finite economy is non-empty in Koutsougeras (1998).

## 4.1 The sequential core and the size of blocking coalitions

Now we formulate a version of the theorem on the blocking power of small and big coalitions for the sequential core. Its proof follows from the corresponding Theorem 3.4 proved in Subsection 3.1 for the ex-ante core and is given in Appendix A.

**Theorem 4.4.** *Assume that the economy $\mathscr{E}$ is atomless and $f$ is an assignment such that $f(t,\omega)$ is an interior point of $X_t(\omega)$ for all $(t,\omega) \in T \times \Omega$. Let $g$ be a feasible assignment in $\mathscr{E}(f)$ not belonging to the ex post core of $\mathscr{E}(f)$. Under* $(\mathbf{A}_1)$-$(\mathbf{A}_8)$*, for any given $0 < \varepsilon < \mu(T)$, there is a coalition $R$ and an assignment $y$ such that $\mu(R) = \varepsilon$ and $g$ is ex post blocked by $R$ via $y$ in $\mathscr{E}(f)$.*

**Corollary 4.5.** *Let $(f, g)$ be a two-period trading plan not in the sequential core. Suppose further that $f(t,\omega)$ is an interior point of $X_t(\omega)$ for all $(t,\omega) \in T \times \Omega$. Under* $(\mathbf{A}_1)$-$(\mathbf{A}_9)$*, for any given $0 < \varepsilon < \mu(T)$, there is a coalition $R$ with $\mu(R) = \varepsilon$ such that either one of the following is satisfied:* (i) *$f$ is ex ante blocked by $R$ via some allocation in $\mathscr{E}$; or* (ii) *$g$ is ex post blocked by $R$ via some assignment in $\mathscr{E}(f)$.*

The following example shows that a result similar to Corollary 4.5 is not valid for the dominance relation defining the two-stage core although all the conditions listed in Corollary 4.5 are satisfied. The example is formulated within the special case of asymmetric information economies.

**Example 4.6.** Consider an asymmetric information economy whose space of agents is the probability measure space $(T, \mathscr{T}, \mu)$, where $T = [0,1]$ is endowed with the Borel $\sigma$-algebra $\mathscr{T}$ and the Lebesgue probability measure $\mu$. The exogenous uncertainty is described by a measurable space $(\Omega, \mathscr{F})$, where $\Omega = \{a, b\}$ and $\mathscr{F}$ is the power set of $\Omega$. The commodity space is $\mathbb{R}^2$. Let $X_t(\omega) = \mathbb{R}^2_+$ and $\mathbb{P}_t(\omega) = \frac{1}{2}$ for all $(t,\omega) \in T \times \Omega$. Let $\mathscr{F}_t = \mathscr{F}$ for all $t \in T$. Put,

$$e(t,\omega) := \begin{cases} (6,2), & \text{if } (t,\omega) \in [0, \frac{1}{4}) \times \Omega; \\ (2,6), & \text{if } (t,\omega) \in [\frac{1}{4}, \frac{1}{2}) \times \Omega; \\ (8,8), & \text{if } (t,\omega) \in [\frac{1}{2}, 1] \times \Omega, \end{cases}$$

and

$$u_t(\omega, x) := \begin{cases} x_1 \mathbf{1}_a(\omega) + x_2 \mathbf{1}_b(\omega), & \text{if } (t,\omega) \in [0, \frac{1}{4}) \times \Omega; \\ x_2 \mathbf{1}_a(\omega) + x_1 \mathbf{1}_b(\omega), & \text{if } (t,\omega) \in [\frac{1}{4}, \frac{1}{2}) \times \Omega; \\ x_1 + x_2, & \text{if } (t,\omega) \in [\frac{1}{2}, 1] \times \Omega, \end{cases}$$

where $\mathbf{1}_{\omega_0}(\omega) = 1$ if $\omega_0 = \omega$; and $\mathbf{1}_{\omega_0}(\omega) = 0$, otherwise. Note that the aggregate endowment in each state is $(6,6)$. Define a feasible allocation $f$ by

$$f(t,\omega) := \begin{cases} (11,11), & \text{if } (t,\omega) \in [0, \frac{1}{2}) \times \Omega; \\ (1,1), & \text{if } (t,\omega) \in [\frac{1}{2}, 1] \times \Omega. \end{cases}$$

Let $g$ be a feasible assignment such that

$$g(t,\omega) := \begin{cases} (22,0)\mathbf{1}_a + (0,22)\mathbf{1}_b, & \text{if } t \in [0,\frac{1}{4}); \\ (0,22)\mathbf{1}_a + (22,0)\mathbf{1}_b, & \text{if } t \in [\frac{1}{4},\frac{1}{2}); \\ (1,1), & \text{if } t \in [\frac{1}{2},1]. \end{cases}$$

Note that $g \in \mathscr{C}(\mathscr{E}; f)$. It can be easily verified that $g$ can be blocked in the sense of (ii) of Definition 4.3 by any sub-coalition of $[\frac{1}{2},1]$. However, the same cannot be done by a coalition whose measure is close to 1. Notice also that, since the allocation $(f,g)$ does not belong to the two-stage core, then a fortiori it does not belong to the sequential core. Consequently, the conclusion of Theorem 4.4 for this allocation holds true.

## 4.2 Coalitional fairness of sequential core allocations

The following definitions formalize coalitional fairness properties of allocations at the ex-post stage. When paired with the corresponding ex-ante notions, they allow us to formulate sequential fairness properties of allocations.

**Definition 4.7.** Given an assignment $f$, an assignment $g$ is *ex post* $\mathscr{C}_{(\mathscr{T}_0,\mathscr{T}_1)}(\mathscr{E}; f)$*-fair* if there do not exist a state $\omega_0$, two disjoint elements $S_1 \in \mathscr{T}_0$, $S_2 \in \mathscr{T}_1$ and an assignment $h$ in $\mathscr{E}(f;\omega_0)$ such that $u_t(\omega_0, h(t)) > u_t(\omega_0, g(t,\omega_0))$ $\mu$-a.e. on $S_1$, and

$$\int_{S_1} (h - f(\cdot,\omega_0))d\mu = \int_{S_2} (g(\cdot,\omega_0) - f(\cdot,\omega_0))d\mu.$$

Also in the ex post case, symmetric definitions can be formulated by interchanging the role of $\mathscr{T}_1$ and $\mathscr{T}_0$.

**Definition 4.8.** Given an assignment $f$, an assignment $g$ is *ex post* $\mathscr{C}_{(\mathscr{T}_1,\mathscr{T}_0)}(\mathscr{E}; f)$*-fair* if there do not exist a state $\omega_0$, two disjoint elements $S_1 \in \mathscr{T}_1$, $S_2 \in \mathscr{T}_0$ and an assignment $h$ in $\mathscr{E}(f;\omega_0)$ such that $u_t(\omega_0, h(t)) > u_t(\omega_0, g(t,\omega_0))$ $\mu$-a.e. on $S_1$, and

$$\int_{S_1} (h - f(\cdot,\omega_0))d\mu = \int_{S_2} (g(\cdot,\omega_0) - f(\cdot,\omega_0))d\mu.$$

The joint use of conditions contained in Definitions 4.7 and 4.8 gives us a notion of coalitional fairness of allocations under which coalitions of small agents and those containing all large agents are immune from reciprocal envy at time $\tau = 1$.

**Definition 4.9.** Given an assignment $f$, an assignment $g$ is *ex post* $\mathscr{C}_{\{\mathscr{T}_1,\mathscr{T}_0\}}(\mathscr{E}; f)$*-fair* if it is ex post $\mathscr{C}_{(\mathscr{T}_0,\mathscr{T}_1)}(\mathscr{E}; f)$-fair as well as ex post $\mathscr{C}_{(\mathscr{T}_1,\mathscr{T}_0)}(\mathscr{E}; f)$-fair.

Finally, a two-period trading plan $(f, g)$ which is *sequential* $\mathscr{C}_{\{\mathscr{T}_0,\mathscr{T}_1\}}$*-fair* ensures that coalitions of small agents do not envy coalitions containing all large agents and vice versa when the state of nature is unknown as well as when it has been revealed.

**Definition 4.10.** A two-period trading plan $(f, g)$ is called *sequential* $\mathscr{C}_{\{\mathscr{T}_0,\mathscr{T}_1\}}$*-fair* if it is ex ante $\mathscr{C}_{\{\mathscr{T}_0,\mathscr{T}_1\}}$-fair and ex post $\mathscr{C}_{\{\mathscr{T}_0,\mathscr{T}_1\}}$-fair.

Our main results show that sequential core allocations of the mixed market satisfy sequential coalitional fairness.

**Theorem 4.11.** *Assume that $f$ is an assignment such that $f(t, \omega)$ is an interior point of $X_t(\omega)$ for all $(t, \omega) \in T \times \Omega$, and $g \in \mathscr{C}(\mathscr{E}; f)$. Under* $(\mathbf{A}_1)$*-*$(\mathbf{A}_8)$*, $g$ is ex post* $\mathscr{C}_{\{\mathscr{T}_0,\mathscr{T}_1\}}$*-fair.*

**Corollary 4.12.** *Let $(f, g)$ be in the sequential core. Suppose further that $f(t, \omega)$ is an interior point of $X_t(\omega)$ for all $(t, \omega) \in T \times \Omega$. Under* $(\mathbf{A}_1)$*-*$(\mathbf{A}_9)$*, $(f, g)$ is sequential* $\mathscr{C}_{\{\mathscr{T}_0,\mathscr{T}_1\}}$*-fair.*

Analogous properties of coalitional fairness can be formulated assuming that the dominance relation inducing envy between coalitions is consistent with the dominance relation introduced in Definition 4.3 to define the two-stage core.

**Definition 4.13.** A two-period trading plan $(f, g)$ is called *two-stage* $\mathscr{C}_{(\mathscr{T}_0,\mathscr{T}_1)}$*-fair* if the following are satisfied:

(i) $\displaystyle\int_T f(\cdot, \omega) d\mu = \int_T e(\cdot, \omega) d\mu$ for all $\omega \in \Omega$ and $g$ is ex post $\mathscr{C}_{(\mathscr{T}_0,\mathscr{T}_1)}(\mathscr{E}; f)$-fair;

(ii) there do not exist two disjoint elements $S_1 \in \mathscr{T}_0$, $S_2 \in \mathscr{T}_1$ and an assignment $h$ such that

1. $h(t, \cdot) \in \mathscr{A}_t$ and $V_t(h(t, \cdot)) > V_t(g(t, \cdot))$ $\mu$-a.e. on $S_1$;
2. $\displaystyle\int_{S_1} (h(\cdot, \omega) - e(\cdot, \omega)) d\mu = \int_{S_2} (f(\cdot, \omega) - e(\cdot, \omega)) d\mu$, for each $\omega \in \Omega$.

**Definition 4.14.** A two-period trading plan $(f, g)$ is called *two-stage* $\mathscr{C}_{(\mathscr{T}_1,\mathscr{T}_0)}$*-fair* if the following conditions hold:

(i) $\displaystyle\int_T f(\cdot, \omega) d\mu = \int_T e(\cdot, \omega) d\mu$ for all $\omega \in \Omega$ and $g$ is expost $\mathscr{C}_{(\mathscr{T}_1,\mathscr{T}_0)}(\mathscr{E}; f)$-fair; and

(ii) there do not exist two disjoint elements $S_1 \in \mathscr{T}_1$, $S_2 \in \mathscr{T}_0$ and an assignment $h$ such that

1. $h(t,\cdot) \in \mathscr{A}_t$ and $V_t(h(t,\cdot)) > V_t(g(t,\cdot))$ $\mu$-a.e. on $S_1$;
2. $\displaystyle\int_{S_1} (h(\cdot,\omega) - e(\cdot,\omega))d\mu = \int_{S_2} (f(\cdot,\omega) - e(\cdot,\omega))d\mu$, for each $\omega \in \Omega$.

**Definition 4.15.** A two-period trading plan $(f, g)$ is called *two-stage $\mathscr{C}_{\{\mathscr{T}_0,\mathscr{T}_1\}}$-fair* if it is two-stage $\mathscr{C}_{(\mathscr{T}_0,\mathscr{T}_1)}$-fair and two-stage $\mathscr{C}_{(\mathscr{T}_1,\mathscr{T}_0)}$-fair.

However, the following example shows that a result similar to Corollary 4.12 cannot be stated for the two-stage core of the mixed market. Again, we make use of the specification given by asymmetric information.

**Example 4.16.** Consider an asymmetric information economy whose space of agents is the probability measure space $(T, \mathscr{T}, \mu)$, where $T = [0, \frac{1}{2}] \cup \{1\}$. It is assumed that $[0, \frac{1}{2}]$ is endowed with the Borel $\sigma$-algebra $\mathscr{B}([0, \frac{1}{2}])$ and the Lebesgue measure $\nu$; $\mathscr{T} = \mathscr{B}([0, \frac{1}{2}]) \otimes \{1\}$ and the Lebesgue probability measure $\mu$ is defined by $\mu(A) = \nu(A)$ if $A \in \mathscr{B}([0, \frac{1}{2}])$ and $\mu(A) = \nu(A \cap [0, \frac{1}{2}]) + \frac{1}{2}$, otherwise. It particular, $\mu(\{1\}) = \frac{1}{2}$. The exogenous uncertainty is described by a measurable space $(\Omega, \mathscr{F})$, where $\Omega = \{a, b\}$ and $\mathscr{F}$ is the power set of $\Omega$. The commodity space is $\mathbb{R}^2$. Let $X_t(\omega) = \mathbb{R}^2_+$ and $\mathbb{P}_t(\omega) = \frac{1}{2}$ for all $(t, \omega) \in T \times \Omega$. Let $\mathscr{F}_t = \mathscr{F}$ for all $t \in T$. Put,

$$e(t,\omega) := \begin{cases} (6,2), & \text{if } (t,\omega) \in [0, \frac{1}{4}) \times \Omega; \\ (2,6), & \text{if } (t,\omega) \in [\frac{1}{4}, \frac{1}{2}] \times \Omega; \\ (4,4), & \text{if } (t,\omega) \in \{1\} \times \Omega, \end{cases}$$

and

$$u_t(\omega, x) := \begin{cases} x_1 \mathbf{1}_a(\omega) + x_2 \mathbf{1}_b(\omega), & \text{if } (t,\omega) \in [0, \frac{1}{4}) \times \Omega; \\ x_2 \mathbf{1}_a(\omega) + x_1 \mathbf{1}_b(\omega), & \text{if } (t,\omega) \in [\frac{1}{4}, \frac{1}{2}] \times \Omega; \\ x_1 + x_2, & \text{if } (t,\omega) \in \{1\} \times \Omega, \end{cases}$$

where $\mathbf{1}_{\omega_0}(\omega) = 1$ if $\omega_0 = \omega$; and $\mathbf{1}_{\omega_0}(\omega) = 0$, otherwise. Note that the aggregate endowment in each state is $(4, 4)$. Define a feasible allocation $f$ by

$$f(t,\omega) := \begin{cases} (2,2), & \text{if } (t,\omega) \in [0, \frac{1}{2}) \times \Omega; \\ (6,6), & \text{if } (t,\omega) \in \{1\} \times \Omega, \end{cases}$$

Let $g$ be a feasible assignment such that

$$g(t,\omega) := \begin{cases} (4,0)\mathbf{1}_a + (0,4)\mathbf{1}_b, & \text{if } t \in [0, \frac{1}{4}); \\ (0,4)\mathbf{1}_a + (4,0)\mathbf{1}_b, & \text{if } t \in [\frac{1}{4}, \frac{1}{2}]; \\ (6,6), & \text{if } t = 1. \end{cases}$$

It can be easily verified that $(f, g)$ is in the two-stage core. To show that it is not two-stage $\mathscr{C}_{\{\mathscr{T}_0.\mathscr{T}_1\}}$-fair, define

$$h(t,\omega) := \begin{cases} (8,4), & \text{if } (t,\omega) \in [0, \frac{1}{4}) \times \Omega; \\ (4,8), & \text{if } (t,\omega) \in [\frac{1}{4}, \frac{1}{2}] \times \Omega; \\ (4,4), & \text{if } (t,\omega) \in \{1\} \times \Omega, \end{cases}$$

Let $S_1 = [0, \frac{1}{2}]$ and $S_2 = \{1\}$. It is clear that $h(t, \cdot) \in \mathscr{A}_t$ and $V_t(h(t, \cdot)) > V_t(g(t, \cdot))$ $\mu$-a.e. on $S_1$, and

$$\int_{S_1} (h(\cdot,\omega) - e(\cdot,\omega))d\mu = (1,1) = \int_{S_2} (f(\cdot,\omega) - e(\cdot,\omega))d\mu$$

for all $\omega \in \Omega$. So, the two-period trading plan $(f, g)$ is not two-stage $\mathscr{C}_{\{\mathscr{T}_0.\mathscr{T}_1\}}$-fair. Notice that the example shows also that the sequential core is a proper subset of the two-stage core.

## 4.3 A comparison with other sequential core notions

Suppose that the Definition 4.3 is reformulated with no requirement for the allocation $h$ improving $g$ to be consistent with the constraints imposed on ex-ante trade, that is removing the condition $h(t, \cdot) \in \mathscr{A}_t$, $\mu$-a.e. on $S$. The corresponding core notion is the idea of a *strong sequential core* proposed by Predtetchinski et al. (2002).

The strong sequential core, denoted by $SSC(\mathscr{E})$, is included in the two-stage core. Both definitions of strong sequential core and two-stage core, and the notion of sequential core, coincide in the requirement that $g$ is robust to deviations when the state of nature is realized. In all cases, the dominance relation employed is the standard one.

This part of the definition is the same for the notion of *weak sequential core* proposed by Herings et al (2006). Weak sequential core is defined based on the Definition 4.3 when the requirement $h(t, \cdot) \in \mathscr{A}_t$, $\mu$-a.e. on $S$ is replaced by the requirement that $h \in \mathscr{C}(\mathscr{E}; S; y)$, where $y(t, \cdot) \in \mathscr{A}_t$, $\mu$-a.e. on $S$, and $\mathscr{C}(\mathscr{E}; S; y)$ denotes the ex-post core of the economy with initial endowment $y$ and the set of agents $S$. Therefore, the weak sequential core, denoted $WSC(\mathscr{E})$, is defined by requiring that the deviation $h$ of a coalition $S$ from $g$ at time $\tau = 0$ is credible in the sense that $h$ cannot be counterblocked by some subcoalition of $S$ when the state of nature is realized.

It is easy to show the inclusion $WSC(\mathscr{E}) \subseteq TSC(\mathscr{E})$: for a two-period trading plan $(f, g)$ which belongs to the weak sequential core and not to the two-stage core, it can be assumed that $g$ is improved by $S$ via $h$ at time $t = 0$, as in Definition 4.3, (ii). Then, any allocation $y$ in the ex-post core $\mathscr{C}(\mathscr{E}; S; h)$ defined by $h$ is individually rational and allows a credible deviation from $g$. Thus, under the standard assumptions which render the ex-post core non-empty, we have that $SSC(\mathscr{E}) \subseteq WSC(\mathscr{E}) \subseteq TSC(\mathscr{E})$, while $SSC(\mathscr{E})$ and $SC(\mathscr{E})$ may be distinct subsets of $TSC(\mathscr{E})$.

Also, it is clear that, if we remove the ex-ante trade restrictions defined by the correspondence $\mathscr{G}_t$, the strong sequential core and the two-stage core coincide, so that $SSC(\mathscr{E}) = WSC(\mathscr{E}) = TSC(\mathscr{E})$.

The notions of coalitional fairness formulated in this section could be applied to the strong and weak sequential core, leading to sets of *strong sequential* $\mathscr{C}_{\{\mathscr{T}_0,\mathscr{T}_1\}}(\mathscr{E})$*-fair* and *weak sequential* $\mathscr{C}_{\{\mathscr{T}_0,\mathscr{T}_1\}}(\mathscr{E})$*-fair* allocations. An analysis similar to that applied to the two-stage core would follow.

# 5 Applications to asset markets and asymmetric information economies

We close the paper with the specific applications of our results. We start from the case of asset markets. Since, in the first period, individuals choose portfolio rather than choosing actual bundles for different states, we now propose a new definition of the ex ante core in this setting.

**Definition 5.1.** An *asset trading plan* is a Lebesgue integrable function $\theta : T \to \mathbb{R}^J$ and it is called *feasible* if $\displaystyle\int_T \theta d\mu = 0$. An asset trading plan $\theta$ is *ex ante blocked by a coalition* $S$ if there is an asset trading plan $\varphi$ such that $\displaystyle\int_S \varphi d\mu = 0$ and $W_t(\varphi(t)) > W_t(\theta(t))$ $\mu$-a.e. on $S$. The *ex ante core* in the asset market economy $\{\mathscr{E}, (S^j)_{j=1}^J\}$ is the set of feasible asset trading plans that are not ex ante blocked by any coalition.

We denote by $\Xi[\theta]$ the allocation generated by the asset trading plan $\theta$, where $\Xi[\theta]$ is defined by

$$\Xi[\theta](t,\omega) = e(t,\omega) + \sum_{j=1}^{J} \theta^j(t) \cdot S^j(\omega).$$

Conversely, for any coalition $S$ and an assignment $g$ with $g(t,\cdot) - e(t,\cdot) \in \mathscr{G}_t$ $\mu$-a.e. on $S$, there is some $\varphi = \Phi[g] : S \to \mathbb{R}^J$ such that

$$g(t,\omega) = e(t,\omega) + \sum_{j=1}^{J} \varphi^j(t) \cdot S^j(\omega).$$

Note that $\varphi$ is not necessarily Lebesgue integrable, and for any asset trading plan $\theta$ with $\int_S \theta d\mu = 0$, we must have that the allocation $\Xi[\theta]$ is feasible.

Consider now an asset market economy $\{\mathscr{E}, (S^j)_{j=1}^J\}$ and let $\{\mathscr{E}, \mathscr{G}\}$ be the economy that is generated by $\{\mathscr{E}, (S^j)_{j=1}^J\}$, that is, $\mathscr{G}_t = \left\{\sum_{j=1}^J \theta_t^j \cdot S^j : \theta_t \in \mathbb{R}^J\right\}$ for all $t \in T$. Clearly, $(\mathbf{A}_1)$ and $(\mathbf{A}_2)$ are satisfied; $(\mathbf{A}_8)$ follows from $(\mathbf{A}_7)$ and the structure of $\mathscr{G}_t$ for all $t \in T$ while $(\mathbf{A}_9)$ is trivially satisfied. In the light of $(\mathbf{A}_4)$-$(\mathbf{A}_7)$, one can derive the following facts:

$(\mathbf{B}_1)$ For all $t \in T$, $W_t$ is continuous and for all $\theta \in \mathbb{R}^J$, $t \mapsto W_t(\theta)$ is $\mathscr{T}$-measurable.

$(\mathbf{B}_2)$ For all $t \in T$, $W_t$ is monotone in the sense that $W_t(\theta+\varphi) > W_t(\theta)$ for all $\theta, \varphi \in \mathbb{R}^J$ with $\varphi \geq 0$ with $\varphi \neq 0$.

$(\mathbf{B}_3)$ For all $t \in T_1$, $W_t$ is concave.

An asset market economy $\{\mathscr{E}, (S^j)_{j=1}^J\}$ is said to be *complete* if $\sum_{j=1}^J \theta^j \cdot S^j(\omega) = 0$ for all $\omega \in \Omega$ implies $\theta^j = 0$ for all $j \in J$.

For simplicity, we formulate our main results in the case of complete asset markets. The general case is discussed in Remark 5.13

**Lemma 5.2.** *Suppose that the asset market economy $\{\mathscr{E}, (S^j)_{j=1}^J\}$ is complete and $\theta$ is a trading plan. Let $\widetilde{\theta}_0 : T \to \mathbb{R}_+^J$ be a Lebesgue integrable function such that $\theta(t) \in B(t) = \{\psi \in \mathbb{R}^J : -\widetilde{\theta}_0(t) \leq \psi \leq \widetilde{\theta}_0(t)\}$ $\mu$-a.e. on $T$. Under $(\mathbf{A}_3)$-$(\mathbf{A}_7)$, if $\Xi[\theta]$ is ex ante blocked by some coalition $S$ via some assignment $h$ in $\{\mathscr{E}, \mathscr{G}\}$ such that $\varphi(t) = \Phi[h](t) \in B(t)$ $\mu$-a.e. on $S$ then $\theta$ must be ex ante blocked by $S$ in $\{\mathscr{E}, (S^j)_{j=1}^J\}$.*

*Proof.* Define a correspondence $F : S \rightrightarrows \mathbb{R}^J$ by letting

$$F(t) = \left\{\psi \in \mathbb{R}^J : \sum_{j=1}^J \psi^j S^j = h(t,\cdot) - e(t,\cdot)\right\} \cap B(t).$$

Since $\varphi(t) \in F(t)$, one has $F(t) \neq \emptyset$ $\mu$-a.e. on $S$. It is easy to show that $F$ has measurable graph. By the measurable projection theorem, $F$ has a Lebesgue integrable selection $\xi$. By the completeness of the asset market economy and the fact that $\int_S h(\cdot,\omega)d\mu = \int_S e(\cdot,\omega)d\mu$ for all $\omega \in \Omega$, one concludes $\theta$ is ex ante blocked by $S$ via $\xi$. $\square$

The following theorem is an immediate consequence of Theorem 3.4.

**Theorem 5.3.** *Suppose that the asset market economy $\{\mathscr{E}, (S^j)_{j=1}^J\}$ is complete, atomless and $\theta$ is a feasible asset trading plan not beloging to the ex ante core. Under* $(\mathbf{A}_3)$*-*$(\mathbf{A}_7)$*, for any given $0 < \varepsilon < \mu(T)$, there is a coalition $R$ with $\mu(R) = \varepsilon$ such that $\theta$ is ex ante blocked by $R$.*

*Proof.* Suppose that $\theta$ is a feasible asset trading plan not belonging to the ex ante core of $\{\mathscr{E}, (S^j)_{j=1}^J\}$. It follows that there is a coalition $D$ and an asset trading plan $\varphi$ such that $\theta$ is ex ante blocked by $D$ via $\varphi$. Then there is a countable valued integrable function $\widetilde{\theta} : T \to \mathbb{R}_+^J$ such that $-\widetilde{\theta} \ll \theta, \varphi \ll \widetilde{\theta}$ . Let $\{T_i : i \geq 1\}$ be collection of a measurable subsets of $T$ such that the values of $\widetilde{\theta}$ are constant on $T_i$ for $i \geq 1$. Let $\widetilde{\theta}(t) = x_i$ if $t \in T_i$. Define the set $B_i$ by letting

$$B_i = \left\{\psi \in \mathbb{R}^J : -x_i \leq \psi \leq x_i\right\}.$$

Let

$$\mathscr{H}_i = \left\{\sum_{j=1}^{J} \psi^j \cdot S^j : \psi \in B_i\right\}$$

for all $i \geq 1$ and $\mathscr{H}_t = \mathscr{H}_i$ if $t \in T_i$. Thus, $\mathscr{H}_i$, $f := \Xi[\theta]$ and $h := \Xi[\varphi]$ satisfy all the hypothesis listed in the statement of Theorem 3.4. By Theorem 3.4, for any $0 < \varepsilon < \mu(T)$, there is a coalition $R$ such that $\Xi[\theta]$ is ex ante blocked by $R$ via some assignment $g$ with $g(t, \cdot) - e(t, \cdot) \in 4\mathscr{H}_t$ $\mu$-a.e. on $T$ in $\{\mathscr{E}, \mathscr{G}\}$. Hence, by Lemma 5.2, $\theta$ is ex ante blocked by $R$ in $\{\mathscr{E}, (S^j)_{j=1}^J\}$. $\square$

We now introduce the concept of coalitional fairness in the framework of asset markets. As in the general case, with the first notion, we require that no coalition of small agents envies a disjoint coalition containing all large agents because aggregate portfolio choosen by the second coalition would make members of the first coalition better off.

**Definition 5.4.** A feasible asset trading plan $\theta$ is called *ex-ante $\mathscr{C}_{(\mathscr{T}_0, \mathscr{T}_1)}(\mathscr{E})$-fair* if there do not exist two disjoint elements $S_1 \in \mathscr{T}_0$, $S_2 \in \mathscr{T}_1$ and an asset trading plan $\varphi$ such that $\mu$-a.e. on $S_1$:

(i) $W_t(\varphi(t)) > W_t(\theta(t))$;

(ii) $\displaystyle\int_{S_1} \varphi d\mu = \int_{S_2} \theta d\mu.$

In the next notion, the role of coalitions is exchanged. The fair criterion requires that no coalition containing all large agents envies portfolios choosen by a disjoint coalition of small agents.

**Definition 5.5.** A feasible asset trading plan $\theta$ is called *ex-ante* $\mathscr{C}_{(\mathscr{T}_1,\mathscr{T}_0)}(\mathscr{E})$*-fair* if there do not exist two disjoint elements $S_1 \in \mathscr{T}_1$, $S_2 \in \mathscr{T}_0$ and an asset trading plan $\varphi$ such that $\mu$-a.e. on $S_1$:

(i) $W_t(\varphi(t)) > W_t(\theta(t))$;

(ii) $\displaystyle\int_{S_1} \varphi d\mu = \int_{S_2} \theta d\mu.$

Our final fairness requirement states that no coalition of small agents envies the aggregate portfolio of a disjoint coalition containing all large agents and vice versa.

**Definition 5.6.** A feasible allocation $f$ is called *ex-ante* $\mathscr{C}_{\{\mathscr{T}_0,\mathscr{T}_1\}}(\mathscr{E})$*-fair* if it is ex-ante $\mathscr{C}_{(\mathscr{T}_0,\mathscr{T}_1)}(\mathscr{E})$-fair and ex-ante $\mathscr{C}_{(\mathscr{T}_1,\mathscr{T}_0)}(\mathscr{E})$-fair.

The following theorem is simply obtained as a corollary of Theorem 3.15.

**Theorem 5.7.** *Let* $(\mathbf{A}_3)$-$(\mathbf{A}_7)$ *be satisfied. Then any asset trading plan in the ex-ante core of the asset market economy* $\{\mathscr{E}, (S^j)_{j=1}^J\}$ *is ex-ante* $\mathscr{C}_{\{\mathscr{T}_0,\mathscr{T}_1\}}(\mathscr{E})$*-fair.*

*Proof.* Let $\theta$ be an asset trading plan belonging to the ex ante core of the asset market economy $\{\mathscr{E}, (S^j)_{j=1}^J\}$. Assume for contradiction $\theta$ is not ex-ante $\mathscr{C}_{\{\mathscr{T}_0,\mathscr{T}_1\}}(\mathscr{E})$-fair. Without loss of generality, we assume that $\theta$ is not ex-ante $\mathscr{C}_{(\mathscr{T}_0,\mathscr{T}_1)}(\mathscr{E})$-fair. Hence, there exist two disjoint elements $S_1 \in \mathscr{T}_0$, $S_2 \in \mathscr{T}_1$ and an asset trading plan $\varphi$ such that $\mu$-a.e. on $S_1$:

(i) $W_t(\varphi(t)) > W_t(\theta(t))$;

(ii) $\displaystyle\int_{S_1} \varphi d\mu = \int_{S_2} \theta d\mu.$

Then there is a countable valued integrable function $\widetilde{\theta} : T \to \mathbb{R}^J_+$ such that $-\widetilde{\theta} \ll \theta, \varphi \ll \widetilde{\theta}$ . Let $\{T_i : i \geq 1\}$ be collection of a measurable subsets of $T$ such that the values of $\widetilde{\theta}$ are constant on $T_i$ for $i \geq 1$. Let $\widetilde{\theta}(t) = x_i$ if $t \in T_i$. Define the set $B_i$ by letting

$$B_i = \left\{\psi \in \mathbb{R}^J : -x_i \leq \psi \leq x_i\right\}.$$

Let

$$\mathscr{H}_i = \left\{\sum_{j=1}^J \psi^j \cdot S^j : \psi \in B_i\right\}$$

for all $i \geq 1$ and $\mathscr{H}_t = \mathscr{H}_i$ if $t \in T_i$. Moreover, it follows from (ii) that

$$\sum_{j=1}^{J} \int_{S_1} \varphi^j \cdot S^j(\omega) = \sum_{j=1}^{J} \int_{S_2} \theta^j \cdot S^j(\omega),$$

which further yields

$$\int_{S_1} (\Xi[\varphi](\cdot, \omega) - e(\cdot, \omega)) d\mu = \int_{S_1} (\Xi[\theta](\cdot, \omega) - e(\cdot, \omega)) d\mu.$$

On the other hand, it follows from (i) that $V_t(\Xi[\varphi](t, \cdot)) > V_t(\Xi[\theta](t, \cdot))$ $\mu$-a.e. Thus, $\mathscr{H}_t$, $f := \Xi[\theta]$ and $h := \Xi[\varphi]$ satisfy all the hypothesis listed in the statement of Theorem 3.9. So, by Theorem 3.9, there is a coalition $R$ such that $\Xi[\theta]$ is ex ante blocked by $R$ via some assignment $g$ with $g(t, \cdot) - e(t, \cdot) \in 4\mathscr{H}_t$ $\mu$-a.e. on $T$ in $\{\mathscr{E}, \mathscr{G}\}$. Hence, by Lemma 5.2, $\theta$ is ex ante blocked by $R$ via some asset trading plan in $\{\mathscr{E}, (S^j)_{j=1}^J\}$.

$\square$

Finally, we extend our results on Vind's theorem and fairness to the sequential core of asset markets.

**Definition 5.8.** A two-period trading plan $(\theta, g)$ is in the *sequential core* of the asset market economy $\{\mathscr{E}, (S^j)_{j=1}^J\}$ if $\theta$ is in the ex ante core and $g \in \mathscr{C}(\mathscr{E}; \Xi[\theta])$.

The proof of the following theorem is now immediate.

**Theorem 5.9.** *Let $(\theta, g)$ be a two-period trading plan not in the sequential core. Suppose further that $\Xi[\theta](t, \omega)$ is an interior point of $X_t(\omega)$ for all $(t, \omega) \in T \times \Omega$. Under $(\mathbf{A}_3)$-$(\mathbf{A}_7)$, for any given $0 < \varepsilon < \mu(T)$, there is a coalition $R$ with $\mu(R) = \varepsilon$ such that either of the following is satisfied:* (i) *$\theta$ is ex ante blocked by $R$ in $\{\mathscr{E}, (S^j)_{j=1}^J\}$; or* (ii) *$g$ is ex post blocked by $R$ in $\mathscr{E}(f)$.*

**Definition 5.10.** Given an asset trading plan $\theta$, an assignment $g$ is *ex post $\mathscr{C}_{(\mathscr{T}_0, \mathscr{T}_1)}(\mathscr{E}; \theta)$-fair* if there do not exist a state $\omega_0$, two disjoint elements $S_1 \in \mathscr{T}_0$, $S_2 \in \mathscr{T}_1$ and an assignment $h$ in $\mathscr{E}(\omega_0)$ such that $u_t(\omega_0, h(t)) > u_t(\omega_0, g(t, \omega_0))$ $\mu$-a.e. on $S_1$, and

$$\int_{S_1} (h - \Xi[\theta](\cdot, \omega_0)) d\mu = \int_{S_2} (g(\cdot, \omega_0) - \Xi[\theta](\cdot, \omega_0)) d\mu.$$

**Definition 5.11.** Given an asset trading plan $\theta$, an assignment $g$ is *ex post $\mathscr{C}_{(\mathscr{T}_1, \mathscr{T}_0)}(\mathscr{E}; \theta)$-fair* if there do not exist a state $\omega_0$, two disjoint elements $S_1 \in \mathscr{T}_1$, $S_2 \in \mathscr{T}_0$ and an assignment $h$ in $\mathscr{E}(\omega_0)$ such that $u_t(\omega_0, h(t)) > u_t(\omega_0, g(t, \omega_0))$ $\mu$-a.e. on $S_1$, and

$$\int_{S_1} (h - \Xi[\theta](\cdot, \omega_0)) d\mu = \int_{S_2} (g(\cdot, \omega_0) - \Xi[\theta](\cdot, \omega_0)) d\mu.$$

As before we can define the *sequential* $\mathscr{C}_{\{\mathscr{T}_0,\mathscr{T}_1\}}$*-fairness* of a two-period trading plan and in the light of Theorem 4.11 and Theorem 5.7, formulate the following easy consequence.

**Theorem 5.12.** *Let $(\theta, g)$ be in the sequential core. Suppose further that $\Xi[\theta](t,\omega)$ is an interior point of $X_t(\omega)$ for all $(t,\omega) \in T \times \Omega$. Under $(\mathbf{A}_3)$-$(\mathbf{A}_7)$, $(\theta, g)$ is sequential $\mathscr{C}_{\{\mathscr{T}_0,\mathscr{T}_1\}}$-fair.*

**Remark 5.13.** Notice that we can also also guarantee that our results hold true for a general asset market economy by specifying the linear restriction introduced in section 7. More precisely, we choose in this case $m = J$ and $\pi(\omega,\theta) = \sum_{j=1}^{J} \theta^j \cdot S^j(\omega)$ for all $\theta \in \mathbb{R}^J$. Details are given in section 7.

We close this section with few remarks about applications to asymmetric information economies. For these models, restrictions in trades at the ex-ante stage are asymmetric and defined by the requirement that each agent only subscribes those contract that he is able to distinguish according to his private information. Thus, for each agent $t \in T$, we set $\mathscr{G}_t = \mathscr{G}_i$ if $t \in J_i$ and $1 \leq i \leq m$, where $\mathscr{G}_i$ and $J_i$ are defined in Subsection 2.2. This leads to the concept of *private core* (see Yannelis, 1991), *privately fair allocation* (refer to Bhowmik, 2015), *sequential core*, *two-stage core*, *sequentially* $\mathscr{C}_{\{\mathscr{T}_0,\mathscr{T}_1\}}(\mathscr{E})$*-fair allocations* under the private information setting. Thus, our main results (Corollary 3.5, Theorem 3.15, Theorem 4.4, Corollary 4.12) extend the existing results for the private core in the literature to the general model including the case of sequential notions.

**Remark 5.14.** An appropriate modification of the definition of $\mathscr{G}_t$, may allow the ex-ante restriction for agent $t$'s trade to vary with the coalition that agent $t$ joins. This model, also known as model with information sharing rule, permits to include in our analysis the concepts of *coarse core*, *fine core*, *weak fine core*. For more detailed discussion of these core notions, we refer to Bhowmik (2015).

# 6 Appendix A

Suppose that $\mathscr{H}$ is a sub-restriction of $\mathscr{G}$, and $f$ and $h$ are two $\mathscr{H}$-allocations such that $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$ $\mu$-a.e. on some coalition $S$. We say that $(f, h, S, \mathscr{H})$ satisfies the property **(P)** if the following is true:
There exist some $0 < \lambda < 1$ and some $0 < \eta < 1$ such that for all

$$z \in \bigcap \{\eta \mathscr{H}_i : i \geq 1\} \cap \mathbb{B}(0,\eta)^{\Omega},$$

there is an assignment $y^z$ such that $y^z(t,\cdot) - e(t,\cdot) \in 4\mathscr{H}_t$, $V_t(y^z(t,\cdot)) > V_t(f(t,\cdot))$ $\mu$-a.e. on $S$, and

$$\int_S (y^z - e)d\mu + z = (1-\lambda)\int_S (h-e)d\mu.$$

Analogously, we say that $(f,h,S,\mathscr{H})$ satisfies the property **(Q)** if the following is true: There exist some $0 < \eta < 1$ and a sub-coalition $R$ of $S$ such that for all

$$z \in \bigcap \{\eta \mathscr{H}_i : i \geq 1\} \cap \mathbb{B}(0,\eta)^\Omega,$$

there is an assignment $\xi^z$ such that $\xi^z(t,\cdot) - e(t,\cdot) \in 4\mathscr{H}_t$ and $V_t(\xi^z(t,\cdot)) > V_t(f(t,\cdot))$ $\mu$-a.e. on $R$, and

$$\int_R (\xi^z - e)d\mu + z = \frac{1}{2}\int_S (h-e)d\mu.$$

**Lemma 6.1.** *Let $\mathscr{H}$ be a sub-restriction of $\mathscr{G}$ and $(\mathbf{A}_9')$ be satisfied for $\mathscr{H}$. Suppose also that $(\mathbf{A}_1)$-$(\mathbf{A}_4)$, $(\mathbf{A}_6)$ and $(\mathbf{A}_7)$ hold true. Assume that $f$ and $h$ are two $\mathscr{H}$-allocations such that $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$ $\mu$-a.e. on some coalition $S$. Then $(f,h,S,\mathscr{H})$ satisfies the property "**(P)**" if either of the following two conditions is satisfied:*

(i) $a^S > 0$.

(ii) $a^S = 0$, *there is an $\varepsilon > 0$ such that $e(t,\omega) + \mathbb{B}(0,\varepsilon) \subseteq X_t(\omega)$ for all $(t,\omega) \in T \times \Omega$, and, moreover, there are some $i_0 \in \Lambda^S$ and $\delta > 0$ such that*

$$\frac{2}{\mu(S_{i_0})}\int_D (h-e)d\mu \in \mathscr{H}_{i_0}$$

*for any measurable subset $D$ of $S$ with $\mu(D) < \delta$.*

*Proof.* Let $\{\delta_m : m \geq 1\} \subseteq (0,1)$ be a decreasing sequence converging to 0. For all $i \in \Lambda^S$ and $r \geq 1$, define the set

$$S_i^r := \left\{ t \in S_i : e(t,\omega) + \mathbb{B}\left(0, \frac{1}{r}\right) \subseteq X_t(\omega) \text{ for all } \omega \in \Omega \right\}.$$

Consider the function $\psi : S \times \Omega \to \mathbb{R}$, defined by $\psi(t,\omega) := \operatorname{dist}\big(e(t,\omega), \mathbb{R}^\ell \setminus X_t(\omega)\big)$. Since $\psi(\cdot,\omega)$ is $\mathscr{T}$-measurable for all $\omega \in \Omega$, the set

$$S_i^r = \left\{ t \in S_i : \psi(t,\omega) \geq \frac{1}{r} \text{ for all } \omega \in \Omega \right\}$$

is $\mathscr{T}$-measurable. Moreover, $\{S_i^r : r \geq 1\}$ is increasing and $S_i \sim \bigcup\{S_i^r : r \geq 1\}$ for all $i \in \Lambda^S$[12]. If $\Lambda^S$ is finite, choose an integer $r' \geq 1$ with $\mu(S_i^{r'}) > \frac{2\mu(S_i)}{3}$ for all $i \in \Lambda^S$. Define $\kappa$ to be $\frac{1}{r'}$ if (i) is satisfied; and $\varepsilon$, otherwise. Let

$$d \in \mathbb{R}^\ell_{++} \cap \mathbb{B}\left(0, \frac{\kappa}{3}\right).$$

Choose some $\zeta < \frac{\kappa}{3}$ such that $d - \mathbb{B}(0,\zeta) \subseteq \mathbb{R}^\ell_{++}$. In case (ii) holds, pick a finite subset $\Theta$ of $\Lambda^S$ such that $i_0 \in \Theta$, $\sum_{i\in\Lambda^S\setminus\Theta} \mu(S_i) < \delta$ and

$$\frac{2}{\mu(S_{i_0})}\int_{\bigcup\{S_i : i\in\Lambda^S\setminus\Theta\}} (h-e)d\mu \in \mathbb{B}(0,\zeta)^\Omega.$$

Define $\Psi := \Lambda^S$ if (i) is satisfied; and $\Theta$, if (ii) is satisfied. Let $E := \bigcup\{S_i^{r'} : i \in \Psi\}$ if (i) is satisfied; and $S$, otherwise. For all $m \geq 1$, define the function $h^m : S\times\Omega \to \mathbb{R}^\ell$ by letting

$$h^m(t,\omega) := \begin{cases} (1-\delta_m)h(t,\omega) + \delta_m(e(t,\omega) - 3d), & \text{if } (t,\omega) \in E\times\Omega; \\ (1-\delta_m)h(t,\omega) + \delta_m e(t,\omega), & \text{otherwise.} \end{cases}$$

Put,

$$R^m := \left\{t \in S : V_t(h^k(t,\cdot)) > V_t(f(t,\cdot)) \text{ for all } k \geq m\right\}.$$

By $(\mathbf{A}_4)$ and $(\mathbf{A}_6)$, the mapping $\xi^k : S \to \mathbb{R}$, defined by

$$\xi^k(t) := V_t(h^k(t,\cdot)) - V_t(f(t,\cdot)),$$

is $\mathscr{T}$-measurable and so is $R^m$. It is obvious that $\{R^m : m \geq 1\}$ is increasing and $S \sim \bigcup\{R^m : m \geq 1\}$. It follows from the absolute continuity of Lebesgue integral that there is some $\gamma > 0$ such that

$$\frac{2}{b}\int_D (h-e)d\mu \in \mathbb{B}(0,\zeta)^\Omega$$

for any measurable subset $D$ of $S$ with $\mu(D) < \gamma$, where $b := \min\{\mu(S_i) \,|\, i \in \Psi\}$. For each $i \in \Psi$, define $G_i := S_i^{r'}$ if (i) is satisfied; and $S_i$, if (ii) is satisfied. Choose some $m'$ such that $\mu(S \setminus R^{m'}) < \min\{\gamma, \frac{c}{4}\}$, where $c := \min\{\mu(G_i) \,|\, i \in \Psi\}$. Hence,

$$\mu(G_i \cap R^{m'}) > \frac{3\mu(G_i)}{4} > \frac{\mu(S_i)}{2} \geq \frac{b}{2}$$

[12]We denote by $A \sim B$ the fact that $B \subseteq A$ and $\mu(A \setminus B) = 0$.

and

$$\frac{2}{b}\int_{S_i\setminus R^{m'}}(h-e)d\mu \in \mathbb{B}\left(0,\zeta\right)^{\Omega}$$

for all $i \in \Psi$. Thus, one obtains

$$\frac{1}{\mu(G_i \cap R^{m'})}\int_{S_i\setminus R^{m'}}(h-e)d\mu \in \mathbb{B}\left(0,\zeta\right)^{\Omega}$$

for all $i \in \Psi$. Put, $\lambda := \delta_{m'}$ and

$$\eta := \min\left\{\lambda\zeta\mu(G_i \cap R^{m'}) : i \in \Psi\right\}.$$

Let $z \in \bigcap\{\eta\mathscr{H}_i : i \in \mathbb{K}\} \cap \mathbb{B}\left(0,\eta\right)^{\Omega}$ and define

$$\hat{z}_i := \frac{1}{\lambda|\Psi|\mu(G_i \cap R^{m'})}z,$$

where $|\Psi|$ denotes the number of elements in $\Psi$. Assuming $z = \eta z'$ for some $z' \in \bigcap\{\mathscr{H}_i : i \in \mathbb{K}\}$, one obtains

$$\hat{z}_i = \frac{\alpha_i\zeta z'}{|\Psi|}$$

for some $0 < \alpha_i \leq 1$ and for all $i \in \Psi$. Since $\mathscr{H}_i$ is convex and $0 \in \mathscr{H}_i$, one obtains $\hat{z}_i \in \mathscr{H}_i$ for all $i \in \Psi$. On the other hand, $z(\omega) \in \mathbb{B}(0,\eta)$ implies $\hat{z}_i(\omega) \in \mathbb{B}(0,\zeta)$ for all $i \in \Psi$. By Lemma 5 in Shitovitz (1973), one has

$$\frac{1}{\mu(S_i \setminus R^{m'})}\int_{S_i\setminus R^{m'}}(h-e)d\mu \in \mathscr{H}_i$$

for all $i \in \Psi$. Since $\mu(S_i \setminus R^{m'}) < \mu(G_i \cap R^{m'})$, the above together with the fact that $\mathscr{H}_i$ is convex and $0 \in \mathscr{H}_i$ further yield

$$\frac{1}{\mu(G_i \cap R^{m'})}\int_{S_i\setminus R^{m'}}(h-e)d\mu \in \mathscr{H}_i \cap \mathbb{B}\left(0,\zeta\right)^{\Omega}$$

for all $i \in \Psi$. Define

$$c_i^z := \hat{z}_i + \frac{1}{\mu(G_i \cap R^{m'})}\int_{S_i\setminus R^{m'}}(h-e)d\mu$$

for all $i \in \Psi \setminus \{i_0\}$; and

$$c_{i_0}^z := \hat{z}_{i_0} + \frac{1}{\mu(G_{i_0} \cap R^{m'})}\left[\int_{S_{i_0}\setminus R^{m'}}(h-e)d\mu + \int_{\bigcup\{S_i : i\in\Lambda^S\setminus\Theta\}}(h-e)d\mu\right],$$

if (ii) is satisfied. Obviously, $c_i^z \in 3\mathscr{H}_i \cap \mathbb{B}(0,\kappa)^\Omega$ for all $i \in \Psi$. Thus, $e(t,\omega) - c_i^z \in X_t(\omega)$ for all $(t,\omega) \in G_i \times \Omega$ and $i \in \Psi$. For all $i \in \Psi$, consider an assignment $g_i^z : S_i \times \Omega \to \mathbb{R}_+$ defined by

$$g_i^z(t,\omega) := \begin{cases} (1-\lambda)h(t,\omega) + \lambda(e(t,\omega) - c_i^z(\omega)), & \text{if } (t,\omega) \in (G_i \cap R^{m'}) \times \Omega; \\ (1-\lambda)h(t,\omega) + \lambda e(t,\omega), & \text{if } (t,\omega) \in ((S_i \cap R^{m'}) \setminus G_i) \times \Omega; \\ h(t,\omega), & \text{otherwise.} \end{cases}$$

It is easy to verify that $g_i^z(t,\omega) \in X_t(\omega)$ for all $(t,\omega) \in S_i \times \Omega$. Since $-c_i^z \in 3\mathscr{H}_i$, one must have $g_i^z(t,\cdot) - e(t,\cdot) \in 4\mathscr{H}_i$ for all $t \in G_i \cap R^{m'}$. Likewise, it can be shown that $g_i^z(t,\cdot) - e(t,\cdot) \in 4\mathscr{H}_i$ for all $t \in S_i$. Moreover, $g_i^z(t,\omega) \gg h^{m'}(t,\omega)$ for all $t \in G_i \cap R^{m'}$. It follows that

$$V_t(g_i^z(t,\cdot)) > V_t(h^{m'}(t,\cdot)) > V_t(f(t,\cdot))$$

for all $t \in G_i \cap R^{m'}$. Hence, $V_t(g_i^z(t,\cdot)) > V_t(f(t,\cdot))$ for all $t \in S_i$. It can be checked that

$$\int_{S_i} (g_i^z - e)d\mu + \frac{z}{|\Psi|} = (1-\lambda)\int_{S_i} (h-e)d\mu.$$

for all $i \in \Psi \setminus \{i_0\}$; and

$$\int_{S_{i_0}} (g_{i_0}^z - e)d\mu + \frac{z}{|\Psi|} = (1-\lambda)\int_{S_{i_0}} (h-e)d\mu - \lambda \int_{\bigcup\{S_i : i \in \Lambda^S \setminus \Theta\}} (h-e)d\mu,$$

if (ii) is satisfied. Define the assignment $y^z : T \times \Omega \to \mathbb{R}^\ell$ by letting

$$y^z(t,\omega) := \begin{cases} g_i^z(t,\omega), & \text{if } (t,\omega) \in S_i \times \Omega, i \in \Psi; \\ h(t,\omega), & \text{otherwise.} \end{cases}$$

It can be simply checked that $y^z$ satisfies the required condition. $\square$

**Proposition 6.2.** *Suppose that $\mathscr{H}$ is a sub-restriction of $\mathscr{G}$ that satisfies* $(\mathbf{A}_9')$. *Assume further that $f$ and $h$ are two $\mathscr{H}$-allocations such that $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$ $\mu$-a.e. on some coalition $S \in \mathscr{T}_0$. Then $(f,h,S,\mathscr{H})$ satisfies the property* **(Q)**.

*Proof.* Choose an $i_0 \in \mathbb{K}$ such that $\mu(S \cap I_{i_0}) > 0$. By Lemma 6.1, $(f,h,S\cap I_{i_0},\mathscr{H})$ satisfies the property **(P)**. Then there exist some $0 < \lambda < 1$ and some $0 < \eta < 1$ such that for all

$$z \in \bigcap\{\eta\mathscr{H}_i : i \geq 1\} \cap \mathbb{B}(0,\eta)^\Omega,$$

there is an assignment $y^z$ such that $y^z(t,\cdot) - e(t,\cdot) \in 4\mathscr{H}_t$ and $V_t(y^z(t,\cdot)) > V_t(f(t,\cdot))$ $\mu$-a.e. on $S \cap I_{i_0}$ and

$$\int_{S\cap I_{i_0}} (y^z - e)d\mu + z = (1-\lambda)\int_{S\cap I_{i_0}} (h-e)d\mu.$$

Let $\eta_0 = \frac{\eta}{2}$ and take any $z \in \bigcap\{\eta_0 \mathscr{H}_i : i \geq 1\} \cap \mathbb{B}(0,\eta_0)^\Omega$. Then $\hat{z} = 2z \in \bigcap\{\eta\mathscr{H}_i : i \geq 1\} \cap \mathbb{B}(0,\eta)^\Omega$. So, there is an assignment $y^{\hat{z}}$ such that $y^{\hat{z}}(t,\cdot) - e(t,\cdot) \in 4\mathscr{H}_t$ and $V_t(y^{\hat{z}}(t,\cdot)) > V_t(f(t,\cdot))$ $\mu$-a.e. on $S \cap I_{i_0}$ and

$$\int_{S\cap I_{i_0}} (y^{\hat{z}} - e)d\mu + \hat{z} = (1-\lambda)\int_{S\cap I_{i_0}} (h-e)d\mu.$$

By the Lyapunov convexity theorem, there is a coalition $R_1 \subseteq S \cap I_{i_0}$ such that

$$\int_{R_1} (h-e)d\mu = \lambda \int_{S\cap I_{i_0}} (h-e)d\mu.$$

Define the correspondence $\Gamma_f : R_1 \rightrightarrows (\mathbb{R}^\ell)^\Omega$ by letting

$$\Gamma_f(t) := \{z \in X_t : z - e(t,\cdot) \in 4\mathscr{H}_t \text{ and } V_t(z) > V_t(f(t,\cdot))\}.$$

Clearly, $\displaystyle\int_{R_1} y^{\hat{z}} d\mu \in \int_{R_1} \Gamma_f d\mu$ and $\displaystyle\int_{R_1} h d\mu \in \int_{R_1} \Gamma_f d\mu$. Since $\displaystyle\int_{R_1} \Gamma_f d\mu$ is convex, one obtains

$$\frac{1}{2}\int_{R_1} y^{\hat{z}} d\mu + \frac{1}{2}\int_{R_1} h d\mu \in \int_{R_1} \Gamma_f d\mu.$$

So, there is an assignment $\varphi^{\hat{z}}$ such that $\int_{R_1} \varphi^{\hat{z}} d\mu \in \int_{R_1} \Gamma_f d\mu$ and

$$\int_{R_1} \varphi^{\hat{z}} d\mu = \frac{1}{2}\int_{R_1} y^{\hat{z}} d\mu + \frac{1}{2}\int_{R_1} h d\mu.$$

This implies that

$$\int_{R_1} (\varphi^{\hat{z}} - e)d\mu = \frac{1}{2}\int_{R_1} (y^{\hat{z}} - e)d\mu + \frac{1}{2}\int_{R_1} (h-e)d\mu.$$

Again, by the Lyapunov convexity theorem, there are coalitions $R_2 \subseteq (S \cap I_{i_0}) \setminus R_1$ and $R_3 \subseteq S \setminus I_{i_0}$ such that

$$\int_{R_2} (y^{\hat{z}} - e)d\mu = \frac{1}{2}\int_{(S\cap I_{i_0})\setminus R_1} (y^{\hat{z}} - e)d\mu$$

and

$$\int_{R_3} (h-e)d\mu = \frac{1}{2}\int_{S\setminus I_{i_0}} (h-e)d\mu.$$

Let $R := R_1 \cup R_2 \cup R_3$ and define an assignment $\xi^z : T \times \Omega \to \mathbb{R}^\ell$ such that

$$\xi^z(t,\omega) := \begin{cases} \varphi^{\hat{z}}(t,\omega), & \text{if } (t,\omega) \in R_1 \times \Omega; \\ y^{\hat{z}}(t,\omega), & \text{if } (t,\omega) \in R_2 \times \Omega; \\ h(t,\omega), & \text{otherwise.} \end{cases}$$

Thus, $\xi^z(t,\cdot) - e(t,\cdot) \in 4\mathscr{H}_t$ and $V_t(\xi^z(t,\cdot)) > V_t(f(t,\cdot))$ $\mu$-a.e. on $R$, and

$$\int_R (\xi^z - e)d\mu + z = \frac{1}{2}\int_S (h-e)d\mu.$$

$\square$

**Proof of Theorem 3.4** Choose an $\varepsilon \in (0, \mu(S))$. Let $\alpha \in (0,1)$ be such that $\varepsilon = \alpha\mu(S)$. It follows from the Lyapunov convexity theorem that there exists a coalition $E$ such that $\mu(E) = \alpha\mu(S)$ and $\int_E (h-e)d\mu = \alpha \int_S (h-e)d\mu = 0$. Thus, there is a coalition $E$ with $\mu(E) = \varepsilon$ and $\int_E (h-e)d\mu = 0$. If $\mu(S) = \mu(T)$ then nothing is remaining to prove. So, assume that $\mu(S) < \varepsilon < \mu(T)$. By Proposition 6.2, there exist an $\eta > 0$ and a sub-coalition $R$ of $S$ such that for all $z \in \bigcap\{\eta\mathscr{H}_i : 1 \le i \le n\} \cap \mathbb{B}(0,\eta)^\Omega$, there is an assignment $y^z$ such that $y^z(t,\cdot) - e(t,\cdot) \in 4\mathscr{H}_t$ and $V_t(y^z(t,\cdot)) > V_t(f(t,\cdot))$ $\mu$-a.e. on $R$, and

$$\int_R (y^z - e)d\mu + z = \frac{1}{2}\int_S (h-e)d\mu. \tag{6.1}$$

Let

$$\beta := 1 - \frac{\varepsilon - \mu(R)}{\mu(T \setminus R)}.$$

By the Lyapunov convexity theorem, there is a coalition $B \subseteq T \setminus R$ such that $\mu(B) = (1-\beta)\mu(T \setminus R)$ and

$$\int_B (f-e)d\mu = (1-\beta)\int_{T\setminus R} (f-e)d\mu.$$

Let

$$D := \bigcap\{\beta\eta\mathscr{H}_i : 1 \le i \le n\} \cap \mathbb{B}(0,\beta\eta)^\Omega,$$

and define $g : T \times D \to \mathbb{R}$ by

$$g(t,z) := V_t(f(t,\cdot)+z) - V_t(f(t,\cdot)).$$

Since $g(\cdot,z)$ is $\mathscr{T}$-measurable for all $z \in D$ and $g(t,\cdot)$ is continuous with respect to the subspace topology of the usual topology on $D$ for all $t \in T$, $g$ is $\mathscr{T} \otimes \mathscr{B}(D)$-measurable. Define the correspondence $\Lambda_f : T \rightrightarrows D$ by letting

$$\Lambda_f(t) := \{z \in D : g(t,z) > 0\}.$$

By our assumption, $\Lambda_f(t) \neq \emptyset$ for all $t \in T$. Moreover, $\mathrm{Gr}_{\Lambda_f}$ is $\mathscr{T} \otimes \mathscr{B}(D)$-measurable. Consider the correspondence $\Phi_f : T \rightrightarrows D$ defined by

$$\Phi_f(t) := \{z \in D : f(t,\omega) + z(\omega) \in X_t(\omega) \text{ for all } \omega \in \Omega\}.$$

As $X_t(\omega)$ is closed, $\Phi_f(t)$ can be equivalently expressed as

$$\Phi_f(t) = \{z \in D : \mathrm{dist}(f(t,\omega) + z(\omega), X_t(\omega)) = 0 \text{ for all } \omega \in \Omega\}.$$

Since $0 \in D$, $\Phi_f(t) \neq \emptyset$ for all $t \in T$. As before, one can show that $\mathrm{Gr}_{\Phi_f}$ is $\mathscr{T} \otimes \mathscr{B}(D)$-measurable. Let $\Psi_f : T \rightrightarrows D$ be a correspondence defined by $\Psi_f(t) := \Lambda_f(t) \cap \Phi_f(t)$ for all $t \in T$. By our assumptions, $\Psi_f(t) \neq \emptyset$ for all $t \in T$. Moreover, $\mathrm{Gr}_{\Psi_f}$ is $\mathscr{T} \otimes \mathscr{B}(D)$-measurable. By the Aumann-Saint-Beuve measurable selection theorem, there is a $\mathscr{T}$-measurable selection $\xi$ of $\Psi_f$ such that $g(t, \xi(t)) > 0$ for all $t \in T$. Define

$$d := \frac{1}{\mu(B)} \int_B \xi d\mu.$$

By Lemma 5 in Shitovitz (1973), one has

$$d \in \bigcap \{\beta\eta\mathscr{H}_i : 1 \leq i \leq n\} \cap \mathbb{B}\,(0, \beta\eta)^{\Omega}.$$

So,

$$c := d\mu(B) \in \bigcap \{\beta\eta\mathscr{H}_i : 1 \leq i \leq n\} \cap \mathbb{B}\,(0, \beta\eta)^{\Omega}$$

and

$$c_0 := \frac{c}{\beta} \in \bigcap \{\eta\mathscr{H}_i : 1 \leq i \leq n\} \cap \mathbb{B}\,(0, \eta)^{\Omega}.$$

Thus, by Equation (6.1), there is an assignment $y^{c_0}$ such that $y^{c_0}(t,\cdot) - e(t,\cdot) \in 4\mathscr{H}_t$ and $V_t(y^{c_0}(t,\cdot)) > V_t(f(t,\cdot))$ $\mu$-a.e. on $R$, and

$$\int_R (y^{c_0} - e) d\mu + c_0 = \frac{1}{2} \int_S (h - e) d\mu = 0.$$

As in the proof of Proposition 6.2, one can show that there is an assignment $\varphi$ such that $\varphi(t,\cdot) - e(t,\cdot) \in 4\mathscr{H}_t$ and $V_t(\varphi(t,\cdot)) > V_t(f(t,\cdot))$ $\mu$-a.e. on $R$, and

$$\int_R (\varphi - e) d\mu = \beta \int_R (y^{c_0} - e) d\mu + (1 - \beta) \int_R (f - e) d\mu.$$

Let $E := R \cup B$. Consider an assignment $\psi : T \times \Omega \to \mathbb{R}^\ell$ defined by[13]

$$\psi(t,\omega) := \begin{cases} \varphi(t,\omega), & \text{if } (t,\omega) \in R \times \Omega; \\ f(t,\omega) + \xi(t,\omega), & \text{otherwise.} \end{cases}$$

[13] $\xi(t,\omega)$ denotes the $\omega^{\text{th}}$-coordinate of $\xi(t)$.

Obviously, $\psi$ is an assignment with $\psi(t,\cdot) - e(t,\cdot) \in 4\mathscr{H}_t$ $\mu$-a.e. on $T$ and $\mu(E) = \varepsilon$. Furthermore, $V_t(\psi(t,\cdot)) > V_t(f(t,\cdot))$ $\mu$-a.e. on $E$. It can be easily verified that $\int_E (\psi - e) d\mu = 0$. This completes the proof. $\square$

**Proof of Theorem 3.9** Suppose that the hypothesis of the theorem is satisfied. The rest of the proof is decomposed into two cases:

*Case 1.* $\mu(S_1 \cup S_2) = \mu(T)$. It is not difficult to show that there is an assignement $\varphi$ such that $\varphi(t,\cdot) - e(t,\cdot) \in \mathscr{H}_t$ and $V_t(\varphi(t,\cdot)) > V_t(f(t,\cdot))$ $\mu$-a.e. on $S_1$, and

$$\int_{S_1} (\varphi - e) d\mu = \frac{1}{2} \int_{S_1} (h - e) d\mu + \frac{1}{2} \int_{S_1} (f - e) d\mu.$$

This implies that

$$\int_{S_1} (\varphi - e) d\mu = \frac{1}{2} \int_{S_1 \cup S_2} (f - e) d\mu = 0.$$

This is a contradiction to that fact that $f$ is in the ex ante core.

*Case 2.* $\mu(S_1 \cup S_2) < \mu(T)$. By the Lyapunov convexity theorem, there is a coalition $B \subseteq T \setminus S_2$ such that $\mu(B) = \frac{1}{2}\mu(T \setminus S_2)$ and

$$\int_B (f - e) d\mu = \frac{1}{2} \int_{T \setminus S_2} (f - e) d\mu.$$

By Proposition 6.2, there exist an $\eta > 0$ and a sub-coalition $R$ of $S$ such that for all $z \in \bigcap \{\eta \mathscr{H}_i : 1 \le i \le n\} \cap \mathbb{B}(0,\eta)^{\Omega}$, there is an assignment $y^z$ such that $y^z(t,\cdot) - e(t,\cdot) \in 4\mathscr{H}_t$ and $V_t(y^z(t,\cdot)) > V_t(f(t,\cdot))$ $\mu$-a.e. on $R$, and

$$\int_R (y^z - e) d\mu + z = \frac{1}{2} \int_{S_1} (h - e) d\mu. \tag{6.2}$$

Applying an argument similar to that in the proof of Theorem 3.4, one can show that there exists an element

$$c \in \bigcap \{\eta \mathscr{H}_i : 1 \le i \le n\} \cap \mathbb{B}(0,\eta)^{\Omega}$$

such that $c = \displaystyle\int_B \xi d\mu$, where

(i) $\xi(t) \in \bigcap \{\eta \mathscr{H}_i : 1 \le i \le n\} \cap \mathbb{B}(0,\eta)^{\Omega}$;

(ii) $f(t,\cdot) + \xi(t) \in X_t$; and

(iii) $V_t(f(t,\cdot) + \xi(t)) > V_t(f(t,\cdot))$

for all $t \in B$. Thus, by Equation (6.2), there exists an assignment $y^c$ such that $y^c(t, \cdot) - e(t, \cdot) \in 4\mathscr{H}_t$ and $V_t(y^c(t, \cdot)) > V_t(f(t, \cdot))$ $\mu$-a.e. on $R$, and

$$\int_R (y^c - e)d\mu + c = \frac{1}{2}\int_{S_1} (h - e)d\mu = \frac{1}{2}\int_{S_2} (f - e)d\mu.$$

Thus,

$$\int_R (y^c - e)d\mu + \int_B (f - e)d\mu + c = 0,$$

which further implies

$$\int_R (y^c - e)d\mu + \int_B (f + \xi - e)d\mu = 0.$$

Define the assignment $\psi : T \times \Omega \to \mathbb{R}^\ell$ by letting

$$\psi(t, \omega) := \begin{cases} y^c(t, \omega), & \text{if } (t, \omega) \in R \times \Omega; \\ f(t, \omega) + \xi(t, \omega), & \text{if } (t, \omega) \in (B \setminus R) \times \Omega; \\ f(t, \omega), & \text{otherwise.} \end{cases}$$

Note that $\psi(t, \cdot) - e(t, \cdot) \in 4\mathscr{H}_t$ $\mu$-a.e. on $T$. If $\mu(R \cap B) = 0$ then $f$ is ex ante blocked by $R \cup B$ via $\psi$, which is a contradiction. So, assume that $\mu(R \cap B) > 0$ and rewrite the above equality as

$$\frac{1}{2}\int_{R \setminus B} (\psi - e)d\mu + \int_{R \cap B} (\psi - e)d\mu + \frac{1}{2}\int_{B \setminus R} (\psi - e)d\mu = 0.$$

Thus, there exist $R_1, R_2 \in \mathscr{T}$ such that $R_1 \subseteq R \setminus B$ and $R_2 \subseteq B \setminus R$ such that

$$\int_{R_1} (\psi - e)d\mu + \int_{R \cap B} (\psi - e)d\mu + \int_{R_2} (\psi - e)d\mu = 0.$$

Thus, the coalition $E := R_1 \cup (R \cap B) \cup R_2$ ex ante blocks $f$ via $\psi$, which is a contradiction.

**Proof of Theorem 3.12** Suppose that the hypothesis of the theorem is true. The rest of the proof is completed by considering the following two cases:

*Case 1.* $\mu(S_1 \cup S_2) = \mu(T)$. This case can be done analogous to Case 1 in the proof of Theorem 3.9.

*Case 2.* $\mu(S_1 \cup S_2) < \mu(T)$. By Lemma 6.1, there exist a $\lambda \in (0, 1)$ and an $\eta > 0$ such that for all $z \in \bigcap\{\eta\mathscr{G}_i : 1 \leq i \leq n\} \cap \mathbb{B}(0, \eta)^\Omega$, there is an assignment $y^z$ such that $y^z(t, \cdot) - e(t, \cdot) \in 4\mathscr{H}_t$ and $V_t(y^z(t, \cdot)) > V_t(f(t, \cdot))$ $\mu$-a.e. on $S_1$, and

$$\int_{S_1} (y^z - e)d\mu + z = (1 - \lambda)\int_{S_1} (h - e)d\mu. \tag{6.3}$$

By the Lyapunov convexity theorem, there are coalitions $B_1 \subseteq S_2$ and $B_2 \subseteq T \backslash (S_1 \cup S_2)$ such that

$$\int_{B_1} (f-e) d\mu = \frac{\lambda}{2} \int_{S_2} (f-e) d\mu$$

and

$$\int_{B_2} (f-e) d\mu = \frac{1}{2} \int_{T \backslash (S_1 \cup S_2)} (f-e) d\mu.$$

Let $B := B_1 \cup B_2$. Applying an argument similar to that in the proof of Theorem 3.4, one can show that there exists an element

$$c \in \bigcap \left\{ \frac{\eta}{2} \mathscr{H}_i : 1 \leq i \leq n \right\} \cap \mathbb{B} \left( 0, \frac{\eta}{2} \right)^{\Omega}$$

such that $c = \int_B \xi d\mu$, where

(i) $\xi(t) \in \bigcap \left\{ \frac{\eta}{2} \mathscr{H}_i : 1 \leq i \leq n \right\} \cap \mathbb{B} \left( 0, \frac{\eta}{2} \right)^{\Omega}$;

(ii) $f(t, \cdot) + \xi(t) \in X_t$; and

(iii) $V_t(f(t, \cdot) + \xi(t)) > V_t(f(t, \cdot))$

for all $t \in B$. Define

$$c_0 := 2c \in \bigcap \{ \eta \mathscr{H}_i : 1 \leq i \leq n \} \cap \mathbb{B} \left( 0, \eta \right)^{\Omega}.$$

Thus, by Equation (6.3), there exists an assignment $y^{c_0}$ such that $y^{c_0}(t, \cdot) - e(t, \cdot) \in 4\mathscr{H}_t$ and $V_t(y^{c_0}(t, \cdot)) > V_t(f(t, \cdot))$ $\mu$-a.e. on $S_1$, and

$$\int_{S_1} (y^{c_0} - e) d\mu + c_0 = (1 - \lambda) \int_{S_1} (h - e) d\mu = (1 - \lambda) \int_{S_2} (f - e) d\mu.$$

Thus, one has

$$\int_{S_1} (y^{c_0} - e) d\mu + \lambda \int_{S_2} (f - e) d\mu + \int_{T \backslash S_2} (f - e) d\mu + c_0 = 0.$$

Hence, one can find an assignment $\varphi$ such that $\varphi(t, \cdot) - e(t, \cdot) \in 4\mathscr{H}_t$ and $V_t(\varphi(t, \cdot)) > V_t(f(t, \cdot))$ $\mu$-a.e. on $S_1$, and

$$\int_{S_1} (\varphi - e) d\mu = \frac{1}{2} \int_{S_1} (y^{c_0} - e) d\mu + \frac{1}{2} \int_{S_1} (f - e) d\mu.$$

As a result, one obtains

$$\int_{S_1} (\varphi - e) d\mu + \int_B (f - e) d\mu + c = 0,$$

which is equivalent to

$$\int_{S_1} (\varphi - e)d\mu + \int_B (f + \xi - e)d\mu = 0,$$

Thus, the coalition $R := S_1 \cup B$ ex ante blocks $f$ via $\psi$, defined by

$$\psi(t, \omega) := \begin{cases} \varphi(t, \omega), & \text{if } (t, \omega) \in S_1 \times \Omega; \\ f(t, \omega) + \xi(t, \cdot), & \text{if } (t, \omega) \in B \times \Omega; \\ f(t, \omega), & \text{otherwise.} \end{cases}$$

Since $\psi(t, \cdot) - e(t, \cdot) \in 4\mathscr{H}_t$ $\mu$-a.e. on $T$, one arrives at a contradiction. $\square$

**Proof of Theorem 4.4** Suppose that $g$ is a feasible assignment in $\mathscr{E}(f)$ not belonging to the ex post core of $\mathscr{E}(f)$. Then there are some state $\omega_0$, coalition $S$ and assignment $h$ in $\mathscr{E}(f; \omega_0)$ such that $u_t(\omega_0, h(t)) > u_t(\omega_0, g(t, \omega_0))$ $\mu$-a.e. on $S$, and

$$\int_S h d\mu = \int_S f(\cdot, \omega_0)d\mu.$$

Since Corollary 3.5 is valid when $\Omega = \{\omega_0\}$ and $\mathscr{G}_t = \mathbb{R}^\ell$ for all $t \in T$, we can conclude the following: for any $0 < \varepsilon < \mu(T)$, there must exist some coalition $R$ and assignment $y$ in $\mathscr{E}(f; \omega_0)$ such that $u_t(\omega_0, y(t)) > u_t(\omega_0, g(t, \omega_0))$ $\mu$-a.e. on $R$, and

$$\int_R y d\mu = \int_R f(\cdot, \omega_0)d\mu.$$

This completes the proof. $\square$

**Proof of Theorem 4.11** Suppose that $g \in \mathscr{C}(\mathscr{E}; f)$. Without loss of generality, suppose that $g$ is not ex post $\mathscr{C}_{(\mathscr{T}_0, \mathscr{T}_1)}(\mathscr{E}; f)$-fair. Thus, there exist a state $\omega_0$, two disjoint elements $S_1 \in \mathscr{T}_0$, $S_2 \in \mathscr{T}_1$ and an assignment $h$ in $\mathscr{E}(f; \omega_0)$ such that $u_t(\omega_0, h(t)) > u_t(\omega_0, g(t, \omega_0))$ $\mu$-a.e. on $S_1$, and

$$\int_{S_1} (h - f(\cdot, \omega_0))d\mu = \int_{S_2} (g(\cdot, \omega_0) - f(\cdot, \omega_0))d\mu.$$

Since Corollary 3.5 is valid when $\Omega = \{\omega_0\}$ and $\mathscr{G}_t = \mathbb{R}^\ell$ for all $t \in T$, we can conclude that $g$ is not in the core of $\mathscr{E}(f; \omega_0)$, which is is a contradiction. This completes the proof. $\square$

# 7 Appendix B

The additional material contained in this appendix allows us to apply our main results to asset markets which are not necessarily complete. The proofs reproduce those of previous theorems and are omitted.

Assume that $X_t(\omega) = \mathbb{R}^\ell$ for all $(t, \omega) \in T \times \Omega$. Thus, $(\mathbf{A}_1)$ and $(\mathbf{A}_2)$ are satisfied trivially. Let $\pi : \Omega \times \mathbb{R}^m \to \mathbb{R}^\ell$ be a function such that $\pi(\omega, \cdot)$ is linear, $\pi(\omega, x) \in \mathbb{R}^\ell_+ \setminus \{0\}$ for all $x \in \mathbb{R}^m_{++}$, and $\pi(\omega, 0) = 0$. Define

$$\mathscr{H}_t = \mathscr{G}_t = \{(\pi(\omega, x(\omega)) : \omega \in \Omega) : x(\omega) \in \mathbb{R}^m \text{ for all } \omega \in \Omega\}$$

for all $t \in T$. It is evident that $(\mathbf{A}_8)$ and $(\mathbf{A}_9)$ are satisfied. Let $a : T \times \Omega \to \mathbb{R}^m$ and $b : T \times \Omega \to \mathbb{R}^m$ be such that $a(\cdot, \omega)$ and $b(\cdot, \omega)$ are Lebesgue integrable for all $\omega \in \Omega$,

$$f(t, \omega) - e(t, \omega) = \pi(\omega, a(t, \omega)) \text{ and } h(t, \omega) - e(t, \omega) = \pi(\omega, b(t, \omega)).$$

The following fact is a consequence of the construction of the assignment $y^z$ in the proof of Lemma 6.1: there are some $0 < \lambda < 1$ and $0 < \eta < 1$ such that for all $d \in \mathbb{R}^m_{++} \cap \mathbb{S}(0, \eta)$[14], there is some $c^d : T \times \Omega \to \mathbb{R}^m$ such that $c^d(\cdot, \omega)$ is Lebesgue integrable function for all $\omega \in \Omega$ and $y^z(t, \omega) - e(t, \omega) = \pi(\omega, c^d(t, \omega))$ for all $(t, \omega) \in T \times \Omega$, and

$$\int_S c^d(\cdot, \omega) d\mu + d = (1 - \lambda) \int_S b(\cdot, \omega) d\mu \tag{7.1}$$

for all $\omega \in \Omega$. On the other hand, defining in the proof of Proposition 6.2 the correspondence $\Gamma_f : R_1 \rightrightarrows (\mathbb{R}^\ell)^\Omega$ as

$$\Gamma_f(t) := \left\{x \in (\mathbb{R}^m)^\Omega : V_t(\pi(\cdot, x(\cdot)) + e(t, \cdot)) > V_t(f(t, \cdot))\right\}$$

and applying similar arguments, one can find some sub-coalition $R$ of $S$ and $0 < \eta < 1$ such that for all $d \in \mathbb{R}^m_{++} \cap \mathbb{S}(0, \eta)$, there exists a function $\xi^d : T \times \Omega \to \mathbb{R}^m$ such that $\xi^d(\cdot, \omega)$ is Lebesgue integrable function for all $\omega \in \Omega$ and $\xi^z(t, \omega) - e(t, \omega) = \pi(\omega, \xi^d(t, \omega))$ for all $(t, \omega) \in T \times \Omega$, and

$$\int_R \xi^d(\cdot, \omega) d\mu + d(\omega) = \frac{1}{2} \int_S b(\cdot, \omega) d\mu \tag{7.2}$$

for all $\omega \in \Omega$.

In the light of above remarks and equations, one can obtain the following results.

[14] $\mathbb{S}(0, \eta)$ denoted the open ball centered at 0 and radius $\eta$.

**Theorem 7.1.** *Assume that the economy $\mathscr{E}$ is atomless (that is, $T_1 = \emptyset$). Let $a : T \times \Omega \to \mathbb{R}^m$ and $b : T \times \Omega \to \mathbb{R}^m$ be such that $a(\cdot, \omega)$ and $b(\cdot, \omega)$ are Lebesgue integrable for all $\omega \in \Omega$,*

$$f(t, \omega) - e(t, \omega) = \pi(\omega, a(t, \omega)) \text{ and } h(t, \omega) - e(t, \omega) = \pi(\omega, b(t, \omega)).$$

*Suppose further that there is a coalition $S$ such that $V_t(h(t, \cdot)) > V_t(f(t, \cdot))$ $\mu$-a.e. on $S$, $\int_T a(\cdot, \omega) d\mu = 0$ and $\int_S b(\cdot, \omega) d\mu = 0$ for all $\omega \in \Omega$. Suppose that $f$ is not in the ex ante core. Under $(\mathbf{A}_3)$-$(\mathbf{A}_7)$, for any given $0 < \varepsilon < \mu(T)$, there is a coalition $R$, a function $c : T \times \Omega \to \mathbb{R}^m$ and an assignment $y$ such that $\mu(R) = \varepsilon$ and $f$ is ex ante blocked by $R$ via $y$; and $y(t, \cdot) - e(t, \cdot) = \pi(\omega, c(t, \omega))$ for all $(t, \omega) \in T \times \Omega$. Moreover, if $a(t, \cdot) : \Omega \to \mathbb{R}^m$ and $b(t, \cdot) : \Omega \to \mathbb{R}^m$ are constant $\mu$-a.e. on $T$ then $c(t, \cdot) : \Omega \to \mathbb{R}^m$ can be chosen to be constant $\mu$-a.e. on $T$, and $\int_R c(\cdot, \omega) d\mu = 0$.*

**Theorem 7.2.** *Suppose that $a : T \times \Omega \to \mathbb{R}^m$ is a function such that $a(t, \cdot)$ is constant $\mu$-a.e. on $T$, $\int_T a(\cdot, \omega) d\mu = 0$ for all $\omega \in \Omega$, and*

$$f(t, \omega) - e(t, \omega) = \pi(\omega, a(t, \omega)).$$

*Let there be no coalition $S$ and function $b : T \times \Omega \to \mathbb{R}^m$ such that $b(t, \cdot)$ is constant $\mu$-a.e. on $T$, $\int_S b(\cdot, \omega) d\mu = 0$ for all $\omega \in \Omega$ and*

$$V_t(e(t, \cdot) + \pi(\omega, b(t, \omega))) > V_t(f(t, \cdot))$$

*$\mu$-a.e. on $S$. Under $(\mathbf{A}_3)$-$(\mathbf{A}_7)$, there do not exist two disjoint elements $S_1 \in \mathscr{T}_0$ and $S_2 \in \mathscr{T}_1$, and a function $c : T \times \Omega \to \mathbb{R}^m$ such that $\mu$-a.e. on $S_1$ and for each $\omega \in \Omega$:*

(i) *$c(t, \cdot)$ is constant;*

(ii) *$V_t(e(t, \cdot) + \pi(\omega, c(t, \omega))) > V_t(f(t, \cdot))$;*

(iii) *$\int_{S_1} c(\cdot, \omega) d\mu = \int_{S_2} a(\cdot, \omega) d\mu$.*

**Theorem 7.3.** *Suppose that $a : T \times \Omega \to \mathbb{R}^m$ is a function such that $a(t, \cdot)$ is constant $\mu$-a.e. on $T$, $\int_T a(\cdot, \omega) d\mu = 0$ for all $\omega \in \Omega$, and*

$$f(t, \omega) - e(t, \omega) = \pi(\omega, a(t, \omega)).$$

*Let there be no coalition $S$ and function $b : T \times \Omega \to \mathbb{R}^m$ such that $b(t, \cdot)$ is constant $\mu$-a.e. on $T$, $\int_S b(\cdot, \omega) d\mu = 0$ for all $\omega \in \Omega$ and*

$$V_t(e(t, \cdot) + \pi(\omega, b(t, \omega))) > V_t(f(t, \cdot))$$

*$\mu$-a.e. on $S$. Under $(\mathbf{A}_3)$-$(\mathbf{A}_7)$, there do not exist two disjoint elements $S_1 \in \mathscr{T}_1$ and $S_2 \in \mathscr{T}_0$, and a function $c : T \times \Omega \to \mathbb{R}^m$ such that $\mu$-a.e. on $S_1$ and for each $\omega \in \Omega$:*

(i) *$c(t, \cdot)$ is constant;*

(ii) *$V_t(e(t, \cdot) + \pi(\omega, c(t, \omega))) > V_t(f(t, \cdot))$;*

(iii) *$\int_{S_1} c(\cdot, \omega) d\mu = \int_{S_2} a(\cdot, \omega) d\mu$.*

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